Lemma: *If $W : M_0 \to M_1$ is a cobordism, there is a Morse function $f : W \to [0,1]$ such that $f^{-1}(0) = M_0, f^{-1}(1) = M_1$.*

Proof: We first choose an open cover $\mathcal{U} = \{U_1, \ldots, U_k, U'_1, \ldots, U'_l, V\}$ of W satisfying the following conditions:

1. The U_i's cover M_0 and the U_i's cover M_1.
2. $U_i \cap U'_j = \emptyset$ for all i, j.
3. Each U_i is the domain of a chart $\psi_i : U_i \to \mathbb{R}^n \times R^{\geq 0}$.
4. Each U'_i is the domain of a chart $\psi'_i : U_i \to \mathbb{R}^n \times R^{\geq 0}$.
5. $\overline{V} \cap \partial W = \emptyset$.

Let $\lambda : [0, \infty) \to \mathbb{R}$ be a smooth function which satisfies $\lambda(x) = x$ for $x \leq 1/4$ and $\lambda(x) = 1/2$ for $x \geq 1/2$. Define $f_0, f_1 : \mathbb{R}^n \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ by $f_0(\mathbf{x}) = \lambda(x_{n+1})$, and $f_1(\mathbf{x}) = 1 - \lambda(x_{n+1})$.

Now let $g_i = f_0 \circ \psi_i$ and $g'_i = f_1 \circ \psi'_i$. Let $g_V : V \to \mathbb{R}$ be the constant function with value $1/2$. Choose a partition of unity $\{\phi_1, \ldots, \phi_k, \phi'_1, \ldots, \phi'_l, \phi_V\}$ subordinate to $\mathcal{U}$, and let

$$g = \phi_V g_V + \sum_{i=1}^{k} \phi_i g_i + \sum_{i=1}^{l} \phi'_i g'_i.$$

By construction, g is a weighted average of functions whose image is contained in $[0, 1]$, so its image is contained in $[0, 1]$.

Suppose $g(p) = 0$. Then every term in the sum defining it must be 0. Now either $\phi_i(p) \neq 0$ for some i, or $\phi'_i(p) \neq 0$, or $\phi_V(p) \neq 0$. Since f_1 and g_V are both strictly positive, the latter two cases are impossible. Thus $\phi_i(p) \neq 0$. It follows that $g_i(p) = 0$, which implies $p \in M_0$. Thus $g^{-1}(0) = M_0$. A similar argument shows that $g^{-1}(1) = M_1$.

Suppose $p \in M_1$, and $\mathbf{v} \in T_p(W)$ points into W. Then $dg'_i(\mathbf{v}) < 0$ whenever $p \in U'_i$. Note that $\sum_{i=1}^{l} \phi'_i(q) \equiv 1$ for all q in an open neighborhood of p, so $\sum_{i=1}^{l} d\phi'_i|_p = 0$. Thus

$$\begin{aligned} dg(v) &= \sum_{i=1}^{l} (\phi'_i(p) dg'_i(\mathbf{v}) + g_i(p) d\phi'_i(\mathbf{v})) \\ &= \sum_{i=1}^{l} \phi'_i(p) dg'_i(\mathbf{v}) < 0. \end{aligned}$$

A similar argument shows that $dg \neq 0$ on M_0. By compactness of ∂W, we conclude that there is an $\epsilon > 0$ and an open set $U \subset M$ containing ∂W such that $|\nabla g| > \epsilon$ on U.

We now perturb g to obtain a Morse function f. Embed W into R^N for some $N \gg 0$. Then as shown in class, given $\delta > 0$ we can find a linear function $L : \mathbb{R}^N \to \mathbb{R}$ such that $|L|, |\nabla L| < \delta$ on W and $g + L$ is Morse. Consider perturbations of the form $f = g + \rho L$, where $\rho : W \to [0, 1]$ is a function such that $\rho \equiv 1$ on a compact set K containing the complement of U, and $\rho \equiv 0$ on an open set V containing ∂W. By choosing δ small enough, we can arrange that f is Morse on K and that $df \neq 0$ on $W - K$. Thus f is Morse. By choosing δ sufficiently small we can also ensure that $f^{-1}(1) = g^{-1}(1) = M_1$, $f^{-1}(0) = g^{-1}(0) = M_0$, and im $f = [0, 1]$.