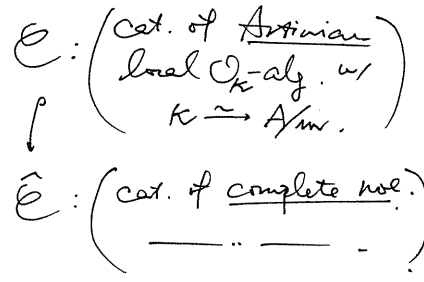


o 2nd (better) definition of deformations.

$\mathfrak{o} \subset \mathcal{O} \subset K$. $\mathcal{O}/\mathfrak{o} = \bar{k} \cong \mathbb{F}_q$. $\bar{k} \subset K \subset \bar{K}$.

\mathcal{O}_K : complete unram. ext. of \mathcal{O} w/ res. field K .

Σ_0 : formal \mathcal{O} -mod. of ht h/K .

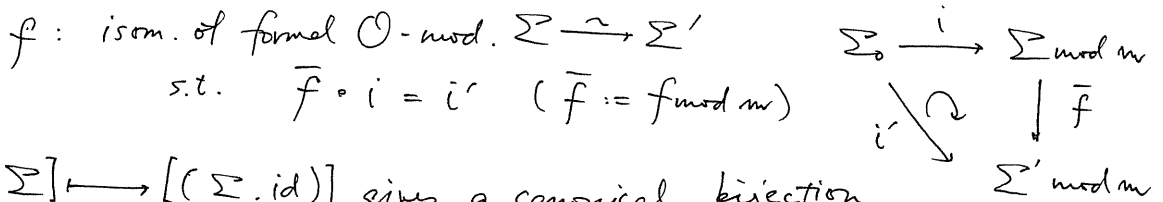


Def: $(A, m) \in \hat{\mathcal{E}}$.

Deformation of Σ_0 to $A := A$ pair (Σ, i) s.t.

- Σ : formal \mathcal{O} -mod. / A . $\bar{\Sigma}$
- i : an isom. $\Sigma_0 \xrightarrow{\sim} \Sigma \bmod m (= \Sigma \otimes_A A/m)$

Isomorphism of deformations := $f: (\Sigma, i) \xrightarrow{\sim} (\Sigma', i')$...



Lemma! $[\Sigma] \mapsto [(\Sigma, id)]$ gives a canonical bijection

between the deform. in the 1st def. to the set of isom. classes in this sense.

set $Def(A)$ of isom. classes of

\therefore) Map is clearly well-def'd. Inject... $f: (\Sigma, id) \xrightarrow{\sim} (\Sigma', id) \Rightarrow f \bmod m = id$.

Surj... Take (Σ, i) . then $\Sigma_0 = i^{-1} \bar{\Sigma} i$. Lift i to any $\tilde{i} \in A[[X]]$ w/ $\tilde{i}(0) \in A^\times$ and define $\Sigma' := \tilde{i}^{-1} \Sigma \tilde{i} / A$. then $\Sigma' \bmod m = \Sigma_0$. $\tilde{i}: \Sigma' \xrightarrow{\sim} \Sigma$ w/ $\tilde{i} \circ id = i$. i.e. $\tilde{i}: (\Sigma', id) \xrightarrow{\sim} (\Sigma, i)$.

Cor. \uparrow Isom. of formal \mathcal{O} -mod./ K $f: \Sigma_0 \xrightarrow{\sim} \Sigma'_0$ gives an isom. of functors $f^*: Def(\Sigma'_0) \xrightarrow{\sim} Def(\Sigma_0)$ by:

$f^*(A): Def(\Sigma'_0)(A) \ni [(\Sigma', i')] \mapsto [(\Sigma, i \circ f)] \in Def(\Sigma_0)(A)$.

In particular. $Aut(\Sigma_0)$ acts on $Def(\Sigma_0)$ from the right $[g^* \circ f^* = (f \circ g)^*]$

Rem. If we choose $\phi: Spf \mathcal{O}_K[[T_1, \dots, T_{h-1}]] \xrightarrow{\sim} Def(\Sigma_0)$. this translates into the left action on $\mathcal{O}_K[[T_1, \dots, T_{h-1}]]$.

If $K = \bar{\mathbb{F}}_p$. $\exists!$ isom class of Σ_0 . (Lec. 6) $\Rightarrow Def(\Sigma_0)$: canonical up to isom.

In this case. $Aut(\Sigma_0) = \mathcal{D}^\times$, where \mathcal{D} is the maximal order of the division alg./ K w/ Hasse inv. $1/h$.

② Deformation w/ Drinfeld level str. From now on. $\text{ht}_0 \Sigma_0 = n$.

