

Non Abelian Lubin-Tate theory
and beyond

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§ 1. GL_n の Langlands 対応

$n \geq 1$.

代数的整教論

(\mathbb{Q} の絶対 Galois 群)
 $G_{\mathbb{Q}}$ を調べる

$G_{\mathbb{Q}}$ の n 次元表現

étale
cohomology

L 関数

代数的幾何 / \mathbb{Q}

\mathbb{Q} 上の代数的体 (n=4-7)

表現論

(保型形式)

GL_n / \mathbb{Q} の保型表現

rank n .

量子力学 / 場の量子論

関数

(可換対象)

$\text{tr}(AB) = \text{tr}(BA)$

有限次元

↑
観測できる対象

作用素

(演算子)

非可換

$AB \neq BA$

無限次元

↑
背後に隠れているもの

群の表現

関数 ... 空間の性質を反映

幾何

整教論

Spec \mathbb{Z} の関数

{ (0), 2, 3, 5, ... }

$\neq -1$

↑
隠れた 調和
rigid.

Euler 積

$$\prod_p (1 - a_p p^{-s})^{-1} \quad \left(= \sum_{n \geq 1} \frac{a_n}{n^s} \right)$$

⇒ 解析接続 + 関数等式 ($\text{Re } s > 1$
⇒ $O(n^{-\nu})$)

値
"a_p" ← 何かのトレース

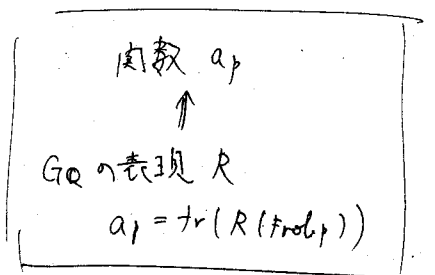
② Langlands: Euler 積は GL_n の保型表現からくる.

② 代数的整教論

(\mathbb{Q} の絶対 Galois 群 $G_{\mathbb{Q}}$ を用いる)

$$G_{\mathbb{Q}} = \varprojlim_{\substack{K/\mathbb{Q} \\ \text{fin. Gal.}}} \text{Gal}(K/\mathbb{Q})$$

$\forall p$. Frobp ... $\text{Gal}(K/\mathbb{Q})$ の共役類



② 代教幾何

Weil 予想

X/\mathbb{Q} : 代教多様体

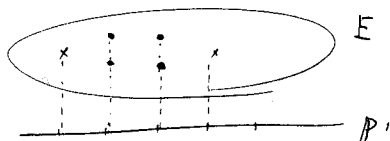
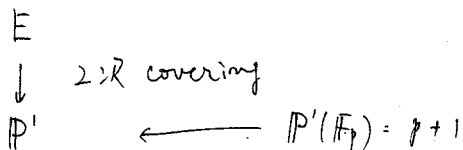
$$\Rightarrow X_p := X \text{ mod } \mathbb{F}_p \quad \forall p$$

$p \longmapsto \# \text{-有理点の個数}$
 $\# X_p(\mathbb{F}_p)$

← Frobp の X の étale コホモロジーへの作用のトレースと解釈できる.
(Lefschetz trace formula)

3.

例) E/\mathbb{Q} : 楕円曲線
 $y^2 = (x \text{ の } 3 \text{ 次式})$



$\Rightarrow \# E(\mathbb{F}_p) \doteq p+1.$

Weil予想
 $|a_p| \leq 2\sqrt{p}.$

$a_p := \# E(\mathbb{F}_p) - (p+1).$

$\rightarrow L(s, E) = \prod_p (1 - a_p p^{-s} + p^{1-2s})^{-1}$
 (各 $2L$ と $\sim 70p$)

$L(s, E)$ を与えら

- Galois 表現 (2次元)
- GL_2 の保型表現

の対応は確立. (谷山-志村予想, 1999 解決)

$R : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{Q}_\ell)$
 \uparrow ℓ 進数.

$\mathbb{Z} \ni a_p = \text{tr}(R(\text{Frob}_p))$
 $= \alpha_p + \beta_p.$ (Weil予想)

$\alpha_p, \beta_p \in \overline{\mathbb{Q}}$ $|\alpha_p| = |\beta_p| = \sqrt{p}$

$\alpha_p = \sqrt{p} e^{i\theta}, \beta_p = \sqrt{p} e^{-i\theta} \Rightarrow a_p = 2\sqrt{p} \cos \theta_p.$

$p \longmapsto \theta_p$ の分布が...
 正確に予想された.

(佐藤-Tate予想 (cos))

$j - \text{inv} \notin \mathbb{Z}$
 1975 解決. 2006.

GL_n の Langlands 対応の系

GL_1 の Langlands 対応 = 類体論
 ↑
 局所 Langlands 対応 } 大域
 } 局所

④ 大域対応

= ほとんどすべての ρ の
 様子で決定してしまう。
 ← 「有限」を除いてすべて

Galois 表現 ... Chebotarev Density Thm.
 保型表現 ... Multiplicity One Thm

$\text{Spec}(\mathbb{Z})$ は rigid

- 「素数の値」(素) 以上の情報が悪い素数 (特異点) には隠れている。
- 局所 Langlands 対応は証明された (1999) が、大域理論を用いる証明しか知られていない...
 ← 局所的証明が欲しい。

④ ところから GL_1 -case (局所類体論) は局所的に証明できる。
 (Lubin-Tate 理論 ⇒ 非可換できる)

§ 2.

p : prime.

$$\mathbb{Z}_p := \varprojlim_m \mathbb{Z}/p^m\mathbb{Z} \quad \mathbb{O}_p := \mathbb{Z}_p\left[\frac{1}{p}\right]$$

$$[K:\mathbb{O}_p] < \infty.$$

K : CDVF. complete disc. val.

\mathcal{O}_K : integral closure of \mathbb{Z}_p in K .

$\mathcal{O}_K \supset \mathfrak{p}_K$: $\exists!$ maximal ideal.

\mathfrak{p}_K : uniformizer.

$$k := \mathcal{O}_K/\mathfrak{p}_K \cong \mathbb{F}_3.$$

$$G_K := \text{Gal}(\bar{K}/K).$$

$K(\mu_N)$: decomposition field of $X^N - 1$.

Hensel's lemma $p \nmid N$.

$$\mu_N \subset k \iff \mu_N \subset K.$$

このとき

$$\begin{array}{ccc} \mu_N & \xrightarrow{\sim} & \mu_N \\ \mathcal{O}_K & \longrightarrow & \hat{k} \end{array}$$

$$p \nmid N. \quad \text{Gal}(K(\mu_N)/K)$$

$\downarrow \cong$

$$\text{Gal}(k(\mu_N)/k) = \langle \alpha \mapsto \alpha^g \rangle \cong \mathbb{Z}/f\mathbb{Z}$$

$$N = 8^f - 1 \quad \mathbb{F}_8^* \quad \mathbb{F}_3$$

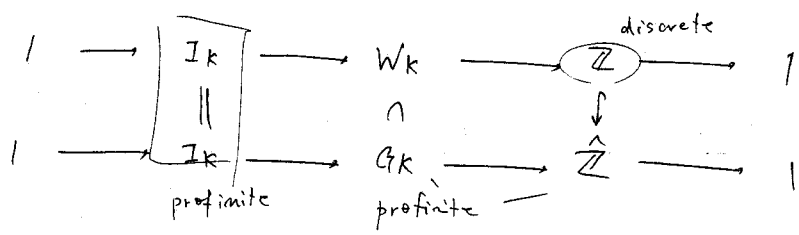
$K^{ur} := \bigcup_{p \nmid N} K(\mu_N)$: maximal unramified ext'n of K .

$$\text{Gal}(K^{ur}/K) \cong \text{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}.$$

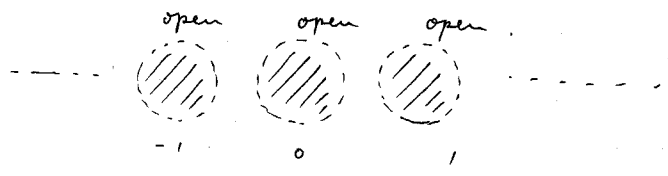
normalization.

$$\begin{aligned} \varphi: G_K &\longrightarrow \text{Gal}(K^{\text{alg}}/K) \cong \text{Gal}(\bar{K}/K) \cong \hat{\mathbb{Z}} \\ &\quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ &\text{Frob}_K \longmapsto (\alpha \mapsto \alpha^q)^{-1} \longmapsto 1 \\ &\text{(geometric Frobenius)} \end{aligned}$$

