Def. A : $\mathcal{O}$-alg. $f \in (X) \subset A[[X]]$. $\exists f^{-1} \in A[[X]]$ (i.e. $f'(0) \in A^\times$)

Σ : formal $\mathcal{O}$-mod./A $\Rightarrow$ $f\Sigma f^{-1} := (f\circ F \circ f^{-1}.\ f\circ[\cdot]\circ f^{-1})$
"$(F.[\cdot])$" $\cdots$ formal $\mathcal{O}$-mod./A (check!)

$f : \Sigma \xrightarrow{\sim} f\Sigma f^{-1}$ $[\,f\circ\Sigma = (f\Sigma f^{-1})\circ f\,]$

"change of coordinates"

$$X.Y \xrightarrow{F} F(X.Y) \qquad f(X).f(Y) \xrightarrow{f\circ F\circ f^{-1}} f(F(X.Y))$$
$$X \xrightarrow{[a]} [a](X) \qquad f(X) \xrightarrow{f\circ[a]\circ f^{-1}} f([a](X))$$

Lemma 1. $f(X) := X - cX^n$ ($n>1$. $c\in A$) $\Rightarrow$ $f\Sigma f^{-1} \equiv \Sigma - c\cdot B_n^{\mathcal{O}}$ (mod deg $n+1$)

where $B_n^{\mathcal{O}} := (B_n(X.Y),\ (a^n-a)X^n)$ $\left[\text{i.e. } \begin{array}{l} fFf^{-1} \equiv F - cB_n \\ f[a]f^{-1} \equiv [a] - c(a^n-a)X^n \quad (\forall a\in\mathcal{O})\end{array}\right]$

$\because$)
$$\left.\begin{array}{l} f\circ F = F(X.Y) - cF(X.Y)^n \equiv F(X.Y) - c(X+Y)^n \\ F\circ f = F(X-cX^n.\ Y-cY^n) \equiv F(X.Y) - c(X^n+Y^n)\end{array}\right\} f\circ F \equiv F\circ f - c\cdot B_n$$
$$\left.\begin{array}{l} f\circ[a] = [a](X) - c([a](X))^n \equiv [a](X) - c(aX)^n \\ [a]\circ f = [a](X-cX^n) \equiv [a](X) - a\cdot cX^n\end{array}\right\} \begin{array}{l} f\circ[a] \equiv [a]\circ f - c(a^n-a)X^n \\ \text{(mod deg } n+1)\end{array}$$

substitute $f^{-1}(X).\ f^{-1}(Y)$ $[\,f^{-1}(X) \equiv X \text{ mod deg } 2\,]$ $\Rightarrow$ $f\Sigma f^{-1} \equiv \Sigma - c\cdot B_n^{\mathcal{O}}$. $\lrcorner$

$\mathcal{O}$-Lazard's Lemma. $\Sigma_1.\ \Sigma_2$: formal $\mathcal{O}$-mod./A $[\,\varpi\in\mathfrak{p}\subset\mathcal{O}.\ \mathcal{O}/\mathfrak{p}\cong\mathbb{F}_q\,]$

$\Sigma_1 \equiv \Sigma_2$ (mod deg n) $\Rightarrow$ $\Sigma_1 \equiv \Sigma_2 + c\cdot A_n^{\mathcal{O}}$ (mod deg $n+1$) for some $c\in A$.

where:
$$A_n^{\mathcal{O}} := (A_n.\ a_nX^n) := \begin{cases} \left(\frac{1}{\varpi}B_n.\ \frac{a^n-a}{\varpi}X^n\right) & (n=q^s) \\ (B_n.\ (a^n-a)X^n) & (\text{otherwise})\end{cases}$$

Cor. 1. Let $\Sigma_1 \equiv \Sigma_2$ (mod deg n). Then:

$[\varpi]_1 \equiv [\varpi]_2$ (mod $(\mathfrak{p}.X^{n+1})$) $\Longleftrightarrow$ $\left\{\begin{array}{l} \exists f(X) = X - cX^n \ (\text{i.e. } \exists c\in A) \\ f\Sigma_1 f^{-1} \equiv \Sigma_2 \ (\text{mod deg } n+1)\end{array}\right.$.

