A Stroll Through Completeness and Compactness

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Ce n'est pas à l'architecture, à la maçonerrie, qu'il faut comparer la géometrie ou l'analyse, mais à la botanique, à la géographie, aux sciences physiques même. Il s'agit de décrire un monde, de la découvrir et non de le construire ou de l'inventer, car il existe en dehors de l'esprit humain et indépendant de lui.

[You should not compare geometry or analysis to architecture or building but to botany, geography or even the natural sciences. It seeks to describe a world, to discover this world and not to construct it or invent it, for this world exists outside human thought and independent of it.]

Raymond Queneau *Odile*

[Motto for Kahane and Salem *Ensembles parfaits et séries trigonometriques*]

I myself, a professional mathematician, on re-reading my own work find it strains my mental powers to recall to mind from the figures the meanings of the demonstrations, meanings which I myself originally put into the figures and the text from my mind. But when I attempt to remedy the obscurity of the material by putting in extra words, I see myself falling into the opposite fault of becoming chatty in something mathematical.

Kepler, introduction to *Astronimia Nova*

La satisfaction du maître n'est pas l'unique objet de lenseignement . . . La tâche de l'éducateur est de faire repasser l'esprit de l'enfant par où a passé celui de ses pères, en passant rapidement par certaines étapes mais en n'en supprimant aucune.

[The satisfaction of the master is not the only object of education. . . . The task of the educator is to take the imagination of the child along the road that their ancestors took, passing rapidly over certain parts but omitting none of them.]

Poincaré in *L'enseignement mathématique*, Vol. 1, (1899), p. 157-162.

We go upon the practical mode of teaching, Nickleby; the regular education system. C-l-e-a-n, clean, verb active, to make bright, to scour. W-i-n, win, d-e-r, der, winder, a casement. When the boy knows this out of book, he goes and does it.

Dickens *Nicholas Nickleby*

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Introduction

Suppose that you were transported back in time to the late 19th century and asked to give a series of lectures at the Collège de France setting out some future developments in analysis. You would, I think, need to begin with a couple of dull lectures setting up the notations of set theory followed by three or four interesting lectures on countability. From time to time in your remaining lectures you would need to pause the exposition to discuss various abstract ideas like 'isomorphism' and 'equivalence relation', but the able and experienced mathematicians in your audience would be intrigued rather than puzzled by these notions. Your audience would not know modern undergraduate linear algebra (which is shaped by ideas coming from modern analysis) but would have plenty of insight into the importance of linearity.

It would be natural to start with completeness and compactness and develop each in turn. However, your main problem would be to convince your auditors that the additional abstraction involved had a pay-off commensurate with the work involved. Naturally, you would seek to make this task easier by dealing with metric spaces rather than topological spaces and concentrating your attention on the real line and similar spaces.

Finally, but very importantly, you would illustrate the power of the new methods by showing how they could be used to resolve hard problems already known, or at least easily appreciated by your late 19th century audience.

It should be clear by now that I have written these notes to please myself and any resemblance to any university syllabus is entirely accidental.

The exercises that within the main text are an integral part of the text and the reader is asked to read and think about each one. In some cases the desired argument will be clear to the reader without writing anything down and in others a few scribbled notes will suffice to make things obvious. None of the exercises are intended as brain teasers, but from time to time, in the nature of things, the reader may wish to consult the accompanying sketch solutions. In my view, it will usually be more profitable to do this after you have thought about the matter for a bit. The sketch solutions are not intended as models. The final section of each chapter contains a mixture of exercises, some hard and some easy, which take matters a little bit further, but are not essential.

These notes are in a VERY preliminary state. The kindest thing you can do is to send me lists of errors, problems etc to my e-mail:–

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Chapter 1

Continuous functions

1.1 Differential equations

In the 19th century, analysts studied concrete objects like differential equations. A great deal of ingenuity was devoted to finding solutions of differential equations like

$$\frac{dy}{dx} = x^3y^2$$

or

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3.$$

Exercise 1.1.1. *Even in these degenerate days, the reader ought to be able to solve these equations. If not, take down the battered copy of Piaggio's* Differential Equations *from the library shelves and see how it is done. (Rigour is not required.)*

If, by solving a differential equation, we mean writing down a solution in terms of known functions, then it has been clear since the 17th century that this is a hopeless task. If we only know functions of the type $P(x)/Q(x)$ where P and Q are polynomials, then we cannot even solve the differential equation

$$l'(x) = \frac{1}{x}$$

for $x > 0$.

Exercise 1.1.2. *Here are two possible proofs of the result just stated.*

(i) Suppose that $a_N, b_M \neq 0$ *and that* $P(x) = \sum_{j=0}^{N} a_j x^j$ *and* $Q(x) = \sum_{k=0}^{M} b_k x^k$ *are such that*

$$\frac{d}{dx}\frac{P(x)}{Q(x)} = \frac{1}{x}.$$

Show that

$$x(P'(x)Q(x) - Q'(x)P(x)) = Q(x)^2.$$

Derive a contradiction by equating coefficients.

(ii) Suppose that

$$l'(x) = \frac{1}{x}$$

for $x > 0$. By using the mean value theorem, show that

$$1 \geq l(2^{n+1}) - l(2^n) \geq \frac{1}{2}.$$

Deduce that $l(2^n) \to \infty$ as $n \to \infty$, but $2^{-n} l(2^n) \to 0$ as $n \to \infty$. Conclude that l cannot be written as the ratio of two polynomials.

Of course, we can simply declare that the solution of

$$l'(x) = \frac{1}{x}$$

for $x > 0$ with $l(1) = 0$ is to be written $l(x) = \log x$, but, to do this, we must show that the supposed solution exists and is unique.

Exercise 1.1.3. *Explain why the solution exists and is unique.*

Once we have the logarithm function, we can solve new differential equations.

Exercise 1.1.4. *Let*

$$l'(x) = \frac{1}{x}$$

for $x > 0$ with $l(1) = 0$.

(i) Explain why l is a strictly increasing function and use part (ii) of Exercise 1.1.2 to show that $l(x) \to \infty$ as $x \to \infty$.

(ii) If y is fixed with $y > 0$ show that

$$\frac{d}{dx}(l(yx) - l(x)) = 0$$

for all $x > 0$. Deduce that that $l(x) + l(y) = l(xy)$ for all $x, y > 0$ and in particular that $l(x) = -l(1/x)$ for all $x > 0$.

(iii) Conclude that $l : (0, \infty) \to \mathbb{R}$ is a bijection.

(iv) If we write $e = l^{-1}$ (so that e is a function from $\mathbb{R}$ to $(0, \infty)$), explain why e is differentiable and

$$e'(x) = e(x)$$

for all x and $e(0) = 1$.

(v) Suppose that $E : \mathbb{R} \to (0, \infty)$ is differentiable with

$$E'(x) = E(x)$$

for all x and $E(0) = 1$. By considering $\dfrac{d}{dx}\dfrac{e(x)}{E(x)}$, or otherwise, show that $E = e$.

As the reader knows, we write $\log x = l(x)$ and $\exp x = e(x)$. However, even armed with polynomials, logarithms and exponentials, Liouville showed that we cannot solve the very simple differential equation

$$f'(x) = \exp(-x^2).$$

Liouville's theorem and later theorems, which show, that (as one might expect) the process of constructing such new functions never ends, belong more to algebra than analysis (look at part (i) of Exercise 1.1.2) and we shall not pursue the matter further[1].

Before leaving this set of ideas, we look at a very nice way of defining the sine and cosine using differential equations. If the reader has not met it before, she should find it a pleasure to work through the exercise. If she has met it before, she should find it a pleasure to recall (without necessarily going through the details).

Exercise 1.1.5. *Assume that $s : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function satisfying the conditions*

$$s''(x) + s(x) = 0 \text{ for all } x,\ \ s(0) = 0, s'(0) = 1.$$

Explain why s is infinitely differentiable and why, if we set $c(x) = s'(x)$, we have

$$c''(x) + c(x) = 0 \text{ for all } x,\ \ c(0) = 1, c'(0) = 0.$$

Show also that $c'(x) = -s(x)$

(i) Let a be fixed and set $f(x) = s(a - x)c(x) + c(a - x)s(x)$. By considering f' and using the mean value theorem, show that

$$s(a - x)c(x) + c(a - x)s(x) = s(a)$$

for all x. By choosing appropriate a and x, deduce that

$$s(u + v) = s(u)c(v) + c(u)s(v)$$

for all $u,\ v \in \mathbb{R}$.

[1]The theorems belong to Galois Theory and are not easy to prove.

(ii) Prove similarly that $s(u-v) = s(u)c(v) - c(u)s(v)$. *and deduce that* $s(-x) = -s(x)$. *State and prove similar formulae for* $c(u+v)$, $c(u-v)$ *and* $c(-x)$.

(iii) Show that $s(x)^2 + c(x)^2 = 1$ *for all* x *and deduce that .*

(iv) Suppose that $c(x) \geq 0$ *for all* $0 \leq x \leq b$. *Explain why* s *is increasing on this interval and* c *decreasing. Show that there exists a* u *with* $c(u) = 3/5$, $s(u) = 4/5$ *and that* $c(2u) < 0$.

(v) Deduce that there exists a $w > 0$ *such that* $c(w) = 0$. *If we write*

$$\omega = \inf\{w > 0 \ : \ c(w) = 0\},$$

explain why $c(\omega) = 0$ *and* $\omega > 0$. *Show that* $s(\omega) = 1$.

(vi) Show that $s(x+\omega) = c(x)$, $c(x+\omega) = -s(x)$. *Show that* $s(x+4\omega) = s(x)$, $c(x+4\omega) = c(x)$.

(vii) Show that, if $s(x+4\rho) = s(x)$ *and* $c(x+4\rho) = c(x)$ *for all* $x \in \mathbb{R}$, *then* ρ *is an integral multiple of* 4ω.

Again, this elegant approach only works if we can show a solution exists and is unique.

Exercise 1.1.6. *(i) Show that the power series*

$$S(x) = \sum_{r=1}^{\infty} \frac{1}{(2r+1)!}(-1)^r x^{2r+1}$$

has infinite radius of convergence. By quoting appropriate theorems, prove that S *is twice differentiable and*

$$S''(x) + S(x) = 0 \text{ for all } x, \ S(0) = 0, S'(0) = 1.$$

(ii) State a version of Taylor series with remainder and, use it to prove that, if

$$s''(x) + s(x) = 0 \text{ for all } x, \ s(0) = 0, s'(0) = 1,$$

then $s = S$. *(Note that you must prove that the remainder term in your series tends to zero.)*

In the cases we have discussed we have been able to produce ad hoc proofs of the existence and uniqueness of the solutions of our differential equations. Could we replace them by a general theorem? The following standard examples show that this is not entirely straight-forward.

Exercise 1.1.7. *Consider the problem of finding a once differentiable function* $y : \mathbb{R} \to \mathbb{R}$ *satisfying the conditions*

$$y'(t) = 3y(t)^{2/3}, \ y(0) = 0.$$

(i) Show that $y(t) = t^3$ *and* $y(t) = 0$ *give two solutions.*
(ii) Show that, if $a \leq 0 \leq b$, *the function given by*

$$y(t) = \begin{cases} (t-a)^3 & \text{for } t \leq a \\ 0 & \text{for } a < t < b \\ (t-b)^3 & \text{for } t \geq b \end{cases}$$

is a solution.

(iii) We have not exhausted the list of solution types. Can you find two more? (They are variations on the theme of (ii).)

Exercise 1.1.8. *State a solution to the system*

$$y'(t) = (1 + y(t))^2, \ y(0) = 0.$$

What is the range over which your solution is valid?

Our approach to the concrete problem of existence and uniqueness of solutions will involve the abstract notion of a metric space and the ideas of completeness and compactness.

1.2 Complete metric spaces

The notion of a metric space generalises the idea of distance.

Definition 1.2.1. *We say that* (X, d) *is a* metric space *if* X *is a non-empty set and* $d : X^2 \to \mathbb{R}$ *is a function satisfying the following conditions.*
(i) $d(x, y) \geq 0$ *for all* $x, y \in X$.
(ii) $d(x, y) = 0$ *if and only if* $x = y$.
(iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
(iv) **[The triangle inequality]** $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$.

If we think of $d(x, y)$ as the 'distance between x and y' then the definition codifies some of our natural intuitions about distance. (However, the definition of a metric space only dates back to Fréchet in 1906.)

Many of our metrics will be derived from norms. We shall write $\mathbb{F}$ to mean $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.2.2. *Let* V *be a vector space over* $\mathbb{F}$. *We say that a map* $N : V \to \mathbb{R}$ *is a* norm *if, writing* $N(\mathbf{u}) = ||\mathbf{u}||$, *the following conditions hold.*
(i) $||\mathbf{u}|| \geq 0$ *for all* $\mathbf{u} \in V$.
(ii) $||\mathbf{u}|| = 0$ *implies* $\mathbf{u} = \mathbf{0}$.
(iii) $||\lambda \mathbf{u}|| = |\lambda| ||\mathbf{u}||$ *for all* $\mathbf{u} \in V$ *and* $\lambda \in \mathbb{F}$.
(iv) **[The triangle inequality]** $||\mathbf{u}|| + ||\mathbf{v}|| \geq ||\mathbf{u} + \mathbf{v}||$ *for all* $\mathbf{u}, \mathbf{v} \in V$.
We say that 'V *is a normed vector space with norm* $|| \, ||$*'.*

Exercise 1.2.3. *The reader is assumed to be familiar with the idea a vector space. Recall, or check the following two results which we shall use over and over again.*

(i) If V is a vector space over $\mathbb{F}$ and $U \subseteq V$ has the following properties

(a) $\mathbf{0} \in U$,

(b) $\boldsymbol{x}, \boldsymbol{y} \in U \Rightarrow \mathbf{x} + \mathbf{y} \in U$,

(c) $\lambda \in \mathbb{F},\ \boldsymbol{x} \in U \Rightarrow \lambda\mathbf{x} \in U$,

then U itself is a vector space over $\mathbb{F}$. We say that U is a subspace of V.

(ii) If V is a vector space (so, in particular, if $V = \mathbb{F}$) and X is a non-empty set then the collection V^X of functions $f : X \to V$ equipped with the pointwise *operations*

$$(f + g)(x) = f(x) + g(x),\ (lambdaf)(x) = \lambda(f(x))\, for\ x \in X$$

is a vector space.

Use these results to show that the collection $C_{\mathbb{R}}([0, 1])$ of continuous real valued functions on $[0, 1]$ forms a vector space under pointwise addition and scalar multiplication. I will usually leave such checking implicit.

