Lecture 10 29/10/07

∘ 2nd (better) definition of deformations.

$\mathfrak{p} \subset \mathcal{O} \subset K$. $\mathcal{O}/\mathfrak{p} = k \cong \mathbb{F}_q$. $k \subset \kappa \subset \bar{k}$.

$\mathcal{O}_\kappa$: complete unram. ext. of $\mathcal{O}$ w/ res. field κ.

Σ_0 : formal $\mathcal{O}$-mod. of ht h / κ.

$\mathcal{C}$: (cat. of Artinian local $\mathcal{O}_\kappa$-alg. w/ $\kappa \xrightarrow{\sim} A/\mathfrak{m}$.)

$\downarrow$

$\hat{\mathcal{C}}$: (cat. of complete noe. —— .. —— .)

Def. $(A, \mathfrak{m}) \in \hat{\mathcal{C}}$.

- Deformation of Σ_0 to A := A pair (Σ, i) s.t.
  - Σ : formal $\mathcal{O}$-mod. $/A$.
  - i : an isom. $\Sigma_0 \xrightarrow{\sim} \overline{\Sigma} = \Sigma \bmod \mathfrak{m}$ $(= \Sigma \otimes_A A/\mathfrak{m})$
- Isomorphism of deformations := $f : (\Sigma, i) \xrightarrow{\sim} (\Sigma', i')$ ...

  f : isom. of formal $\mathcal{O}$-mod. $\Sigma \xrightarrow{\sim} \Sigma'$ s.t. $\bar{f} \circ i = i'$ ($\bar{f} := f \bmod \mathfrak{m}$)

$$\begin{tikzcd} \Sigma_0 \arrow[r, "i"] \arrow[dr, "i'"'] & \Sigma \bmod \mathfrak{m} \arrow[d, "\bar{f}"] \arrow[dl, phantom, "\circlearrowright"] \\ & \Sigma' \bmod \mathfrak{m} \end{tikzcd}$$

Lemma 1 $[\Sigma] \longmapsto [(\Sigma, \mathrm{id})]$ gives a canonical bijection between the set $\mathrm{Def}(A)$ of isom. classes of deform. in the 1st def. to the set of isom. classes in this sense.

∵) Map is clearly well-def'd. Inj... $f : (\Sigma, \mathrm{id}) \xrightarrow{\sim} (\Sigma', \mathrm{id}) \Rightarrow f \bmod \mathfrak{m} = \mathrm{id}$.

Surj... Take (Σ, i). then $\Sigma_0 = i^{-1} \overline{\Sigma} i$. Lift i to any $\tilde{i} \in A[[X]]$ w/ $\tilde{i}'(0) \in A^\times$. and define $\Sigma' := \tilde{i}^{-1} \Sigma \tilde{i} / A$. then $\Sigma' \bmod \mathfrak{m} = \Sigma_0$. $\tilde{i} : \Sigma' \xrightarrow{\sim} \Sigma$ w/ $\bar{\tilde{i}} \circ \mathrm{id} = i$. i.e. $\tilde{i} : (\Sigma', \mathrm{id}) \xrightarrow{\sim} (\Sigma, i)$. ∎

Cor. $\forall$ Isom. of formal $\mathcal{O}$-mod. $/\kappa$ $f : \Sigma_0 \xrightarrow{\sim} \Sigma_0'$ gives an isom. of functors $f^* : \mathrm{Def}(\Sigma_0') \xrightarrow{\sim} \mathrm{Def}(\Sigma_0)$ by :

$$f^*(A) : \mathrm{Def}(\Sigma_0')(A) \ni [(\Sigma, i)] \longmapsto [(\Sigma, i \circ f)] \in \mathrm{Def}(\Sigma_0)(A).$$

In particular. $\mathrm{Aut}(\Sigma_0)$ acts on $\mathrm{Def}(\Sigma_0)$ from the right $[g^* \circ f^* = (f \circ g)^*]$

Rem.

- If we choose $\phi : \mathrm{Spf}\, \mathcal{O}_\kappa[[T_1, \dots, T_{h-1}]] \xrightarrow{\sim} \mathrm{Def}(\Sigma_0)$. this translates into the left action on $\mathcal{O}_\kappa[[T_1, \dots, T_{h-1}]]$.
- If $\kappa = \overline{\mathbb{F}}_p$. $\exists!$ isom class of Σ_0. (Lec. 6) $\Rightarrow \mathrm{Def}(\Sigma_0)$: canonical up to isom.

  In this case. $\mathrm{Aut}(\Sigma_0) = \mathcal{D}^\times$, where $\mathcal{D}$ is the maximal order of the division alg. $/K$ w/ Hasse inv. $1/h$.

