ALGEBRAIC TOPOLOGY (PART III) MICHAELMAS 2022

EXAMPLE SHEET 1

Examples Class 1: Tuesday November 1, 3:30-5:30 in MR 2. I will mark problems 6 and 10.

1. Let $\sigma_1, \sigma_2 : [0,1] \to \mathbb{R}$ be given by $\sigma_1(x) = x, \sigma_2(x) = 1 - x$. Identifying $[0,1]$ with Δ^1 gives a cycle $\sigma_1 + \sigma_2$ in $C_1(\mathbb{R})$. Find an $x \in C_2(\mathbb{R})$ with $dx = \sigma_1 + \sigma_2$.

2. If $0 \to H \to G \to \mathbb{Z}^n \to 0$ is an exact sequence of abelian groups, show that $G \simeq H \oplus \mathbb{Z}^n$. What are the possible isomorphism types of G in the exact sequences below?

$$0 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/4 \to 0 \qquad\qquad 0 \to \mathbb{Z} \to G \to \mathbb{Z}/4 \to 0$$

3. If $f : (C, d) \to (C', d')$ is a chain map, the *mapping cone* of f is the chain complex $(M(f), d_f)$ whose underlying group is given by $M(f)_n = C_{n-1} \oplus C'_n$, and whose differential is given by

$$(d_f)_n = \begin{pmatrix} d_{n-1} & 0 \\ (-1)^n f_{n-1} & d'_n \end{pmatrix}.$$

Show that $(M(f), d_f)$ is a chain complex, and that if $f \sim g$, then $M(f) \sim M(g)$. Show that $H_*(M(f)) = 0$ if and only if the map $f_* : H_*(C) \to H_*(C')$ is an isomorphism.

4. If X is a space, the *cone on* X is defined to be $CX := X \times I / X \times 0$. Show that CX is contractible. The *suspension of* X is defined to be $\Sigma X := CX / X \times 1$. Express $\widetilde{H}_*(\Sigma X)$ in terms of $\widetilde{H}_*(X)$. If ΣX is an n-manifold, show that $H_*(\Sigma X) \cong H_*(S^n)$.

5. Let S_g be a surface of genus g. Taking $H_*(T^2)$ as given, use the long exact sequence of a pair to compute $H_*(S_g)$ for $g > 1$. If $A \subset S_g$ is a non-separating simple closed curve, what is $H_*(S_g, A)$?

6. Show S^{n+m+1} can be decomposed as the union of $S^n \times D^{m+1}$ and $D^{n+1} \times S^m$ along their common boundary $S^n \times S^m$. Compute $H_*(S^n \times S^m)$ and $H_*(D^{n+1} \times S^m, S^n \times S^m)$ when $n, m > 0$.

7. Let $V \subset S^3$ be an open subset of S^3 homeomorphic to $S^1 \times \operatorname{int} D^2$, and let $K = S^1 \times 0 \subset S^1 \times \operatorname{int} D^2 \simeq V$. Compute $H_*(S^3, K)$ and $H_*(S^3 - K)$.

8. A *vector field* on S^n is a continuous map $\mathbf{v} : S^n \to \mathbb{R}^{n+1}$ such that $\langle \mathbf{x}, \mathbf{v}(\mathbf{x}) \rangle = 0$ for all $\mathbf{x} \in S^n$. If $\mathbf{v}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in S^n$, show that id_{S^n} is homotopic to the antipodal map. Deduce that if S^n admits a nonvanishing vector field, then n is odd.

9. Show that $A \in GL_2(\mathbb{Z})$ defines a homeomorphism $A : T^2 \to T^2$. Compute the induced map $A_* : H_*(T^2) \to H_*(T^2)$. (Hint: for H_2, consider $H_2(T^2, T^2 - 0)$.)

10. If $f : X \to X$ is a homeomorphism, let Y_f be the quotient of $X \times [0,1]$ obtained by identifying $(x,0)$ and $(f(x),1)$. Show there is a long exact sequence

$$\longrightarrow H_{n+1}(Y_f) \longrightarrow H_n(X) \xrightarrow{1-f_*} H_n(X) \longrightarrow H_n(Y_f) \longrightarrow$$

Compute $H_*(Y_f)$ when $X = S^n$ and f is the antipodal map; when $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and f is multiplication by $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$.

11. Suppose C and C' are free chain complexes over $\mathbb{Z}$ and that $C_i = C'_i = 0$ for $i < 0$. If $H_*(C) = 0$, show that C is contractible, in the sense that $1_C \sim 0$. If $f : C \to C'$ is a chain map which induces an isomorphism on homology, show that f is a chain homotopy equivalence. (Hint: use question 4).

12. Given $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$, define

$$g : (D^n \times D^m, \partial(D^n \times D^m)) \to (D^n \times D^m, \partial(D^n \times D^m))$$

by $g(x,y) = (f(x), y)$. Show that $\deg g = \deg f$.

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