ALGEBRAIC TOPOLOGY (PART III) **MICHAELMAS 2012**

EXAMPLE SHEET 4

1. Let M be a closed orientable 4-manifold, and write $H_i(M) \simeq F_i \oplus T_i$, where F_i is free and T_i is torsion. Find all relations between F_i and F_j, T_i and T_j for differing values of i and j.

2. Suppose M is a closed n-manifold, and $f : S^{m-1} \to M$, $1 \leq m \leq n$. Can $M \cup_f D^m$ be homotopy equivalent to a closed n-manifold for $n = 3$? $n = 4$?

3. Let $E = T\mathbb{CP}^n$. Compute $H_*(S(E))$.

4. Suppose that M is a compact odd-dimensional manifold with boundary. Show that $\chi(M) = \frac{1}{2}\chi(\partial M)$. Conclude that $\mathbb{RP}^2$ does not bound a compact 3-manifold. Does $\mathbb{RP}^3$ bound a compact 4-manifold?

5. Show that there is no orientation reversing homeomorphism of $\mathbb{CP}^2$.

6. Show that there are orientable real vector bundles with zero Euler class which do not have a nonvanishing section.

7. Suppose $\pi : E \to B$ is a fibre bundle with fibre F, and that B and F are finite cell complexes. Show that E can be made into a finite cell complex whose cells are in bijection with pairs (σ_B, σ_F), where σ_B and σ_F are cells of B and F.

8. If $\dim H_*(X; Q)$ is finite, the Euler characteristic $\chi(X)$ is $\chi(X) = \sum_i (-1)^i \dim H_*(X; \mathbb{Q})$.

   If X is a finite cell complex with n_i cells of dimension i, show that $\chi(X) = \sum_i (-1)^i n_i$.

   Conclude that if E is as in problem 7, then $\chi(E) = \chi(F)\chi(B)$.

9. Suppose M_1 and M_2 are closed connected oriented n-manifolds, and let M_i' be the manifold obtained by removing an open n-ball from M_i. The orientation of M_i induces an orientation on $\partial M_i \simeq S^{n-1}$. The connected sum $M_1 \# M_2$ is the result of identifying the boundary S^{n-1} of M_1 with that of M_2 by a standard orientation reversing homeomorphism (e.g a reflection.) Show that $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ is not homotopy equivalent to $\mathbb{CP}^2 \# \mathbb{CP}^2$. If M is a closed 4-manifold containing an embedded sphere S with $S \cdot S = 1$, show that $M = M' \# \mathbb{CP}^2$ for some M'.

10. Let

$$Fl_n = \{0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n \,|\, V_i \text{ is a linear subspace of dimension } i\}$$

be the variety of complete flags in $\mathbb{C}^n$. Show that Fl_n has the structure of a cell complex, all of whose cells are even dimensional. (Hint: write Fl_n as a bundle over $\mathbb{CP}^{n-1}$.) Show that its Poincare polynomial is

$$[n]! := [n][n-1]\cdots[1],$$

where $n = (1-t^{2n})/(1-t^2)$.

11. Use the previous problem to show that the Poincare polynomial of $G_{\mathbb{C}}(k,n)$ is

$$\frac{[n]!}{[k]![n-k]!}.$$

J.Rasmussen@dpmms.cam.ac.uk