Harvard 3/7/07

§1. Intro.
§2. Langlands Corresp.
[i) local, ii) global.]
§3. LC + Motives
[i) global obj's are motivic, ii) motivic obj / $\mathbb{F}_q$, iii) motivic local obj.]

Converse of Weight-Monodromy Conj.

§1.

LC: Ⓐ Algebraic Number Theory (Galois rep's) ↔ Ⓑ Representation Theory (Autom. rep's of GL_n)

Ⓒ Algebraic Geometry (Pure Motives of alg. var.'s)

Base field $[L:\mathbb{Q}]<\infty$ $\qquad$ $[K:\mathbb{Q}_p]<\infty$

$$\begin{tikzcd} A \arrow[rr, dashed] & & B \arrow[dl, dashed] \\ & C \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, leftrightarrow] \arrow[dr] & & B \arrow[dl] \\ & C \arrow[ul, dotted, bend left, "WMC?"] & \end{tikzcd}$$

$\longrightarrow$ known

$\dashrightarrow$ partially known.

§2. i) Local.

$[K:\mathbb{Q}_p]<\infty$. $\mathcal{O}_K/\mathfrak{p}_K = k \simeq \mathbb{F}_q$. $K^{ur} := \bigcup_{p\nmid N} K(\mu_N)$.

Ⓐ $\varphi: Gal(\overline{K}/K) \twoheadrightarrow Gal(K^{ur}/K) \xrightarrow{\sim} Gal(\overline{k}/k) \xrightarrow{\sim} \hat{\mathbb{Z}}$

$\mathrm{Frob}_K \mapsto (x\mapsto x^q)^{-1} \longmapsto 1$

$W_K := \varphi^{-1}(\mathbb{Z})$. $I_K := \mathrm{Ker}\,\varphi$. $\Omega := \mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$ (ℓ: prime)

Def. Weil-Deligne rep (r,N,V) $/\Omega$ of W_K: $\begin{cases} r: W_K \to GL(V) & (\dim_\Omega V<\infty) \\ N\in \mathrm{End}(V) \end{cases}$

s.t. $\begin{cases} \mathrm{Ker}(r|_{I_K}) \triangleleft I_K : \text{fin. index.} \\ r(\sigma)N = \chi(\sigma)N r(\sigma) \quad (\forall \sigma\in W_K) \end{cases}$ $\qquad \chi: W_K \to \Omega^\times$, $\sigma \mapsto q^{-\varphi(\sigma)}$

$\forall \ell\neq p$.

Th'm. $\{n\text{-dim WD rep's}/\overline{\mathbb{Q}}_\ell \text{ of } W_K\} \xleftrightarrow{\text{bij.}} \{\text{cont. rep } W_K \to GL_n(\overline{\mathbb{Q}}_\ell)\}$.

$(r,N,V) \mapsto (\sigma \mapsto r(\sigma)\cdot\exp(t(\sigma)N) \in GL(V))$

$t: I_K \to \mathbb{Z}_\ell$

Rem. $\{\text{WD rep}/\overline{\mathbb{Q}}_\ell\} \simeq \{\text{WD rep}/\mathbb{C}\}$.

ex. $Sp_n := (r, N, \langle e_1,\cdots,e_n\rangle)$. $r(\sigma)e_i := \chi(\sigma)^{i-1}e_i$ $\quad Ne_i := e_{i+1}$

Prop. If r: s.s. (we say Frob-s.s.), then $\exists n_i, r_i$.

$(r,N) \cong \bigoplus_i Sp_{n_i}\otimes(r_i, 0, V_i)$ $\quad (r_i: W_K \to GL(V_i) : \text{irred.})$

Ⓑ $\pi : GL_n(K) \longrightarrow GL(V)$. V : Ω-vect. sp.

Def. (π, V) : smooth $\Leftrightarrow$ $\forall v \in V$. $\exists U \subset GL_n(K)$: open cpt. s.t.
$v \in V^U := \{ v \in V \mid \pi(\sigma)v = v \ \ \forall \sigma \in U \}$.

Thm (LLC).

$$\underset{Ⓐ}{\{\text{Frob-s.s. } n\text{-dim. WD-rep}/\mathbb{C} \text{ of } W_K\}} \overset{\text{bij.}}{\longleftrightarrow} \underset{Ⓑ}{\{\text{irred. sm. rep}/\mathbb{C} \text{ of } GL_n(K)\}}$$
$$r = (r, N, V) \longleftrightarrow \pi.$$

ex. $n=1$: $\chi : W_K \longrightarrow \mathbb{C}^\times \longleftrightarrow \chi \circ \mathrm{Art}_K$. $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{ab}$

$r \otimes \chi \longleftrightarrow \pi \otimes (\chi \circ \mathrm{Art}_K \circ \det)$

$r_1 \oplus r_2$, $\overset{\text{irred}}{r} \longmapsto Sp_n \otimes r \longleftrightarrow \pi_1 \boxplus \pi_2$. $\pi \longmapsto St_n(\pi)$

r : irred $\longleftrightarrow \pi$: cuspidal.