$W_K := \varphi^{-1}(\mathbb{Z})$: Weil group of K .
 $I_K := \ker \varphi$: inertia group of K .



$$W_K = \coprod_{i \in \mathbb{Z}} I_K \cdot \text{Frob}_K^i$$

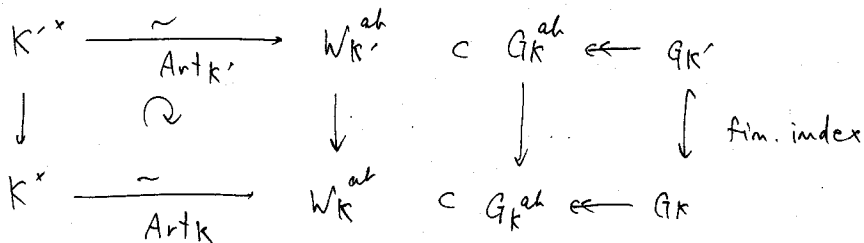


→ topology on W_K
 (stronger than the induced topo from G_K)

2) comes from

Base change

$$K'/K : \text{fin.}$$



$$(K^{\times} : N_{K'/K}(K'^{\times})) < \infty \\
 \text{index} = [K' : K]$$

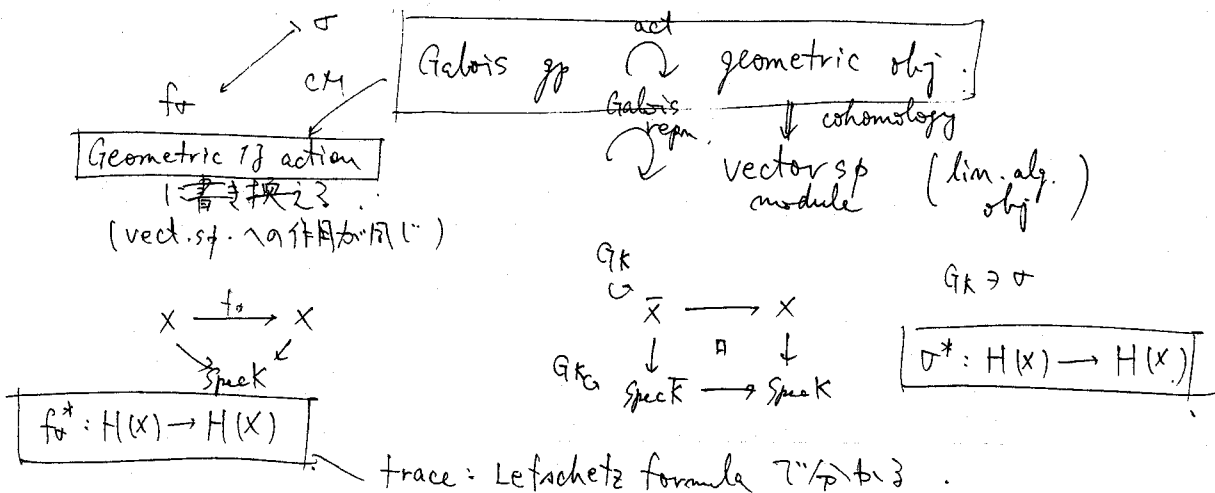
Rem

Art_K : characterized by BC (functoriality).
 Local Langlands Correspond should be characterized by functoriality.

Lubin-Tate theory . "LCFT via LT theory by T.T."

... construction of Art_K using formal groups.

Philosophy Complex Multiplication



ex 1) $K = \mathbb{F}_p$, X/\mathbb{F}_p 気合: coefficient を 乗.
 SGA5 $\text{Frob}_p^{-1} \sim \text{Hét}(X, \mathcal{O}_X)$

$F: X \rightarrow X$ relative Frob. (coordinate を 乗)
 \rightarrow Lefschetz trace formula
 $F^* \sim \text{Hét}(X, \mathcal{O}_X)$

こちらの作用が同じ

($\because F \geq \text{Frob}_p^{-1}$ 気合 X abs. Froh \rightarrow cohomology
 (= is identity.)

ex 2) K/\mathbb{Q} imag. quad. $\mathcal{O}_K \hookrightarrow \text{End}(E)$ ell. curve.
 CM-theory E/H : H : Hilbert Class field of K .

$\mathcal{O}_K \sim \text{Hét}(E, \mathcal{O}_E) \supset GH$

dim 2. \leftarrow Galois repn $\mathcal{O}_K \otimes \mathcal{O}_E - \text{rk } 1$ } rk 1 i.e. abel.
 GH は G_K に の ぶ る.

H の ぼ と ん と す べ っ と の 素 点 で ex 1 の 議 論.

$\Rightarrow GH$ の Froh の 作用 は $E \bmod \mathfrak{p}$ の F (relative Froh) Galois geometric

\uparrow \mathfrak{p} の F に お け る K の 素 点 π 上
 $\mathfrak{f} = (\varpi) \in H$ $\mathfrak{f} \subset \mathcal{O}_K$
 \exists \mathfrak{f} $N_{H/K}(\varpi) \in \mathcal{O}_K \subset \text{End}(E)$ geometric
 \mathfrak{f} の E 上 の action τ
 τ は \mathfrak{f} の 素 点 上

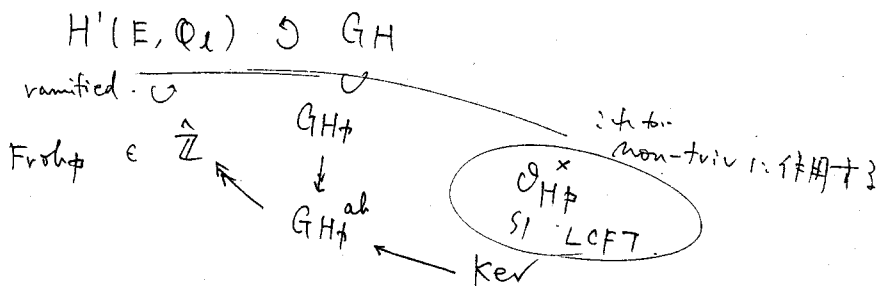
こちが類体論の実現

G_K の char $\iff A_K^*/K^*$ の char

CM theory 7.

\mathfrak{p} to E a bad prime. ($H_{\mathfrak{p}}$: local field)

$\implies E \bmod \mathfrak{p}$ to singular.



Lubin-Tate theory.

\ni a ramified Galois action \ni geometric (書)

$l \neq p$: prime.

$H_{\text{ét}}^1(E, \mathbb{Q}_\ell) := H^1(E, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$

$H_{\text{ét}}^1(E, \mathbb{Z}_\ell) = (T_\ell E)^\vee$

E/F perfect $T_\ell E := \varprojlim_m E[\ell^m]$ Tate module

$GF \hookrightarrow E[\ell^m] := \{ \alpha \in E(F) \mid L_m(\alpha) = 0 \}$

abel gp $\cong (\mathbb{Z}/\ell\mathbb{Z})^2$ \cong 冪分点

ramified の場合. 冪分点 の E の formal completion at origin $T_\ell E$ 冪分点 \ni 冪分点.

CM elliptic curve $E \longrightarrow$ formal comp.
 Lubin-Tate formal gp
 (冪分点 冪分点 の場合)

K/\mathbb{Q}_p fin.

Def. $A: \mathcal{O}_K$ -alg.

formal \mathcal{O}_K -module $\Sigma := (F, [\cdot]) / A$.

... $F \in A[[X, Y]]$, s.t.

- 1) $F(X, Y) \equiv X + Y \pmod{\text{deg } 2}$
- 2) $F(F(X, Y), Z) = F(X, F(Y, Z))$.
- 3) $F(X, Y) = F(Y, X)$.

