EXAMPLE SHEET 1

1. If $X_1 \sim X_2$ and $Y_1 \sim Y_2$, show there is a bijection between the sets $[X_1, Y_1]$ and $[X_2, Y_2]$.

2. Let $\sigma_1, \sigma_2 : [0,1] \to \mathbb{R}$ be given by $\sigma_1(x) = x, \sigma_2(x) = 1 - x$. Identifying $[0,1]$ with Δ^1 gives a cycle $e_{\sigma_1} + e_{\sigma_2}$ in $C_1(\mathbb{R})$. Find an $x \in C_2(\mathbb{R})$ with $dx = e_{\sigma_1} + e_{\sigma_2}$.

3. What are the possible isomorphism types of the abelian group G in the following exact sequences?

$$0 \to \mathbb{Z} \to G \to \mathbb{Z} \to 0 \qquad\qquad 0 \to \mathbb{Z} \to G \to \mathbb{Z}/4 \to 0$$
$$0 \to \mathbb{Z}/4 \to G \to \mathbb{Z} \to 0 \qquad\qquad 0 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/4 \to 0$$

4. Let X be the genus 2 surface shown in Figure 1.
   (a) Use the Mayer Vietoris sequence to compute $H_*(X)$. (Hint: divide X along A.)
   (b) Let A, B and C be curves as shown in the figure. What are $H_*(X - A), H_*(X - B)$ and $H_*(X - C)$?
   (c) Use the exact sequence of a pair to compute $H_*(X, A), H_*(X, B)$ and $H_*(X, C)$.

5. Let $i : S^1 \times D^2 \to S^3$ be an embedding, and let U be the interior of its image. Use the Mayer-Vietoris sequence to compute $H_*(S^3 - U)$.

6. Show S^{n+m+1} can be decomposed as the union of $S^n \times D^{m+1}$ and $D^{n+1} \times S^m$ along their common boundary $S^n \times S^m$. Compute $H_*(S^n \times S^m)$ and $H_*(D^{n+1} \times S^m, S^n \times S^m)$.

7. Suppose $f : T^2 \to T^2$ is a homeomorphism. Show that $f_* : H_1(T^2) \to H_1(T^2)$ defines an element of $GL(2, \mathbb{Z})$, and that any element of $GL(2, \mathbb{Z})$ can be realized by a homeomorphism of T^2.

8. (*Cancellation*) Suppose (C, d) is a chain complex, that $C_n = C'_n \oplus A$, $C_{n-1} = C'_{n-1} \oplus A$, and that the component of d_n mapping A to A is the identity map. Show that (C, d) is chain homotopy equivalent to (C', d'), where $C'_i = C_i$ for $i \neq n, n-1$, $d'_i = d_i$ for $i \neq n-1, n, n+1$, and d'_{n+1} is the composition of d_{n+1} with the projection onto C'_n. (Hint: use $d^2 = 0$ to determine d_n.)

9. If $f : X \to X$ is a homeomorphism, let Y be the quotient of $X \times [0,1]$ obtained by identifying $(x, 0)$ and $(f(x), 1)$. Show there is a long exact sequence

$$\longrightarrow H_{n+1}(Y) \longrightarrow H_n(X) \xrightarrow{1-f_*} H_n(X) \longrightarrow H_n(Y) \longrightarrow$$

Compute $H_*(Y)$ when $X = S^n$ and f is the antipodal map; when $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and f is multiplication by $\begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$.

10. If $H_*(X)$ is a free abelian group, show that $H_*(X \times S^1) \cong H_*(X) \oplus H_{*-1}(X)$. (In fact, this is true even if $H_*(X)$ is not free.) Compute $H_*(T^n)$.

11. Show that if $f : D^n \to D^n$ is any continuous map, there is some $x \in D^n$ with $f(x) = x$. (Hint: if not, you can construct a map $D^n \to S^{n-1}$ which restricts to the identity on S^{n-1}.)

12. If $f : (C, d) \to (C', d')$ is a chain map, the *mapping cone* of f is the chain complex $(M(f), d_f)$ whose underlying group is given by $M(f)_n = C_{n-1} \oplus C'_n$, and whose differential is given by

$$(d_f)_n = \begin{pmatrix} d_{n-1} & 0 \\ (-1)^n f_{n-1} & d'_n \end{pmatrix}.$$

Show that $(M(f), d_f)$ is a chain complex, and that if $f \sim g$, then $M(f) \sim M(g)$. By considering an appropriate mapping cone, give a proof of the Five Lemma. If C and C' are free finitely generated chain complexes over $\mathbb{Z}$, prove that $H_*(M(f)) = 0$ if and only if f is a chain homotopy equivalence.

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