EXAMPLE SHEET 1

1. If M is a topological n-manifold and $p \in M$, show that $H_*(M, M-p) = \mathbb{Z}$ if $* = n$ and is 0 for $* \neq n$. Deduce that if Y is a space obtained by starting with a finite set of tetrahedra and identifying faces in pairs, Y is a topological manifold if and only if $\mathrm{lk}(v) = S^2$ for every vertex $v \in Y$.

2. Identify S^3 with the set $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. Let A and B be the subsets of S^3 where $|z| \geq |w|$ and $|z| \leq |w|$, respectively.
   (a) Use the above decomposition to show that $L(1,0) \simeq S^3$.
   (b) More generally, show that $L(p,q) \simeq S^3/(\mathbb{Z}/p)$ where $1 \in \mathbb{Z}/p$ acts by sending (z,w) to $(\omega z, \omega^q w)$, where ω is a primitive pth root of unity.
   (c) Deduce that $L(p,q) \simeq L(p,p+q) \simeq L(p,q')$, where $qq' \cong 1 \mod p$.
   (d) Show that $L(p,-q) \simeq -L(p,q)$.

3. Suppose that γ_1 and γ_2 are two simple closed curves in a surface S which intersect in a single point. Show that $\tau_{\gamma_1}\tau_{\gamma_2}\tau_{\gamma_1} = \tau_{\gamma_2}\tau_{\gamma_1}\tau_{\gamma_2}$ in $MCG(S)$.

4. If P is the complement of three open balls in S^2, show that $(T^2 - B^2) \times I \simeq P \times I$.

5. Let $L \subset S^3$ be an ℓ component link, and let $Y = S^3 - \nu(L)$. Show that $H_1(Y) = \mathbb{Z}^\ell$ and $H_2(Y) = \mathbb{Z}^{\ell-1}$ and that $H_1(Y)$ is generated by the meridians of L. More generally, if $K \subset Y$ is a null-homologous knot in a closed 3-manifold, show that $H_1(Y - \nu(K)) \cong H_1(Y) \oplus \mathbb{Z}$. What is $H_1(Y - \nu(K))$ if $[K] \neq 0$ in $H_1(Y, \mathbb{Q})$?

6. Let Y be the manifold obtained by doing 0-surgery on each of the three components of the link in Figure 1, which is known as the Borromean rings. Draw a Heegaard diagram of Y, and use it to show that $Y \simeq T^3$. Show that there is a null-homologous knot $B_g \subset \#^{2g}(S^1 \times S^2)$ so that 0 surgery on B_g is $S^1 \times \Sigma_g$.

7. If Y is a compact 3-manifold with boundary, show that $\chi(\partial X) = 2\chi(X)$. Deduce that $\#^n\mathbb{RP}^2$ bounds a compact 3-manifold if and only if n is even.

8. Let K_1 and K_2 be the knots shown in Figure 2. Show that the result of -1 surgery on K_1 is the same as the result of $+1$ surgery on K_2. (Hint: consider the link at the bottom of the figure.)

9. Let $K \subset Y$ be an embedding of the Klein bottle into an orientable 3-manifold Y, and let Z be a closed regular neighborhood of K. Show that $\partial Y \simeq T^2$, and compute the map $i_* : H_*(\partial Z) \to H_1(Z)$. Deduce that $H_1(Y)$ must have order ≥ 4. Show that the Klein bottle embeds in the lens space $L(4,1)$.

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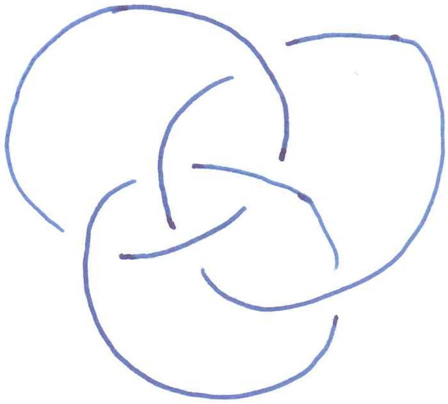
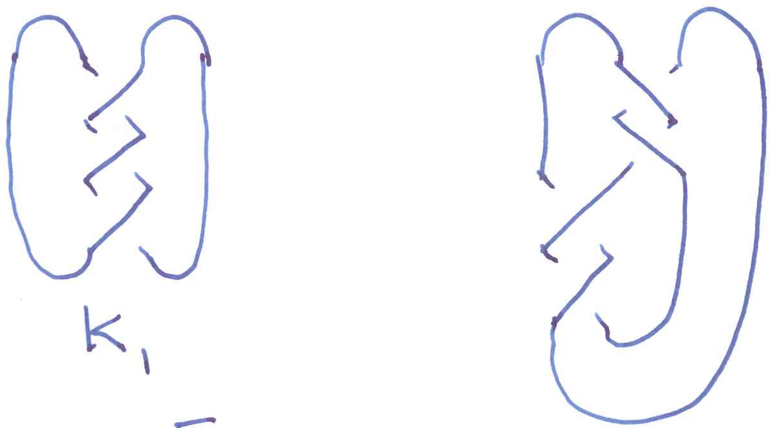


Figure 1



K_1

K_2

Figure 2