I leave it to the reader to check that a norm gives rise to a metric in a natural manner.

Exercise 1.2.4. *Let V be a vector space over $\mathbb{F}$ with norm $\|\ \|$. Show that, if we write $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$, d is a metric on V. (Exercise 1.2.11 gives an example of a metric which cannot be obtained in this way.)*

Exercise 1.2.5. *Check that*

$$\|\mathbf{x}\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$$

defines a norm on $\mathbf{R}^n$. We shall call this the usual *or the* Euclidean *norm* .

I shall not pause to give other examples, since these notes are filled with examples of metrics and norms.

We have obvious generalisations of the notion of a limit and of a Cauchy sequence. (Note that, when we define such notions for metric spaces, we silently extend them to normed spaces by using the standard metric associated with our norm given in Exercise 1.2.4.)

Definition 1.2.6. *(i) If (X, d) is a metric and $x_n \in X$, we say that $x_n \to x_0$ (or, more explicitly, $x_n \underset{d}{\to} x_0$) if $d(x_n, x_0) \to 0$.*

(ii) If (X, d) is a metric and $x_n \in X$, we say that the x_n form a Cauchy sequence *if, given any $\epsilon > 0$, we can find an $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N(\epsilon)$.*

It is easy to prove results along the following lines.

Exercise 1.2.7. *(i) [Uniqueness] If (X, d) is a metric space, $x, y \in X$, $x_n \in X$ and $x_n \underset{d}{\to} x$, $x_n \underset{d}{\to} y$, show that $x = y$.*

(ii) [Any convergent sequence satisfies Cauchy's condition.] If (X, d) is a metric space, $x_n \in X$ and $x_n \underset{d}{\to} x$, show that the x_n form a Cauchy sequence.

(iii) If (X, d) is a metric space and we have a Cauchy sequence x_n in X, then if we can find $n(j) \to \infty$ and $x \in X$ such that $x_{n(j)} \underset{d}{\to} x$ as $j \to \infty$, it follows that $x_n \underset{d}{\to} x$ as $n \to \infty$. (Thus any Cauchy sequence with a convergent subsequence must converge.)

(iv) If $(V, \|\ \|)$ is a normed vector space over $\mathbb{F}$ and $\mathbf{u}_n, \mathbf{v}_n, \mathbf{u}, \mathbf{v} \in V$ with

$$\mathbf{u}_n \underset{\|\ \|}{\to} \mathbf{u} \text{ and } \mathbf{v}_n \underset{\|\ \|}{\to} \mathbf{v},$$

show that $\mathbf{u}_n + \mathbf{v}_n \underset{\|\ \|}{\to} \mathbf{u} + \mathbf{v}$.

(v) If $(V, \|\ \|)$ is a normed vector space over $\mathbb{F}$ and $\mathbf{u}_n, \mathbf{u} \in V$, $\lambda_n, \lambda \in \mathbb{F}$ with

$$\mathbf{u}_n \underset{\|\ \|}{\to} \mathbf{u}_0 \text{ and } \lambda_n \to \lambda,$$

show that $\lambda_n \mathbf{u}_n \underset{\|\ \|}{\to} \lambda_0 \mathbf{u}_0$.
[Hint: If you cannot do any of these, look at the case $X = V = \mathbb{R}$, $\|x\| = |x|$, $d(x, y) = |x - y|$.]

If we want to do analysis, then we need a richer structure than a mere metric.

Definition 1.2.8. *We say that a metric space (X, d) is* complete *if, whenever we have a Cauchy sequence $x_n \in X$, it follows that there exists an $x_0 \in X$ such that $x_n \underset{d}{\to} x$. (More concisely, every Cauchy sequence converges.)*

The main norms and metrics discussed in these notes will be complete. It may, however, be worth giving an example of a normed space which is not complete.

We will need the following result which, with variations, is used throughout these notes[2].

Exercise 1.2.9. *Let $f : [a, b] \to \mathbb{R}$ be continuous and $f(t) \geq 0$ for all $t \in [a, b]$. If $b > a$ and*

$$\int_a^b f(t)\, dt = 0$$

show that $f(t) = 0$ for all $t \in [a, b]$.

Give examples to show that the result is false if we only assume f integrable and positive or if we only assume f continuous.

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[2]The reader may be nervous about which integral to use. She may be assured that any integral that she is happy with will do.

Exercise 1.2.10. *Consider the space* $C([-1,1]) \to \mathbb{R}$ *of real valued continuous functions on the closed interval* $[-1,1]$. *Explain briefly how* $C([-1,1])$ *forms a vector space*[3]. *If* $f \in C([-1,1])$, *we write*

$$\|f\|_1 = \int_{-1}^{1} |f(t)|\, dt.$$

(i) Show that $\| \ \|_1$ *is a norm on* $C([-1,1])$.

(ii) (Why we need to be careful in our proof.) If $f_n(t) = t^{2n}$ *for* $n \geq 1$ *and* $f(t) = 0$, *show that* $f_n \underset{\| \ \|_1}{\to} f$, *but it is not true that,* $f_n(t) \to f(t)$ *as* $n \to \infty$ *for all* $t \in [-1,1]$*? Why?*

(iii) Let

$$f_n(t) = \begin{cases} -1 & \text{if } t \leq -n^{-1}, \\ nt & \text{if } -n^{-1} < t < n^{-1}, \\ 1 & \text{if } n^{-1} \leq t. \end{cases}$$

Show that the f_n *form a Cauchy sequence.*

Suppose, if possible, that $f_n \underset{\| \ \|_1}{\to} f$ *for some* $f \in C([-1,1])$. *By observing that*

$$\int_{\delta}^{1} |f_n(t) - f(t)|\, dt \leq \|f_n - f\|_1,$$

or otherwise, show that $f(t) = 1$ *for all* $\delta \leq t \leq 1$ *whenever* $\delta > 0$. *Show that* $f(t) = 1$ *for* $0 < t \leq 1$ *and* $f(t) = -1$ *for* $-1 \leq t < 0$. *Conclude that no* f *with our required properties can exist and so* $C([-1,1], \| \ \|_1)$ *is not complete.*

The idea of a complete normed space seems so natural now that it is hard to believe that the notion was only isolated by Banach in 1920. To the great majority of mathematicians of the time the question 'what objects should we study in analysis' would have appeared absurd and the minority who disagreed gave answers which were either too narrow or or too broad. Banach spotted a useful answer and complete normed spaces are called Banach spaces in his honour. It took a long time to convince classical analysts that the idea was useful for anything except 'the ease with which it has furnished doctoral theses'[4]. Zygmund's great work *Trigonometric Series* published in 1959 is probably the last major analysis text which does not use the concept explicitly.

The following example of a complete metric space is mainly useful as a source of counter-examples.

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[3] In future I will usually assume that the reader will silently check that the spaces of functions we consider are vector spaces without being asked.

[4] The quotation is from Wiener in *Ex-Prodigy*. However, Wiener does not neglect to tell the reader that he discovered Banach spaces independently.

Exercise 1.2.11. *If X is a non-empty set, define*

$$\Delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Show that (X, Δ) is a complete metric space. We call Δ the discrete metric *on X.*

If X is a non-trivial vector space over $\mathbb{F}$ show that we cannot find a norm $\| \; \|$ on X with $\|x - y\| = \Delta(x, y)$ for $x \neq y$. (We give a much stronger version of this result in Exercise 1.5.30.)

Let us write $C_{\mathbb{F}}([a, b])$ for the space of continuous functions $f : [a, b] \to \mathbb{F}$. (Often we just write $C([a, b]) = C_{\mathbb{F}}([a, b])$, but, sometimes, as in Section 4.4, when we discuss the Stone–Weierstrass theorem, the distinction between $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ is important.) There is a natural norm associated with this space.

Recall that a continuous function on a closed bounded interval is bounded and attains its bounds. (We shall state and prove this result in a more general context as Theorem 4.3.15.)

Exercise 1.2.12. *(i) Explain why*

$$\|f\|_\infty = \sup\{|f(t)| : t \in [a, b]\}$$

exists for all $f \in C_{\mathbb{F}}([a, b])$.

(ii) Show that $\| \; \|_\infty$ is a norm on $C_{\mathbb{F}}([a, b])$.

The classical theorem that the uniform limit of continuous functions is continuous and the general principle of uniform convergence now give us the key fact about the *uniform norm* $\| \; \|_\infty$.

Theorem 1.2.13. $(C_{\mathbb{F}}([a, b]), \| \; \|_\infty)$ *is complete.*

We shall state and reprove this result in a more general context as Theorem 4.4.2.

The following long but easy sequence of exercises, make explicit several useful facts.

Exercise 1.2.14. *(i) If (X, d) is a metric space, use the triangle inequality to show that any sequence of points $x_n \in X$ such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ converges must be a Cauchy sequence.*

(ii) If (X, d) is a complete metric space, deduce that any sequence of points $x_n \in X$ such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ converges must converge.

(iii) If $(V, \| \; \|)$ is a complete normed space, show that, if a sequence of points $\mathbf{v}_r \in V$ satisfies the condition $\sum_{r=1}^{\infty} \|\mathbf{v}_r\|$ converges, then $\sum_{r=1}^{n} \mathbf{v}_r$ converges in norm as $n \to \infty$.

(iv) Deduce the standard result that absolute convergence implies convergence in $\mathbb{C}$. *(That is to say, if* $\sum_{n=1}^{\infty} |a_n|$ *converges, so does* $\sum_{n=1}^{\infty} a_n$.*)*

(v) Also deduce the Weierstrass M-test: If $M_n \in \mathbb{R}$ *and* $f_n \in C([a,b])$, *then, if* $|f_n(x)| \leq M_n$ *for all* $x \in [a,b]$ *and all n, and if* $\sum_{n=1}^{\infty} M_n$ *converges then* $\sum_{n=1}^{N} f_n$ *converges uniformly to some* $f \in C([a,b])$.

Exercise 1.2.15. *(i) By extracting subsequences, or otherwise, prove the following result. Let* ϵ_j *be a fixed sequence with* $\epsilon_j > 0$ *and* $\epsilon_n \to 0$. *If* (X,d) *is a metric space such that every sequence* x_j *with* $d(x_{j-1}, x_j) < \epsilon_j$ *converges, then* (X,d) *is complete.*

(ii) Let ϵ_j *be a fixed sequence with* $\epsilon_j > 0$ *and* $\epsilon_n \to 0$. *If* $(X, \| \ \|)$ *is a normed space such that, whenever* $\|\mathbf{y}_j\| \leq \epsilon_j$ *for all* $j \geq 1$, *it follows that* $\sum_{j=1}^{n} \mathbf{y}_j$ *tends to a limit, show that* $(X, \| \ \|)$ *is complete.*
[*The advantage of results (i) and (ii) is that we can choose our test sequence* ϵ_j *tending to zero very rapidly.*]

(iii) Use (i) to prove the converse of Exercise 1.2.14 (ii) :- Suppose (X,d) *is a metric space such that* x_n *tends to a limit whenever* $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ *converges. Then* (X,d) *is complete.*

State and prove the converse of Exercise 1.2.14 (iii).

Exercise 1.2.16. *(i) Let* $C^1([a,b])$ *be the space of once continuously differentiable functions* $f : [a,b] \to \mathbb{R}$. *Show (quoting any theorems that you need) that the equation*

$$\|f\|_{(1)} = \|f\|_\infty + \|f'\|_\infty$$

defines a complete norm on $C^1([a,b])$.

(ii) Let $C^p([a,b])$ *be the space of p times continuously differentiable functions* $f : [a,b] \to \mathbb{R}$. *Show that the equation*

$$\|f\|_{(p)} = \sum_{j=0}^{p} \|f^{(j)}\|_\infty$$

defines a complete norm on $C^p([a,b])$.

(iii) Is $\|f\|_A = \|f'\|_\infty$ *a norm on* $C^1([a,b])$? *If it is a norm, is it complete? Is* $\|f\|_\infty$ *a norm on* $C^1([a,b])$? *If it is a norm, is it complete? Is* $\|f\|_B = |f(a)| + \|f'\|_\infty$ *a norm on* $C^1([a,b])$? *If it is a norm, is it complete? Give reasons.*

1.3 The contraction mapping theorem

In 1916 the Polish mathematician Steinhaus was taking an evening stroll when

> I overheard the words 'Lebesgue measure'. I approached the park bench and introduced myself to the two young apprentices of mathematics. They told me they had another companion by the name of Witold Wilkosz, whom they extravagantly praised. The youngsters were Stefan Banach and Otto Nikodym. Steinhaus *Reminiscences*

Steinhaus was a major figure in 20th century mathematics, but it is hard to dissent from his view that 'My greatest discovery was Stefan Banach.'

At the beginning of the 20th century, analysts had acquired a large collection of tricks that usually worked or often worked. Specialists recognised that many of the arguments they used fell into almost standard patterns. Banach was able to spot exactly why such techniques worked and replace pages of 'semi-automatic argument' by simple general arguments.

For centuries, mathematicians have used various versions of the following heuristic. Suppose we wish to solve the equation $F(x) = 0$. If we can rewrite our problem so that we need to solve $x = g(x)$, then we can take an initial 'trial solution' x_0 and look at $x_1 = g(x_0)$. With luck the new 'trial solution' x_1 will be closer to the true solution. We now look at $x_2 = g(x_1)$ and so on.

Exercise 1.3.1. *The following method of finding the square root of a positive number b was written down by the brilliant Greek engineer Heron in 60 AD, but may well have been known earlier:-*

Make a first guess $x_0 > 0$. Now find x_r iteratively by taking

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{b}{x_r}\right).$$

If you have not met Heron's method before, try it on an example.

It often happens, as in Heron's method that x_n converges to the true solution x_0 and it sometimes happens that we can prove it. This technique is most likely to work if

$$g(x_0 + t) = x_0 + \epsilon(t)$$

where the 'error term' $\epsilon(t)$ is expected to be small compared with t. From the pure mathematician's point of view the importance of this argument is that, sometimes, it can be used to show that the *existence* of a true solution.

Banach replaced many (but not all) versions of the method by a single simple theorem (sometimes called Banach's fixed point theorem, but, more usually, the contraction mapping theorem)[5].

[5] Banach's s first version of this result occurs in his PhD thesis which also laid the ground work for Banach spaces. Since Banach was largely self taught, he had no previous university degree and a special case had to be made by Steinhaus and others to allow him to proceed. It is a sign of a good university that such an exemption from the ordinary rules is very, very hard to obtain. It is a sign of a less good institution that such exemptions are impossible.

