EXAMPLE SHEET 3

1. Suppose Y is a 3-manifold with $\partial Y \simeq T^2$ and $b_1(Y) = 1$, and let $\widetilde{Y}$ be the map corresponding to the map $\pi_1(Y) \to H_1(Y)^{fr}$. Choose a generalized Heegaard decomposition of Y with n 2–handles, $n+1$ 1–handles, and 1 0–handle, and let $C_*^{cell}(\widetilde{Y})$ be the cellular chain complex. Let A and B be the matrices of the boundary operators d_2 and d_1 in this complex (so that A is a matrix with $n+1$ rows and n columns.)

   (a) Let A_i be A with the ith row omitted, and let b_i be the ith element of B. Use the relation $BA = 0$ to show that $(\det A_i)/b_i = (\det A_j)/b_j$ whenever $b_i \neq 0 \neq b_j$.

   (b) Show that if $b_i \neq 0$, $T(Y) = (\det A_i)/b_i$.

   (c) Deduce that $T(Y) = \Delta_Y/(t-1)$ for some $\Delta_Y \in \mathbb{Z}[t^{\pm 1}]$. (Hint: choose a handle decomposition in which some 1-handle represents the generator of $H_1(Y)$.) If K is a knot in S^3, then $\Delta_K := \Delta_{S^3-\nu(K)}$ is the *Alexander polynomial* of K.

2. Suppose Y is 3-manifold whose boundary is a disjoint union of two or more tori. Show that $b_1(Y) > 1$. Arguing as in the previous problem, deduce that $T(Y) = \Delta_Y \in \mathbb{Z}[H_1(Y)^{fr}]$. If $L \subset S^3$ is a link, $\Delta_L := T(S^3 - \nu(L))$ is the *multivariable Alexander polynomial* of L.

3. Draw generalized Heegaard diagrams of the knot and link shown in Figure 1. Use them to compute Δ_K and Δ_L.

4. Let Σ be a compact surface with boundary. Show that $T(\Sigma \times S^1) = (t-1)^{\chi(\Sigma)}$.

5. Write $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. Let $K_1 = \{(z,0) \in S^3 \mid |z| = 1\}$, $K_2 = \{(0,w) \in S^3 \mid |w| = 1\}$, and let $T(p,q) = \{(z,w) \in S^3 \mid z^p = w^q\}$ be the (p,q) *torus knot.*

   (a) Sketch the relative position of K_1, K_2 and $T(p,q)$ in S^3.

   (b) Let L be the union of K_1, K_2, and $T(p,q)$. Show that $S^3 - \nu(L) \simeq \Sigma \times S^1$, where Σ is a 3-punctured sphere.

   (c) Use this to show that
   $$\Delta_K = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}.$$

6. Compute the multivariable Alexander polynomial of the link L shown in Figure 2. By doing surgery on the component labeled K, compute the multivariable Alexander polynomial of the link $T(2,2n)$ shown in the figure.

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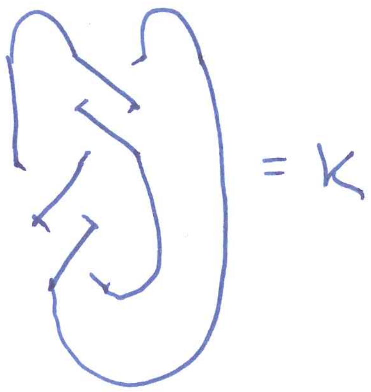
Figure 1



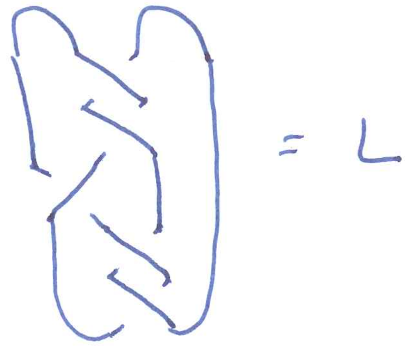


Figure 2