EXAMPLE SHEET 1

1. Show that any orientable two-manifold has an orientation reversing diffeomorphism.

2. (a) Let $V \subset \mathbb{R}^4$ be the lattice $D_4 \cup (D_4 + v)$, where $v = (1/2, 1/2, 1/2, 1/2)$. Let Q be the quadratic form on V defined by $Q(x, x) = x \cdot x$. Show that $Q \simeq 4\,(1)$.
   (b) Show there is a positive definite unimodular form on $\mathbb{Z}^{12}$ which is not isomorphic to $12\,(1)$.

3. Show that $(S^2 \times S^2)\#\overline{\mathbb{CP}}^2$ is diffeomorphic to $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. (It might help to understand the isomorphism of intersection forms first.)

4. Let D be the unit disk in $\mathbb{C}$, and let $X = D \times 0 \cup 0 \times D \subset D \times D$. Show that X is homologous to a smoothly embedded annulus A with $A \cap \partial(D \times D) = X \cap \partial(D \times D)$. (Hint: replace $zw = 0$ with $zw = \epsilon$.)

5. Let E_n be the complex line bundle over S^2 with Euler number n, and let D_n be its unit disk bundle. Find all the groups and maps in the long exact sequence on homology for the pair $(D_n, \partial D_n)$. Show that the universal cover of ∂D_n is S^3. Describe the action of the group of deck transformations.

6. Compute the genus of a smooth algebraic curve of bidegree (m, n) (*i.e.* representing the homology class (m, n) with respect to the standard basis $\{[\mathbb{CP}^1] \times a, a \times [\mathbb{CP}^1]\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

7. Show that there are two isomorphism classes of three-dimensional real vector bundles over S^2. Show that the unit sphere bundle of the nontrivial bundle is diffeomorphic to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.

8. Show that n dimensional real vector bundles over S^m are classified by $\pi_{m-1}(O(n))$. Show that $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Explain why this makes sense in terms of characteristic classes. Find two four-dimensional vector bundles over S^4 whose images in $\mathbb{Z} \oplus \mathbb{Z}$ are linearly independent. Conclude that there are infinitely many vector bundles over S^4 with Euler number 1. This is how Milnor constructed the first exotic 7-spheres. (NB: There are only finitely many diffeomorphism classes of manifolds homeomorphic to S^7. Different vector bundles can have diffeomorphic unit sphere bundles.)

9. Show that a smooth degree 2 hypersurface in $\mathbb{CP}^3$ is diffeomorphic to $S^2 \times S^2$, and that a smooth cubic hypersurface is diffeomorphic to $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$.

10. What is the intersection form of a smooth hypersurface of degree 5 in $\mathbb{CP}^3$? Same question for degree 6.

11. (a) Let Q be a unimodular quadratic form on a real vector space V. Suppose that V has a subspace L whose dimension is half that of V and that $Q(x, y) = 0$ for all $x, y \in L$. Show that $\sigma(Q) = 0$.

    (b) Suppose that M is a $4k$-manifold which bounds a $(4k+1)$-manifold W. Use the long exact sequence of the pair (W, M) to show that $\sigma(M) = 0$.

12. Let $K \subset S^3$ be a smoothly embedded knot with tubular neighborhood U, and let $Y = S^3 - U$ be its complement. Compute all groups and maps in the long exact sequence for the homology of the pair $(Y, \partial Y)$. Show that up to isotopy there is a unique embedded curve ℓ on ∂Y which bounds in Y. This curve is called the *longitude* of K.

13. Let $X = S^1 \times D^2$, and let $a = S^1 \times 1$ and $b = 1 \times \partial D^2$ be curves on $\partial X = T^2$. Show that a diffeomorphism $f : \partial X \to \partial X$ extends to X if and only if $f_*([b]) = [b]$. (If you like, you may assume f is given by a linear map on $\mathbb{R}^2$.)

14. In the notation of problems 7 and 8, let $M = Y \cup_f X$, where $f : \partial X \to \partial Y$ is an orientation reversing diffeomorphism with $f_*([b]) = p[m] + q[\ell]$. Show that if f' is any other such f, the resulting manifold M' is diffeomorphic to M. (M is called p/q surgery on K.) Compute $H_*(M)$.

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