Theorem 1.3.2. [The contraction mapping theorem] *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a map such that there exists a* $K < 1$ *with* $d(Tx, Ty) \leq Kd(x, y)$ *for all* $x,\ y \in X$. *Then there exists a unique* $a \in X$ *such that* $Ta = a$.

Another way of stating the result is to say that T has a unique fixed point. Fieed point theorems are rare but often extremely useful.

Maps T such that $d(Tx, Ty) \leq Kd(x, y)$ for all $x,\ y \in X$ and some $K < 1$ are called *contraction maps*. The following exercise sheds some light on the proof that follows.

Exercise 1.3.3. *Let* (X, d) *be a (not necessarily complete) metric space and* $T : X \to X$ *a map such that there exists a* $K < 1$ *with* $d(Tx, Ty) \leq Kd(x, y)$ *for all* $x,\ y \in X$. *Suppose that there exists an* $a \in X$ *with* $Ta = a$. *Show that, if* x_0 *is any point in* X *and we define* $x_1 = Tx_0,\ x_2 = Tx_1,\ \dots,$ *we have* $x_n \to a$.

Proof of Theorem 1.3.2. **(Uniqueness)** Suppose that $a = Ta$ and $b = Tb$. Then

$$0 \leq d(a, b) = d(Ta, Tb) \leq Kd(a, b)$$

and, since $K < 1$, it follows that $d(a, b) = 0$ and $a = b$.
(Existence) Pick any $x_0 \in X$ and define $x_n = Tx_{n-1}$ for $n \geq 1$. If $r \geq 1$, we have

$$d(x_{r+1}, x_r) = d(Tx_r, Tx_{r-1}) \leq Kd(x_r, x_{r-1})$$

and so, using this result r times, $d(x_{r+1}, x_r) \leq K^r d(x_1, x_0)$. It follows that, if $m > n$, then, using the triangle inequality,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{r=n}^{m-1} d(x_{r+1}, x_r) \leq \sum_{r=n}^{m-1} K^{r-1} d(x_1, x_0) \\ &\leq \frac{K^{n-1}}{1-K} d(x_1, x_0) \to 0 \end{aligned}$$

as $n \to \infty$. Thus the x_n form a Cauchy sequence and we can find an $a \in X$ such that $x_n \to a$. (We could have used Exercise 1.2.14 (ii).)

We complete the proof by showing that a is, indeed, a fixed point. Observe that

$$\begin{aligned} d(Ta, a) &\leq d(Ta, x_{n+1}) + d(x_{n+1}, a) = d(Ta, Tx_n) + d(x_{n+1}, a) \\ &\leq Kd(a, x_n) + d(x_{n+1}, a) \to 0 + 0 = 0. \end{aligned}$$

Thus $d(Ta, a) = 0$ and $a = Ta$. ■

Here is a very simple example of the contraction mapping theorem in action. We shall give a deeper example in the next section.

Exercise 1.3.4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a once differentiable function such that there is an $0 < k < 1$ such that $|f'(x) - 1| < k$ for all x.*

(i) Define $T : \mathbb{R} \to \mathbb{R}$ by $Tx = f(x) - x$. By using the mean value theorem, show that T is a contraction mapping on $\mathbb{R}$ with the usual metric. Deduce that there is a unique $a \in \mathbb{R}$ such that $f(a) = a$.

(ii) Prove the statement in the previous sentence directly without using the contraction mapping theorem.

Exercise 1.3.4 provides a prelude to an example which shows that we cannot replace the condition that 'there exists a $K < 1$ with $d(Tx, Ty) \leq Kd(x, y)$' by the weaker condition '$d(Tx, Ty) < d(x, y)$ for $x \neq y$'.

Exercise 1.3.5. *(i) Find a differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $1 > g'(x) > 0$ and $g(x) < 0$ for all $x \in \mathbb{R}$.*

(ii) If g has the properties stated in (i) and $Tx = x - g(x)$, show that $|Tx - Ty| < |x - y|$ for all $x \neq y$, but the equation $Tx = x$ has no solution.

1.4 Differential equations

We now return to the question posed in Section 1.1. When can we prove the existence and uniqueness of solutions of a differential equation? Our first task is to decide what is meant by a differential equation. Most people would agree that a differential equation takes the form

$$y^{(n)}(t) = g(y(t), y'(t), y''(t) \ldots, y^{(n-1)}(t), t)$$

(with suitable initial conditions) and even those who disagree might well accept that this is a good place to start.

Exercise 1.4.1. *Go through the differential equations that you have met seeing if you can put them in the form just given.*

If we take $x_1(t) = y(t)$, then the proposed problem can be rewritten in the form

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= x_3(t) \\ &\vdots \\ x_{n-2}'(t) &= x_{n-1}(t) \\ x_{n-1}'(t) &= g(x_1(t), x_2(t), \ldots, x_{n-1}(t), t), \end{aligned}$$

that is to say, in the form

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t), t)$$

with

$$\mathbf{f}(s_1, s_2, \ldots, s_n) = (s_2, s_3, \ldots, s_{n-1}, g(s_1, s_2, \ldots, s_n)).$$

I shall consider the one dimensional problem

$$x'(t) = f(x(t), t)$$

and leave the easy task of generalising to n dimensions to the reader (see Exercise 1.4.6).

The first thing we do is change the question from one on *differentiation* to one on *integration*. To see why this might be advantageous observe that the derivative of a function is often less well behaved than the function itself whilst the integral of the function is often better behaved. (In the same way 'differencing' is a more dangerous procedure than 'averaging' when we want to make sense of a table of figures.

Lemma 1.4.2. *Let $f : \mathbb{R} \times [a, b] \to \mathbb{R}$ be continuous and suppose $x_0 \in \mathbb{R}$, $t_0 \in (a, b)$.*

(i) If $x : [a, b] \to \mathbb{R}$ is continuous and

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\, ds$$

for all $t \in [a, b]$, then x is differentiable on (a, b), $x(t_0) = x_0$ and

$$x'(t) = f(x(t), t)$$

for all $t \in (a, b)$.

(ii) If $x : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), $x(t_0) = x_0$ and

$$x'(t) = f(x(t), t)$$

for all $t \in (a, b)$, then

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\, ds$$

for all $t \in [a, b]$.

Proof. Just use the fundamental theorem of the calculus. ∎

We now use the contraction mapping theorem to give a very satisfactory answer to the question posed in Section 1.1.

Theorem 1.4.3. [Picard's existence theorem] *Suppose that $x_0, t_0 \in \mathbb{R}$, $\delta > 0$, $k > 0$ and $k\delta < 1$. If the continuous function $f : \mathbb{R} \times [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ satisfies the* Lipschitz condition

$$|f(u,t) - f(v,t)| \leq k|u - v|$$

for all $t \in [t_0 - \delta, t_0 + \delta]$ and $u,\ v \in \mathbb{R}$, then there is one and only one differentiable function $x : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$ such that

$$x'(t) = f(x(t), t) \text{ and } x(t_0) = x_0.$$

Proof. By translation, we may suppose $x_0 = t_0 = 0$. By Lemma 1.4.2, it is sufficient to prove that there is one and only one continuous function $x : [-\delta, \delta] \to \mathbb{R}$ such that

$$x(t) = \int_0^t f(x(s), s)\, ds$$

for all $|t| \leq \delta$.

We now look at $C([-\delta, \delta])$ with the uniform norm. If $w \in C([-\delta, \delta])$ and we set

$$(Tw)(t) = \int_0^t f(w(s), s)\, ds$$

for all $t \in [-\delta, \delta]$, then $Tw \in C([-\delta, \delta])$. Thus T is a map from $C([-\delta, \delta])$ to itself, Further, if $u,\ v \in C([-\delta, \delta])$ and $0 \leq t \leq \delta$, then

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &= \left| \int_0^t f(u(s), s) ds - \int_0^t f(v(s), s) ds \right| \\ &= \left| \int_0^t f(u(s), s) ds - f(v(s), s) ds \right| \\ &\leq \int_0^t |f(u(s), s) ds - f(v(s), s)| ds \leq \int_0^t k|u(s) - v(s)| ds \\ &\leq \int_0^t k||u - v||_\infty = kt||u - v||_\infty \leq k\delta||u - v||_\infty. \end{aligned}$$

Similarly

$$|(Tu)(t) - (Tv)(t)| \leq k\delta||u - v||_\infty$$

whenever $-\delta \leq t \leq 0$. Thus

$$||Tu - Tv||_\infty \leq k\delta||u - v||_\infty$$

and we have shown that T is a contraction mapping on the complete normed space $(C([-\delta, \delta], ||\ ||_\infty)$. The contraction mapping theorem tells us that there is a unique $u \in C([-\delta, \delta])$ with $Tu = u$ and this is precisely what we set out to prove. ■

Exercise 1.4.4. *(i) If the partial derivative*

$$\frac{\partial f}{\partial u}(u,t)$$

exists and is bounded in absolute value by k *for all* $(u,t) \in \mathbb{R} \times [t_0 - \delta, t_0 + \delta]$, *show that* f *satisfies the Lipschitz condition*

$$|f(u,t) - f(v,t)| \leq k|u - v|$$

for all $t \in [t_0 - \delta, t_0 + \delta]$ *and* $u, v \in \mathbb{R}$.

(ii) If $f(u,t) = |u|$ *show that* f *satisfies a Lipschitz condition, but that the partial derivative with respect to the first variable does not exist everywhere.* [*Informally, we may say that the Lipschitz condition is close to, though slightly weaker than, differentiability.*]

Our Picard theorem is a *local* theorem which states that (under appropriate conditions) we can solve our problem on some 'small patch'. We then try and combine the patches in some way to get a *global* solution. Here is a simple example of such a patching.

Exercise 1.4.5. *In this question we take* $k > 0$ *and suppose that the continuous function* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies the* Lipschitz condition

$$|f(u,t) - f(v,t)| \leq k|u - v|$$

everywhere.

(i) Suppose $y : (a,b) \to \mathbb{R}$ *satisfies the differential equation*

$$y'(t) = f(y(t), t)$$

for all $t \in (a,b)$. *Suppose that* $b - a > \delta > 0$ *and* $4k\delta < 1$. *By considering the problem of finding a differentiable function* $x : (b - 3\delta, b + \delta)$ *such that*

$$x'(t) = f(x(t), t) \text{ for } t \in (b - 3\delta, b + \delta) \text{ and } x(b - \delta) = y(b - \delta),$$

show that there exists a unique differentiable function $\tilde{y} : (a, b + \delta) \to \mathbb{R}$ *satisfying the differential equation*

$$\tilde{y}'(t) = f(\tilde{y}(t), t)$$

for all $t \in (a, b + \delta)$ *and the equation*

$$\tilde{y}(t) = y(t)$$

for all $t \in (a,b)$.

(ii) Show that if $x_0, t_0 \in \mathbb{R}$, *there is one and only one differentiable function* $x : \mathbb{R} \to \mathbb{R}$ *such that*

$$x'(t) = f(x(t), t) \text{ for all } t \in \mathbb{R} \text{ and } x(t_0) = x_0.$$

In Exercise 1.8.3 we give an account of these matters which is closer to Picard's original proof. The reader will observe that Picard's original proof gives more information, but that Banach's proof is more flexible. We shall consider further generalisations of Picard's theorem in Section 5.1.

Exercise 1.4.6. *Without necessarily writing much down, convince yourself that, by generalising the proof of theorem 1.4.3 in the appropriate manner, we can prove the following natural n dimensional version of Picard's theorem.*

Suppose that $\mathbf{x}_0 \in \mathbb{R}^n$*,* $t_0 \in \mathbb{R}$ *and* $\delta > 0$*,* $k > 0$ *and* $k\delta < 1$*. If the continuous function* $\mathbf{f} : \mathbb{R}^n \times [t_0 - \delta, t_0 + \delta] \to \mathbb{R}^n$ *satisfies the* Lipschitz condition

$$\|\mathbf{f}(\mathbf{u}, t) - \mathbf{f}(\mathbf{v}, t)\| \leq k\|\mathbf{u} - \mathbf{v}\|$$

for all $t \in [t_0 - \delta, t_0 + \delta]$ *and* $\mathbf{u}$*,* $\mathbf{v} \in \mathbb{R}^n$*, then there is one and only one differentiable function* $\mathbf{x} : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$ *such that*

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t), t) \text{ and } \mathbf{x}(t_0) = \mathbf{x}_0.$$

[*If you are the kind of person who worries that we have not defined vector integrals or said what* $\|\mathbf{u}\|$ *means (take it to be the Euclidean length of* $\mathbf{u}$*), do not do this exercise. If you are not, you will find the exercise a useful revision of the one dimensional ideas.*]

1.5 Open and closed sets

This section consists of a long series of definitions and exercises. Most readers should find them comfortably familiar (at least in the case of $\mathbb{R}^n$ with the usual metric). Any reader who finds them neither familiar nor easy is probably reading the wrong exposition.

Definition 1.5.1. *If* (X, d) *is a metric space,* $a \in X$ *and* $r > 0$*, we write*

$$B(a, r) = \{x \in X : d(x, a) < r\}.$$

Definition 1.5.2. *If* (X, d) *is a metric space, we say that a subset* U *of* X *is* open *if, given any* $x \in U$*, we can find a* $\delta > 0$ *such that* $B(x, \delta) \subseteq U$*.*

Exercise 1.5.3. *If* (X, d) *is a metric space,* $a \in X$ *and* $r > 0$*, show that* $B(a, r)$ *is open.*

Naturally, we call $B(a, r)$ the *open ball* centre a and radius r.

Exercise 1.5.4. *If (X, d) is a metric space, show that the collection τ of open sets has the following properties.*

(i) X, $\varnothing \in \tau$.

(ii) If $U_\alpha \in \tau$ for all $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.

(iii) If $U_j \in \tau$ for all $1 \leq j \leq 1$, then $\bigcap_{j=1}^n U_j \in \tau$.

Exercise 1.5.5. *Give $\mathbb{R}$ its usual metric. Show that the interval (a, b) is open for all $a < b$. Show that $\bigcap_{n=1}^\infty (-1/n, 1/n)$ is not open.*

Exercise 1.5.6. *Let (X, d) be a metric space. Show that the following two conditions on a subset E of X are equivalent.*

(i) If $x_n \in E$, $x \in X$ and $x_n \to x$, then $x \in E$.

(ii) The set $X \setminus E$ is open.

Definition 1.5.7. *If E obeys the two equivalent conditions of Exercise 1.5.6, we say that E is* closed.

Exercise 1.5.8. *If (X, d) is a metric space, $a \in X$ and $r > 0$, let us write*

$$\bar{B}(a, r) = \{x \in X \,:\, d(a, x) \leq r\}.$$

Show that $\bar{B}(a, r)$ is closed.