Def. $(A, \mathfrak{m}) \in \hat{\mathcal{E}}$. $\Sigma := \text{formal } \mathcal{O}\text{-mod}/A$. Let \mathfrak{m}_Σ be the set \mathfrak{m} w/ $\mathcal{O}\text{-mod. str.}$ defined by $x \pm y := F(x, y)$, $[a](x)$ ($a \in \mathcal{O}$).
 [different from the $\mathcal{O}\text{-mod str.}$ induced by the $A\text{-mod. str.}$!!] unless $\Sigma = \hat{G}_a$

A Drinfeld level \mathbb{F}^m -structure ($m \geq 1$) on Σ is an

$\mathcal{O}\text{-mod. hom.}$ $\alpha : (\mathbb{F}^m/\mathcal{O})^n \rightarrow \mathfrak{m}_\Sigma$

st. $[\mathbb{F}] = \left(\prod_{x \in (\mathbb{F}^m/\mathcal{O})^n} (x - \alpha(x)) \right)$ as ideals in $A[X]$.

where $[\mathbb{F}] := (\text{ideal gen. by } \{[a](x) \mid a \in \mathbb{F}\}) = (\text{ideal gen. by } [\mathbb{F}](x))$
 $\mathbb{F} \in \mathcal{O} : \text{unif.}$

Rem. This shows, for $\forall c \in \mathfrak{m}$,

$$\exists x \in (\mathbb{F}^m/\mathcal{O})^n. \alpha(x) = c \iff [\mathbb{F}](c) = 0 \iff c \in \Sigma[\mathbb{F}] := \left\{ \begin{array}{l} c \in \mathfrak{m} \\ [a](c) = 0 \\ (\forall a \in \mathbb{F}) \end{array} \right\}$$

i.e. $\alpha : (\mathbb{F}^m/\mathcal{O})^n \rightarrow \Sigma[\mathbb{F}]$. $\mathcal{O}\text{-module of } \mathbb{F}\text{-torsion pts of } \Sigma$.
 (roots of $[\mathbb{F}](x)$)

Def. Fix $\Sigma_0 : \text{formal } \mathcal{O}\text{-mod}/k$ of ht n . Let $\mathcal{F}_0 := \text{Def}(\Sigma_0)$.

For $\forall m \geq 1$. Define a functor :

$$\mathcal{F}_m : \hat{\mathcal{E}} \rightarrow (\text{Sets})$$

$(A, \mathfrak{m}) \mapsto \text{isom. classes of triples } (\Sigma, i, \alpha)$

where $[(\Sigma, i)] \in \mathcal{F}_0(A)$, $\alpha : \text{level } \mathbb{F}^m\text{-str. on } \Sigma$.

$$f : (\Sigma, i, \alpha) \xrightarrow{\sim} (\Sigma', i', \alpha') : \text{isom} \iff \left\{ \begin{array}{l} \cdot f : (\Sigma, i) \xrightarrow{\sim} (\Sigma', i') \\ \cdot f \circ \alpha = \alpha' \end{array} \right.$$

↑ deformation w/ level $\mathbb{F}^m\text{-str.}$ $f : \mathfrak{m}_\Sigma \xrightarrow{\sim} \mathfrak{m}_{\Sigma'}$: isom of $\mathcal{O}\text{-mod. induced by } c \mapsto f(c)$

Rem. \exists natural forgetting mor. of functors $\mathcal{F}_m \rightarrow \mathcal{F}_0$.

Let $\mathcal{F}_0 = \text{Spf } R_0$. [we can always choose $R_0 \xrightarrow{\sim} \mathcal{O}_k[[T_1, \dots, T_{n-1}]]$]

Thm (Drinfeld) 1) \mathcal{F}_m is represented by $R_m \in \hat{\mathcal{E}}$, which is a finite flat R_0 -alg. by $\mathcal{F}_m \rightarrow \mathcal{F}_0 (\iff R_0 \rightarrow R_m)$.

2). Let $[(\Sigma^{\text{univ}}, i^{\text{univ}}, \alpha^{\text{univ}})]$ be the universal object over R_m (i.e. the one corresponding to $\text{id} \in \text{Hom}_{\hat{\mathcal{E}}}(R_m, R_m) = \mathcal{F}_m(R_m)$). If we choose a basis (e_1, \dots, e_n) of $\mathbb{F}^m/\mathcal{O} \cong \mathbb{F}^m$ -module, then $X_i := \alpha^{\text{univ}}(e_i)$ ($1 \leq i \leq n$) is a set of regular parameters of R_m . (In particular, R_m is a regular local ring.)

