The Axiom of Choice

Zachiri McKenzie

I hope that one day I will see a free Palestinian state!

> *"Remember the solidarity shown to Palestine here and everywhere... and remember also that there is a cause to which many people have committed themselves, difficulties and terrible obstacles notwithstanding. Why? Because it is a just cause, a noble ideal, a moral quest for equality and human rights."*
> *- Edward Said*

The Axiom of Choice

(AC): If $\mathcal{S}$ is a collection of non-empty pairwise disjoint sets, then there exists a C such that:

$$C \cap X = \{u\} \text{ for all } X \in \mathcal{S}.$$

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(ii) (Zermelo's Well-Ordering Principle) Every set can be well-ordered.

(iii) (Tukey's Lemma) If $\mathcal{F}$ is a family of non-empty sets satisfying the following condition:

$$X \in \mathcal{F} \text{ if and only if } (S \subseteq X) \wedge (S \text{ is finite}) \Longrightarrow S \in \mathcal{F}.$$

Then $\mathcal{F}$ has a maximal element with respect to inclusion ($\subseteq$).

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Theorem

If $(M, \in)$ is a model of ZFA and the set of atoms, $A \subseteq M$, is infinite, then there is no ordinal definable well-ordering of A. $\square$

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(i) $F \neq \emptyset$;

(ii) *if* $p \leq q$ *and* $p \in F$, *then* $q \in F$;

(iii) *if* $p, q \in F$, *then there exists* $r \in F$ *such that* $r \leq p$ *and* $r \leq q$.

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Definition

Let $(P, \leq)$ be a partial order. A set $D \subseteq P$ is dense in P if and only if for all $p \in P$ there exists a $q \in D$ such that $q \leq p$.

Definition

Let $(P, \leq)$ *be a partial order. A set* $G \subseteq P$ *is a generic filter over* M *if and only if*

(i) G *is a filter;*

(ii) *if* $D \subseteq P$ *is dense in* P *and* $D \in M$*, then* $D \cap G \neq \emptyset$.

Definition

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The idea is that given a model of ZFC, $(M, \in)$, and a partial order $(P, \leq)$ in M, we can produce an interpretation of an extended model $(M[G], \in)$ that contains G; a filter on $(P, \leq)$ that is generic over M.

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Definition

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Definition

Let $(P, \leq)$ *be a partial order and let* $G \subseteq P$ *be a generic filter over* M*. For each* P*-name* $x \in M$*, define the evaluation of* x *at* G *by recursion:*

$$x^G = \{y^G \mid (y, p) \in x \wedge p \in G\}.$$

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$$\check{x} = \{(\check{y}, \mathbb{1}_P) \mid y \in x\}.$$

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Let M be a transitive model of ZFC. For any partial order $(P, \leq)$ (with a maximal element $\mathbb{1} \in P$) in M and $G \subseteq P$ a generic filter over M we define the following classes:

$$M^P = \{x \in M \mid x \text{ is a } P\text{-name } \}$$

$$M[G] = \{x^G \mid x \in M^P\}.$$

Theorem

(Generic Model Theorem) Let M be a transitive model of ZFC and let $(P, \leq)$ be a partial order in M. If $G \subseteq P$ is a generic filter over M, then $M[G]$ is a model of ZFC such that:

(i) $M \subseteq M[G]$;

(ii) $G \in M[G]$;

(iii) $\mathrm{NO}^{M[G]} = \mathrm{NO}^{M}$;

(iv) *if N is a transitive model of ZF such that $M \subseteq N$ and $G \in N$, then $M[G] \subseteq N$.*

□

An important feature of the extended model $M[G]$ is that there exists a definable relation in the ground model M relating truth in $M[G]$ to G.

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Theorem

(The Forcing Theorem) Let M be a transitive model of ZFC and let $(P, \leq)$ be a partial order in M. If σ is a sentence of the forcing language, then for every $G \subseteq P$ generic over M,

$$M[G] \models \sigma \textit{ if and only if } (\exists p \in G) p \Vdash \sigma.$$

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Definition

Let $(P, \leq)$ be a partial order. Let $M[G]$ be an extended model generated by a generic filter G. For each $x \in M[G]$ define $\dot{x} \in M$ to be the P-name such that $\dot{x}^G = x$.

Symmetric Submodels of Generic Extensions

Definition

Let π *be an automorphism of* $(P, \leq)$*. Define the 'automorphism'* π' *of* M^P *by* $\in$*-recursion:*

$$\pi'(x) = \{(\pi'(y), \pi(p)) \mid (y, p) \in x\} \textit{ for all } x \in M^P.$$

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Definition

Let $\mathcal{G}$ *be a group of automorphisms of* M^P. *A set of subgroups* $\mathcal{F}$ *of* $\mathcal{G}$ *is a filter if and only if*

(i) $\mathcal{G} \in \mathcal{F}$;

(ii) *if* $H \in \mathcal{F}$ *and* $H \subseteq K$, *then* $K \in \mathcal{F}$;

(iii) *if* $H, K \in \mathcal{F}$, *then* $H \cap K \in \mathcal{F}$;

(iv) *if* $\pi \in \mathcal{G}$ *and* $H \in \mathcal{F}$, *then* $\pi H \pi^{-1} \in \mathcal{F}$.

Definition

Let $\mathcal{G}$ be a group of automorphisms. For any set x define,

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Let HS be the class of all hereditarily symmetric P-names.

$$\text{Let } N = \{x^G \mid x \in HS\}.$$

Theorem

The class $N \subseteq M[G]$ defined above is a model of ZF Set Theory. $\square$

Jech-Sochor Theorem

Theorem

(Jech-Sochor) Let $(M, \in)$ *be a model of ZFA with* $A \subseteq M$ *the set of atoms. Let* α *be an ordinal number. There exists a symmetric model* N *of ZF and an embedding* $x \mapsto \tilde{x}$ *into* N *witnessing that:*

$$(P_\alpha(A))^U \textit{ is } \in \textit{-isomorphic to } (P_\alpha(\tilde{A}))^N.$$

□