Naturally, we call $\bar{B}(a, r)$ the *closed ball* centre a and radius r.

Exercise 1.5.9. *By first using condition (i) of Exercise 1.5.6 and then by using (ii), give two proofs of the following statement.*

If (X, d) is a metric space, then the set $\mathcal{F}$ of closed sets has the following properties.

(a) X, $\varnothing \in \mathcal{F}$.

(b) If $F_\alpha \in \mathcal{F}$ for all $\alpha \in A$, then $\bigcap_{\alpha \in A} F_\alpha \in \mathcal{F}$.

(c) If $F_j \in \mathcal{F}$ for all $1 \leq j \leq n$, then $\bigcap_{j=1}^n F_j \in \mathcal{F}$.

Exercise 1.5.10. *(i) Suppose that (X, d) is a metric space and E a subset of X. If we define $d_E : E^2 \to \mathbb{R}$ by $d_E(x, y) = d(x, y)$ for all x, $y \in E$, show that (E, d_E) is a metric space.*

(ii) Show that, if (E, d_E) is complete, then E is closed.

(iii) If (X, d) is a complete metric space, show that (E, d_E) is complete if E is closed.

(iv) Consider the collection V of continuous functions $f : [a, b] \to \mathbb{F}$ such that $f(a) = f(b)$. Show that V forms a complete normed space under the uniform norm.

We shall usually follow the mathematical custom of writing d instead of d_E.

Exercise 1.5.11. *Let (X, d) and (Y, ρ) be metric spaces. Show that the following three conditions on a function $f : X \to Y$ are equivalent.*

(i) If $x_n \in X$, $x \in X$ and $x_n \underset{d}{\to} x$, then $f(x_n) \underset{\rho}{\to} f(x)$.

(ii) Given $x \in X$ and $\epsilon > 0$, we can find a $\delta > 0$ such that $d(x, x') < \delta$ implies $\rho(f(x), f(x')) < \epsilon$.

(iii) If U is open in (Y, ρ), then $f^{-1}(U)$ is open in (X, d).

Definition 1.5.12. *If f obeys the three equivalent conditions of Exercise 1.5.11, we say that f is* continuous.

The reader should check that our extended definition of continuous is compatible with her previous notions.

Exercise 1.5.13. *Use each of the three conditions of Exercise 1.5.11 in turn to produce three proofs of the following result.*

Suppose (X_1, d_1), (X_2, d_2) and (X_3, d_3) are metric spaces and $f : X_1 \to X_2$, $g : X_2 \to X_3$ are maps. If f and g are continuous, then so is $gf : X_1 \to X_3$.

Exercise 1.5.14. *Let (X, d) and (Y, ρ) be metric spaces. Show that $f : X \to Y$ is continuous if and only if, whenever E is closed in (Y, ρ), then $f^{-1}(E)$ is closed in (X, d).*

Definition 1.5.15. *If (X, d) is a metric space, we define the* closure $\operatorname{Cl} E$ *and* interior $\operatorname{Int} E$ *by*

$$\operatorname{Cl} E = \bigcap \{F : F \text{ is closed and } F \supseteq E\}$$

and

$$\operatorname{Int} E = \bigcap \{U : U \text{ is open and } U \subseteq E\}.$$

Exercise 1.5.16. *We work in a metric space (X, d).*

(i) Show that $\operatorname{Cl} E$ is closed and $\operatorname{Cl} E \supseteq E$. Show that, if F is closed and $F \supseteq E$, then $F \supseteq \operatorname{Cl} E$. (Thus '$\operatorname{Cl} E$ is the smallest closed set containing E.')

(ii) State and prove corresponding results for $\operatorname{Int} E$.

(iii) Show that $\operatorname{Cl}(X \setminus E) = X \setminus \operatorname{Int} E$ and $\operatorname{Int}(X \setminus E) = X \setminus \operatorname{Cl} E$.

Exercise 1.5.17. *Let (X, d) be a metric space and E a subset of X.*

(i) Show that

$$x \in \operatorname{Cl} E \Leftrightarrow \text{there exist } x_n \in E \text{ with } x_n \to x \text{ as } n \to \infty.$$

(ii) Show that

$$x \in \operatorname{Int} E \Leftrightarrow \text{there exists a } \delta > 0 \text{ such that } B(x, \delta) \subseteq E.$$

Exercise 1.5.18. *We use the notation for open and closed balls introduced earlier. If* $(V, \| \|)$ *is a normed space space,* $\mathbf{a} \in X$ *and* $r > 0$, *show that*

$$B(\mathbf{a}, r) = \operatorname{Int} \bar{B}(\mathbf{a}, r) \text{ and } \bar{B}(\mathbf{a}, r) = \operatorname{Cl} B(\mathbf{a}, r).$$

By considering the discrete metric of Exercise 1.2.11, or otherwise, show that the corresponding result may be false for general metric spaces.

Definition 1.5.19. *We say that a subset* E *of a metric space* X *is* dense *if* $\operatorname{Cl} E = X$.

Exercise 1.5.20. *(i) If we give* $\mathbb{R}$ *the usual metric, show that* $\mathbb{Z}$ *is not dense in* $\mathbb{R}$.

(ii) The proof that $\mathbb{Q}$ *is dense in* $\mathbb{R}$ *is will be tightly linked to whatever axiom (or axioms) we choose as basic to analysis. Throughout these notes we have quoted the result that* $1/n \to 0$ *(a version of the axiom of Archimedes*[6]*) which in turn is usually deduced from the principle that every non-empty bounded set has a supremum.*

Show that $\mathbb{Q}$ *is dense in* $\mathbb{R}$ *from the fact that* $1/n \to 0$ *(or some other foundational principle).*

Sometimes we can extend properties from a dense subset to the whole space and sometimes we cannot.

Exercise 1.5.21. *Consider* $\mathbb{R}$ *with the usual metric. Show that* $E = \mathbb{R} \setminus \{0\}$ *is dense in* $\mathbb{R}$ *and that the function* $f : E \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

is continuous on E. *Show carefully that there does not exist a continuous function* $\tilde{f} : \mathbb{R} \to \mathbb{R}$ *such that* $\tilde{f}(x) = f(x)$ *for all* $x \in E$.

Exercise 1.5.22. *Suppose that* E *is a dense subset of* $\mathbb{R}$ *with the usual metric and* f *is a uniformly continuous function* $f : E \to \mathbb{R}$.

(i) If $x \in \mathbb{R}$, $x_n \in E$ *and* $x_n \to x$ *show that* $f(x_n)$ *is a Cauchy sequence. Explain why* $f(x_n) \to \bar{f}(x)$ *where the value of* $\bar{f}(x)$ *depends only on* x *and not on the sequence* x_n.

(ii) Deduce that there exists a unique continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ *such that* $\tilde{f}(x) = f(x)$ *for all* $x \in E$. *Is* $\tilde{f}$ *necessarily uniformly continuous and why?*

The possibility that (under the right circumstances) we can extend results from dense subsets to entire spaces explains why we shall be particularly interested in *separable* spaces.

[6] Who credited to Eudoxus.

Definition 1.5.23. *We say that a metric space* (X, d) *is* separable *if it has a countable dense subset.*

Exercise 1.5.24. *(i) Show that* $\mathbb{R}^n$ *with the usual metric is separable.*

(ii) Suppose Δ *is the discrete metric on a space* X. *State, with reasons, a necessary and sufficient condition for* (X, Δ) *to be separable..*

Exercise 1.5.25. *Show that the following are all metrics on* $\mathbb{R}^2$.

$$\begin{aligned}
d_1(\mathbf{x}, \mathbf{y}) &= \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)^{1/2},\\
d_2(\mathbf{x}, \mathbf{y}) &= \max\{|x_1 - y_1|, |x_2 - y_2|\},\\
d_3(\mathbf{x}, \mathbf{y}) &= |x_1 - y_1| + |x_2 - y_2|,\\
d_4(\mathbf{x}, \mathbf{y}) &= \min\{1, d_1(\mathbf{x}, \mathbf{y})\},\\
d_5(\mathbf{x}, \mathbf{y}) &= \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y},\\ 1 & \text{otherwise.}\end{cases}
\end{aligned}$$

The reader will probably be familiar with the fact that it hardly matters whether we use d_1, d_2 or d_3 to study $\mathbb{R}^2$ and it is pretty clear that, in many circumstances, it will not matter if we use d_1 or d_4. However, it is also clear that d_5 (the discrete metric) has a very different behaviour to the other metrics. The easy exercises that follow discuss how we can make these ideas precise.

Exercise 1.5.26. *Let* d *and* ρ *be metrics on a space* X. *We write*

$$B_d(a, r) = \{x \in X : d(a, x) < r\} \text{ and } B_\rho(a, r) = \{x \in X : \rho(a, x) < r\}.$$

Show that the following statements are equivalent.

(i) $x_n \underset{d}{\rightarrow} x \Rightarrow x_n \underset{\rho}{\rightarrow} x$.

(ii) If $a \in X$ *and* $\delta > 0$, *we can find an* $\epsilon > 0$ *such that* $B_\rho(a, \delta) \supseteq B_d(a, \epsilon)$.

(iii) If U *is an open set in* (X, ρ), *then* U *is an open set in* (X, d).

(iv) If we define the identity map $\iota : (X, d) \to (X, \rho)$ *by* $\iota x = x$ *for all* $x \in X$, *then* ι *is continuous.*

Exercise 1.5.27. *Let* d *and* ρ *be metrics on a space* X. *Show that the following statements are equivalent.*

(i) $x_n \underset{d}{\rightarrow} x \Leftrightarrow x_n \underset{\rho}{\rightarrow} x$.

(ii) If $a \in X$ *and* $\delta > 0$, *we can find* ϵ, $\epsilon' > 0$ *such that* $B_\rho(a, \delta) \supseteq B_d(a, \epsilon)$ *and* $B_d(a, \delta) \supseteq B_\rho(a, \epsilon')$.

(iii) The set U *is open in* (X, ρ) *if and only if* U *is open in* (X, d).

(iv) If we define the identity map $\iota : (X, d) \to (X, \rho)$ *by* $\iota x = x$ *for all* $x \in X$, *then* ι *and* ι^{-1} *are continuous.*

Exercise 1.5.27 suggests the following definition which parallels the idea of isomorphism in algebra.

Definition 1.5.28. *We say that two metric spaces* (X, d) *and* (Y, ρ) *are* homeomorphic *if there exists a bijection* $f : (X, d) \to (Y, \rho)$ *such that* f *and* f^{-1} *are both continuous. We say that* f *is a metric space homeomorphism.*

Exercise 1.5.29. *Show that homeomorphism is an equivalence relation on any collection of metric spaces* (X, d).

Informally, we may say that two metric spaces are homeomorphic if they have the same open set structure or, equivalently, if the have the same behaviour with respect to limits.

Exercise 1.5.30. *Let* V *be a non-trivial vector space with a norm* $\| \, \|$. *Show that* V *with the distance derived from the norm cannot be homeomorphic with any* (X, Δ) *where* Δ *is the discrete metric (see Exercise 1.2.11). (Informally, we say that not every metric can be derived from a norm.)*

Note that homeomorphic spaces need not have the same behaviour with respect to Cauchy sequences.

Exercise 1.5.31. *(i) Show that, if we use the usual metrics, the map* $f : (-1, 1) \to \mathbb{R}$ *given by* $f(x) = \tan(\pi x)/2$ *is a homeomorphism. However,* $\mathbb{R}$ *with the usual metric is complete, but the open interval* $(-1, 1)$ *is not. (You should explain why* $(-1, 1)$ *is not complete.)*

(ii) Find a metric d *on* $\mathbb{R}$ *such that*

$$d(x_n, x) \to 0 \Leftrightarrow |x_n - x| \to 0$$

but d *is not complete. Prove that* d *is indeed a metric with the required properties.*

There is natural way of ensuring that metric spaces behave the same way with respect to Cauchy sequences

Definition 1.5.32. *(i) We say that two metric spaces* (X, d) *and* (Y, ρ) *are* Lipschitz equivalent *if there exists a bijection* $f : (X, d) \to (Y, \rho)$ *and a constant* $K > 0$ *such that*

$$Kd(x, y) \geq \rho(f(x), f(y)) \geq K^{-1}d(x, y).$$

(ii) We say that two metrics d *and* ρ *on a space* X *are* Lipschitz equivalent *if there exists a constant* $K > 0$ *such*

$$Kd(x, y) \geq \rho(x, y) \geq K^{-1}d(x, y),$$

that is to say, the identity map is Lipschitz.

Exercise 1.5.33. *(i) Show that Lipschitz equivalent metric spaces are homeomorphic.*

(ii) Show that Lipschitz equivalence is an equivalence relation on any collection of metric spaces (X, d).

(iii) Explain briefly why Lipschitz equivalence in the sense of 1.5.32 (ii) is an equivalence relation on any collection of metrics on a given space.

(iv) Show that, if (X, d) *and* (Y, ρ) *are Lipschitz equivalent and* (X, d) *is complete, then* (Y, ρ) *is complete.*

(v) Show that $\rho(x, y) = |x^3 - y^3|$ *defines a complete metric on* $[-1, 1]$. *Using this result, or otherwise show that we can have two metrics* d *and* ρ *on a space* X *such that*

$$d(x_n, x) \to 0 \Leftrightarrow \rho(x_n, x) \to 0,$$

and d *and* ρ *are complete, but* d *and* ρ *are not Lipschitz equivalent. (We give another example in Exercise 4.7.3 (ii).)*

So far, in this section we have dealt only with pure metric structures. Once we add a vector space structure and consider the metric defined by a norm we get much more rigid behaviour.

Many of my readers will be familiar with the idea of the operator norm. This will play an increasingly important role as the notes progress.

Exercise 1.5.34. *Let* $(U, \| \|_U)$, $(V, \| \|_V)$ *and* $(W, \| \|_W)$ *be normed spaces over* $\mathbb{F}$.

(i) If $T : U \to V$ *is a continuous linear map, explain why we can find a* $\delta > 0$ *such that*

$$\|T\mathbf{u}\|_V \leq 1 \text{ for all } \|\mathbf{u}\|_U \leq \delta$$

and deduce that there exists a $K > 0$ *such that*

$$\|T\mathbf{u}\|_V \leq K \text{ for all } \|\mathbf{u}\|_U \leq 1.$$

Explain why this means that we can define the operator norm *of* T *by*

$$\|T\| = \sup\{\|T\mathbf{u}\|_V : \|\mathbf{u}\|_U \leq 1\}.$$

Show that $\|T\mathbf{w}\|_V \leq \|T\|\|\mathbf{w}\|_U$ *for all* $\mathbf{w} \in U$.