◎ Deformation w/ Drinfeld level str. $\boxed{\text{From now on. } ht_{\mathcal{O}}\Sigma_0 = n.}$

Def. $(A.\mathfrak{m}) \in \hat{\mathcal{C}}$. $\Sigma \underset{:=}{} (F.[.])$: formal $\mathcal{O}$-mod$/A$. Let $\mathfrak{m}_\Sigma$ be the set $\mathfrak{m}$ w/ $\mathcal{O}$-mod. str. defined by $x +_\Sigma y := F(x.y)$, $[a](x)$ $(a \in \mathcal{O})$.
[different from the $\mathcal{O}$-mod str. induced by the A-mod. str !! $\underbrace{\ }_{\text{unless } \hat{\Sigma} = \hat{\mathbb{G}}_a}$]
A Drinfeld level $\mathfrak{p}^m$-structure $(m \geq 1)$ on Σ is an $\mathcal{O}$-mod. hom.
$$\alpha : (\mathfrak{p}^{-m}/\mathcal{O})^n \longrightarrow \mathfrak{m}_\Sigma$$
s.t.
$$[\mathfrak{p}] = \Big(\prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O})^n} (X - \alpha(x)) \Big) \quad \text{as ideals in } A[\![X]\!].$$
where $[\mathfrak{p}] := (\text{ideal gen. by } \{[a](X) \mid a \in \mathfrak{p}\}) = (\text{ideal gen. by } [\varpi](X))$ $\varpi \in \mathcal{O}$: unif.

Rem. This shows, for $^\forall c \in \mathfrak{m}$,
$$^\exists x \in (\mathfrak{p}^{-1}/\mathcal{O})^n.\ \alpha(x) = c \iff [\varpi](c) = 0 \iff c \in \Sigma[\mathfrak{p}] := \left\{ c \in \mathfrak{m} \,\middle|\, \begin{array}{l}[a](c) = 0\\ (^\forall a \in \mathfrak{p})\end{array} \right\}.$$
i.e. $\alpha : (\mathfrak{p}^{-1}/\mathcal{O})^n \twoheadrightarrow \Sigma[\mathfrak{p}]$. $\qquad$ $\Sigma[\mathfrak{p}]$: $\mathcal{O}$-module of $\mathfrak{p}$-torsion pts of Σ. (roots of $[\varpi](X)$).

Def. Fix Σ_0 : formal $\mathcal{O}$-mod$/k$ of ht n. Let $\mathcal{F}_0 := \mathrm{Def}(\Sigma_0)$.
For $^\forall m \geq 1$. Define a functor :
$$\mathcal{F}_m : \hat{\mathcal{C}} \longrightarrow (\text{Sets})$$
$$(A.\mathfrak{m}) \longmapsto \left\{ \begin{array}{l}\text{isom. classes of}\\ \text{triples } (\Sigma.i.\alpha)\end{array} \right\} \ni [(\Sigma.i.\alpha)]$$
where $[(\Sigma.i)] \in \mathcal{F}_0(A)$, α : level $\mathfrak{p}^m$-str. on Σ.
$f : (\Sigma.i.\alpha) \xrightarrow{\sim} (\Sigma'.i'.\alpha')$: isom $\iff \begin{cases} \cdot\ f : (\Sigma.i) \xrightarrow{\sim} (\Sigma'.i') \\ \cdot\ f \circ \alpha = \alpha'. \text{ where } f : \mathfrak{m}_\Sigma \xrightarrow{\sim} \mathfrak{m}_{\Sigma'} : \text{isom of } \mathcal{O}\text{-mod. induced by } c \mapsto f(c) \end{cases}$
$(\Sigma.i.\alpha)$ $\Uparrow$ deformation w/ level $\mathfrak{p}^m$-str.

Rem. $\cdot$ $^\exists$natural forgetting mor. of functors $\mathcal{F}_m \longrightarrow \mathcal{F}_0$.
$\cdot$ Let $\mathcal{F}_0 = \mathrm{Spf}\, R_0$. [we can always choose $R_0 \xrightarrow{\sim} \mathcal{O}_K[\![T_1.\cdots.T_{n-1}]\!]$]

Thm (Drinfeld) 1) $\mathcal{F}_m$ is represented by $R_m \in \hat{\mathcal{C}}$, which is a finite flat R_0-alg. by $\mathcal{F}_m \to \mathcal{F}_0$ ($\Leftrightarrow R_0 \to R_m$).
2). Let $[(\Sigma^{univ}.i^{univ}.\alpha^{univ})]$ be the universal object over R_m (i.e. the one corresponding to $id \in \mathrm{Hom}_{\hat{\mathcal{C}}}(R_m.R_m) = \mathcal{F}_m(R_m)$). If we choose a basis $(e_1, \dots, e_n)$ of $\mathfrak{p}^{-m}/\mathcal{O}$ as $\mathcal{O}/\mathfrak{p}^m$-module. then $X_i := \alpha^{univ}(e_i)$ $(1 \leq i \leq n)$ is a set of regular parameters of R_m. (In particular. R_m is a regular local ring.)