$\because$) $\Leftarrow$: $f[\varpi]_1 f^{-1} \equiv [\varpi]_1 - c(\varpi^n-\varpi)X^n$ by Lem. 1. $\Rightarrow$: By Lem. 1, enough to show $\Sigma_1 \equiv \Sigma_2 + cB_n^{\mathcal{O}}$. use $\mathcal{O}$-Lazard. if $n=q^s$. $\Sigma_1 \equiv \Sigma_2 + c'\cdot A_n^{\mathcal{O}}$ for $c'\in A$.
$[\varpi]_1 \equiv [\varpi]_2 + c'\left(\frac{\varpi^n-\varpi}{\varpi}\right)X^n$. but our assumption says $c'\in\mathfrak{p}A$. i.e. $c' = c\varpi$.
$\Rightarrow \Sigma_1 \equiv \Sigma_2 + cB_n^{\mathcal{O}}$. $\lrcorner$

Recall : $\hat{\mathbb{G}}_a := (X+Y.\ [a](X) := aX \ (\forall a\in\mathcal{O}))$.

Cor. 2
i) A : K-alg. $\Rightarrow$ $\Sigma \cong \hat{\mathbb{G}}_a$.
ii) A : k-alg.
$\text{ht}_{\mathcal{O}}\Sigma = h \Leftrightarrow \left\{\begin{array}{l} \exists f : \Sigma \xrightarrow{\sim} f\Sigma f^{-1} \equiv \hat{\mathbb{G}}_a \ (\text{mod deg } q^h) \\ f\Sigma f^{-1} \equiv \hat{\mathbb{G}}_a + c\cdot A_n^{\mathcal{O}}.\ c\neq 0 \ (\text{mod deg } q^h+1)\end{array}\right.$
[in particular. $\text{ht}\,\Sigma = \infty \Rightarrow \Sigma \cong \hat{\mathbb{G}}_a$].
iii) $A \subseteq \mathbb{F}_{q^h}$. $[\varpi]_1 = [\varpi]_2 = X^{q^h}$ $\Rightarrow$ $\Sigma_1 \cong \Sigma_2$ for $\Sigma_1.\ \Sigma_2$.

$\because$) Set $\Sigma_2 := \widehat{\mathbb{G}}_a$ in i). ii). For each $n \geq 1$ [in ii). $n < q^h$]

construct. by induction on n:

$$f_n : \Sigma_1 \xrightarrow{\sim} \Sigma_1^{(n)} := f_n \Sigma_1 f_n^{-1} \equiv \Sigma_2 \left(\substack{\text{mod}\\ \deg n+1}\right), \quad f_n \equiv f_{n-1} \left(\substack{\text{mod}\\ \deg n}\right)$$

$f_1 := id$. Assume we have f_{n-1}.

Use Cor. 1 i): $pA = A$. ii): $[\varpi]_2 = 0$. $n < q^h$. $[\varpi]_1 \equiv 0 \left(\substack{\text{mod}\\ \deg q^h}\right)$ $\left(\Rightarrow f_{n-1}[\varpi]_1 f_{n-1}^{-1} \equiv 0 \left(\substack{\text{mod}\\ \deg q^h}\right)\right)$ iii): $[\varpi]_1 = X^{q^h}$. $f_{n-1} \in \mathbb{F}_{q^h}[[X]]$ $\Rightarrow \underbrace{f_{n-1}[\varpi]_1 f_{n-1}^{-1}}_{\substack{\| \\ f_{n-1}^{-1}(X^{q^h})}} = X^{q^h} = [\varpi]_2$

on $\Sigma_1^{(n-1)}$ & Σ_2.

$$\Rightarrow \exists g_n = X - cX^n. \quad \Sigma_1^{(n)} := g_n \Sigma_1^{(n-1)} g_n^{-1} \equiv \Sigma_2 \quad (\text{mod} \deg n+1).$$

$f_n := g_n \circ f_{n-1}$ $\qquad \Sigma_1^{(n)} = f_n \Sigma_1 f_n^{-1}$.