Show also that if $T : U \to V$ *is a linear map, with* $\|T\mathbf{u}\|_V \leq K\|\mathbf{u}\|_U$ *for all* $\mathbf{u} \in U$, *then* T *is continuous.*

(ii) We know from algebra that the space $\mathcal{L}(U, V)$ *of linear maps* $T : U \to V$ *can be made into a vector space in a natural manner*[7] *by setting*

$$(\lambda T + \mu S)\mathbf{u} = \lambda T\mathbf{u} + \mu S\mathbf{u}.$$

[7] If you are unfamiliar with this fact, carry out the easy check.

Show that the set $\mathcal{L}_C(U, V)$ of continuous linear maps is a subspace of $\mathcal{L}(U, V)$ and so a vector space.

(iii) Show that the operator norm is indeed a norm on $\mathcal{L}_C(U, V)$. Show also that, if $T \in \mathcal{L}_C(U, V)$ and $S \in \mathcal{L}_C(V, W)$, then $ST \in \mathcal{L}_C(U, W)$ and $\|ST\| \leq \|S\|\|T\|$.

(iv) If $T_n \in \mathcal{L}_C(U, V)$ and $T : U \to V$ is a function such that $\|T_n\mathbf{u} - T\mathbf{u}\|_V \to 0$ as $n \to \infty$ for each $\mathbf{u} \in U$, show that T is linear. If, in addition, there exists a K such that $\|T_n\| \leq K$ for all n show that $T \in \mathcal{L}_C(U, V)$.

(v) If $\mathbf{u} \in U$ and the sequence T_n is Cauchy in the operator norm, show that $T_n\mathbf{u}$ is Cauchy in $(V, \| \|_V)$.

(vii) If $(V, \| \|_V)$ is a complete normed space, show by using (v) and (iv) that $\mathcal{L}_C(U, V)$ with the operator norm is a complete normed space.

Definition 1.5.35. *In the particular case when $V = \mathbb{F}$ with its standard structure, we write $U' = \mathcal{L}_C(U, \mathbb{F})$ and call U' the* dual space *of U.*

We sometimes refer to the elements of the dual space as *functionals*.

Exercise 1.5.36. *Let c_{00} be the vector space of sequences $\mathbf{a} = (a_1, a_2, \ldots)$ with the $a_j \in \mathbb{F}$ and only finitely many a_j non-zero. (We use the standard coordinate-wise definition of addition and scalar multiplication.)*

Show that

$$\|\mathbf{a}\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$$

defines a norm on c_{00}. Show that, if we use this norm, the equation

$$T(\mathbf{a}) = \sum_{j=1}^{\infty} j a_j$$

gives a well defined linear map $T : c_{00} \to \mathbb{F}$ (where $\mathbb{F}$ has its usual norm) which is not continuous.

[*The alert reader will observe that $(c_{00}, \| \|_\infty)$ is not a complete normed space (the unalert reader should prove this) and ask for an example with a complete norm. Her question is very proper, but requires discussion of the Axiom of Choice.*]

We have answered the question of when two metric spaces are the same from the point of view of their open set structure by introducing the notion of homeomorphism. The reader will probably be familiar with the fact that a 'vector space isomorphism' means that two vector spaces are the same from the point of view of their algebraic structure. What happens if we ask whether two normed vector spaces are the same from the point of view of normed vector space theory? It is natural to use the following definition.

Definition 1.5.37. *The normed vector spaces* $(U, \| \|_U)$ *and* $(V, \| \|_V)$ *are* isomorphic *if and only if there exists a linear bijection* $T : U \to V$ *such that* T *and* T^{-1} *are continuous. We call* T *an isomorphism of the two normed vector spaces.*

Exercise 1.5.38. *(i) If* U *and* V *are vector spaces and* $T : U \to V$ *is both bijective and linear, show that* $T^{-1} : V \to U$ *is linear.*

(ii) If $(U, \| \|_U)$ *and* $(V, \| \|_V)$ *are normed vector spaces show that a linear bijection* $T : U \to V$ *is a norm vector space isomorphism if and only if there exists a* $K > 0$ *with*

$$K\|\mathbf{u}\|_U \geq \|T\mathbf{u}\|_V \geq K^{-1}\|\mathbf{u}\|_U$$

for all $\mathbf{u} \in U$.

(iii) If $(U, \| \|_U)$ *and* $(V, \| \|_V)$ *are normed vector spaces and* $T : U \to V$ *is a norm vector space isomorphism, show that* $\|T\|\|T^{-1}\| \geq 1$.

(iv) Let $U = V = \mathbb{R}^2$ *and let* $\| \|_U$ *and* $\| \|_V$ *be the usual norms. Give, with proof a normed vector space isomorphism* $T : (U, | \|_U) \to (V, | \|_V)$ *with* $\|T\|\|T^{-1}\| > 1$.

(v) Explain why, if one of a pair of isomorphic normed vector spaces is complete, the other must be.

1.6 Baire's category theorem

The next theorem is a profound triviality.

Theorem 1.6.1. [Baire's category theorem] *If* (X, d) *is a complete non-empty metric space, then* X *cannot be written as the union of a countable collection of closed sets with empty interior.*

One way of thinking of a closed set E with empty interior is the following. The property of belonging to E is *unstable*, since arbitrarily small changes take us outside E, but the the property of not belonging to E is *stable* since, if we are at a point outside E, all sufficiently small changes keep us outside E.

Exercise 1.6.2. *Let* (X, d) *be a metric space. Show that a subset* E *of* X *has empty interior if and only if* $X \setminus E$ *is dense.*

We shall prove a slightly stronger version of Baire's theorem.

Theorem 1.6.3. *Let* (X, d) *be a complete non-empty metric space. If* $E_1, E_2, \ldots$ *are closed sets with empty interiors, then* $X \setminus \bigcup_{j=1}^{\infty} E_j$ *is dense in* X.

Proof. Suppose that $x_0 \in X$ and $\delta_0 > 0$. We shall show that there exists a $y \in B(x_0, \delta_0)$ such that $y \notin \bigcup_{j=1}^{\infty} E_j$.

To this end, we perform the following inductive construction. Given $x_{n-1} \in X$ and $\delta_n > 0$, we can find $x_n \in X$ such that $x_n \in B(x_{n-1}, \delta_n/4)$, but $x_n \notin E_n$. (For, if not, we would have $B(x_{n-1}, \delta_{n-1}/4) \subseteq E_n$ and E_n would have a non-empty interior.) Since E_n is closed and $x_n \notin E_n$, we can now find a δ_n with $\delta_{n-1}/2 > \delta_n > 0$ such that $B(x_n, \delta_n) \bigcap E_n = \emptyset$.

Now observe that

$$\delta_m \leq 2^{-1}\delta_{m-1} \leq 2^{-2}\delta_{m-2} \leq \ldots \leq 2^{m-n}\delta_n$$

for all $n \geq m \geq 0$. It follows that, if $r \geq s$, then

$$d(x_r, x_s) \leq \sum_{j=r}^{s-1} d(x_{j+1}, x_j) \leq \sum_{j=r}^{s-1} \delta_j/4 \leq 4^{-1}\delta_0 \sum_{j=r}^{s-1} 2^{-j} \leq 2^{-r-1}\delta_0.$$

Thus the x_r form a Cauchy sequence and converge in (X, d) to some point y.

The same kind of calculation as in the last paragraph gives

$$d(x_r, x_s) \leq \sum_{j=r}^{s-1} d(x_{j+1}, x_j) \leq \sum_{j=r}^{s-1} \delta_r/4 \leq 4^{-1}\delta_r \sum_{j=0}^{s-r-1} 2^{-r} \leq \delta_r/2,$$

whenever $s \geq r$ and so

$$d(x_r, y) \leq d(x_r, x_s) + d(x_s, y) \leq \delta_r/2 + d(x_s, y) \to \delta_r/2$$

as $s \to \infty$. We thus have $d(x_r, y) \leq \delta_r/2$, so $y \in B(x_0, \delta_r)$ and $y \notin E_r$ for each $r \geq 1$ as required. ■

For historical reasons, Baire's category theorem is associated with some rather peculiar nomenclature.

Definition 1.6.4. *Let (X, d) be a metric space. We say that a subset A of X is* meagre *(or* of the first category*) if it is a subset of the union of a countable collection of closed sets with empty interior*[8]*. We say that* quasi-all *points of X belong to the complement $X \setminus A$ of X.*

Exercise 1.6.5. *Consider $[0, 1]$ with the usual metric. Show that $[0, 1/2]$ is neither meagre nor the complement of a meagre set.*

Theorem 1.6.3 thus states that the complement of a meagre set in a non-empty complete metric space is dense in that space. The next exercise gives a simple but very useful property of meagre sets.

[8]Not surprisingly, a subset of the union of a countable collection of closed sets with empty interior need not itself be a countable collection of closed sets with empty interior, but we shall not prove this here.

Exercise 1.6.6. *Show that the countable union of meagre sets is itself meagre.*

The reader will have met the following theorem before, but, perhaps, not the proof given here.

Theorem 1.6.7. $\mathbb{R}$ *is uncountable.*

Proof. If we give $\mathbb{R}$ its usual metric then point sets $\{x\}$ are closed and have empty interior. It follows that, if E is a countable subset of $\mathbb{R}$, then $E = \bigcup_{e\in E}\{e\}$ is meagre and so $E \neq \mathbb{R}$. ■

The standard undergraduate proof of Theorem 1.6.7 involves decimal expansions, but the proof given here avoids having to talk about the relation between real numbers and decimals. It is also much closer to Cantor's original proof.

Exercise 1.6.8. *If* (X, d) *is a metric space, we say that a point* $x \in X$ *is* isolated *if we can find a* $\delta > 0$ *such that* $B(x, \delta) = \{x\}$.

(i) Show that a point $x \in X$ *is isolated if and only if* $\{x\}$ *is open.*

(ii) Show that any complete non-empty metric space without isolated points is uncountable.

(iii) Give an example of a complete infinite metric space which is countable.

(vi) Give an example of a uncountable complete metric space with every point isolated.

Banach used Baire's category theorem to give a very illuminating proof of the existence of continuous nowhere differentiable functions.

Theorem 1.6.9. *Consider* $C([0, 1])$ *the space of continuous functions* $f : [0, 1] \to \mathbb{R}$ *equipped with the uniform norm. Quasi-all continuous functions are nowhere differentiable.*

The proof revolves round the following lemma.

Lemma 1.6.10. *Consider* $C([0, 1])$ *the space of continuous functions* $f : [0, 1] \to \mathbb{R}$ *equipped with the uniform norm. If* m *is strictly positive integer, then the set* E_m *consisting of those* $f \in C([0, 1])$ *such that there exists an* $x \in [0, 1]$ *with* $|f(x) - f(y)| \leq m|x - y|$ *for all* $y \in [0, 1]$ *is a closed set with empty interior.*

Proof. We first show that E_m is closed. Suppose that $f_n \in E_m$, $f \in C([0, 1])$ and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. By the definition of E_m, we can find $x_n \in [0, 1]$ such that

$$|f_n(x_n) - f_n(y)| \leq m|x_n - y| \text{ for all } y \in [0, 1].$$

By the theorem of Bolzano–Weierstrass, we can find an $x \in [0,1]$ and a sequence $n(j) \to \infty$ such that $x_{n(j)} \to x$ as $j \to \infty$. We observe that

$$\begin{aligned}|f(x)-f(y)| &\leq |f(x) - f(x_{n(j)})| + |f(x_{n(j)}) - f_{n(j)}(x_{n(j)})| \\ &\qquad + |f_{n(j)}(x_{n(j)}) - f_{n(j)}(y)| + |f_{n(j)}(y) - f(y)| \\ &\leq |f(x) - f(x_{n(j)})| + \|f - f_{n(j)}\|_\infty + m|x_{n(j)} - y| + \|f - f_{n(j)}\|_\infty \\ &\leq |f(x) - f(x_{n(j)})| + 2\|f - f_{n(j)}\|_\infty + m|x_{n(j)} - x| + m|x - y| \\ &\to m|x - y|\end{aligned}$$

as $j \to \infty$ and so $|f(x) - f(y)| \leq m|x - y|$ for all $y \in [0,1]$. Thus $f \in E_m$ and E_m is closed.

We now show that E_m has empty interior. We do this by showing that, given $g \in C([0,1])$ and $\delta > 0$, we can find an $f \notin E_m$ with $\|f - g\|_\infty < \delta$. Observe first that, since g is a continuous function on a closed bounded interval, it is uniformly continuous. In particular, we can find an integer $N \geq 1$ such that $|g(x)-g(y)| \leq \delta/8$ whenever $x, y \in [0,1]$ and $|x-y| \leq N^{-1}$. If we now let $\tilde{g}$ be the simplest continuous piecewise linear function with $\tilde{g}(r/N) = g(r/N)$, then $\|g - \tilde{g}\|_\infty \leq \delta/4$.

Since $\tilde{g}$ is piecewise linear, we can find a K such that

$$|\tilde{g}(x) - \tilde{g}(y)| \leq K|x - y|$$

for all $x, y \in [0,1]$. Choose an integer M such that $2M > K + m$. If we let $\tilde{h}$ be the simplest continuous piecewise linear function with $h(2r/M) = \delta/2$ and $h((2r+1)/M) = -\delta/2$, then $\|h\|_\infty = \delta/2$ and, given any $x \in [0,1]$, we can find a y_x with $|x - y_x| = 1/(2M)$ such that $|h(x) - h(y_x)| = \delta/2$. Setting $f = \tilde{g} + h$, we see that

$$\|f - g\|_\infty \leq \|g - \tilde{g}\|_\infty + \|h\|_\infty \leq 3\delta/4$$

and yet

$$\begin{aligned}|f(x) - f(y_x)| &\geq |h(x) - h(y_x)| - |\tilde{g}(x) - \tilde{g}(y_x)| \\ &\geq \delta - K|x - y_x| = 2M|x - y_x| - K|x - y_x| > m|x - y_x|,\end{aligned}$$

so $f \notin E_m$. ■

*N*ote on proof methods The last two paragraphs of the proof above illustrate a very important idea. Suppose we wish to find a nasty object close to some given object. The natural idea is to kick the given object. But if our original object was already nasty then the kick might improve the nasty object.

Instead we show that there is a *nice* object close to the original object and then show that there is a nasty object close to every nice object.

