FOUR-MANIFOLDS **LENT 2009**

EXAMPLE SHEET 3

1. Let $\Gamma = D_{12} \cup (D_{12} + (1/2, \ldots, 1/2))$ be the odd unimodular lattice of problem 1, Example Sheet 1. Find a characteristic vector for Γ with length less than 12.

2. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be given by $f(z_1, z_2) = z_1^2 + z_2^2$. For $w \in \mathbb{C}$ let $F_w = f^{-1}(w)$. Show that F_w is homeomorphic to $S^1 \times \mathbb{R}$ for $w \neq 0$, and that F_0 is homeomorphic to a cone. For $\epsilon \in \mathbb{R}^+$, describe what happens as $\epsilon \to 0$.

3. For the genus two Heegaard diagrams shown in Figure 1a and b, find all generators of $\widehat{CF}$. Divide them into equivalence classes according to which $Spin^c$ structure they represent. Determine the relative homological grading in each equivalence class and compute $\widehat{HF}$.

4. The Heegaard diagram (Σ, α, β) shown in Figure 2 represents the complement of the trefoil knot in S^3. To be specific, start with $\Sigma \times [-1, 1]$ and attach disks to each of the α circles in $\Sigma \times -1$ and to the β in $\Sigma \times 1$. The result is a 3-manifold whose top boundary is T^2 and whose lower boundary is S^2. Filling in the lower boundary with B^3 gives $S^3 - T$.

   (a) Draw curves in $\Sigma - \beta$ representing the longitude and the meridian of T. (Hint: to find the meridian, look for a curve in $\Sigma - \beta$ which when you attach a 2 handle along it, gives a manifold with trivial π_1.)

   (b) Explain how to draw a Heegaard diagram representing n surgery on T.

   (c) When $n \gg 0$, show that most equivalence classes of generators contain exactly three elements.

   (d) Describe the differentials in a generic equivalence class. How do they change as we vary the basepoint?

5. Let $U \subset S^3$ be the unknot, and consider the HF^- exact triangle for 1-surgery on U. Each map in this triangle is induced by a cobordism. Represent each cobordism by a genus 1 triple Heegaard diagram, and work out the induced maps on HF^- explicitly from this diagram. (Remember to keep track of $Spin^c$ structures!)

6. Describe the HF^- exact triangle for n surgery on U, where $n > 0$. Two of the maps in this triangle are induced by cobordisms. Describe the set of of $Spin^c$ structures on these cobordisms, and the map induced by each $Spinc^c$ structure.

7. Suppose $\Sigma \subset M$ is a smoothly embedded surface in a closed 4-manifold, that $\Sigma \cdot \Sigma = k \geq 0$, and that $\Phi(M, \mathfrak{s}) \neq 0$. By considering $M \#^k \overline{\mathbb{CP}}$, show that $|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + k \leq 2g(\Sigma) - 2$.

8. Suppose M is a closed 4-manifold with $b_2^+ \geq 3$. Use the absolute grading together with the results of the previous exercise to show that there are only finitely many $Spin^c$ structures $\mathfrak{s}$ on M for which $\Phi(M, \mathfrak{s} \neq 0$.

9. If $b_1(Y) = 0$, then $HF^-(Y, \mathfrak{s}) \cong R_+ \oplus T$, where T is a torsion module over $\mathbb{Z}[U]$ and $R_+ = \mathbb{Z}[[U, U^{-1}]]/\mathbb{Z}[[U]]$. Define $d(Y, \mathfrak{s})$ to be the absolute grading of $U^{-1} \in R_+$ in this decomposition. (This makes $d(S^3) = 0$.)

   (a) Use exercise 6 to compute $d(L(p,1), \mathfrak{s})$. Conclude that for $p > 3$, $L(p,1)$ does not have an orientation reversing diffeomorphism.

   (b) Suppose Y_1, Y_2 are homology spheres and that $W : Y_1 \to Y_2$ is a homology cobordism (*i.e.* the inclusion maps $\iota_{i*} : H_*(Y_i) \to H_*(W)$ are isomorphisms.) Show that $d(Y_1) = d(Y_2)$.

10. Write $S^3 = \{(z, w) \mid |z|^2 + |w|^2 = 2\}$. For relatively prime integers p and q, let

$$T(p,q) = \{(e^{ip\theta}, e^{iq\theta}) \mid (\theta \in [0, 2\pi]\}$$

be the (p,q) torus knot. (So called because it lies on the torus $|z| = |w| = 1$.) Let C_1 and C_2 be the circles where $|z| = 0$, $|w| = 0$ and write $X = S^3 - T(p,q) - C_1 - C_2$. Show that there is a map from X to S^2- three points with fibre S^1. If m and ℓ are the meridian and longitude of $T(p,q)$, show that the homology class of the fibre is $pqm + \ell$. Conclude that $pq \pm 1$ surgery on $T(p,q)$ is a lens space.

11. Work out all the groups and maps in the exact triangle for n-surgery on the right-handed trefoil. ($T(2,3)$ in the notation of the previous problem.) This can be done either using exercise 4, or exercise 8. For the latter, note that $+5$ surgery on $T(2,3)$ is the lens space $L(5,4) = -L(5,1)$, so $d(L(5,4), \mathfrak{s}) = -d(L(5,1), \mathfrak{s})$. (If T_0 is the result of 0-surgery on the trefoil, you should find that $HF^-(T_0) \cong R_+ \oplus R_+$, where the lowest elements in each R_+ have grading $-1/2$ and $-3/2$ respectively. With twisted coefficients $T = a \neq 1 \in \mathbb{C}$, $HF^+(T_0) \cong \mathbb{Z}$ in absolute grading $-1/2$.)

12. Let B be the 3 component link shown in Figure 3 (aka the Borromean rings.) Let $B(p,q,r)$ be the manifold obtained by doing p, q, and r surgery on the three components of B. Show that $B(0,0,0) = T^3$ and that $B(1,1,n)$ is n-surgery on the trefoil. By using the exact triangle repeatedly, shown that $\widehat{HF}(B(1,0,0)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2$ and $\widehat{HF}((B(0,0,0)) \cong \mathbb{Z}^3 \oplus \mathbb{Z}^3$. (Hint: Use the previous problem and keep track of absolute gradings.)

13. Rational Blowdown

(a) Let H be a generator of $H_2(\mathbb{CP}^2)$. Show that $2H$ is represented by an embedded sphere of self-intersection 4. Let N be a tubular neighborhood of this sphere, and let $W = \mathbb{CP}^2 - N$. Show that $H_*(W;\mathbb{Q}) \cong H_*(B^4;\mathbb{Q})$. W is an example of a *rational homology ball.* Compute the long exact sequence of the pair $(W, \partial W)$ with $\mathbb{Z}$ coefficients.

(b) Show that there are exactly two $Spin^c$ structures on W and that they are exchanged by the conjugation symmetry.

(c) Suppose M is a closed 4-manifold containing a sphere Σ of self-intersection -4, and let N' be a tubular neighborhood of Σ. Then we can form the closed four-manifold $M' = (M - N') \cup W$. Show that if $\mathfrak{s}$ is a $Spin^c$ structure on W with $\langle c_1(\mathfrak{s}), [\Sigma]\rangle = \pm 2$, then $\mathfrak{s}|_{M-N'}$ extends to a unique $Spin^c$ structure $\mathfrak{s}'$ on M' and $\Phi(M', \mathfrak{s}') = \Phi(M, \mathfrak{s})$.

Problems 12 and 13 are not examinable; the rest are.

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