ALGEBRAIC TOPOLOGY (PART III) **MICHELMAS 2012**

EXAMPLE SHEET 2

1. Let M be the Mobius bundle over S^1. Show that $M \oplus M$ is the trivial bundle.

2. Let $E = TS^2$ be the tangent bundle of S^2. Show that the unit sphere bundle $S(E)$ is homeomorphic to $SO(3)$, which is also homeomorphic to $\mathbb{RP}^3$. What is the Euler class of E?

3. Identify $S^3 - 0$ with $\mathbb{R}^3$ by stereographic projection. Describe what the fibres of the Hopf fibration look like under this identification. Sketch three distinct fibres.

4. If $E \to B$ is a real vector bundle, let E^* be the vector bundle $\mathrm{Hom}(E, \mathbb{R})$. Show that $E \cong E^*$. If E is a complex vector bundle, let E^* be the vector bundle $\mathrm{Hom}(E, \mathbb{C})$. Give an example where $E \ncong E^*$.

5. Let $E \to B$ be a real vector bundle equipped with a Riemannian metric, and let $F \subset E$ be a subbundle. Show that $F^\perp$ is a vector bundle, and that $F \oplus F^\perp \cong E$.

6. Let $\pi : E \to B$ be a fibration over a path connected base B. Show that $\pi^{-1}(x) \sim \pi^{-1}(y)$ for all $x, y \in B$.

7. Show that the rank of $\pi_7(S^4)$ is nonzero. (Hint: quaternions.)

8. Prove that S^∞ is contractible.

9. Suppose that L_1 and L_2 are complex line bundles over B which are both locally trivial with respect to an open cover $\{U_\alpha\}$ of B. Describe how transition functions for L_1 and L_2 with respect to this cover are related to transition functions for $L_1 \otimes L_2$. Show that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. (Hint: consider a map $BU(1) \times BU(1) \to BU(1)$.)

10. Let $i_n : U(n) \to U(n+1)$ be the map which sends a matrix A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Show that the map $i_{n*} : \pi_k(U(n)) \to \pi_k(U(n+1))$ is an isomorphism for sufficiently large n relative to k; in other words, that the group $\pi_k(U(n))$ stabilizes as $n \to \infty$. (A famous theorem of Bott says that the limiting group $\pi_k(U)$ is $\mathbb{Z}$ if k is odd and 0 if k is even.)

11. Let G be a connected Lie group. By considering transition functions, show that the set of principal G-bundles over S^n up to isomorphism is in bijection with $\pi_{n-1}(G)$.

12. Suppose that $EG \to BG$ is a classifying bundle for G. Use the preceding problem to show that $\pi_k(EG) = 0$ for all $k > 0$.

13. Given $\gamma \in \pi_{n-1}(SO(n))$, let E_γ be the principal $SO(n)$ bundle over S^n associated to γ in problem 11, and let E'_γ be the associated vector bundle. Show that the map $\pi_{n-1}(SO(n)) \to H^n(S^n)$ which sends γ to $e(E'_\gamma)$ is a homomorphism.

14. Show that $SO(4) \simeq (SU(2) \times SU(2))/(\pm(I, I))$. (Hint: quaternions.) Deduce that the set of $SO(4)$ bundles over S^4 is naturally in bijection with $\mathbb{Z} \oplus \mathbb{Z}$. Show that there are infinitely many distinct real 4-dimensional vector bundles over S^4 whose unit sphere bundle is homotopy equivalent to S^7. (All of these unit sphere bundles are homeomorphic to S^7, but they are not all diffeomorphic.)

15. Let $E_i \subset \mathbb{R}^n$ $(1 \leq i \leq n)$ be the subspace spanned by $e_1, \dots, e_i$. Define a (discontinuous) map $f : G_{\mathbb{C}}(k, n) \to \mathbb{Z}^n$ by $f_n(H) = \mathbf{a}$ where $a_i = \dim H \cap E_i$. Show that $G_C(k, n)$ can be given the structure of a finite cell complex, in which the open cells are the sets are of the form $f^{-1}(p)$ for $p \in \mathbb{Z}^n$. Deduce that $H_*(G_{\mathbb{C}}(k, n))$ (ignoring gradings) is free of rank $\binom{n+k}{n}$. Compute $H_*(G_{\mathbb{C}}(2, 4))$ (with gradings).

16. Show that a map $\phi : G_1 \to G_2$ induces a map $B\phi : BG_1 \to BG_2$. Taking $G_1 = U(1)^n$ and $G_2 = U(n)$ and ϕ to be be the map whose image is the diagonal matrices in $U(n)$, show that if $a \in H^*(BU(n))$,

$$B\phi^*(a) \in H^*(BU(1)^n) \simeq \mathbb{Z}[x_1, \dots, x_n]$$

is invariant under the action of S_n which permutes the x_i. If you know something about Lie groups, formulate an analogous statement for an arbitrary connected Lie groups G. Describe the corresponding ring of invariant polynomials for $G = SO(n)$.

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