Lemma. $\mathcal{F}_m|_{\mathcal{E}}$ is a deformation functor. [not needed in the rest]

\dots $\mathcal{F}_m(\kappa) = \{0\}$... need to show there is a unique level \mathbb{F}^m -str. on Σ_0/κ . but $[\mathcal{D}](x) = g(x^{\delta^{nm}})$. $g(0) \neq 0$. $g \in \kappa[x]$ by the def'n of ht. hence $\Sigma_0[\mathbb{F}^m] = \{0\}$. and $\alpha(x) = 0 \ \forall x \in \mathbb{F}^m/\mathcal{O}$. ($[\mathcal{D}](x) = h(x^{\delta^{nm}})$ w/ $h(0) \neq 0$)

$$\left[\begin{array}{l} \alpha : (\mathbb{F}^m/\mathcal{O})^n \rightarrow \Sigma[\mathbb{F}^m] := \{c \in \mathfrak{m} \mid \alpha(c) = 0 \ (\forall a \in \mathbb{F}^m)\} \\ \underbrace{\mathbb{F}^m\text{-torsion pts of } \Sigma}_{\cong} = \{c \in \mathfrak{m} \mid [\mathcal{D}^m](c) = 0\}. \end{array} \right]$$

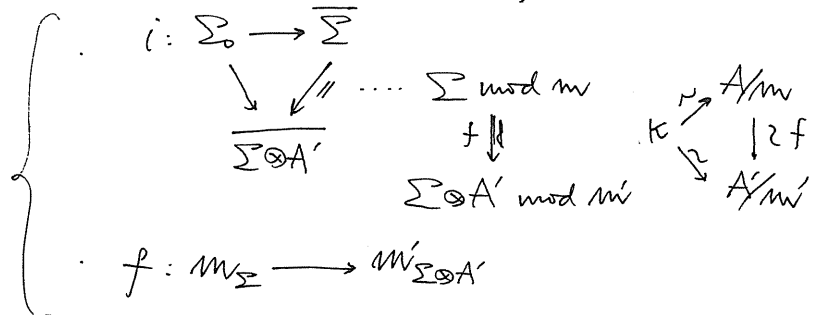
$\mathcal{F}_m(A \times B) \xrightarrow{\sim} \mathcal{F}_m(A) \times \mathcal{F}_m(B) \ni ([(\Sigma_A, i_A, \alpha_A)], [(\Sigma_B, i_B, \alpha_B)])$

\downarrow $[\Sigma_{A \times B}, (i_{A \times B}, \alpha_{A \times B})]$... follows by $MW_{\Sigma_{A \times B}} = MW_{\Sigma_A} \times_{MW_{\Sigma_B}} MW_{\Sigma_C}$

divisibility of power series can be seen componentwise.

Rem. $\mathcal{F}_0, \mathcal{F}_m$ are functors by:

$f: A \rightarrow A'$ in $\mathcal{E} \rightsquigarrow \mathcal{F}_m(f): [(\Sigma, i, \alpha)] \mapsto [(\Sigma \otimes_{A_f} A', i, f \circ \alpha)]$



$\text{Aut}_{\kappa}(\Sigma_0) \overset{\text{right}}{\curvearrowright} \mathcal{F}_m$ by $[(\Sigma, i, \alpha)] \mapsto [(\Sigma, i \circ d, \alpha)]$

$\text{Aut}_{\mathcal{O}}((\mathbb{F}^m/\mathcal{O})^n) \overset{\text{right}}{\curvearrowright} \mathcal{F}_m$ by $[(\Sigma, i, \alpha)] \mapsto [(\Sigma, i, \alpha \circ g)]$

$GL_n(\mathcal{O}/\mathbb{F}^m)$ \cong $\text{Aut}_{\mathcal{O}}((\mathbb{F}^m/\mathcal{O})^n)$. These 2 actions commute w/ each other:

$\mathcal{F}_m \curvearrowright \text{Aut}_{\kappa}(\Sigma_0) \times \text{Aut}_{\mathcal{O}}((\mathbb{F}^m/\mathcal{O})^n) \cong GL_n(\mathcal{O}/\mathbb{F}^m)$

$\mathcal{F}_{m'} \xrightarrow{\text{restrict } \alpha} \mathcal{F}_m \xrightarrow{\text{forget } \alpha} \mathcal{F}_0 \quad (m \leq m')$

$GL_n(\mathcal{O}/\mathbb{F}^{m'}) \rightarrow GL_n(\mathcal{O}/\mathbb{F}^m)$
 $\times \quad \times$
 $\text{Aut}_{\kappa}(\Sigma_0) = \text{Aut}_{\kappa}(\Sigma_0)$