$[\cdot] : \mathcal{O}_K \hookrightarrow \text{End}_A(F)$

$$\left\{ \begin{array}{l} f \in A[[X]] \\ f \equiv 0 \pmod{X} \\ F(f(x), f(\gamma)) = f(F(x, \gamma)) \end{array} \right\}$$

... non commutative ring by.

$$\left\{ \begin{array}{l} f +_F g := F(f, g) \\ f \circ g := f(g(x)) \end{array} \right.$$

$$F(f +_F g(x), f +_F g(\gamma)) = \dots \\ = f +_F g(F(x, \gamma))$$

s.t. $\forall a \in \mathcal{O}_K$.

$$[a](x) \equiv ax \pmod{x^2} \\ \uparrow \\ \text{via } \mathcal{O}_K \rightarrow A$$

Lubin-Tate group

= formal \mathcal{O}_K -mod of height ①

i.e.

$$[\varpi](x) \equiv X^q \pmod{\mathfrak{p}_K}$$

$$q = |\mathcal{O}_K/\mathfrak{p}_K|$$

GL₁

$$\varphi: G_k \longrightarrow \hat{\mathbb{Z}} \quad \tau \in \Gamma_k, \quad \tau^{-1} \tau \in k.$$

φ^{-1} Arith. Frob. Frob $_k^{-1}$ ~~と表す~~.

$$L = \begin{cases} \text{fin. unram. ext. of } K & \leftarrow \text{residue field} \\ \text{or} & k \text{ と同大} \\ \text{completion } \widehat{K^{ur}} & \text{of } \underline{K^{ur}} \end{cases}$$

integral closure of \mathcal{O}_k

residue field: $\bar{k} = \overline{F_f}$

uniformizer ϖ of k is also a uniformizer of L .

i.e. $\phi_k \mathcal{O}_L = \phi_L$.

Lubin-Tate group の構成

($L = \widehat{K^{ur}}$ なら $\mathcal{O}_L \cong \mathbb{Z}$ unique)

Prop. Fix $\varpi \in \mathcal{O}_L$: unif.

Let $f \in \mathcal{O}_L[x]$ satisfy

$$\textcircled{*} \quad \begin{cases} f(x) \equiv \varpi x \pmod{\text{deg } 2} \\ f(x) \equiv x^q \pmod{\phi} \end{cases}$$

$\Rightarrow \exists F_f / \mathcal{O}_L$: formal group s.t.

$$f \in \text{Hom}(F_f, F_f^q)$$

(F_f^q で F の係数に φ を作用させた q -べき級数を表す)

Lemme. $\varpi, \varpi' \in \mathcal{O}_L$: uniformizer.

Let f, f' satisfy for ϖ, ϖ' (resp.)

Let $x_1, \dots, x_n \in \mathcal{O}_L$ satisfy

$$\varpi' x_i = \varpi x_i^q \quad (1 \leq i \leq n) \quad \leftarrow [\textcircled{*} \text{ mod } 2]$$

$\Rightarrow \exists F \in \mathcal{O}_L[x_1, \dots, x_n]$ s.t.

$$\begin{cases} F \equiv \varpi x_1 + \dots + \varpi x_n \pmod{\text{deg } 2} \\ f' \circ F = F^q \circ f \end{cases} \quad \text{--- } \textcircled{*}$$

(pf) For each $m \geq 1$.

$\exists!$ F_m : pol. of degree $\leq m$.
satisfying $\textcircled{*}$ mod deg $m+1$.

$m=1$ OK.

$$G_{m+1} := f' \cdot F_m - F_m^q \cdot f$$

$$G_{m+1} = F_m^q - F_m^q(x_1^q, \dots, x_m^q) \\ = 0 \quad \text{mod } \phi$$

($\because \phi \text{ mod } \phi + 1$ の繰り返し像)

$$\Downarrow \\ \omega' \mid G_{m+1}$$

$$H_{m+1} := F_{m+1} - F_m \text{ 在 } \mathbb{Z} \text{ の } \}$$

$$\textcircled{*} \quad f' \cdot F_{m+1} - F_{m+1}^q \cdot f \\ = G_{m+1} + (f' \cdot H_{m+1} - H_{m+1}^q \cdot f) \\ = G_{m+1} + (\omega' \cdot H_{m+1} - \omega^{m+1} H_{m+1}^q) \quad \text{mod deg } m+2$$

For each monomial X of deg $m+1$.

$\omega' \beta$: coeff of X in G_{m+1}

α : coeff of X in H_{m+1} .

$$\omega' \beta + (\omega' \alpha - \omega^{m+1} \alpha^q) = 0$$

$$\alpha = -\beta - \sum_{i=1}^{\infty} \left(\frac{\omega^{m+1}}{\omega'} \right)^{1+q+\dots+q^{i-1}} \beta^q \quad \square$$

completeness

pf of Prop.

Apply lemma to the case

$$\omega = \omega'$$

$$f = f'$$

$$a = 2$$

$$d_1 = d_2 = 1$$

$$\Rightarrow \exists! F_f \quad f \circ F_f = F_f^q \cdot f$$

- $F(x, F(Y, Z)), F(F(x, Y), Z)$
both satisfy the lemma for $n=3, d_1=d_2=d_3=1$
 \Rightarrow coincide.
- $F(x, Y) = F(Y, x)$: same. ◻

Next. \mathcal{O}_k -action.

$\omega, \omega' \in \mathcal{O}_L$: unit.

$$A_{\omega, \omega'}^L := \{ \theta \in \mathcal{O}_L \mid \omega' \theta = \omega \theta^\psi \} \quad \leftarrow \textcircled{A_{\omega, \omega'}^L = \mathcal{O}_k}$$

Let f, f' satisfy (*) for ω, ω' (resp.)

Then for $\forall \theta \in A_{\omega, \omega'}^L$

Lemma gives $[\theta]_{f, f'} \in \mathcal{O}_L[[X]]$.

$$\text{s.t. } \begin{cases} [\theta]_{f, f'} \equiv \partial X \pmod{\text{deg } 2} \\ f' \circ [\theta]_{f, f'} = [\theta]_{f, f'} \circ f. \end{cases}$$

Prop

1) $[\theta]_{f, f'} \in \text{Hom}_{\mathcal{O}_L}(F_f, F_{f'})$.

2) $[\cdot]_{f, f'} : A_{\omega, \omega'}^L \rightarrow \text{Hom}_{\mathcal{O}_L}(F_f, F_{f'})$ is injective and

$$[\theta]_{f, f'} +_{F_{f'}} [\theta']_{f, f'} = [\theta + \theta']_{f, f'}$$

$$[\theta]_{f, f'} \circ [\theta']_{f, f'} = [\theta \theta']_{f, f'}$$

In particular, we have

$$[\cdot]_f := [\cdot]_{f, f} : \mathcal{O}_k \rightarrow \text{End}_{\mathcal{O}_L}(F_f)$$

$$\text{When } f=f' \quad \left(\uparrow A_{\omega, \omega}^L = \mathcal{O}_k \right)$$

pt 1) Check. $[\theta] \cdot F_f = F_f' \cdot [\theta]$.
 both sides satisfy the condition of Lemma for

$$\bar{\omega} = \bar{\omega}', \quad n=2, \quad \alpha_1 = \alpha_2 = \theta.$$

$$\begin{aligned} f' \cdot ([\theta] \cdot F_f) &= [\theta]^q \cdot f' \cdot F_f \\ &= [\theta]^q \cdot F_f^q \cdot f = ([\theta] \cdot F_f)^q \cdot f \quad \text{etc.} \end{aligned}$$

2) $[\theta'] \cdot [\theta]$ satisfies the condition of Lemma.
 which characterizes $[\theta' \cdot \theta]$ etc. \square

[Later we will see that
 $\Sigma_f = \Sigma_{f'}$ over $\mathcal{O}_{\bar{K}}^{\text{ur}}$ ($v_f, v_{f'}$)]

Next Construct abelian extns. (Lubin-Tate extns) of L
 by adjoining division pts of Σ_f .