Lemma 1.6.11. *Consider $C([0,1])$ the space of continuous functions $f : [0,1] \to \mathbb{R}$ equipped with the uniform norm. There is a meagre set E such that, if $f \notin E$ then given any $x \in [0,1]$ and any $K > 0$, we can find a $y \in [0,1]$ such that*

$$|f(x) - f(y)| > K|x - y|.$$

Proof. Define E_m as in Lemma 1.6.10. If we set $E = \bigcup_{m=1}^{\infty} E_m$, then since the countable union of meagre sets is meagre, it follows that E is a meagre set with the desired property. ■

Theorem 1.6.9 now follows from the following observation.

Lemma 1.6.12. *Let E have the properties given in Lemma 1.6.11. Then, if $f \notin E$,*

$$\limsup_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| = \infty$$

for all $x \in [0,1]$.

Proof. Let $f \notin E$, $x \in [0,1]$, $K > 0$ and $\delta > 0$ be given. Since any continuous function on a closed bounded interval is bounded, we can find a $K' > 0$ such that

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq K'$$

for all $x + h \in [0,1]$, $|h| \geq \delta$. Since $f \notin E$ we can find a $y \in [0,1]$ such that

$$|f(x) - f(y)| > (K + K')|x - y|.$$

Set $k = y - x$. Automatically $0 < |k| < \delta$ and

$$\left| \frac{f(x+k) - f(x)}{k} \right| > K.$$

Since x, K and δ were arbitrary, the lemma follows. ■

1.7 Completing metric spaces

Since we will mainly be concerned with *complete* metric spaces, it is useful background information to know that ‘every metric space (X, d) can be considered as a dense subset of a complete metric space (Y, ρ) with $\rho_X = d$ in an essentially unique manner’. We talk about (Y, ρ) as *the completion of* (X, d) and extend the definition of d to Y by taking $d = \rho$.

In this short section we shall see what this statement means and prove it.

We start with a result which recalls Exercise 1.5.22.

Lemma 1.7.1. *Suppose that E_1 is a dense subset of the complete metric space (X_1, d_1) and E_2 is a dense subset of the complete metric space (X_2, d_2). If $\theta : E_1 \to E_2$ is a bijective map with*

$$d_2(\theta x, \theta y) = d_1(x, y) \text{ for all } x,\ y \in E_1,$$

then there exists a unique $\tilde{\theta} : X_1 \to X_2$ with

$$d_2(\tilde{\theta} x, \tilde{\theta} y) = d_1(x, y) \text{ for all } x,\ y \in X_1$$

and $\tilde{\theta} x = \theta x$ for all $x \in E_1$.

The map $\tilde{\theta} : X_1 \to X_2$ is bijective,

Proof. We start by proving uniqueness. Suppose that we have two maps $\tilde{\theta}_j : X_1 \to X_2$ with

$$d_2(\tilde{\theta}_j x, \tilde{\theta}_j y) = d_1(x, y) \text{ for all } x,\ y \in X_1$$

and $\tilde{\theta}_j x = \theta x$ for all $x \in E_1$ $[j = 1,\ 2]$. If $x \in X_1$ we can find a sequence $x_n \in E_1$ such that $d_1(x_n, x) \to 0$. The hypotheses show that

$$d_2(\tilde{\theta}_1 x, \tilde{\theta}_2 x) \leq d_2(\tilde{\theta}_1 x, \tilde{\theta} x_n) + d_2(\tilde{\theta}_2 x, \tilde{\theta} x_n)) = 2 d_1(x, x_n) \to 0.$$

as $n \to \infty$, so $d_2(\tilde{\theta}_1 x, \tilde{\theta}_2 x) = 0$ and $\tilde{\theta}_1 x = \tilde{\theta}_2 x$. We have shown that $\theta_1 = \theta_2$.

Next we prove existence. Again we start by observing that, given $x \in X_1$, we can find a sequence $x_n \in E_1$ such that $d_1(x_n, x) \to 0$. Since the x_n form a convergent sequence in X_1, they must form a Cauchy sequence. Since $d_2(\theta x_n, \theta x_m) = d_1(x_n, x_m)$, the θx_n form a Cauchy sequence in in X_2 and so converge to a limit. If $z_n \in E_1$ is such that $d_1(z_n, x) \to 0$, then

$$d_2(\theta x_n, \theta z_n) = d_1(x_n, z_n) \leq d_1(x, x_n) + d_1(x, z_n) \to 0$$

as $n \to \infty$ so θz_n tends to the same limit as θx_n. We denote this unique limit by $\tilde{\theta}(x)$. Note that, if $x \in E_1$, we may take $x_n = x$, so $\tilde{\theta}(x) = \theta x$. If $x,\ y \in X_1$ we can find sequences $x_n,\ y_n \in E_1$ such that $d_1(x_n, x)$, $d_1(y_n, y) \to 0$. We observe that

$$|d_1(x_n, y_n) - d_1(x, y)| \leq d_1(x, x_n) + d_1(y, y_n) \to 0$$

so $d_1(x_n, y_n) \to d_1(x, y)$ as $n \to \infty$ and similarly $d_2(\theta x_n, \theta y_n) \to d_1(\theta x, \theta y)$ as $n \to \infty$. Since $d_1(x_n, y_n) = d_2(\theta x_n, \theta y_n)$ we have

$$d_2(\tilde{\theta} x, \tilde{\theta} y) = d_1(x, y)$$

as required.

Since $\theta : E_1 \to E_2$ is a bijective map it has an inverse $\phi = \theta^{-1} : X_2 \to X_1$. Automatically

$$d_1(\phi x, \phi y) = d_2(\theta\phi x, \theta\phi y) = d_2(x, y)$$

so, by what we have already shown, there exists a unique $\tilde{\phi} : X_2 \to X_1$ with

$$d_1(\tilde{\phi}x, \tilde{\phi}y) = d_2(x, y) \text{ for all } x,\ y \in X_2$$

and $\tilde{\phi}x = \phi x$ for all $x \in E_2$. If $x \in X_1$ we can choose $x_n \in E_1$ with $d_1(x_n, x) \to 0$. We observe that

$$d_1(x_n, \tilde{\phi}\tilde{\theta}x) = d_1(\phi\theta x_n, \tilde{\phi}\tilde{\theta}x) = d_1(\tilde{\phi}\theta x_n, \tilde{\phi}\tilde{\theta}x) = d_2(\theta x_n, \tilde{\theta}x) \to 0$$

as $n \to 0$ so $x = \tilde{\phi}\tilde{\theta}x$ for all $x \in X_1$ and similarly $y = \tilde{\theta}\tilde{\phi}y$ for all $y \in X_2$. Thus $\tilde{\theta}$ is bijective with inverse $\tilde{\phi}$. ■

If we take $E_1 = E_2 = E$, $d_1(x, y) = d_2(x, y) = d(x, y)$ for $x = y$ and $\theta = \iota$, where ι is the identity mapping $\iota : E \to E$, we see that (X_1, d_1) and (X_2, d_2) are 'naturally identified with each other' by the map $\tilde{\iota}$. Thus the completion of a metric space, if it exists, is unique in the sense of Lemma 1.7.1, that is to say 'up to isometry'.

Definition 1.7.2. *Let (X, d) and (Y, ρ) be metric spaces. We say that a surjective function $f : X \to Y$ is an* isometry *if $\rho(f(a), f(b)) = d(a, b)$ for all a, $b \in X$. If such an f exists, we say that (X, d) and (Y, ρ) are isometric.*

Note that two isometric spaces are automatically Lipschitz equivalent.

The proof that such a complete space exists is outlined in a long, but straightforward, exercise.

Exercise 1.7.3. *Let (X, d) be a metric space.*

(i) Consider the space Z of all Cauchy sequences in X. (More formally, Z consists of the functions $\mathbf{x} : \mathbb{N} \to X$ such that given any $\epsilon > 0$ there exists an $N(\epsilon)$ with $d(x_n, x_m) < \epsilon$ for all m, $n \geq N(\epsilon)$.) Show that

$$\rho(\mathbf{x}, \mathbf{y}) = \lim_{n\to\infty} d(x_n, y_n)$$

is a well defined function on Z^2.

(ii) Show that $\rho(\mathbf{x}, \mathbf{y}) \geq 0$, $\rho(\mathbf{x}, \mathbf{x}) = 0$, $\rho(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{x}, \mathbf{y})$ and $\rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}) \geq \rho(\mathbf{x}, \mathbf{z})$ for all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in Z$. Show, by means of an example, that ρ need not be a metric.

(iii) If $\mathbf{x}$, $\mathbf{y} \in Z$ write $\mathbf{x} \sim \mathbf{y}$ if $\rho(\mathbf{x}, \mathbf{y}) = 0$. Show that $\sim$ is an equivalence relation on Z.

(iv) If $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{y}$, $\mathbf{y}' \in Z$ and $\mathbf{x} \sim \mathbf{x}'$, $\mathbf{y} \sim \mathbf{y}'$, show that

$$\rho(\mathbf{x}, \mathbf{x}') = \rho(\mathbf{y}, \mathbf{y}').$$

(v) Explain why, given $\mathbf{x} \in Z$ *and* $\epsilon > 0$ *we can find an* $\mathbf{x}' \in Z$ *with* $\mathbf{x}' \sim \mathbf{x}$ *and*

$$d(x'_n, x'_m) < \epsilon \text{ for all } m,\ n.$$

(vi) Consider $\tilde{X} = Z/\sim$*, the space of equivalence classes*

$$[\mathbf{x}] = \{\mathbf{x}' \in Z \ :\ \mathbf{x}' \sim \mathbf{x}\}.$$

Show that the equation

$$\tilde{d}([\mathbf{x}], [\mathbf{y}]) = \rho(\mathbf{x}, \mathbf{y})$$

gives a well defined metric.

(vii) Suppose that the sequence $\mathbf{x}(n) \in Z$ *is such that*

(a) $d(x_j(n), x_k(n)) < 2^{-N}$ *for all* j, k *and all* $n \geq N$.

(b) $\rho(\mathbf{x}(n), \mathbf{x}(m)) < 2^{-N}$ *for all* n, $m \geq N$.

Show that if we set $y_n = x_n(n)$ *then* $\mathbf{y} \in Z$ *and* $\rho(\mathbf{x}(n), \mathbf{y}) \to 0$ *as* $n \to \infty$.

(viii) Deduce that $(\tilde{X}, \tilde{d})$ *is complete (that is to say that every Cauchy sequence converges).*

(ix) Let $\theta : X \to \tilde{X}$ *be given by*

$$\theta(x) = \{[\mathbf{a}] \ :\ a_n = x \text{ for all } n\}.$$

Show that θ *is well defined and that*

$$\tilde{d}(\theta(x), \theta(y)) = d(x, y)$$

(that it to say, θ *is an isometry). Show that* $\theta(X)$ *is dense in* $(\tilde{X}, \tilde{d})$.

Remark At the start of this section I claimed that 'every metric space (X, d) can be considered as a dense subset of a complete metric space (Y, ρ) with $\rho_X = d$ in an essentially unique manner'. What we showed is that given a metric space (X, d) we can find a complete metric space (Y, ρ) containing a dense subset $\tilde{X}$ such that $(\tilde{X}, d_{\tilde{X}})$ is an exact copy of (X, d). Moreover, all such (Y, d) are exact copies of each other.

Philosophers can fruitfully debate whether, and in what sense, the two statements of the previous paragraph say the same thing. In practice, mathematicians slip between saying that 'the symmetry group of the equilateral triangle is S_3' and 'the symmetry group of the equilateral triangle is isomorphic to S_3' without too many pangs of conscience.

1.8 Further exercises

Exercise 1.8.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and suppose $f(\alpha) = 0$. Suppose further that there exists a $\delta > 0$ and an M with $0 \leq M < 1$ such that $f'(t) \neq 0$ and $|f''(t)f(t)|/|f'(t)|^2 \leq M$. for all $|t - \alpha| \leq \delta$. Consider the map*

$$Tx = x - \frac{f(x)}{f'(x)}.$$

Observe that $T\alpha = \alpha$ and use the mean value theorem to show that

$$x \in [\alpha - \delta, \alpha + \delta] \Rightarrow |Tx - \alpha| \leq M|x - \alpha|.$$

Deduce that, if $x_0 \in [\alpha - \delta, \alpha + \delta]$, and $x_{n+1} = Tx_n$, then $x_n \in [\alpha - \delta, \alpha + \delta]$ for all n and $x_n \to \alpha$ as $n \to \infty$. This is the Newton–Raphson method for finding roots.

By choosing an appropriate f, show that Heron's method (descibed in Exercise 1.3.1) works if (with the notation of that exercise) $|x0 - b| < \delta$ (where $\delta > 0$ and δ depends on b). By using a further argument show that Heron's method works for all $x_0 \geq 0$.

Exercise 1.8.2. *Let (X, d) be a complete metric space. Suppose further that X is bounded (that is to say, there exists a K such that $d(x, y) \leq K$ for all $x, y \in X$). Let E be the set of all continuous maps $f : X \to X$. If $\rho : E \times E \to \mathbb{R}$ is defined by*

$$\rho(f, g) = \sup\{d(f(x), g(x)) \; : \; x \in X\},$$

show that (E, ρ) is complete.

Now let C be the set of contraction mappings in E and define $\theta : C \to X$ to be the mapping which sends each contraction mapping to its fixed point. Show that θ is continuous.

[*Hint: Fix $f \in C$ and consider $d(\theta(g), f(\theta(g)))$.*]

Exercise 1.8.3. *In this exercise we prove Picard's existence theorem (see Theorem 1.4.3) without using the contraction mapping theorem. Suppose that $x_0 \in \mathbb{R}$ and $k > 0$. For simplicity, we shall consider a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies the Lipschitz condition*

$$|f(u, t) - f(v, t)| \leq k|u - v|$$

for all $t, u, v \in \mathbb{R}$.

(i) If $g \in C(\mathbb{R})$, we set

$$(Tg)(t) = x_0 + \int_0^t f(g(s), s)\, ds.$$

Show, by induction, that, if $u,\ v \in C(\mathbb{R})$ and $|u(t) - v(t)| \le M'$ for $|t| \le R$, then

$$|T^n u(t) - T^n v(t)| \le \frac{M'|t|^n}{n!}$$

for all $|t| \le R$. Deduce that, if $|u(t) - Tu(t)| \le M$ for $|t| \le R$, then

$$|T^{n+1} u(t) - T^n u(t)| \le \frac{M|t|^n}{n!}$$

for all $|t| \le R$ and use the Weierstrass M-test to show that

$$T^n u(t) = u(t) + \sum_{j=1}^{n-1} (T^{j+1} u(t) - T^j u(t))$$

converges uniformly in $[-R, R]$. Conclude that $T^n u(t) \to w(t)$ for each $t \in \mathbb{R}$, where w is some continuous function.