$(= g_n \circ \cdots \circ g_1 .\ g_1 = f_1)$ moreover. $g_n \equiv X \left(\substack{\text{mod}\\ \deg n}\right) \Rightarrow f_n \equiv f_{n-1} \left(\substack{\text{mod}\\ \deg n}\right)$.

i). ii) $\left[\substack{\text{when}\\ h=\infty}\right]$. iii): $f_n \xrightarrow{n\to\infty} f$. $\Sigma_1 \xrightarrow{\sim} f\Sigma_1 f^{-1} = \Sigma_2$.

ii) [$h < \infty$]. $\Rightarrow$: $f := f_{q^h - 1}$. get c by $\mathcal{O}$-Lazard. if $c = 0$. then $ht(\Sigma) = ht(f\Sigma f^{-1}) > h$. (look at $[\varpi]$.)

$\Leftarrow$: def'n of ht. (look at $[\varpi]$).

- Classification / $\overline{\mathbb{F}}_q$.

Lemma 2. A: separably closed field. char $= p$. q = some power of p.

$$\left.\begin{array}{l} u, v \in (X) \subset A[[X]]. \\ v(X) = u(X^q).\ u'(0) \neq 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists f \in (X).\ f'(0) \neq 0. \\ f \circ v \circ f^{-1} = X^q. \end{array}\right.$$

$\because$) Will construct: $f_n \in (X)$. $f_n'(0) \neq 0$. s.t. $\left\{\begin{array}{l} f_n \circ v \circ f_n^{-1} \equiv X^q \left(\substack{\text{mod}\\ \deg q(n+1)}\right) \\ f_n \equiv f_{n-1} \quad (\text{mod} \deg n) \end{array}\right.$

$(\Rightarrow f_n \xrightarrow{n\to\infty} \exists f)$.

$n = 1$. $v(X) = aX^q$ (mod deg $2q$). $\qquad f_1 \circ v \circ f_1^{-1} \equiv c \cdot a \cdot (c^{-1}X)^q$

$\exists c \in A$. $c^{q-1} = a$. $f_1(X) := cX$. $\qquad \equiv X^q$ (mod deg $2q$)

$n > 1$. $v_{n-1} := f_{n-1} \circ v \circ f_{n-1}^{-1} \equiv X^q$ (mod deg qn)

$\equiv X^q + aX^{qn}$ (mod deg $q(n+1)$)

$\exists b \in A$. $b^q - b = a$. $g_n(X) := X + bX^n$. $g_n \circ v \circ g_n^{-1} \equiv X^q - (b^q - b - a)X^{qn}$

$f_n := g_n \circ f_{n-1} \equiv f_{n-1}$ (mod deg n). $\qquad \equiv X^q$ (mod deg $q(n+1)$)

Prop. $\forall$ formal $\mathcal{O}$-mod / A: sep. cl. field / k. of ht h $(<\infty)$ is isom to Σ / A s.t. $\left\{\begin{array}{l} 1)\ [\varpi](X) = X^{q^h} \\ 2)\ \Sigma \equiv \widehat{\mathbb{G}}_a \ (\text{mod} \deg q^h) \\ 3)\ \Sigma: \text{def'd} / \mathbb{F}_{q^h}. \end{array}\right.$

$\because$) Lemma 2 $\Rightarrow$ 1) $\Rightarrow$ 3) $\because$) power series / A commuting w/ X^{q^h} has coefficients in $\mathbb{F}_{q^h}$!

$\Rightarrow$ use Cor. 2 ii) over $A = \mathbb{F}_{q^h}$.

to get 2). we still have 3); also 1) by $\uparrow$. or $f[\varpi]f^{-1} = f(f^{-1}(X)^{q^h}) = X^{q^h}$ $(f \in \mathbb{F}_{q^h}[[X]])$

Thm. Σ_1, Σ_2: formal $\mathcal{O}$-mod / A: sep. cl. field / k of ht h. $\Rightarrow \Sigma_1 \simeq \Sigma_2$.

$\because$) Prop. + Cor. 2 iii).