Prop. $\bar{\omega} \in \mathcal{O}_L$: unif.
 $f \in \mathcal{O}_L[X]$: monic pol. satisfying (*).
 (i.e. $f(v) = \underbrace{\bar{\omega}X + \dots + X^d}_{\text{divisible by } \bar{\omega}} \quad \Big)$

For $\forall m \geq 1$, define

$$f_m := f^{\varphi^{m-1}} \circ \dots \circ f^{\varphi} \circ f \in \mathcal{O}_L[X].$$

$L' = L(\mathcal{M}_{f,m}) :=$ splitting field of f_m .

$$\mathcal{O}_{L'} \supset \mathcal{M}_{f,m} := \{ \alpha \in L(\mathcal{M}_{f,m}) \mid f_m(\alpha) = 0 \}$$

- 1) $\mathcal{M}_{f,m}$ is an \mathcal{O}_K -module by $+_{F_f}, [\cdot]_f$.
- 2) $\forall \alpha \in \mathcal{M}_{f,m} \setminus \mathcal{M}_{f,m-1}$ ("primitive division pt")

$$\mathcal{O}_K/\mathfrak{p}^m \ni a \xrightarrow{\sim} [a]_f(\alpha) \in \mathcal{M}_{f,m}$$

- 3) L'/L : tot. ram.

$$-\alpha : \text{unif of } L' \text{ s.t. } N_{L'/L}(-\alpha) = \varpi^{\varphi^{m-1}}$$

$$4) \rho_{f,m} : \text{Gal}(L'/L) \xrightarrow{\sim} (\mathcal{O}_K/\mathfrak{p}^m)^\times$$

$$\text{indep of } \alpha. \quad (\alpha \mapsto [u]_f(\alpha)) \mapsto u \pmod{\mathfrak{p}^m}$$

ex. $L=K=\mathbb{Q}_p$. $f(x) = (1+x)^p - 1$.

$$\Rightarrow f_m(x) = (1+x)^{p^m} - 1. \Rightarrow \mathcal{M}_{f,m} = \{z-1 \mid z \in \mathcal{M}_{\mathfrak{p}^m}\}$$

$$\Rightarrow L(\mathcal{M}_{f,m}) = \mathbb{Q}_p(\mathcal{M}_{\mathfrak{p}^m})$$

pf.

1) 2) $\mathcal{M}_{f,m} \setminus \mathcal{M}_{f,m-1}$ = set of all roots
of $h(x) := f_m(x)/f_{m-1}(x)$

$$\begin{cases} h(x) \equiv \varpi^{\varphi^{m-1}} \pmod{x} \\ h(x) \equiv X^{(p-1)p^{m-1}} \pmod{\mathfrak{p}} \end{cases}$$

$$\therefore \forall \alpha. \quad 0 = h(\alpha) \equiv \alpha^{(p-1)p^{m-1}} \pmod{\mathfrak{p}}$$

$$\Rightarrow \alpha \notin \mathcal{O}_L^\times, \alpha \in \mathfrak{p}_L$$

m was arbitrary

$$\Rightarrow \mathcal{M}_{f,m} \subset \mathfrak{p}_L$$

\Rightarrow can substitute elements of $\mathcal{M}_{f,m}$ into $+_{F_f}, [\cdot]_f$.

by def. $\begin{cases} f \circ F_f = F_f \circ f \\ f \circ [a] = [a] \circ f \end{cases} \quad (a \in \mathcal{O}_K)$

$$\Rightarrow \begin{cases} f_m \circ F_f = F_f \circ f_m \\ f_m \circ [a] = [a] \circ f_m \end{cases} \Rightarrow \mathcal{M}_{f,m} \text{ is closed under } +_{F_f}, [\cdot]_f$$

2) $|\mathcal{M}_{f,m}| \leq \deg f_m = g^m$
 \downarrow
 $\text{Ker} (\mathcal{O}_K \ni a \longmapsto [a](\alpha) \in \mathcal{M}_{f,m})$ \swarrow \mathcal{O}_K -hom
 $\supset \#K^m$ \uparrow 高々 g^m 個
 あとは $\boxed{\#K^m \supset \text{Ker}}$ を示せばよい...

[Lemma. $g \in \mathcal{O}_L[X]$.
 $\forall \alpha \in \mathcal{M}_{f,m}, g(\alpha) = 0 \Rightarrow f_m(X) \mid g(X)$.]

ω_0 : unit of K .

$[\omega_0^m]$ kills $\mathcal{M}_{f,m}$.

\Rightarrow Lemma $f_m \mid [\omega_0^m]$.

\Downarrow coeff of X in $f_m, [\omega_0^m]$ have valuation m in L .

$[\omega_0^m] = f_m \cdot g_m$ $g_m(X)$ has const term $e \in \mathcal{O}_L^*$
 (val 0).

\downarrow
 $\forall \alpha \in \mathcal{M}_{f,m} \quad f_{m-1}(\alpha) \in \mathcal{O}_L^*$, hence non-zero.

$\Rightarrow [\omega_0^{m-1}](\alpha) = 0$ implies $f_{m-1}(\alpha) = 0$

i.e. $\alpha \in \mathcal{M}_{f,m-1}$.

対偶 $\Rightarrow \left(\alpha \in \mathcal{M}_{f,m} \setminus \mathcal{M}_{f,m-1} \Rightarrow [\omega_0^{m-1}](\alpha) \neq 0 \right)$

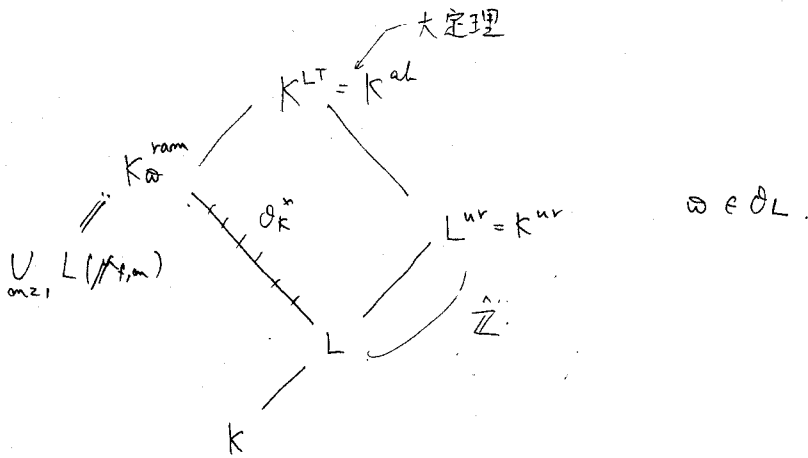
3) 2) 上): $\mathcal{M}_{f,m} = \{ [a](\alpha) \mid a \in \mathcal{O}_K \}$.

$\Rightarrow L(\mathcal{M}_{f,m}) = L(\alpha)$.

α : root of h

$\Rightarrow [L(\alpha): L] \leq \deg h = (g-1)g^{m-1}$.

- \bar{b} . const. term of $h = \omega^{\varphi^{m-1}} = \prod_{\alpha \in \mathcal{M}_{f,m} \setminus \mathcal{M}_{f,m-1}} (-\alpha)$.



$$\text{Gal}(L(K, f, m)/L) \cong (\mathcal{O}_K/\mathfrak{p}^{m^2})^\times$$

$$m^2 \geq m \quad \downarrow \quad \circlearrowright \quad \downarrow$$

$$\text{Gal}(L(K, f, m)/L) = (\mathcal{O}_K/\mathfrak{p}^m)^\times$$

Prop. $\omega, \omega' \in \mathcal{O}_{\hat{K}^{ur}}$, unit.

uniqueness over \hat{K}^{ur}

f, f' satisfy (*) for ω, ω' resp.

\Rightarrow for $\theta \in A_{\omega, \omega'}^{\hat{K}^{ur}} \cap \mathcal{O}_{\hat{K}^{ur}}^\times \neq \emptyset$
we have $[\theta]_{f, f'} \in \text{Isom}_{\mathcal{O}_{\hat{K}^{ur}}}(F_f, F_{f'})$

Prop. L/K : unram. of deg n .

Classification \overline{L}

$\omega, \omega' \in \mathcal{O}_L$, f, f' as usual.

Take $[\theta]_{f, f'} \in \mathcal{O}_{\hat{K}^{ur}}[X]$ as above

$$\Rightarrow [\theta]_{\psi^m} = [\theta] \cdot \underbrace{[N_{L/K}(\omega'/\omega)]}_\text{geometric} \cdot f$$

Gal

In particular,

$$[\theta] \in \mathcal{O}_L[X] \iff N_{L/K}(\omega') = N_{L/K}(\omega)$$

1/12

LCFT の構成.

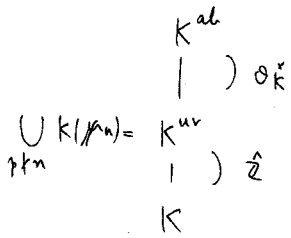
Thm K/\mathbb{Q}_p fin, $G_K := \text{Gal}(K/\mathbb{Q}_p)$.

$\exists!$ $\text{Art}_K : K^\times \rightarrow G_K^{\text{ab}}$

1) $\forall \varpi \in \mathcal{O}_K$ unif. $\Rightarrow \text{Art}_K(\varpi) |_{K^{\text{ur}}} = \text{Frob}_K$

2) $\forall K'/K$ fin. abel. $\forall x \in K'^\times \text{Art}_K(N_{K'/K}(x)) |_{K'} = \text{id}$.