Show that $|T^{n+1} u(t) - T^n u(t)| \to 0$ uniformly on $[-R, R]$ for each $R > 0$ and deduce carefully that $Tf = f$.

(ii) Suppose that

$$u(t) = x_1 + \int_0^t (u(s), s)\, ds.$$

Show that

$$Tu(t) - u(t) = x_0 - x_1$$

and that, if w is the function considered in (i),

$$|w(t) - u(t)| \le |x_0 - x_1| e^{K|t|}.$$

What does this result tell you about the solutions of $x'(t) = f(x(t), t)$ with the two different initial conditions $x(0) = x_0$ and $x(0) = x_1$?

(iii) By looking at the differential equation $x'(t) = Kx(t)$, show that the result of (ii) cannot be improved.

(iv) Take $f(x, t) = Kx$ and $u(t) = x_0$. Find $T^n u(t)$. Does $T^n u$ converge uniformly on $\mathbb{R}$?

[*As often happens, we can extract more information about a* specific *problem by using methods* specific *to that problem. However more* general *methods have the advantage that they can readily be adapted to an entire class of problems.*]

Exercise 1.8.4. *The following extension of the contraction mapping theorem echos part of the previous exercise. Suppose that (X, d) is a metric space and $T : X \to X$ is such that there exists an $N \ge 1$ and a K with $1 > K \ge 0$ such that $d(T^N x, T^N y) \le Kd(x, y)$ for all $x,\ y \in X$. By observing that, if z is a fixed point for T^N, so is Tz, or otherwise, show that T has a unique fixed point.*

Exercise 1.8.5. *Let us say that two metric spaces (X, d) and (Y, ρ) are uniformly equivalent if there is a function $f : X \to Y$ with the following property. Given $\epsilon > 0$ we can find a $\delta > 0$ such that, whenever $d(x, y) < \delta$, we have $\rho(f(x), f(y)) < \epsilon$ and, whenever $\rho(f(x), f(y)) < \delta$, we have $d(x, y) < \epsilon$.*

(i) Show that uniform equivalence is an equivalence relation.

(ii) Show that if (X, d) and (Y, ρ) are uniformly equivalent, then if (X, d) is complete, so is (Y, ρ).

(iii) Show that Lipshitz equivalence implies uniform equivalence and uniform equivalence implies homeomorphism. Show by means of counter-examples, that neither implication can be reversed.

Exercise 1.8.6. *We say that a vector space V over $\mathbb{F}$ has countable dimension if we can find $\mathbf{e}_1, \mathbf{e}_2, \ldots$ such that every $\mathbf{v} \in V$ can be written as*

$$\mathbf{v} = \sum_{j=1}^{\infty} \lambda_j \mathbf{e}_j$$

with $\lambda_j \in K$ and only finitely many of the λ_j non-zero.

(i) Suppose the conditions just stated hold. Let us write

$$E_n = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \ldots \mathbf{e}_n\}.$$

Show that V is finite dimensional if and only if there exists an N with $E_n = E_N$ for all $n \geq N$.

(ii) By using Baire category, or otherwise, show that a norm on a space V of countable dimension is complete if and only if V is finite dimensional.

(iii) If V is a normed space of countable dimension show that every linear map $T : V \to \mathbb{F}$ is continuous if and only if V is finite dimensional.

Exercise 1.8.7. *A metric space is called a* Polish space *if it is homeomorphic to a complete metric space. Explain why Baire's category theorem is true for any Polish space. Show that $[0, 1] \cap \mathbb{Q}$ with the standard metric is not a Polish space. Show that*

$$X = \{(x, y) : x, y \in [0, 1],\ y \in \mathbb{Q}\}$$

is not a Polish space.

Exercise 1.8.8. *(This is a very routine preliminary to Exercise 1.8.9.)*

Let $(V, \| \ \|)$ be a normed vector space and E a dense subset of V. Show that the structure of $(V, \| \ \|)$ is determined by E in the following sense. Suppose $\lambda \in \mathbb{F}$, $\mathbf{x}_n, \mathbf{y}_n \in E$ and $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$ as $n \to \infty$. Then

$$\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y},\ \lambda \mathbf{x}_n \to \lambda \mathbf{x} \text{ and } \|\mathbf{x}_n\| \to \|\mathbf{x}\|$$

as $n \to \infty$.

Exercise 1.8.9. *Suppose that E is a dense subset of a complete metric space (X, d). The object of this question is to show that, if $(E, \| \|)$ is a normed vector space with $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ whenever $\mathbf{x}, \mathbf{y} \in E$, then we can make (X, d) into a normed vector space $(X, \| \|)$ in a consistent manner.*

(i) Let $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x}_n, \mathbf{x}'_n, \mathbf{y}_n, \mathbf{y}'_n \in E$. with $\mathbf{x}_n, \mathbf{x}'_n \to \mathbf{x}$ and $\mathbf{y}_n, \mathbf{y}'_n \to \mathbf{y}$ as $n \to \infty$. Show that $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{z}$ for some $\mathbf{z} \in X$. Show also that $\mathbf{x}'_n + \mathbf{y}'_n \to \mathbf{z}$. Thus we may define $\mathbf{x} \dotplus \mathbf{y} = \mathbf{z}$.

(ii) Show that if $\mathbf{x}, \mathbf{y} \in E$, then $\mathbf{x} + \mathbf{y} = \mathbf{x} \dotplus \mathbf{y}$ so we may set $\mathbf{x} + \mathbf{y} = \mathbf{x} \dotplus \mathbf{y}$ in a consistent manner.

(iii) Show that we can define $\lambda\mathbf{x}$ and $\|\mathbf{x}\|$ for $\lambda \in \mathbb{F}$ and $\mathbf{x} \in X$ in a similar manner.

(iv) Show that $\|\mathbf{x} - \mathbf{y}\| = d(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$.

(v) Check that $(X, \| \|)$ is indeed a normed vector space.

Chapter 2

Convergence and approximation

2.1 The limits of Taylor's theorem

During the 18th century mathematicians tried several ways to make the calculus rigorous. One very promising idea was to make the calculus the study of functions for which Taylor's theorem is true, that is to say, to look at functions f such that $f(x+h) = \sum_{n=0}^{\infty} a_n h^n$ (at least for small h). Another was to make calculus the study of limiting processes. It seems fair to suppose that most mathematicians believed that the two approaches were entirely compatible. This hope was destroyed by Cauchy using the following example. (Note that our definition of differentiability uses the limit approach.)

Exercise 2.1.1. *Define* $E : \mathbb{R} \to \mathbb{R}$ *by*

$$E(x) = \begin{cases} \exp(-1/x^2) & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0. \end{cases}$$

(i) Show, by induction, that E is infinitely differentiable on $\mathbb{R} \setminus \{0\}$ *with*

$$E^{(n)}(x) = P_n(1/x)E(x)$$

where P_n *is a polynomial.*

(ii) Show, by induction, using the definition of the derivative, that E is infinitely differentiable at 0 *with*

$$E^{(n)}(0) = 0.$$

(iii) Deduce that E is infinitely differentiable everywhere, but

$$E(h) \neq \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!} h^n$$

for all $h \neq 0$.

Cauchy's example is supplemented by the following easy observation.

Exercise 2.1.2. *Suppose that* $\delta > 0$ *and*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for all $|x| < \delta$. *Use the fact that a power series can be differentiated term by term within its radius of convergence to show that*

$$a_n = \frac{f^{(n)}(0)}{n!}$$

for all $n \geq 0$.

Of course, Cauchy's example only means that we have to choose between a calculus based on limits and a calculus based on Taylor expansions. It is worth considering what a calculus based on Taylor expansions would look like.

Definition 2.1.3. *Let* E *be a subset of* $\mathbb{R}$. *(We will only be interested in* $E = \mathbb{R}$ *and* $E = [a, b]$.*) We say that* $f : E \to \mathbb{R}$ *is* real analytic *at* $x \in \operatorname{Int} E$ *if we can find a* $\delta(x) > 0$ *and* a_n *such that*

$$f(x + h) = \sum_{n=0}^{\infty} a_n h^n$$

for all $|h| < \delta(x)$.

Notice that, because power series can be differentiated term by term within their circle of convergence, the f just described must be infinitely differentiable on $(x - \delta(x), x + \delta(x))$ with $a_n = f^{(n)}(x)/n!$. In particular, if f is real analytic at each point of the open interval (u, v), it is infinitely differentiable on (u, v).

Real analytic functions have a property familiar to any reader who knows some complex variable theory, but which makes them unsuitable for many purposes.

Theorem 2.1.4. *Suppose that* $f, g : \mathbb{R} \to \mathbb{R}$ *are real analytic at each* $x \in \mathbb{R}$ *and* $f(x) = g(x)$ *for all* $x \in (a, b)$ *where* $b > a$. *Then* $f(x) = g(x)$ *everywhere.*

Proof. By considering $f - g$, we may suppose that $g = 0$. By translation, we may suppose that $a < 0 < b$ and $f(x) = 0$ for $|x| < \eta$ for some $\eta > 0$. Suppose, if possible, that $f(y) \neq 0$ for some $y \in (a, b)$. By reflection we may suppose $y > 0$. We now observe that

$$E = \{w > 0 \,:\, f(t) = 0 \text{ for all } 0 \leq t \leq w\}$$

is a non-empty bounded set and so has a least upper bound $v \leq y$. Since f is real analytic at v and $f^{(n)}(t) = 0$ for all $0 \leq t < v$, continuity tells us that $f^{(n)}(v) = 0$ for all n and

$$f(v+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(v)}{n!} h^n = \sum_{n=0}^{\infty} 0 h^n = 0$$

for all $|h| < \delta(v)$ and some $\delta(v) > 0$. We thus have $v + \delta(v)/2 \in E$, contracting the definition of v. The result follows by contradiction. ■

(A slight strengthening of this result is given in Exercise 2.8.1.)

Cauchy's example enables us to prove the non-obvious fact that infinitely differentiable functions are very much less restricted in their behaviour than their real analytic counterparts.

Exercise 2.1.5. *Let E be the function described in Exercise 2.1.1.*

(i) By considering functions of the form $AE(x-\eta)E(x+\eta)$, or otherwise, show that, given $\eta > 0$, we can find an infinitely differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) \geq 0$ for all x, $h(x) = 0$ for $|x| \geq \eta$ and $\int_{-\infty}^{\infty} h(x)\,dx = 1$.

(ii) By considering functions of the form

$$\int_{-\infty}^{x} h(s-a) - h(s-b)\,ds$$

or otherwise, show that, given $\delta > 0$ and $c < d - 2\delta$, we can find an infinitely differentiable function $k : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} 1 \geq k(x) &\geq 0 && \text{for all } x, \\ k(x) &= 0 && \text{for } x \notin [c-\delta, d+\delta], \\ k(x) &= 1 && \text{for } x \in [c, d]. \end{aligned}$$

(iii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function. Show that, given $\delta > 0$ and $c < d$, we can find an infinitely differentiable function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x \notin [c-\delta, d+\delta] \\ f(x) & \text{if } x \in [c, d]. \end{cases}$$

Cauchy's example gives one way in which an infinitely differentiable function can fail to be real analytic at a point x. Another possible way is for the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n$$

to have radius of convergence zero (so that the Taylor series diverges except when $h = 0$). We use a Baire category argument to to show that this can indeed occur.

We first need an appropriate metric on the space $C^\infty([a,b])$ of infinitely differentiable functions $f : [a,b] \to \mathbb{R}$. (As usual we use left or right derivatives at the the end points of the interval.) We choose a metric of a type that will recur throughout these notes.

Exercise 2.1.6. *(i) If $f,\ g \in C^\infty([a,b])$ explain why the sum on the left hand side of the definition*

$$d(f,g) = \sum_{n=0}^{\infty} 2^{-n} \min\{1, \|f^{(n)} - g^{(n)}\|_\infty\}$$

converges.

(ii) Show that d is a metric on $C^\infty([a,b])$.

(iii) Suppose that f_j is a Cauchy sequence for $(C^\infty([a,b],d)$. Show that $f_j^{(n)}$ is a Cauchy sequence for $(C([a,b],\| \|_\infty)$ and deduce that we can find $F_n \in C([a,b])$ such that $f_j^{(n)} \to F_n$ uniformly on $[a,b]$ as $j \to \infty$ for each n. Using a theorem on uniform limits of derivatives, to be stated precisely, show that $F = F_0$ is infinitely differentiable with $F^{(n)} = F_n$. Show that $d(f_j,F) \to 0$ and so d is a complete metric.

Lemma 2.1.7. *Let $c \in (a,b)$. If we consider the metric space $(C^\infty([a,b],d)$ of Exercise 2.1.6, then quasi-all $f \in C^\infty([a,b])$ satisfy the condition*

$$\limsup_{n\to\infty} (n!)^{-2}|f^{(n)}(c)| = \infty$$

and so cannot be real analytic at c.

Proof. Let

$$E_m = \{f \in C^\infty([a,b]) \,:\, |f^{(n)}(c)| \le 2^n (n!)^2 \text{ for all } n \ge m\}.$$

We claim that E_m is closed with empty interior.

To show that E_m is closed, suppose that $f_j \in E_m$, $f \in C^\infty([a,b])$ and $d(f_j,f) \to 0$ as $j \to \infty$. By the definition of d we have

$$\min\{1, |f^{(n)}(c) - f_j^{(n)}(c)|\} \le \min\{1, \|f^{(n)} - f_j^{(n)}\|_\infty\} \le 2^n d(f_j,f) \to 0$$

as $j \to \infty$ for each n. Since $|f_j^{(n)}(c)| \le 2^n (n!)^2$ for all $n \ge m$, it follows that $|f^{(n)}(c)| \le 2^n (n!)^2$ for all $n \ge m$ and so $f \in E_m$ as required.

