Compatibility of local and global Langlands correspondences

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(joint work with Richard Taylor)

Let L be a number field (finite over $\mathbb{Q}$), n a positive integer, ℓ a fixed prime and $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ a fixed field isomorphism. We denote the absolute Galois group of L by $G_L = \mathrm{Gal}(\overline{L}/L)$. Conjectural global Langlands correspondence predicts a correspondence between (A) algebraic automorphic representation Π of $GL_n(\mathbb{A}_L)$ and (B) n-dimensional ℓ-adic Galois representation $R : G_L \to GL_n(\overline{\mathbb{Q}}_\ell)$ which is (1) almost everywhere unramified and (2) de Rham at ℓ. The cuspidals in (A) should correspond to irreducibles in (B). We are interested in cuspidal Π, and we denote the (conjectural) Galois representation attached to Π by $R = R_{\ell,\iota}(\Pi)$. Up to semisimplification, it is characterized by the property that the eigenvalues of Frobenius Frob_v equal the Satake parameters of Π_v for almost all places of v. One of the most general results in the direction $\Pi \longmapsto R$ is the one obtained by Kottwitz [K], Clozel [C] and Harris-Taylor [HT], which constructs the semisimple representation $R_{\ell,\iota}(\Pi)$ when L is an imaginary CM-field and Π is cuspidal, satisfying (1) Π is conjugate self-dual, (2) Π is regular algebraic and (3) there is a finite place x of L where Π_x is (essentially) square-integrable. We show the compatibility of this correspondence with the local Langlands correspondence at all places outside ℓ; we need to fix notations for the local Langlands correspondence.

Let K be a finite extension of $\mathbb{Q}_p$ and n a positive integer. We denote the maximal unramified extension of K by K^{ur}, the (geometric) Frobenius by Frob, and the Weil group by $W_K = \{\sigma \in G_K \mid \sigma|_{K^{ur}} \in \mathrm{Frob}^{\mathbb{Z}}\}$. The local Langlands correspondence rec (proved by Harris-Taylor [HT] and Henniart [He]) gives the correspondence from (A$'$) irreducible admissible representations (over $\mathbb{C}$) of $GL_n(K)$ to (B$'$) n-dimensional F-semisimple Weil-Deligne representations (over $\mathbb{C}$) of W_K. Recall (see Tate [Ta]) that a Weil-Deligne representation is a pair (r, N) of a finite dimensional representation $r : W_K \to GL(V)$ and an $N \in \mathrm{End}(V)$ such that $r(\sigma)N = \chi(\sigma)Nr(\sigma)$ for all $\sigma \in W_K$, where $\chi : W_K \to \mathbb{Q}^\times$ is the composite of the local reciprocity map $W_K \to W_K^{ab} \cong K^\times$ (sending lifts of Frob to uniformizers) and the normalized absolute value $|\,|_K : K^\times \to \mathbb{Q}^\times$. We can define the F-semisimplification $r^{F\text{-ss}}$ of r, and write $(r, N)^{F\text{-ss}} = (r^{F\text{-ss}}, N)$ and $(r, N)^{\mathrm{ss}} = (r^{F\text{-ss}}, 0)$. The cuspidal (resp. square integrable, tempered) representations in (A$'$) correspond to irreducible (resp. indecomposable, pure) representations in (B$'$). For the definition of pure Weil-Deligne representations, see [TY]. For a prime ℓ and an ℓ-adic Galois representation ρ of G_K (assume de Rham when $\ell = p$), denote the associated Weil-Deligne representation (over $\overline{\mathbb{Q}}_\ell$) of W_K by $\mathrm{WD}(\rho)$ (for $\ell \neq p$ use quasi-unipotence of Grothendieck; for $\ell = p$ use Berger's theorem and Fontaine's functor D_{pst}).

Theorem A. (Harris-Taylor [HT], Taylor-Yoshida [TY]) Let L be a CM-field, ℓ a prime and fix $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. Let Π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ satisfying the three conditions above, and $R_{\ell,\iota}(\Pi)$ be the associated

ℓ-adic representation of G_L. Then, for all finite place v of L not dividing ℓ:

$$\iota \mathrm{WD}(R_{\ell,\iota}(\Pi)|_{G_{L_v}})^{F\text{-ss}} \cong \mathrm{rec}\big(\Pi_v^\vee \cdot |\det|_K^{\frac{1-n}{2}}\big)$$

as Weil-Deligne representations over $\mathbb{C}$ of W_{L_v}.