Moreover, $\text{Art}_K : K^\times \xrightarrow{\sim} W_K^{\text{ab}} \subset G_K^{\text{ab}}$

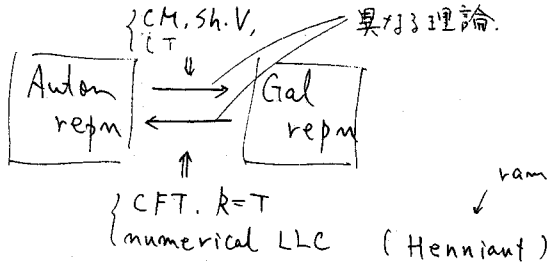


Lubin-Tate theory $\xrightarrow[\text{異対称理論}]{\text{全射}} \text{turns out to be } K^{\text{LT}} = K^{\text{ab}}$

--- construct abelian ext

K^{LT}/K

Local Kronecker-Weber Thm (ramification theory)



Lubin-Tate theory.

$L/K : \left\{ \begin{array}{l} \text{fin. unram. or} \\ L = \hat{K}^{\text{ur}} \end{array} \right.$

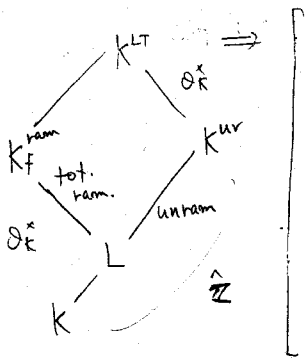
\Rightarrow \exists Lubin-Tate group $\Sigma_f := (F_f, [\cdot]_f) / \mathcal{O}_L$.

for $\forall f \in \mathcal{O}_L[[X]]$ s.t.

$f(x) \equiv \square X \pmod{\mathfrak{m}_L^2}$, $f(x) \equiv X^q \pmod{\mathfrak{m}_L}$

\uparrow (mod deg 2) unif. of L .

Σ_f relative Frobenius 射 (1-非線形)



$$\bigcup_{m \geq 1} L(f, m) \Rightarrow \left\{ \begin{array}{l} \uparrow \\ \{\alpha \mid f_m(\alpha) = 0\} = \{\alpha \mid [f_K^m](\alpha) = 0\}; \\ f^m\text{-division pts of } \Sigma_f. \end{array} \right.$$

$$\text{Gal}(K_f^{\text{ram}}/L) \xrightarrow{\sim} \mathcal{O}_K^x$$

Prop 1 $\Rightarrow K_f^{\text{ram}} \cdot \widehat{K}^{\text{ur}}$: indep of $f, \omega, L \Rightarrow K^{\text{LT}} := K_f^{\text{ram}} \cdot K^{\text{ur}}$, canonical.

特 to: $L = K \geq \mathbb{Z}$ の場合: $\text{Gal}(K^{\text{LT}}/K) = \mathcal{O}_K^x \times \widehat{\mathbb{Z}}$.

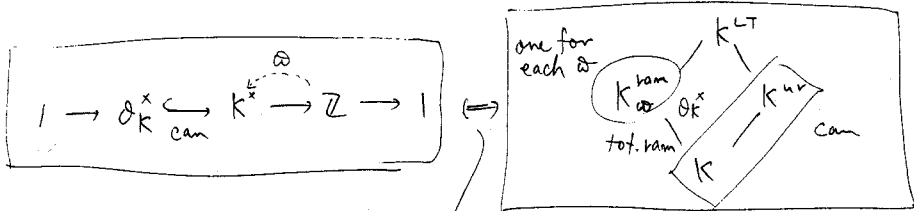
Prop 2 $\Rightarrow K_f^{\text{ram}}$ depends only on $N_{L/K}(\omega)$, write $K_{N_{L/K}(\omega)}^{\text{ram}}$

$$L = K \geq \mathbb{Z} \quad \text{Art}_K^{\omega}: K^x \xrightarrow{\sim} \mathcal{O}_K^x \times \mathbb{Z} \rightarrow \text{Gal}(K_{\omega}^{\text{ram}}/K) \times \text{Gal}(K^{\text{ur}}/K) = \text{Gal}(K^{\text{LT}}/K)$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$u \cdot \omega^n \mapsto (u, n) \mapsto ((f_{i,m}^{-1}(u))_{m \geq 1}, \text{Frob}_K^n)$$

Σ を定める Σ : Σ は \mathbb{Z} indep of ω (Prop 2 を用いる).



ω の choice に対する

さらに: K'/K : f_{im} に対する Base Change を示すには

Coleman Norm Operator を用いる.

(T.Y. LCFT via LT theory).

① LCFT の構成 $\left\{ \begin{array}{l} \text{LT theory} \dots \text{Gal}(\text{abel}) \text{ ext. } K^{\text{LT}} \subset K^{\text{ab}} \text{ の構成} \\ \text{local KW} \dots K^{\text{LT}} = K^{\text{ab}} \end{array} \right. + \left. \begin{array}{l} \text{Art}_K \\ \text{B.C.} \end{array} \right\}$



Local Langlands Corresp (i.e. $\text{Gal}(\overline{K}/K) \sim \text{LT} \dots$)

Non-abelian LT-theory.

-- étale cohomology $H^{n-1}(X \otimes \overline{K}^{\text{ur}}, \overline{\mathbb{Q}}_l)$ of the moduli sp.
 $X / \mathcal{O}_{\overline{K}^{\text{ur}}}$ of [formal \mathcal{O}_K -module of ht n + Drinfeld level str]
 realizes the LLC

-- proven via Sh.V. (global)
 Harris-Taylor

← GL₁:
 CM-theory \Rightarrow LT theory
 $E \supset \mathcal{O}_K$
 K/\mathbb{Q} : imag. quad. $\Rightarrow \left\{ \begin{array}{l} K = \mathbb{Q}_p \\ [K:\mathbb{Q}_p] = 2 \end{array} \right.$

Goal: Local construction of

$\left\{ \begin{array}{l} \text{irred. smooth (adm.)} \\ \text{rep'n of } \text{GL}_n(K) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} n\text{-dim Weil-Deligne} \\ \text{rep'n of } W_K \end{array} \right\}$

$n=1$:

K^\times

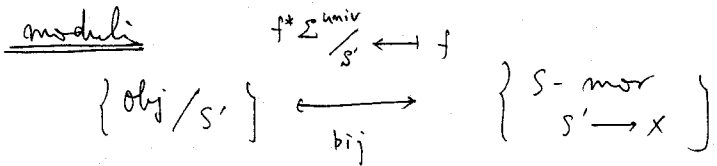
W_K^{ab}

$n=1$. • LT-theory:

$$\begin{array}{ccc} W_K^{\text{ab}} & \xrightarrow{(L^{\text{ur}} \supset \mathbb{Q})} & W_K^{\text{LT}} \xrightarrow{\sim} K^\times \\ & & \text{Art}_K^{-1} \\ \cap & & \cap \\ G_K^{\text{ab}} & \longrightarrow & G_K^{\text{LT}} \end{array}$$

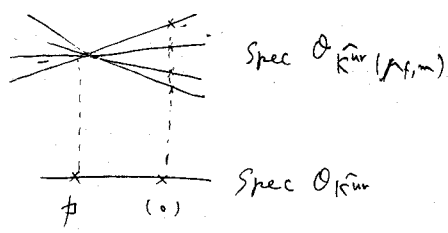
K^\times a char $\chi \Rightarrow W_K$ a char $\chi \circ \text{Art}_K^{-1}$

- Prop 1: $\exists! \Sigma / \mathcal{O}_{\widehat{K}ur} : LT \text{ gp}$
 -- moduli sp X_0 of LT-groups / $\mathcal{O}_{\widehat{K}ur} = S$:
 relative 0-dim
 $X_0 = \text{Spec } \mathcal{O}_{\widehat{K}ur}$

