EXAMPLE SHEET 2

1. If $f_0, f_1 : S^{n-1} \to X$ and $f_0 \sim f_1$, show that $X \cup_{f_0} D^n \sim X \cup_{f_1} D^n$.

2. Consider the cell structure on S^n which has two cells of each dimension between 0 and n, corresponding to the northern and southern hemispheres of S^k. Write out its cellular chain complex and verify that it has the correct homology.

3. Let $X = S^1 \times D^2 / S^1 \times S^1$. Show that X has a cell decomposition with one 2-cell and one 3-cell. Compute its homology using the cellular chain complex.

4. Write $S^{2k-1} = \{(z_1, \dots, z_k) \in \mathbb{C}^k \mid \sum |z_i|^2 = 1\}$ The group $\mathbb{Z}/p$ acts on S^{2k-1} by $a \cdot \mathbf{z} = \lambda^a \mathbf{z}$, where $\lambda = e^{2\pi i/p}$. The *lens space* $L_k(p,1)$ is the quotient $S^{2k-1}/(\mathbb{Z}/p)$. Show that $L_k(p,1)$ has a cell decomposition with one cell of each dimension $0 \le 2k-1$. Find $C_*^{cell}(L_k(p,1))$ and compute its homology.

5. Compute $H_*(L_3(12,1) \times \mathbb{RP}^3)$ with coefficients in $\mathbb{Z}$, $\mathbb{Z}/2$, and $\mathbb{Z}/4$.

6. Let $f : S^n \to S^n$ have degree k, and let $X = S^n \cup_f D^{n+1}$. Let $\pi : X \to X/S^n \simeq S^{n+1}$ be the quotient map. Compute $\pi^* : H^*(S^{n+1}) \to H^*(X)$ and $\pi_* : H_*(X) \to H_*(S^{n+1})$.

7. Let $0 \to A \to B \to C$ be a short exact sequence of abelian groups. Show there is a long exact sequence

$$\dots \to H^n(X,A) \to H^n(X,B) \to H^n(X,C) \to H^{n+1}(X,A) \to \dots$$

   The boundary map in this sequence is known as the Bockstein homomorphism. Compute this boundary map when $X = \mathbb{RP}^3$ and the short exact sequence of groups is $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$. Do the same for $X = L_1(4,1)$.

8. Let $R = \mathbb{C}[x]/(x^3)$, and for $i = 1,2$, let M_i be the R-module $\mathbb{C}[x]/(x^i)$. Find a free resolution of M_1 and use it to compute $\mathrm{Tor}_*^R(M_1, M_1)$ and $\mathrm{Tor}_*^R(M_1, M_2)$.

9. Let Σ_2 be a surface of genus 2. Compute the cup product on $H^*(\Sigma_2)$. (Hint: consider a map $\Sigma_2 \to T^2 \vee T^2$.) Show that any map $T^2 \to \Sigma_2$ has degree 0.

10. Let X be a space obtained by removing copies of int B^4 from two copies of $\mathbb{CP}^2$ and identifying the resulting boundaries by a homeomorphism of S^3. Show that $H_*(X) \simeq H_*(S^2 \times S^2)$, but that X is not homotopy equivalent to $S^2 \times S^2$.

11. Suppose $x \in H_n(X)$, where X is an arbitrary topological space. Show that there is a finite cell complex A and a map $f : A \to X$ so that $x \in \operatorname{im} f_*$.

12. Let X be a finite cell complex, and let A be a subcomplex of X. Let $Y = X \times \{0\} \cup A \times I \subset X \times I$. Show that any map $Y \to Z$ extends to a map $X \times I \to Z$. (Hint: start with the case $X = D^n$, $A = S^{n-1}$.)

13. Let X be a connected finite cell complex with $\pi_i(X) = 0$ for all $0 < i \leq n$, and let $\pi : X \to X/X_n$ be the quotient map. Show that there is a map $f : X/X_n \to X$ so that $f \circ \pi \sim 1_X$. Using the Hurewicz theorem, conclude that if $\pi_1(X) = 1$ and $H_i(X) = 0$ for $i > 0$, then X is contractible.

14. (If you know π_1.) Construct a connected finite cell complex X with $H_i(X) = 0$ for $i > 0$ but which is not contractible. Find a map $f : S^1 \vee S^2 \to S^1 \vee S^2$ which induces the identity map on $H_*(S^1 \vee S^2)$, but is not homotopic to the identity.

15. The *suspension* ΣX of a space X is defined to be $X \times [0,1]/\sim$, where $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for all $x, y \in X$. Show that if $a, b \in H^*(\Sigma X)$ have positive degree, then $a \cup b = 0$.

J.Rasmussen@dpmms.cam.ac.uk