$\{ r$: unram. $(r|I_K = id.\ N=0) \} \longleftrightarrow \left\{ \begin{array}{l} \pi : \text{spherical gen. by } \overset{1\text{-dim.}}{\pi^U} \circlearrowleft.\ U = GL_n(\mathcal{O}_K) \\ \mathbb{C}[U\backslash G/U] \simeq \mathbb{C}[X_1^\pm, \dots, X_n^\pm]^{\mathfrak{S}_n}. \end{array} \right\}$

e.v. of Frob_K $\downarrow \wr$ $\quad \sim$

$(\mathbb{C}^\times)^n / \mathfrak{S}_n$. $\longleftarrow$ e.v. of $X_1 \cdots X_n \curvearrowright \pi^U$ (Satake param.)

$\boxed{\text{ii) global.}}$ $[L:\mathbb{Q}] < \infty$. $G_L := \mathrm{Gal}(\bar{\mathbb{Q}}/L)$. $v | p \Rightarrow [L_v : \mathbb{Q}_p] < \infty$

Ⓐ Galois rep'n : $R : G_L \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell)$ ... irred. cont. s.t.
$R|_{G_{L_v}}$: i) unram. for a.a. v, ii) de Rham if $v | \ell$.

Ⓑ Cuspidal Autom. Rep. $\Pi = \bigotimes_v \Pi_v$ of $GL_n(\mathbb{A}_L)$ in $L^2(GL_n(L)\backslash GL_n(\mathbb{A}_L))_{cusp}$
$(\Pi_v$: irred. sm. rep of $GL_n(L_v)$ if $v \nmid \infty)$ s.t. algebraic if $v | \infty$

Conj. (GLC). For $\forall \iota : \bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$. $\exists$bij. $\{R\} \longleftrightarrow \{\Pi\}$.
s.t. $\iota R|_{W_{L_v}} \overset{\text{LLC}}{\longleftrightarrow} \Pi_v$. $(\forall v \nmid \ell)$.

Rem. · R : determined by $R|_{W_{L_v}}$ for a.a. v (Chebotarev density).
· Π : ———— Π_v ———— (Strong Multiplicity One).
· GLC $\Rightarrow$ {ℓ-adic Gal. rep.} : indep. of ℓ.

$\boxed{\text{§3. i) global obj's are motivic}}$

(C). Thm. (Etale Cohomology). X : proper smooth var. $/L$ $\rightsquigarrow$ R : irred. subgl of $H^i(X\otimes\bar{L}, \bar{\mathbb{Q}}_\ell) \circlearrowleft G_L$.

$\Rightarrow R$: Gal. rep in the above sense

$\because$) for a.a. v. $\exists\, \mathcal{X}/\mathcal{O}_{L_v}$: prop. sm. s.t. $X\otimes L_v \simeq \mathcal{X}\otimes L_v$.

$$H^i(X\otimes\bar{L}, \bar{\mathbb{Q}}_\ell) \xrightarrow{\sim} H^i(X\otimes\bar{L}_v, \bar{\mathbb{Q}}_\ell) \xrightarrow{\sim} H^i(\mathcal{X}\otimes\bar{k}_v, \bar{\mathbb{Q}}_\ell)$$
$$\circlearrowleft G_L \longleftrightarrow G_{L_v}\circlearrowleft \longrightarrow G_{k_v}\circlearrowleft \quad \Rightarrow \text{unram at } v.$$

p-adic Hodge theory $\Rightarrow$ de Rham at $v|\ell$.

(A) $\to$ (C).

Conj. (Fontaine-Mazur) All R arises in this way up to Tate twist. (is motivic)

[$\Rightarrow$ Hodge symmetry for all R]

$$\begin{tikzcd} (A) \arrow[rr, dashed, "\text{Modularity}"] & & (B) \arrow[dl, dashed, "\text{Shimura Var}"] \\ & (C) \arrow[ul, "\text{EC}"] & \end{tikzcd}$$

$\boxed{\text{ii) motivic obj's}/\mathbb{F}_q}$

(C) Thm (Weil Conj.) $\ell\neq p$. $X/\mathbb{F}_q$: proper smooth

$\Rightarrow$ $\forall$ e.v. of $\mathrm{Frob}_q \curvearrowright H^i(X\otimes\bar{\mathbb{F}}_q, \bar{\mathbb{Q}}_\ell)$ are Weil q^i-numbers.