Some remarks on the theorem:

(1) As we assumed the existence of x where Π_x is square-integrable, we obtain the irreducibility of $R_{\ell,\iota}(\Pi)$ if x does not divide ℓ. This is because $R_{\ell,\iota}(\Pi)$ is semisimple by definition, and the theorem tells us that its restriction at x is indecomposable.
(2) The equality of $(\ )^{\mathrm{ss}}$ of both sides in the theorem was one of the main results of Harris-Taylor ([HT], Introduction, Theorem C). The new result in [TY] is the determination of the monodromy operator N.
(3) In [TY], the theorem for v dividing ℓ is shown to follow from the functoriality of the p-adic weight spectral sequence of Mokrane [M].

We sketch the idea of proof of the theorem. First we note that the temperedness of Π_v is shown in [HT] (Introduction, Theorem C). Hence, by (Theorem A)$^{\mathrm{ss}}$, it suffices to prove that the left hand side is pure (in particular, this follows from the Weight-Monodromy Conjecture). Using the global base change, we reduce to the case when Π_v has a fixed vector by the Iwahori subgroup $\mathrm{Iw}_n = \{g \in GL_n(\mathcal{O}_{L,v}) \mid g \bmod v \text{ is upper triangular }\}$. We descend Π to an automorphic representation π of a unitary group G which locally at v looks like GL_n and at infinity looks like $U(1,n-1) \times U(0,n)^{[L:\mathbb{Q}]/2-1}$. Then we realise $R_{\ell,\iota}(\Pi)$ in the cohomology of a Shimura variety X associated to G with Iwahori level structure at v. More precisely (assume the infinitesimal character to be trivial for simplicity), the representation $R_{\ell,\iota}(\Pi)$ appears inside the semisimplification of the π^p-isotypic component of $H^{n-1}(X, \overline{\mathbb{Q}}_\ell)$. We show that X has strictly semistable reduction at v with a nice moduli-theoretic definition of the strata of the special fiber (the reduction of X at v), and use the results of [HT] to compute the cohomology of these (smooth, projective) strata as a virtual $G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_v^{\mathbb{Z}}$-module. This description and the temperedness of Π_v shows that the π^p-isotypic component of the cohomology of any strata is concentrated in the middle degree. This implies that the π^p-isotypic component of the Rapoport-Zink weight spectral sequence ([RZ], [S]) degenerates at E_1, which shows that $\mathrm{WD}\big(H^{n-1}(X, \overline{\mathbb{Q}}_\ell)[\pi^p]|_{G_{L_v}}\big)$ is pure.

In the special case that Π_v is a twist of a Steinberg representation and Π_∞ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I]. After we had posted the first version of this paper, Boyer [B] has announced an alternative proof with presumably stronger results.

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References

[B] P. Boyer, *Faisceau pervers des cycles évanescents des variétés de Drinfeld et filtration de monodromie-locale du modèle de Deligne-Carayol (avec l'appendice de L. Fargues)* (version préliminaire), `http://www.institut.math.jussieu.fr/~boyer/`.

[C] L. Clozel, *Représentations Galoisiennes associées aux représentations automorphes autoduals de $GL(n)$*, Publ. Math. IHES, **73** (1991), 97–145.

[He] G. Henniart, *Une preuve simple des conjectures de Langlands pour $GL(n)$ sure un corps p-adiques*, Invent. Math. **139** (2000), 439–455.

[HT] M. Harris, R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, Ann. of Math. Studies **151**, Princeton Univ. Press, Princeton-Oxford, 2001.

[I] T. Ito, *Weight-monodromy conjecture for p-adically uniformized varieties*, Invent. Math. **159** (2005), 607-656.

[K] R. Kottwitz, *On the λ-adic representations associtated to some simple Shimura varieties*, Invent. Math. **108** (1992), 653–665.

[M] A. Mokrane, *La suite spectrale des poids en cohomologie de Hyodo-Kato*, Duke Math. J. **72** (1993), no. 2, 301–337.

[RZ] M. Rapoport, T. Zink, *Über die lokale Zetafunktion von Shimuravarietäten, Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Invent. Math. **68** (1982), no. 1, 21–101.

[S] T. Saito, *Weight spectral sequences and independence of ℓ*, J. Inst. Math. Jussieu **2** (2003), 583–634.

[Ta] J. Tate, *Number Theoretic Background*, in: A. Borel and W. Casselman, ed., *Automorphic Forms, Representations and L-functions*, Proc. Symp. in Pure Math. **33**–2, AMS, 1979.

[TY] R. Taylor, T. Yoshida, *Compatibility of local and global Langlands correspondences*, December 2004, `math-NT/0412357`.