EXAMPLE SHEET 2

1. Let Y be an orientable 3-manifold with boundary, and let m be the rank of $H_1(\partial Y)$. Use Poincare duality and the long exact sequence of the pair $(Y, \partial Y)$ to show that the kernel of the map $H_1(\partial Y) \to H_1(Y)$ has rank $m/2$.

2. Suppose K is a null-homologous knot in a three-manifold Y. Show that $H_1(Y - K) \cong H_1(Y) \oplus \mathbb{Z}$.

3. Suppose C_* is a finitely generated chain complex defined over the ring $\mathbb{Q}[U]$, where U has homological degree -2.
   (a) Consider C_* as an (infinitely generated) complex defined over $\mathbb{Q}$. Show that the dual (over $\mathbb{Q}$) of C_* is C^+.
   (b) Show that $H_*(C)$ is a direct sum of copies of $\mathbb{Q}[U]$ and $\mathbb{Q}[U]/(U^{n_i})$. (Use the structure theorem for modules over a PID.)
   (c) Give an example of a chain complex C with homology $\mathbb{Q}[U]/(U^n)$. Compute the homology of the chain complexes $\widehat{C}$, C^-, C^+, and C^∞. Write out the long exact sequences discussed in class relating these homology groups.
   (d) Explain how this is formally similar to the homology of the Hopf bundle over $\mathbb{CP}^n$.

4. Let C be the complex defined over $\mathbb{Q}[T, T^{-1}]$ and generated by two elements x and y, with $dx = (T-1)y$ and $dy = 0$. Compute the homology of C as a $\mathbb{Q}[T]$ module. Let $M_a = \mathbb{Q}[T, T^{-1}]/(T-a)$. Compute the homology of the complex $C \otimes_{\mathbb{Q}[T]} M_a$ both directly and by using the universal coefficient theorem. (If you've only seen the universal coefficient theorem for complexes defined over $\mathbb{Z}$, note that $\mathbb{Q}[T, T^{-1}]$, like $\mathbb{Z}$, is a PID. The statement works (with obvious changes) over any PID.)

5. Let $K \subset Y$ be a knot, and let $K(\lambda)$ be the result of integer surgery on K along a longitude λ, and let m be a meridian of K, pushed out far enough so that it is on the boundary of the solid torus we remove to do the surgery. Thus we can view m as a curve on $K(\lambda)$.
   (a) Let $K' = S^1 \times 0 \subset S^1 \times D^2$ be the core of the new torus we glued in to make $K(n)$. Show that m is isotopic to K'. Explain why this is not the case if we replace $K(\lambda)$ with an arbitrary, nonintegral surgery $K(\alpha)$.

(b) Let m' and ℓ' be the meridian and longitude of m in K. (ℓ' is the boundary of a little disk intersecting K.) If we identify $K(\lambda) - m$ with $Y - K$ in the obvious way, show that $m' = \lambda$, $\ell' = -m$.

(c) Now suppose that K is null-homologous. If we do n-surgery on K, followed by p/q surgery on m (with respect to the basis m', ℓ'), show that the resulting three-manifold is $K_{n-\frac{q}{p}}$.

6. The result of p/q surgery on the unknot is called the *lens space* $L(p, -q)$.

   (a) Show that $L(p, -q) = S^1 \times D^2 \cup_f S^1 \times D^2$, where the diffeomorphism f takes ∂D^2 in the second solid torus to $p[S^1] + q\partial[D^2]$ on the boundary of the first torus.

   (b) Compute $\pi_1(L(p, q))$ and $H_*(L(p, -q))$. Show that the universal cover of $L(p, -q)$ is S^3. (Unless $p/q = 0/q$. What happens then?)

   (c) Draw a Heegaard diagram of $L(p - q)$. What is $\widehat{HF}(L(p, -q))$?

   (d) Show that the manifold obtained by surgery on a chain of spheres (as in Figure 2) is a lens space. What are p and q for the chain shown in the figure?

7. Write down the complex $CF^-(H, z_i) \otimes (\mathbb{Z}/2)$ for the Heegaard splitting H shown in Figure 1 and for $i = 1, 2, 3$. In each case, check that $d^2 = 0$ and that the homology is $HF^-(S^3)$.

8. Assuming the existence of handle decompositions for manifolds, give a proof of Poincare duality, as follows.

   (a) Suppose the manifold W^m is built out of n_i m-dimensional i-handles $(1 \leq i \leq m.)$ Show that W is homotopy equivalent to a CW complex built out of n_i m-dimensional cells.

   (b) If $X = D^i \times D^{m-i}$ is an i-handle, let $S_A(X) = \partial D^i \times 0$ and $S_B(X) = 0 \times \partial D^{m-i}$ be its *attaching* and *belt* spheres. Suppose X_i and X_{i+1} are an i-handle and an $i+1$-handle, and let Y_i and Y_{i+1} are the associated cells in the CW decomposition. Show that in the cellular chain complex, the coefficient of Y_i in dY_{i+1} is given by $\pm S_A(X_{i+1}) \cdot S_B(X_i)$. (Hint: how do we compute the degree of a map $S^n \to S^n$?)

   (c) By reversing the roles of ∂_1 and ∂_2, show that W has a handle decomposition with n_{m-i} i-handles.

   (d) Show that the complex associated to this decomposition is the dual of the original cellular complex.

   (e) Explain how the same principle can be used to prove Poincare duality for manifolds with boundary.

9. More about the Kummer surface

(a) Let $\iota : \mathbb{R}^2 \to \mathbb{R}^2$ be the involution which sends x to $-x$. Show that ι descends to a map $T^2 \to T^2$. How many fixed points does this map have? Show that $T^2/(\iota(x) \sim x)$ is homeomorphic to S^2.

(b) Show that the Kummer surface (as defined in class) has a map to S^2 whose generic fibre is T^2.

(c) Show that there are four special fibres, each of which is consists of 5 spheres $F_1, F_2, \ldots F_5$ of self intersection -2, and that $F_i \cdot F_j = 1$ if $i = 1, j > 1$ and is 0 if $i, j > 1$. Express the homology class of the generic fibre in terms of these spheres.

(d) Use this to compute explicitly compute the intersection form on the Kummer surface.

J.Rasmussen@dpmms.cam.ac.uk