ALGEBRAIC TOPOLOGY (PART III) **MICHAELMAS 2014**

EXAMPLE SHEET 4

1. Suppose $\pi : E \to B$ is a real vector bundle. If $f : B' \to B$, let $E' = \{(v, b) \in E \times B' \,|\, \pi(v) = f(b)\}$, and let $p : E' \to B$ be the obvious projection.

   (a) Show that E' is a vector bundle. Describe local trivializations and transition functions for E' in terms of local trivalizations and transition functions for E. E' is called the *pull-back* of E, and is usually denoted by $f^*(E)$.

   (b) If U is Thom class for E, show that $f^*(U)$ is a Thom class for $f^*(E)$. Deduce that $e(f^*(E)) = f^*(e(E))$.

2. Use Poincaré duality to show that $H^*(\mathbb{RP}^n, \mathbb{Z}/2) \simeq \mathbb{Z}/2[x]/(x^{n+1})$. (Hint: induct on n, using the inclusion $\mathbb{RP}^{n-1} \to \mathbb{RP}^n$.) Determine the ring structure on $H^*(\mathbb{RP}^n; \mathbb{Z})$.

3. Let M be a closed orientable 4-manifold, and write $H_i(M) \simeq F_i \oplus T_i$, where F_i is free and T_i is torsion. Find all relations between F_i and F_j, T_i and T_j for differing values of i and j.

4. Show that if M is a manifold with $H_1(M; \mathbb{Z}/2) = 0$, then M is orientable.

5. If M is an orientable manifold of dimension $4n+2$, show that the dimension of $H_{2n+1}(M; \mathbb{Q})$ is even.

6. If M is an oriented $4n$ dimensional manifold, the pairing $(a, b) = \langle a \cup b, [M] \rangle$ defines a symmetric bilinear form on $H^{2n}(M)$. If $H^*(M)$ is free over $\mathbb{Z}$, we can choose a basis of $H^{2n}(M)$ and write $(a, b) = a^T Ab$ for some symmetric matrix A with entries in $\mathbb{Z}$. Show $\det A = \pm 1$.

7. If $p \in \mathbb{C}[z_0, z_1, z_2]$ is a homogenous polynomial, we define

$$V_p = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \,|\, p(z_0, z_1, z_2) = 0\}.$$

   If p and q are chosen such that V_p and V_q are embedded submanifolds of $\mathbb{CP}^2$ which intersect transversely, show that V_p and V_q intersect in precisely $(\deg p)(\deg q)$ points.

8. Suppose $f : M \to N$ is a map of connected oriented manifolds. A point $p \in N$ is a *regular point* of f if it has an open neighborhood U such that $V = f^{-1}(U) \simeq U \times F$, and $f|_V$ is projection on the first factor. (If f is a map of smooth manifolds, Sard's theorem tells us that most $p \in N$ are regular.) We say $F_p = f^{-1}(p)$ is a *regular fibre* of F.

   (a) Show that $PD([F_p]) = f^*([N]^*)$. Deduce that any two regular fibres represent the same homology class in M.

(b) If $f : \mathbb{CP}^2 \# \mathbb{CP}^2 \to \mathbb{CP}^1$, show that any regular fibre is null-homologous. Show there is map $f : \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \to \mathbb{CP}^1$ whose regular fibre is nontrivial.

9. Suppose M is a closed n-manifold, and $f : S^{m-1} \to M$, $1 \leq m \leq n$. Can $M \cup_f D^m$ be homotopy equivalent to a closed n-manifold for $n = 3$? $n = 4$?

10. Let E be the tangent bundle of $\mathbb{CP}^n$. Compute $H_*(S(E))$.

11. Suppose that M is a compact odd-dimensional manifold with boundary. Show that $\chi(M) = \frac{1}{2}\chi(\partial M)$. Conclude that $\mathbb{RP}^2$ does not bound a compact 3-manifold. Does $\mathbb{RP}^3$ bound a compact 4-manifold?

12. Given $\gamma : S^1 \to SO(2)$, let $E_\gamma = D^2_a \times \mathbb{R}^2 \coprod D^2_b \times \mathbb{R}^2 / \sim$, where $\sim$ identifies $(x, v) \in S^1_a \times \mathbb{R}^2$ with $(x, \gamma(x) \cdot v)$. Show that E_γ is an oriented vector bundle over S^2 and compute its Euler class in terms of γ.

13. Let M be an oriented manifold with the property that $H_*(M)$ is generated by embedded submanifolds. Given $f : M \to M$, let $\Lambda = \{(p, f(p)) \in M \times M\}$ be the graph of f. Λ is an embedded submanifold of $M \times M$.

    (a) If $\langle a_i \rangle$ is a basis of $H_i(M; k)$ and $\langle b_i \rangle$ is the dual basis under the cup product pairing, show that $PD(\Lambda) = \sum f^*(a_i) \times b_i$.
    (b) Show that $L(f) := \Lambda \cdot \Delta = \sum_{j=0}^n (-1)^j \operatorname{Tr} f_j^*$, where $f_j^* : H^j(M; k) \to H^j(M; k)$.
    (c) Deduce that if $L(f) \neq 0$, f must have a fixed point. (This is the *Lefshetz fixed point theorem.*)
    (d) Show that any map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ has a fixed point.

14. Construct a 3 dimensional real vector bundle over S^4 which does not have a nonvanishing section.

15. Given $\phi : S^{2n-1} \to S^n$, let $X_\phi = S^n \cup_\phi D^{2n}$. $H^*(X_\phi) = \langle 1, x_n, x_{2n} \rangle$, where $x_i \in H^i(X_\phi)$. Thus $x_n^2 = H(\phi) x_{2n}$ for some $H(\phi) \in \mathbb{Z}$.

    (a) Show that the map $[\phi] \to H(\phi)$ defines a homomorphism $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$. H is known as the *Hopf invariant.*
    (b) Show that H is surjective for $n = 2, 4$.
    (c) By considering a cell decomposition of $S^n \times S^n$, show that H is nontrivial for every even n. Deduce that $\pi_{4m-1}(S^{2m})$ is infinite for all $m > 0$.

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