<u>Lemma</u>. $\mathcal{F}_m|_{\mathcal{C}}$ is a deformation functor. [not needed in the rest]

(∵). · $\mathcal{F}_m(k) = \{\cdot\}$ ... need to show there is a unique level $\mathfrak{p}^m$-str. on Σ_0/k. but $[\varpi](X) = g(X^{q^{n}})$. $g'(0) \neq 0$. $g \in k[\![X]\!]$ by the def'n of ht. hence $\Sigma_0[\mathfrak{p}^m] = \{0\}$. and $\alpha(x) = 0 \quad \forall x \in \mathfrak{p}^{-m}/\mathcal{O}$. $\left([\varpi](x) = h(x^{q^{nm}}) \text{ w/ } h'(0) \neq 0 \right)$

$$\left[\begin{array}{ll} \alpha : (\mathfrak{p}^{-m}/\mathcal{O})^n \longrightarrow \Sigma[\mathfrak{p}^m] := \{ c \in \mathfrak{m} \mid [a](c) = 0 \ (\forall a \in \mathfrak{p}^m) \} \\ \qquad\qquad \underline{\mathfrak{p}^m\text{-torsion pts of } \Sigma} \quad \rotatebox{90}{$\cong$} \quad = \{ c \in \mathfrak{m} \mid [\varpi^m](c) = 0 \} \end{array} \right]$$

· $\mathcal{F}_m(A \times_C B) \xrightarrow{\sim} \mathcal{F}_m(A) \times_{\mathcal{F}_m(C)} \mathcal{F}_m(B) \ni \left([(\Sigma_A, i_A, \alpha_A)], [(\Sigma_B, i_B, \alpha_B)] \right)$

$\qquad \rotatebox{90}{$\in$}$

$[(\Sigma_{A\times B}, i_{A\times B}, \alpha_{A\times B})]$ ... follows by $\mathfrak{m}_{\Sigma_{A\times B}} = \mathfrak{m}_{\Sigma_A} \times_{\mathfrak{m}_{\Sigma_C}} \mathfrak{m}_{\Sigma_B}$,

divisibility of power series can be seen componentwise.

<u>Rem</u>. · $\mathcal{F}_0, \mathcal{F}_m$ are functors by :

$f : A \longrightarrow A'$ in $\mathcal{C}$ $\rightsquigarrow$ $\mathcal{F}_m(f) : [(\Sigma, i, \alpha)] \longmapsto [(\Sigma \otimes_{A, f} A', i, f \circ \alpha)]$.

· $i : \Sigma_0 \to \overline{\Sigma} \searrow \; \swarrow\!/\!/ \; \overline{\Sigma \otimes A'}$ .... $\Sigma \bmod \mathfrak{m} \underset{f}{\overset{\|}{\downarrow}} \Sigma \otimes A' \bmod \mathfrak{m}'$

$$\begin{array}{ccc} & & A/\mathfrak{m} \\ & \nearrow\sim & \downarrow \wr f \\ k & \searrow\sim & A'/\mathfrak{m}' \end{array}$$

· $f : \mathfrak{m}_\Sigma \longrightarrow \mathfrak{m}'_{\Sigma \otimes A'}$

· $\mathrm{Aut}_k(\Sigma_0) \overset{\text{right}}{\curvearrowright} \mathcal{F}_m$ by $[(\Sigma, i, \alpha)] \longmapsto [(\Sigma, i \circ d, \alpha)]$
$\qquad \rotatebox{90}{$\in$}$
$\qquad d.$

can. $\cong$ $\mathrm{Aut}_{\mathcal{O}}((\mathfrak{p}^{-m}/\mathcal{O})^n) \overset{\text{right}}{\curvearrowright} \mathcal{F}_m$ by $[(\Sigma, i, \alpha)] \longmapsto [(\Sigma, i, \alpha \circ g)]$

$GL_n(\mathcal{O}/\mathfrak{p}^m)$ $\qquad \rotatebox{90}{$\in$}$ g. · these 2 actions commute w/ each other :

$\mathcal{F}_m \circlearrowleft \mathrm{Aut}_k(\Sigma_0) \times \mathrm{Aut}_{\mathcal{O}}((\mathfrak{p}^{-m}/\mathcal{O})^n)$.
$\qquad\qquad\qquad\qquad \| \; GL_n(\mathcal{O}/\mathfrak{p}^m)$.

· $\mathcal{F}_{m'} \xrightarrow[\text{restrict}]{\alpha} \mathcal{F}_m \xrightarrow[\text{forget}]{\alpha.} \mathcal{F}_0 \qquad (m \leq m')$.

$$\begin{array}{ccc} \mathcal{F}_{m'} & & \mathcal{F}_m \\ \circlearrowleft & & \circlearrowleft \\ GL_n(\mathcal{O}/\mathfrak{p}^{m'}) & \twoheadrightarrow & GL_n(\mathcal{O}/\mathfrak{p}^m) \\ \times & & \times \\ \mathrm{Aut}_k(\Sigma_0) & = & \mathrm{Aut}_k(\Sigma_0) \end{array}$$