Some remarks.

Ex. E : ell. curve $/\overline{\mathbb{F}}_p$. $\longrightarrow$ $\hat{E}$: formal gp $/\overline{\mathbb{F}}_p$

$\Rightarrow$ formal $\mathbb{Z}_p$-mod $/\overline{\mathbb{F}}_p$

E : $\begin{cases} \underline{\text{ordinary}} & \Longleftrightarrow \ \mathrm{ht}\,\hat{E} = 1 \\ \underline{\text{supersingular}} & \Longleftrightarrow \ \mathrm{ht}\,\hat{E} = 2. \end{cases}$

Rem. K'/K : fin. sep. $\mathcal{O}'/\mathcal{O}$. $\mathfrak{p}'$. $\mathfrak{p}$. q'. q. $[K':K]=n$

formal $\mathcal{O}'$-module of ht h $\Rightarrow$ formal $\mathcal{O}$-mod of ht nh.
(naturally).

$\because$) $n = ef$. $q' = q^f$. $[\varpi'] = g(X^{q'^h})$ $g'(0) \neq 0$

$\varpi = (\varpi')^e \cdot u$ $\Rightarrow$ $[\varpi] = f(X^{(q^f)^{he}})$. $f'(0) \neq 0$. $\lrcorner$

Ex. $L/\mathbb{Q}$ imag. quad. H: Hilb. class field of L.

E: ell. curve $/H$ w/ $\mathcal{O}_L \hookrightarrow \mathrm{End}(E)$. (Complex Multiplication)

$\mathfrak{p}$: place of H. $\mathfrak{p}|p$. $E_\mathfrak{p} := E \bmod \mathfrak{p}$ $\Rightarrow$ $\hat{E}_\mathfrak{p}$: formal $\mathbb{Z}_p$-mod $/k_\mathfrak{p}$.
$k_\mathfrak{p}$: res. field.

Actually $\mathcal{O}_L \otimes \mathbb{Z}_p = \begin{cases} \mathbb{Z}_p \times \mathbb{Z}_p & (p: \text{split in } L) \\ \mathcal{O}_K.\ [K:\mathbb{Q}_p] = 2 & (p: \text{inert or ram. in } L) \end{cases}$

$[\mathfrak{p}|v,\ L_v \cong \mathbb{Q}_p]$.

- p : split in L $\Rightarrow$ $\hat{E}_\mathfrak{p}$: formal $\mathbb{Z}_p$-mod of ht 1 ($E_\mathfrak{p}$: ordinary)
- p : inert or ram'd in L $\Rightarrow$ $\hat{E}_\mathfrak{p}$: formal $\mathcal{O}_K$-mod of ht 1 ($E_\mathfrak{p}$: supersingular)
  ($\Rightarrow$ formal $\mathbb{Z}_p$-mod of ht 2)
  Lubin-Tate gp.

Rem. $f(x) \equiv \begin{cases} \varpi X & (\varpi \in K) \\ X^{q^n} \end{cases}$ $\rightsquigarrow$ $\Sigma_f/\mathcal{O}$: formal $\mathcal{O}$-mod of ht n.

- $\Sigma_f \otimes \mathcal{O}_{K_n}$: formal $\mathcal{O}_{K_n}$-mod of ht 1 $[K_n/K$ unram. of deg $n]$
- $\phi(x) = X^q : \Sigma_f \longrightarrow \Sigma_f$: endom. $/\mathcal{O}$

$\mathcal{O}_D := \mathcal{O}_{K_n}[\phi] \hookrightarrow \mathrm{End}(\Sigma_n)$. where $\Sigma_n := \Sigma_f \otimes_{\mathcal{O}} \overline{\mathbb{F}}_q$

$\uparrow$ non-comm. ring. $\phi x \phi^{-1} = x^q$ for $x \in \mathcal{O}_{K_n}$

D : div. alg $/K$ of dim h^2, invariant $1/h$.

We can show $\mathcal{O}_D = \mathrm{End}_{\overline{\mathbb{F}}_q}(\Sigma_n)$: endom. ring of ht n formal $\mathcal{O}$-module.