Non-abelian Lubin-Tate Theory (M24)

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Goals

A. Statement of NALT. (including Local Langlands [Ha], [Ca], [HT] + Boyer, Dat,...)
B. Exposition of the local approach (tamely ramified case, [Y1], [Y3]).

aside-a. Statement of Weil Conjecture / Weight-Monodromy Conjeture.
(and their relation to Ramanujan Conjecture / Generalized Ramanujan Conjecture)
aside-b. Relation with Global Langlands Correspondence for GL_n.
(and their realization in the theories of Complex Multiplication / Shimura Varieties)

Topics covered (emphasis is on I>II>III)

I. *Algebraic Number Theory*: ℓ-adic Galois representations of local fields.
II. *Algebraic Geometry*: ℓ-adic vanishing cycle cohomology of Lubin-Tate spaces.
III. *Representation Theory*: Smooth representations of GL_n over local fields.

Notions Introduced

I. - Local class field theory / Lubin-Tate groups. (**A**)
- Weil-Deligne representations / their purity. (**A**)

II. - Formal $\mathcal{O}$-modules / their deformations. (**A**)
- Lubin-Tate spaces / Lubin-Tate towers. (**A**) (= "local Shimura varieties" for GL_n)
- ℓ-adic cohomology. (**A**) (étale sheaves / derived categories / vanishing cycles)
- Intersection theory. (**B**)

III. - Smooth representaions of GL_n over local fields. (**A**)
(parabolic inductions / square-integrables / supercuspidals)
- Satake parameters / temperedness. (**A**)
- Hecke algebras and the modules over them (**B**).

Theorems proven or partially proven

I. - Lubin-Tate theory, [LT], [Y2]. (**A**)
- Grothendieck's ℓ-adic monodromy theorem, [ST]. (**A**)
- Classification of Frobenius-semisimple WD rep's, [Ta]. (**A**)

II. - Drinfeld's theorem 1: uniqueness of formal $\mathcal{O}$-modules over $\overline{\mathbb{F}}_q$, [Dr]. (**A**)
- Drinfeld's theorem 2: representability of the deformation functors, [Dr]. (**A**)
- Intersection theory of Hecke correspondences in Iwahori level, [Y3]. (**B**)
- Construction of Deligne-Lusztig model in level $\mathfrak{p}$, [Y1]. (**B**)

III. - Construction of modules over Iwahori Hecke algebras. (**B**)

Theorems stated (plan)

I. - Local class field theory, [Iw], [Y2]. (**A**)
II. - Basic theorems on ℓ-adic cohomology / vanishing cycles, [SGA4.1/2], [SGA7]. (**A**)
(finiteness / proper base change / Lefschetz trace formula / formal theory)
- Vanishing cycles of semistable schemes and its monodromy filtration, [Sai]. (**B**)
- Deligne-Lusztig theory, [DL]. (**B**)
III. - Basic theorems on smooth representation theory of p-adic groups. (**A**)
(admissibility / Jacquet modules / Bernstein-Zelevinsky classification)
- Satake transforms, [Sat]. (**A**)
- Cuspidals of GL_n over finite fields / depth 0 supercuspidals. (**B**)

Tentative Schedule

Week 0 (05/10): Overview.
Week 1 (08/10 - 12/10): Lubin-Tate theory. (**I**)
Week 2 (15/10 - 19/10): Formal $\mathcal{O}$-modules (1). (**II**)
Week 3 (22/10 - 26/10): Formal $\mathcal{O}$-modules (2). (**II**)
Week 4 (29/10 - 02/11): ℓ-adic cohomology of Lubin-Tate spaces. (**II**)
Week 5 (05/11 - 09/11): ℓ-adic Galois rep's and WD-rep's. (**I**)
Week 6 (12/11 - 16/11): Smooth rep's of GL_n and NALT. (**III, A**)
Week 7 (19/11 - 23/11): Iwahori level case. (**B**)
Week 8 (26/11 - 30/11): Level $\mathfrak{p}$ case. (**B**)

Comments

I gave a basic reference list but these papers are by no means easy to read and you should not jump on them and be discouraged; I make haste to confess that many of them I myself could not penetrate. I will try to use two different tones of lecturing - when we are proving the basic results which do not need much prerequisites we'll go through the proof rather carefully, and when I introduce the notions that come from big fat theories I'll make the exposition as simple as possible, but try pointing out the fascinating connections with adjacent fields that you may want to explore in the future. So this course will have a small tight core and lots of open ends.

Our emphasis is definitely on the ℓ-adic Galois representations of local fields, which have very simple structural classification, but can come from mysterious geometric origin (*ℓ-adic étale cohomology*). I will motivate the algebraic geometry part by starting with the generalization of Lubin-Tate groups, namely the *formal $\mathcal{O}$-modules* of higher height. Then its moduli space will have an interesting cohomology groups on which the local Galois group (and other groups, most notably GL_n!) will act. To be sure, the things I try to motivate "locally" would always have global motivation too, which is a more traditional way of introducing them: the Lubin-Tate space is a local ring of Shimura varieties, etc. My anti-historical treatment somewhat purposefully neglects the role of L-functions, which connect the local and global theory. I believe this partial blindfold is helpful - we have enough of "sophisticated" stuff in number theory; sometimes it's nicer if we can isolate a theory from the rest.

References

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[Y2] T. Yoshida, *Local class field theory via Lubin-Tate theory*, `math.NT/0606108`, to appear in Ann. de la Faculte des Sciences de Toulouse.

[Y3] T. Yoshida, *Weight spectral sequence and Hecke correspondences on Shimura varieties*, thesis 2006.