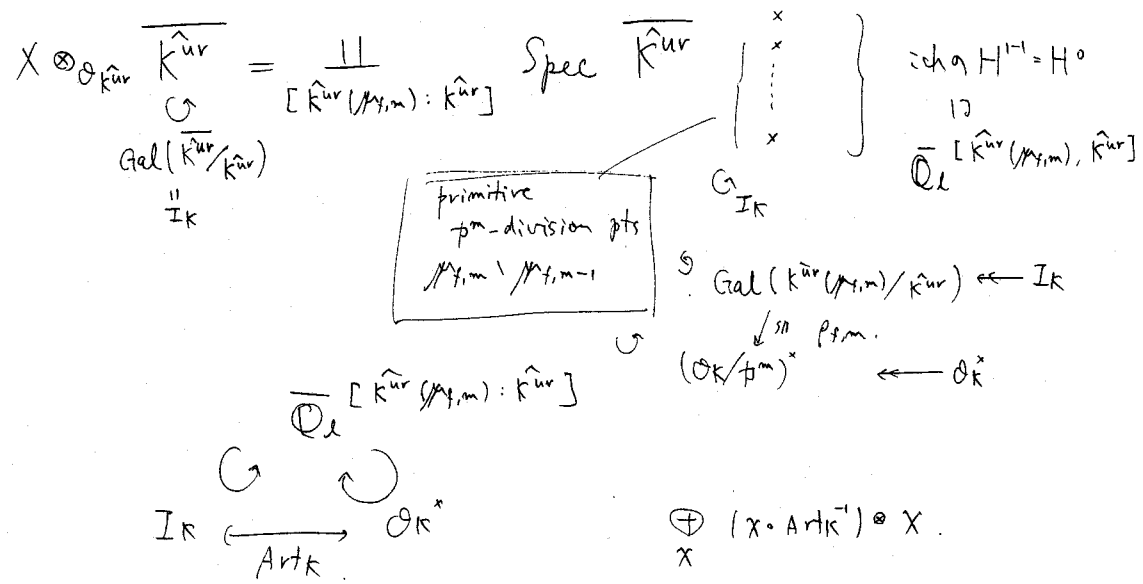


Σ^{univ} / X moduli sp of Obj's.

$\Sigma / \mathcal{O}_{\widehat{K}ur} : LT \text{ groups}$ moduli of {LT groups + Drinfeld p^m -str}
 $= \text{Spec } \mathcal{O}_{\widehat{K}ur}(\overline{\mathcal{M}_{f,m}})$



CDVF obtained by adjoining p^m -division pts $\Sigma[p^m] \mathcal{O}_K$
 $\Sigma \xrightarrow{\sim} \Sigma / \mathcal{O}_{\widehat{K}ur}$



$n \geq 1$

ArtK は Artinian. $X \mapsto X \circ \text{ArtK}$
 に対する対応 $\pi \mapsto \mathcal{L}(\pi)$ を実現する v.s. \mathbb{Q}_k
 $V = \bigoplus_{\pi} \pi \otimes \mathcal{L}(\pi)$
 を作る.

④ X_m の方程式を求めろ.

$$R = \mathcal{O}_{\hat{K}^{\text{ur}}} \quad R/\mathcal{O}_K \quad \mathfrak{g} = \#(\mathcal{O}_K/\mathfrak{p}_K).$$

$n \geq 1$. fix

Def $A: R\text{-alg}$. local $m \subset A$. complete Noetherian. $\overline{\mathbb{F}}_q \cong \mathbb{F}_m$
 Σ : formal \mathcal{O}_K -module / A .

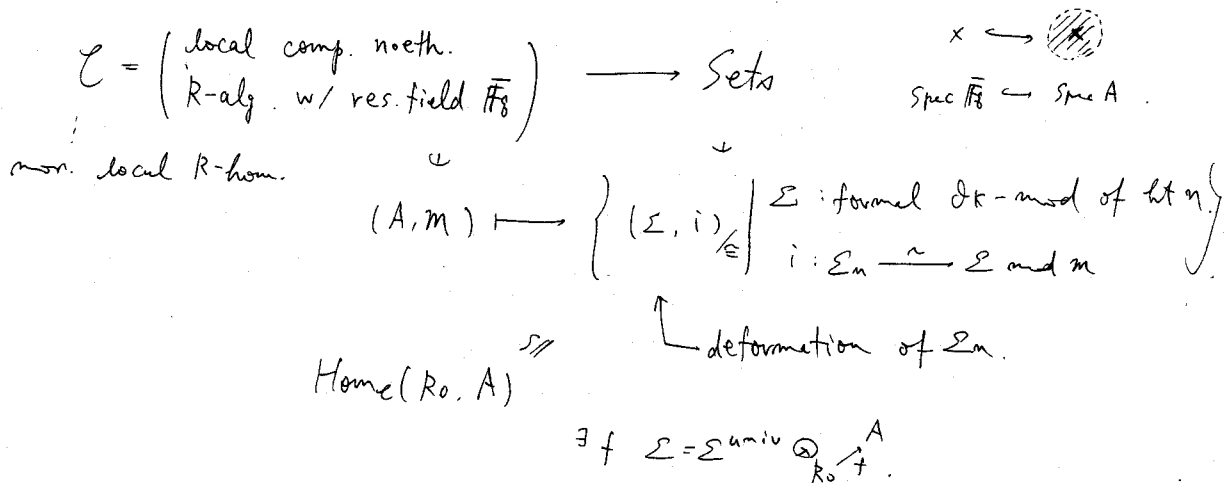
$$\Sigma: \text{height } n \iff [\varpi](X) = X^{\mathfrak{g}^n} \pmod{m + \deg \mathfrak{g}^n}$$

for all unif $\varpi \in \mathcal{O}_K$.

$n \geq 1$. Σ : deformation of Σ_n to A .

Rem $A = \overline{\mathbb{F}}_q \implies \exists! \Sigma_n / \overline{\mathbb{F}}_q$: formal \mathcal{O}_K -mod of ht n .

Thm (Drinfeld) $R_0 = R[[T_1, \dots, T_{n-1}]]$ represents the functor:



Rem • $\Sigma_2 / \overline{\mathbb{F}_3}$ formal comp. of a supersingular elliptic curve / $\overline{\mathbb{F}_3}$
 $k = \mathbb{O}_p$

• Deformation $sp =$ strict complete local ring of modular curve / \mathbb{Z}_p (2-dim sp) at ssing pt.
 Serre-Tate [def. of all curve $E =$ def of p -div $gp \in [p^2]$]

$R[[T]] = \widehat{\mathbb{Z}_p}[[T]] \dots$ 2-dim local ring

Level str. $(A, m) \in \mathcal{C}$. Σ/A : height n .

Drinfeld \mathfrak{p}^m -str. of Σ . \mathcal{O}_K -module by
 $\varphi: (\mathcal{O}_K/\mathfrak{p}^m)^m \xrightarrow{\mathcal{O}_K\text{-hom}} m^2 \quad \left(\frac{+}{\Sigma}, [\cdot] \right)$

s.t. $[\mathcal{O}^m](X) = (\text{unit}) \cdot \prod_{x \in (\mathcal{O}_K/\mathfrak{p}^m)^m} (X - \varphi(x))$

→ higher dim p -div $gp \dots ???$

↪ [Delaunay] interpretation

Thm (Drinfeld). Functor:

$\begin{cases} n \geq 1 \\ m \geq 1 \end{cases}$

$\mathcal{C} \longrightarrow \text{Sets.}$

$(A, m) \longmapsto \{(\Sigma, i, \varphi) / \cong\}$

↙ Drinfeld \mathfrak{p}^m -str.

is represented by a fin-flat R -alg. R_m ,

which is regular, regular parameters $x_i := \varphi(e_i)$

$\{e_1, \dots, e_n\}$: canonical basis of $(\mathcal{O}_K/\mathfrak{p}^m)^m$
 $(1, 0, \dots, 0) \quad (0, \dots, 0, 1)$

φ^{univ} : level \mathfrak{p}^m str on Σ^{univ}/R_m look at the leading term

$$\omega = (\text{unit}) \cdot \prod_{x \in (\mathcal{O}_K/\mathfrak{p})^m \setminus \{0\}} \varphi(x) = (\text{unit}) \prod_{(a_1, \dots, a_n) \in (\mathcal{O}_K/\mathfrak{p})^m \setminus \{0\}} ([a_1](x_1) + \dots + [a_n](x_n))$$

reg. param. of R_1

Spec $R_m = X_m$

↓ fin. flat.

Spec R_0

?

$$R_1 = R[x_1, \dots, x_m] / (\omega - \prod(\dots))$$

$X_m = \text{Spec } R_m$ $m \geq 1$ $R = \mathcal{O}_K \hat{u}$
 $R_m =$ deform ring rep'ting.