[i.e. $c\in\bar{\mathbb{Q}}$ s.t. $\forall\iota : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. $\iota(c)\cdot\overline{\iota(c)} = q^i$]

Rem. Conj... Frob_q : always s.s.

(A) Cor. ($\exists$Weight) R : motivic $\Rightarrow$ $\exists i\in\mathbb{Z}$. for a.a. v. e.v. of $\mathrm{Frob}_v \curvearrowright R|_{W_{L_v}}$ are Weil q^i-#.

(B) Cor. (Ramanujan Conj.) Π : motivic R (assoc. to) $\Rightarrow$ —— Satake param of Π_v ——
[tempered spherical]

$$\begin{tikzcd} \text{(A) Unram reps w/ Weight} \arrow[rr, leftrightarrow, "\boxed{\text{Weil \#'s}}", "\shortparallel"'] \arrow[dr, leftrightarrow, "\text{HT.}"] & & \text{(B) Tempered Spherical rep's.} \arrow[dl] \\ & \text{(C) Motives}/\mathbb{F}_q. \arrow[ul, "\text{WC.}"] & \end{tikzcd}$$

[Motives always come from global obj.]

Thm. (Honda-Tate). All Weil #'s are motivic.

$\because$) realize (B) as Π_v of Hecke char Π / CM-field.
use (B) $\dashrightarrow$ (C) for GL_1/CM-field. (CM abel var. = 0-dim Sh.V.)

$\boxed{\text{(iii) motivic local objects.}}$ $[K:\mathbb{Q}_p]<\infty$. $k \cong \mathbb{F}_q$.

Ⓐ **Def.** $(r,N)=\bigoplus Sp_{n_j}(r_j)$: pure of wt $i \in \mathbb{Z}$ if: $\forall$e.v. of one ($\Rightarrow$ any) lift of Frob_K in r_j are Weil $q^{i+(n_j-1)}$-#.

Ⓒ **Conj.** (WMC) $\ell \neq p$. X/K : pr. sm. $\Rightarrow$ $W_K \curvearrowright H^i(X\otimes\bar{K},\bar{\mathbb{Q}}_\ell)^{F\text{-s.s.}}$: pure of wt i

Ⓐ **Conj.** + Ⓐ→Ⓒ $\forall R$. $\forall v$. $R|_{W_{K_v}}$: pure.

Ⓑ **Conj.** + Ⓑ→Ⓒ $\forall \Pi$. $\forall v$. Π_v : tempered. (Generalized Ramanujan Conj.)
($\Leftrightarrow$ assoc. to pure (r,N)).

Ⓐ Pure WD-rep's $\longleftrightarrow$ Ⓑ Tempered rep's
π^σ: tempered for $\forall\sigma\in\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$
NAHT.
WMC ?
Ⓒ Motives $/K$.

Thm (Non-abelian Honda-Tate). All pure WD rep's are motivic [subqts of $H^i(X\otimes\bar{K},\bar{\mathbb{Q}}_\ell)$, X/K pr. sm.] (up to F-s.s.-fication)

pf. $H^*(X\times Y)=H^*(X)\otimes H^*(Y)$. can twist by (Hecke char's by HT) Weil #'s.
$H^*(X\amalg Y)=H^*(X)\oplus H^*(Y)$. can reduce to $Sp_n\otimes r$.

① $\Longrightarrow$ go to Ⓑ. $Sp_n\otimes r \longleftrightarrow St_n(\pi)$. realize this as Π_v.
Π/CM field. use Ⓑ→Ⓒ. Harris-Taylor. Taylor-Y.

② $\Longrightarrow$ · Sp_n …. use { p-adically uniformized var. (Ito) / some family w/ big monodromy.
· r …. go to Ⓑ. $r \longleftrightarrow \pi = \Pi_v$. use Ⓑ→Ⓒ. Harris-Taylor.

③ $\Longrightarrow$ r … direct summand of $\mathrm{Ind}_{W_{K'}}^{W_K}\mathbb{1}$. up to twisting by char.
$[K':K]<\infty$. $X=\mathrm{Spec}\,K'$. (Honda-Tate).

(Taylor)
Q. characterize geometric p-adic Gal. rep'ns ? (alg. cl. fields) descent.
· ℓ-indep notion of Gal. rep'ns — funct. (top. alg. cl. fields) $\to$ (tensor cat.)
(Mazur)