Notes on Example Sheet 1

5. $i : S^1 \times D^2 \to S^3$ is an embedding. Let $U_1 = \text{int } i(S^1 \times D^2_{1-\epsilon})$ and let $U_2 = S^3 - i(S^1 \times D_{1-2\epsilon})$. Note that U_2 deformation retracts to $S^3 - \text{int } i(S^1 \times D) = S^3 - U$ We'd like to show that the boundary map

$$\partial : H_3(S^3) \to H_2(U_1 \cap U_2) \simeq H_2(T^2)$$

appearing in the Mayer-Vietoris sequence is an isomorphism. To do this consider the map of pairs $I : (S^3, \emptyset) \to (S^3, S^3 - U)$. By excision,

$$H_3(S^3, S^3 - U) \simeq H_3(S^1 \times D^2, T^2) \simeq \mathbb{Z}.$$

We claim that $i_* : H_3(S^3) \to H_3(S^3, S^3 - U)$ is an isomorphism. To see this, consider the further inclusion $j : (S^3, S^3 - U) \to (S^3, S^3 - p)$. The composition $(j \circ i)_*$ is an isomorphism, so i_* must be an isomorphism as well. We have a commutative diagram of Mayer-Vietoris sequences:

$$\begin{array}{ccccccc}
H_2(U_1) \oplus H_3(U_2) & \longrightarrow & H_3(S^3) & \longrightarrow & H_2(U_1 \cap U_2) & \longrightarrow & H_2(U_1) \oplus H_2(U_2) \\
\downarrow & & {\scriptstyle i_*}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow \\
H_3(U_1) \oplus H_3(U_2, S^3 - U) & \longrightarrow & H_3(S^3, S^3 - U) & \xrightarrow{\partial_U} & H_2(U_1 \cap U_2) & \longrightarrow & H_2(U_1) \oplus H_2(U_2, S^3 - U)
\end{array}$$

(This diagram commutes because we can set up a corresponding commuting diagram of SES's of chain complexes). The maps i_* and 1 are isomorphisms, so it suffices to show ∂_U is an isomorphism as well. Now ∂_U is the same map, independent of the choice of embedding i. So to compute ∂_U, we may as well use a nice choice of i — for example, the one from problem 6, where we wrote $S^3 = S^1 \times D^2 \cup^2_T D^2 \times S^1$. In this case $U_2 \simeq S^1 \times D^2$, and it is easy to see that ∂ is an isomorphism.

12. For the last part, we first prove

Lemma 1. *If C is a free finitely generated chain complex over $\mathbb{Z}$ and $H(C) = 0$, then $1_C \sim 0$.*

Proof. We must construct $H : C_* \to C_{*+1}$ with $dH(c) + Hd(c) = c$. Let $A_k = \ker d_k$. $C_k/A_k \simeq \text{im } d_k \subset C_{k-1}$ is free, so we can write $C_k = A_k \oplus B_k$. Since $H_*(C) = 0$, $d_k : B_k \to A_k$ is an isomorphism. We define $H|_{A_k}$ to be the inverse of this map, and $H|_{B_k}$ to be the 0 map. □

Now given $H(M(f)) = 0$, construct $H : M(f) \to M(f)$ as above. In matrix form, write

$$H = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

so $\beta : C' \to C$. Expanding the relation $Hd + dH = 1_M$ in matrix form, we see that β is a chain map and that $\beta \circ f \sim 1_C$ and $f \circ \beta \sim 1'_C$. (The homotopies are given by α and γ.)