$\mathcal{C} \rightarrow \text{Sets}$

$$(A, m) \mapsto \{(\Sigma, i, \varphi) / \cong\}$$

Σ/A : formal \mathcal{O}_K -mod, ht n .

$i : \Sigma_m \xrightarrow{\sim} \Sigma \text{ mod } m / \overline{\mathbb{F}_3}$

$\varphi : (\mathcal{O}_K/\mathfrak{p}_m)^m \rightarrow m$ Drinfeld \mathfrak{p}^m -str.

$\{e_1, \dots, e_n\}$ $I_m(\varphi) = \mathfrak{p}^m$ -division pts.

$x_i := \varphi^{\text{univ}}(e_i) \quad 1 \leq i \leq n$

↑ regular param of R_m

$$R_m = R[x_1, \dots, x_n] / \left(\omega - \prod_{(a_1, \dots, a_n) \in (\mathcal{O}_K/\mathfrak{p}_m)^m \setminus (\neq \mathcal{O}_K/\mathfrak{p}_m)^m} ([a_1](x) + \dots + [a_n](x_m)) \right)$$

$n=1$

$$\omega^{\mathfrak{p}^m} = \prod_{\alpha \in \mathfrak{f}_{\mathfrak{p}_m} \setminus \mathfrak{f}_{\mathfrak{p}_m-1}} (-\alpha)$$

↑

R_m : regular
 $= 1$ eqn defines R_m

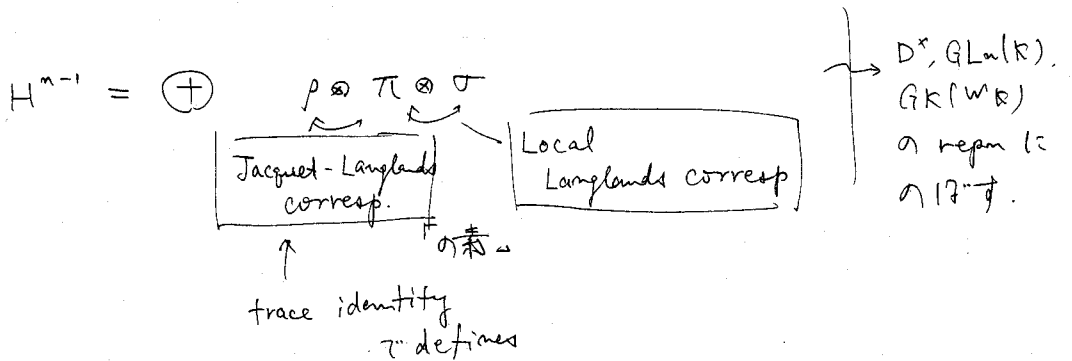
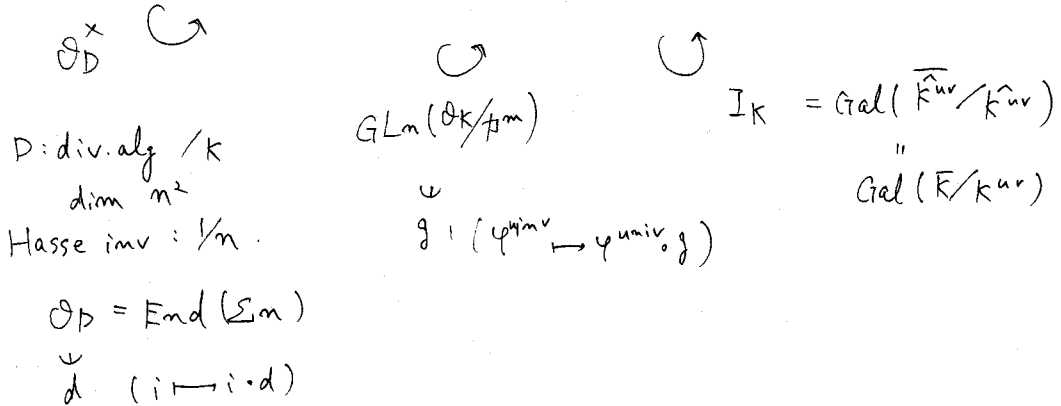
$$H_{\text{ét}}^{n-1}(X_m \otimes_R \overline{K^{ur}}, \overline{\mathbb{Q}_\ell}) \quad \ell \neq p.$$

\uparrow DVR ↖ geom. generic fibre $\{z\}$.

K/\mathbb{Q}_p : fin.

$$H_{\text{ét}}^{n-1}(X_m \otimes_R \overline{K^{ur}}, \overline{\mathbb{Q}_\ell})$$

\uparrow DVR ↖ geom. gen. fibre $\{z\}$.



Local Langlands Corresp

$$\left\{ \begin{array}{l} \text{irred. smooth. (admissible)} \\ \text{rep'n of } GL_n(K) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} n\text{-dim Frobenius s.s.} \\ \text{Weil-Deligne rep'n of } W_K \end{array} \right\}$$

Def $\mathcal{P} = (\pi, V)$: rep'n of $GL_n(K)$, U open cpt $\left\{ \begin{array}{l} \pi : GL_n(K) \rightarrow GL(V) \\ V : \text{vect. sp} / \overline{\mathbb{Q}_e} \text{ (s.t. } \infty : \mathbb{R} \times \mathbb{Z}) \end{array} \right.$

$$V^U := \{ v \in V \mid \pi(\sigma) \cdot v = v \quad (\forall \sigma \in U) \}$$

(π, V) : smooth $\Leftrightarrow \forall v \in U, \exists U, v \in V^U$
 $\Leftrightarrow V = \bigcup_{U \text{ open}} V^U$ $\dots \xrightarrow[\text{discrete}]{G/U \curvearrowright V^U}$ (smooth \Leftrightarrow discrete gp a rep'n)

(π, V) : admissible $\Leftrightarrow \forall U, \dim V^U < \infty$

Rem G : reductive p -adic Lie gp
 irred. smooth \Rightarrow admissible (Casselman)

* Bernstein-Zelvensky

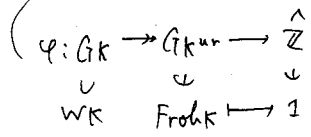
Def $\mathcal{R} = (r, N)$: Weil-Deligne rep of $W_K := \varphi^{-1}(\mathbb{Z})$

$$\left\{ \begin{array}{l} r : W_K \rightarrow GL(V) \\ N \in \text{End}(V) \end{array} \right. \begin{array}{l} \text{discrete} \\ \dim_{\overline{\mathbb{Q}_e}} V < \infty \end{array}$$

s.t. $r(\sigma) \cdot N = \varphi^{-1}(\sigma) \cdot N \cdot r(\sigma)$

- (r, N) : Froh s.s. $\Leftrightarrow r$: s.s.
- $r|_{\mathbb{Z}_K}$: factors thru fin. quot.

$\forall \sigma \in W_K \downarrow \Rightarrow N$ nilp.



Thm
(Grothendieck)
 $l \neq p$.

$$\left\{ \begin{array}{l} n\text{-dim} \\ \text{WD-repr. of } W_k \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{l} n\text{-dim} \\ \text{conti. repr. of } G_k \end{array} \right\}$$

$G_k \rightarrow GL_n(\bar{\mathbb{O}}_l)$
 \cup tame l -adic top.
 $I_k \triangleleft \hat{P}_k$
 (wild inertia pro- p)

$$I_k / \hat{P}_k \cong \prod_{l \neq p} \mathbb{Z}_l. \quad \text{te: } I_k \rightarrow \mathbb{Z}_l$$

tame char.

$$(r, N) \longmapsto \rho \text{ s.t.}$$

$$\rho(\sigma) = r(\sigma) \cdot \exp(\text{te}(\sigma) \cdot N)$$

$$\text{WD}(\rho) \longleftarrow \rho$$

- l -adic monodromy.
- $l = p$, Fontaine p -adic Hodge.
- $\rho \mapsto \text{WD}(\rho)$ p -st-functor.

(R) n -dim Froh s.s. WD repm of W_k
 indecomposable

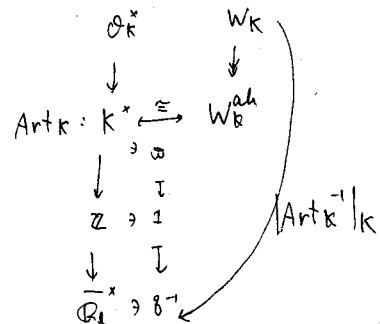
$$R = \bigoplus_{i=1}^t S_{p s_i}(R_i)$$

simi-dim

R_i : irred. $(r_i, 0)$
 n_i -dim WD-rep.

$$n = \sum_{i=1}^t s_i n_i \quad S_p(R) = S_p \otimes R$$

$\left\{ \begin{array}{l} \text{non-triv } N \\ \text{reducible} \\ \text{indecomposable} \end{array} \right\} \rightarrow S_p = \langle e_1, \dots, e_s \rangle$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} r(\sigma) e_i = |\text{Art}_{\mathbb{F}}^{-1}(\sigma)|_K \cdot e_i \\ N e_i = e_{i+1} \end{array}$$


$[\text{Art}_K^{-1}|_K: \text{Froh} \mapsto \varphi^{-1} \dots \text{cyclotomic char } \chi.$

① ρ : irred smooth rep of $GL_n(k)$

$$\rho = \bigoplus_{i=1}^r \text{Sp}_{s_i}(\rho_i)$$

ρ_i : supercuspidal rep of $GL_{s_i}(k)$
 (matrix coeff is cpt supported).



$\text{Sp}_{s_i}(\rho_i)$: square integrable (discrete series) rep of $GL_{s_i}(k)$.
 特別(1,1,1)°の parabolic ind.



$$\bigoplus_{i=1}^r \text{Sp}_{s_i}(\rho_i) = \text{Langlands sum (parabolic ind)} = \text{m-Ind}_H^G \left(\prod_{i=1}^r \text{Sp}_{s_i}(\rho_i) \right)$$

$\left(\prod_{i=1}^r \text{Sp}_{s_i}(\rho_i) \right)$ $GL_{s_i}(k)$ の rep.

Langlands sum (parabolic ind)

$$H = \left(\begin{array}{c|c} \begin{array}{c} s_1 n_1 \\ s_2 n_2 \\ \vdots \\ \theta \end{array} & \\ \hline & \end{array} \right) \subset GL_n(k) = G.$$

② LLC characterization --- 現状では L-factor, ϵ -factor (of pairs) を使う

supercuspidal \longleftrightarrow irred. WD-repm $(r, 0)$

$r: W_K \rightarrow GL(r)$ $\exists r|_{\mathbb{Z}_K} = \rho \otimes \chi$
 \mathbb{Z}_K の fin. quot. \exists factor $\rho \otimes \chi$ repm.

Brauer's thm

$$r|_{\mathbb{Z}_K} = \sum_i \left(\text{Ind}_{\mathbb{Z}_K}^{\mathbb{Z}_K} \chi_i \right)^{m_i}$$

$m_i \in \mathbb{Z}$.

$$\chi_i: \mathbb{Z}_K \rightarrow \overline{\mathbb{Q}}^\times$$

LCFT

$$\mathbb{Z}_K \rightarrow \overline{\mathbb{Q}}^\times$$

Automorphic Induction

??

Induction?

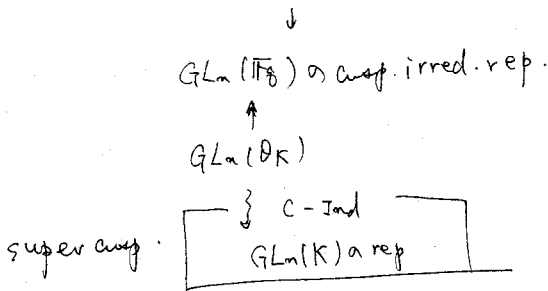
LLC is characterized by functoriality

$$\left. \begin{array}{l} n=1 : \text{LCFT} \\ \text{AI} \longleftrightarrow \text{Ind} \\ \text{PI} \longleftrightarrow \oplus \\ \text{Sp} \longleftrightarrow \text{Sp} \end{array} \right\}$$

さらに、これに Non-abelian LT-theory (geometric) を realize.

$m=1 \dots R_1$ を blow up.

$$\Rightarrow \boxed{\text{Deligne-Lusztig theory}} \\ \text{GL}_n(\mathbb{F}_q) \hookrightarrow \mathbb{F}_q^\times$$



$[L:K] = n$
 \mathbb{F}_q^\times の char
 \uparrow
 $I_K = I_L$ の char
 W_L と W_K に ind $\Rightarrow n$ -dim

Dream \dots

- $X_m = \text{Spec } R_m$ の geometry
- \Rightarrow [char と ζ の Ind に 対応する]
- AI と geometric に τ -exists? (DL の 拡張)
- $\text{Sp}_s \leftrightarrow \text{Sp}_s$ の geometric realization
- \dots [Iwahori Hecke alg の rep'm の $\frac{q}{12}$]
- $\text{Sp}_s(\mathbb{I})$ χ : unram.

Non-abelian L-T for local geometric τ -exists, ...

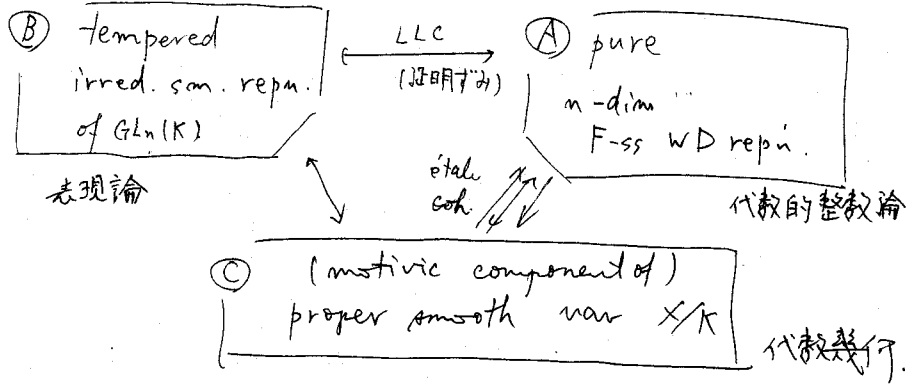
$$I_{W_m} \subset \text{GL}_n(\mathcal{O}_K) \subset \text{GL}_n(K)$$

$$\{g \mid g \equiv \begin{pmatrix} * & \\ 0 & * \end{pmatrix} \pmod{\mathfrak{p}}\}$$

$V = V^{I_{W_m}}$ τ generate \mathcal{Z} of Am. rep.

$V^{I_{W_m}} : \mathbb{Q}_\ell [I_{W_m} \backslash \text{GL}_n(K) / I_{W_m}]$ の f.d. rep.

• Dream 2.
 K/\mathbb{Q}_p : fin



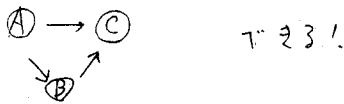
③ → ① ... Weight-Monodromy conj.
 (Weil conj の一般化)

Global is λ -adic Gal rep.

$G_{\mathbb{Q}} \rightarrow GL_n(\overline{\mathbb{Q}_\lambda})$ の local comp ... pure (証明済?)

Autom rep $\Pi = \otimes \Pi_v$ of $GL_n(\mathbb{A})$

の local comp ... tempered
 (Generalized Ramanujan conj)



① → ②: $R = (r, N) \xrightarrow{LLC} P = (\pi, V) \in$ local comp
 autom. rep. $\Pi = \otimes \Pi_p$ of $GL_n(\mathbb{A}_F)$
 ($\Pi_p = \phi$)

Shimura Var の cohomology is
 Global Langlands の実現 $\left\{ \begin{array}{l} F: \text{CM-field} \\ \Pi: \text{regular cuspidal self-dual} \end{array} \right.$

$G_F \times GL_n(\mathbb{A}_F^\times) \backslash H^{n-1}(Sh(V, \overline{\mathbb{Q}_\lambda})[\Pi^\infty])$: λ -adic Galois rep of G_F .

↓ (実現)

↓
 の Galois rep ρ $G_K \subset G_F$ に制限すると K に戻す $K = \mathbb{F}_p$
 (L.T. W.D.E.R.)

(Global-Local Compatibility Taylor-Yoshida)

$\mathcal{X} \otimes K \cong K$ X/K : strictly semistable.

$\Leftrightarrow \exists \mathcal{X}/\mathcal{O}_K$ \mathcal{X} : Zariski local étale cover $\mathcal{O}_K[T_1, \dots, T_m]/(\omega - T_1 \cdots T_m)$

local case NALT \rightarrow

Iwahori-Hecke case