To show that E_m has empty interior, suppose that $f \in C^\infty([a,b])$ and $\epsilon > 0$ are given. Choose N a positive integer such that $4N \ge m$ and $2^{-4N} < \epsilon/2$. Let $K \ge 1$ be a large number to be determined later. If $f^{(4N)}(c) \ge 0$, set

$$g(x) = f(x) + K^{-4N+1/2} \cos K(x-c)$$

and if $f^{(4N)}(c) < 0$ set $g(x) = f(x) - K^{-4N+1/2}\cos K(x-c)$. In either case

$$\begin{aligned} d(f,g) &= \sum_{n=0}^{\infty} 2^{-n}\min\{1, \|f^{(n)} - g^{(n)}\|_\infty\} \\ &\leq \sum_{n=0}^{4N-1} 2^{-n}\|f^{(n)} - g^{(n)}\|_\infty + \sum_{n=4N}^{\infty} 2^{-n} \\ &\leq \sum_{n=0}^{4N-1} 2^{-n}K^{n-4N+1/2} + \sum_{n=4N}^{\infty} 2^{-n} \leq K^{-1/2} + 2^{-4N+1} < \epsilon, \end{aligned}$$

provided only that we take K so large that $K^{-1/2} < \epsilon/2$, whilst

$$|g^{(4N)}(c)| \geq K^{1/2} > 2^N(((4N)!)^2,$$

provided only that we take K sufficiently large. Thus we can ensure that $d(f,g) < \epsilon$ but $g \notin E_m$.

It follows that $E = \bigcup_{m=1}^{\infty} E_m$ is meagre. But, if $f \notin E$, then $|f^{(n)}(c)| > 2^n(n!)^2$ for infinitely many values of n, so

$$\limsup_{n\to\infty}(n!)^{-2}f^{(n)}(c) = \infty.$$

It follows that

$$\limsup_{n\to\infty}\left|\frac{f^{(n)}(c)}{n!}h^n\right| = \infty$$

for all $h \neq 0$ and so f cannot be real analytic at c. ■

Exercise 1.6.6 gave one of the most useful properties of Baire category. Here it enables us to strengthen Lemma 2.1.7 without doing any further work.

Lemma 2.1.8. *Let $c_1, c_2, \ldots$ be a dense subset of (a,b). If we consider the metric space $(C^\infty([a,b],d)$ of Exercise 2.1.6, then quasi-all $f \in C^\infty([a,b])$ satisfy the condition*

$$\limsup_{n\to\infty}(n!)^{-2}|f^{(n)}(c_j)| = \infty$$

and so cannot be real analytic at c_j.

Proof. By Lemma 2.1.7

$$A_j = \{f \in C^\infty([a,b]) \,:\, \limsup_{n\to\infty}(n!)^{-2}|f^{(n)}(c_j)| = \infty\}$$

is meagre. Thus, by Exercise 1.6.6, $\bigcup_{j=1}^{\infty} A_j$ is meagre. ■

Exercise 2.1.9. *Let $M_n > 0$ and let c_1, c_2, ... be a dense subset of (a, b). If we consider the metric space $(C^\infty([a, b], d)$ of Exercise 2.1.6, check (without necessarily writing anything down) that the methods of proof that we used above show that quasi-all $f \in C^\infty([a, b])$ satisfy the condition*

$$\limsup_{n\to\infty} M_n^{-1}|f^{(n)}(c_j)| = \infty$$

for all j.

Theorem 2.1.10. *If we consider the metric space $(C^\infty([a, b], d)$ of Exercise 2.1.6, then quasi-all $f \in C^\infty([a, b])$ fail to be real analytic at every point of (a, b).*

Proof. If f is real analytic at c, then if we can find a $\delta > 0$ and a_n such that

$$f(c + h) = \sum_{n=0}^{\infty} a_n h^n$$

for all $|h| < \delta$. But this means that f is real analytic at each point of $(h-\delta, h+\delta)$ (see Exercise 2.8.2) and so f cannot have the property that it fails to be real analytic on a dense set of points. ∎

We prove a similar result for infinitely differentiable functions on $\mathbb{R}$ with an appropriate metric in Exercise 2.8.28. The reader may ask what remains of Taylor's theorem in the light of the results just proved. We can look at the matter in various ways.

(1) There is a 'local Taylor's theorem' which says that an infinitely differentiable polynomial looks like a polynomial of high degree 'sufficiently close' to a given point. (See Exercise 2.8.4.)

(2) If we look at a *complex analytic* function $f : \mathbb{C} \to \mathbb{C}$ and write $f(x + iy) = u(x, y) + iv(x, y)$ (with x, y, u and v real), then not only are $u, v : \mathbb{R}^2 \to \mathbb{R}^2$ differentiable but they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The Cauchy-Riemann equations are so restrictive that the only f which satisfy them will also satisfy Taylor's theorem.

(3) What results like Theorem 1.6.9 and Lemma 2.1.8 tell us is that good behaviour at the level of the nth derivative is no guarantee of good behaviour at the level of the $n + 1$st derivative. This is reflected in 'real life' numerical analysis by the fact that methods which require good behaviour of high order derivatives or close approximation by high degree polynomials will usually deliver disappointing (and often nonsensical) results. We note also that in 'real life', the

most successful models of stock market prices or wind speed do not use 'smooth' functions.

However, the fact remains that much of analysis is guided by the feeling that 'polynomials are typical smooth functions'. Is there any way to justify this feeling? In Section 2.6 we shall discuss a result of Weierstrass which goes some way to restoring our confidence.

2.2 Divergence of Fourier series

Having shown that there are problems with power series representation of functions, we now show there are also problems with the Fourier series representation.

Instead of working with 2π periodic functions on $\mathbb{R}$, we work with functions on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Most readers will be content to say that 'we work with the real numbers modulo 2π' and accept that we call $\mathbb{T}$ the circle because of the obvious identification $t \leftrightarrow e^{it}$. Those who want a more formal introduction will find it in Exercise 2.8.5.

In accordance with our previous notation, we write $C_{\mathbb{F}}(\mathbb{T})$ for the space of continuous functions $f : \mathbb{T} \to \mathbb{F}$. Unless the choice of $\mathbb{F}$ matters, we will usually just write $C(\mathbb{T}) = C_{\mathbb{F}}(\mathbb{T})$. It should be clear that we can identify $C(\mathbb{T})$ in a natural way with the space of continuous functions $f \in C([-\pi, \pi])$ such that $f(\pi) = f(-\pi)$ and thus, by Exercise 1.5.10 (iv) is a complete space under the uniform norm. If $f \in C(\mathbb{T})$, we write

$$\int_{\mathbb{T}} f(t)\,dt = \int_{-\pi}^{\pi} f(t)\,dt.$$

The reader is probably aware that, if $f \in C(\mathbb{T})$, then we define the rth Fourier coefficient of f by

$$\hat{f}(r) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\exp(-irt)\,dt$$

and that there are plausible reasons for writing

$$f(t) \underset{?}{=} \sum_{r=-\infty}^{\infty} \hat{f}(r)\exp(irt).$$

Dirichlet showed that under quite wide conditions the *partial sums*

$$S_n(f,t) = \sum_{r=-n}^{n} \hat{f}(r)\exp(irt) \to f(t)$$

as $n \to \infty$.

It came as a great surprise when du Bois-Reymond showed that this was not always the case. We shall need to do some preliminary work.

Lemma 2.2.1. *(i) If $f \in C(\mathbb{T})$, then*

$$S_n(f,t) = \frac{1}{2\pi}\int_{\mathbb{T}} f(s)D_n(t-s)\,ds,$$

where

$$D_n(s) = \sum_{r=-n}^{n} \exp(irs).$$

Proof. We have

$$\begin{aligned}
S_n(f,t) &= \sum_{r=-n}^{n} \hat{f}(r)\exp(irt) \\
&= \sum_{r=-n}^{n} \frac{1}{2\pi}\int_{\mathbb{T}} f(s)\exp(-irs)\,ds\exp(irt) \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(s)\sum_{r=-n}^{n}\exp\left(ir(t-s)\right)ds
\end{aligned}$$

as stated. ■

We note that, if we define the *convolution* $F * G$ of two functions F, $G \in C(\mathbb{T})$ by the formula

$$F * G(t) = \frac{1}{2\pi}\int_{\mathbb{T}} F(t-s)G(s)\,ds,$$

then $S_n(f,t) = D_n * f(t)$. We call D_n the *Dirichlet kernel.*

Exercise 2.2.2. *(i) By integrating term by term, show that*

$$\frac{1}{2\pi}\int_{\mathbb{T}} D_n(s)\,ds = 1.$$

(ii) By summing the geometric series (see also Exercise 2.8.6), show that

$$D_n(s) = \begin{cases} \frac{\sin\left((n+\frac{1}{2})s\right)}{\sin\frac{1}{2}s} & \text{if } s \neq 0, \\ 2n+1 & \text{if } s = 0. \end{cases}.$$

(iii) Sketch D_n for various values of n.
(iv) Use the fact that $\sin''(t) \leq 0$ to show that

$$\frac{2t}{\pi} \leq \sin t \leq t$$

for $0 \leq t \leq \pi/2$.

(v) Show that, if $1 \leq r \leq n-1$,

$$\frac{1}{2\pi}\int_{r\pi/(n+\frac{1}{2})}^{(r+1)\pi/(n+\frac{1}{2})} |D_n(t)|\,dt \geq \frac{n+\frac{1}{2}}{r+1}\int_{r\pi/(n+\frac{1}{2})}^{(r+1)\pi/(n+\frac{1}{2})} \left|\sin\left((n+\tfrac{1}{2})t\right)\right| dt = \frac{2\pi}{r+1}.$$

(vi) Deduce that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |D_n(s)|\,ds \geq A\sum_{r=2}^{n}\frac{1}{r}$$

for some $A > 0$ *and conclude that*

$$\frac{1}{2\pi}\int_{\mathbb{T}} |D_n(s)|\,ds \to \infty$$

as $n \to \infty$.

(vii) Explain why we can find a real valued function $f_n \in C(\mathbb{T})$ *such that* $-1 \leq f_n(s) \leq 1$ *for all* $s \in \mathbb{T}$, *but*

$$S_n(f_n, 0) = \frac{1}{2\pi}\int_{\mathbb{T}} f_n(s)D_n(-s)\,ds \geq \frac{1}{2}\left(\frac{1}{2\pi}\int_{\mathbb{T}} |D_n(s)|\,ds\right).$$

We now use this information in a Baire category argument.

Theorem 2.2.3. *Consider* $C(\mathbb{T})$ *with the uniform norm. Then quasi-all* $f \in C(\mathbb{T})$ *have the property that*

$$\limsup_{n\to\infty} |S_n(f, 0)| = \infty.$$

Proof. Consider the set

$$E_m = \{f \in C(\mathbb{T}) \ : \ |S_n(f,t)| \leq m \text{ for all } n \geq 0\}.$$

We claim that E_m is closed with empty interior.

To see that E_m is closed, observe that, if $f_j \underset{\|\ \|_\infty}{\rightarrow} f$, then

$$\begin{aligned}
|\hat{f}_j(r) - \hat{f}(r)| &= \left|\frac{1}{2\pi}\int_{\mathbb{T}} (f_j(t) - f(t))\exp(-irt)\,dt\right| \\
&\leq \frac{1}{2\pi}\int_{\mathbb{T}} \left|(f_j(t) - f(t))\exp(-irt)\right| dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} |f_j(t) - f(t)|\,dt \\
&\leq \|f_j - f\|_\infty \to 0
\end{aligned}$$

and so, if n is fixed, $S_n(f_j, 0) \to S_n(f, 0)$ as $j \to \infty$. In particular, if $f_j \in E_m$ for each j, it follows that $f \in E_m$.

To see that E_m has empty interior, let $f \in C(\mathbb{T})$ and $\epsilon > 0$ be given. If $f \notin E_m$, set $g = f$. If $f \in E_m$, observe that, by Exercise 2.2.2 (vii), we can find an $F \in C(\mathbb{T})$ and an integer $N \geq 1$ such that $\|F\|_\infty \leq 1$ and

$$|S_N(F, 0)| \geq 1 + 4\epsilon^{-1}m.$$

If we set $g = f + (\epsilon/2)F$, then $\|f - g\|_\infty < \epsilon$ and

$$\begin{aligned} |S_N(g, 0)| &= \left|S_N((\epsilon/2)F, 0) + S_N(f, 0)\right| \\ &\geq (\epsilon/2)|S_N(F, 0)| - |S_N(f, 0)| > 2m - m = m. \end{aligned}$$

Thus we can always find a g such that $\|f - g\|_\infty < \epsilon$ and $g \notin E_m$.

We have shown that $\bigcup_{m=1}^{\infty} E_m$ is meagre. Since

$$\limsup_{n\to\infty} |S_n(f, 0)| = \infty,$$

whenever $f \notin \bigcup_{m=1}^{\infty} E_m$, the result follows. ■

The reader who is already primed for the idea of generalisation will observe that although the results of Exercise 2.2.2 are *specific* to the special problem, the method of proof for Theorem 2.2.3 is quite *general*. We shall see in Theorem 2.7.1 how Banach and Steinhaus extracted its essence to produce the principle of uniform boundedness.

Exercise 2.2.4. *(i) If $f \in C(\mathbb{T})$ and $a \in \mathbb{T}$, write $f_a(t) = f(t + a)$. Show that $\hat{f_a}(r) = \exp(ira)\hat{f}(r)$ and $S_n(f_a, 0) = S_n(f, a)$.*

(ii) Consider $C(\mathbb{T})$ with the uniform norm. If $a \in \mathbb{T}$, show that quasi-all $f \in C(\mathbb{T})$ have the property that

$$\limsup_{n\to\infty} |S_n(f, a)| = \infty.$$

(iii) If E is a countable subset of $\mathbb{T}$, show that quasi-all $f \in C(\mathbb{T})$ have the property that

$$\limsup_{n\to\infty} |S_n(f, a)| = \infty \text{ for all } a \in E.$$

(iv) Conclude that quasi-all $f \in C(\mathbb{T})$ diverge on a dense subset of $\mathbb{T}$

Exercise 2.2.5. *(i) Suppose that $f_n \in C(\mathbb{T})$ and there exists a dense sequence a_1, a_2, … of points in $\mathbb{T}$ such that*

$$\limsup_{N\to\infty} f_n(a_j) = \infty.$$

for all $j \geq 1$*. By showing that*

$$E_k = \{t \,:\, |S_n(f,t)| \leq k \text{ for all } n \geq 1\}$$

is closed with dense complement, or otherwise, show that f_n *diverges quasi-everywhere on* $\mathbb{T}$*.*

(ii) Use the result of Exercise 2.2.4 to show that quasi-all (with respect to the uniform norm) continuous functions have Fourier sums $S_n(f)$ *which diverge at quasi-all (with respect to the usual metric) points of* $\mathbb{T}$ *as* $n \to \infty$*. (Although this looks impressive, the ease with which we obtained the result from Exercise 2.2.4 tells us that this result is not as informative as it seems.*[1]*)*

2.3 Fejér's theorem

The results of the previous section cast a shadow over Fourier analysis until the invention of Lebesgue measure and a remarkable discovery of a nineteen year old Hungarian.

Theorem 2.3.1. [Fejér's theorem] *Let* $f \in C(\mathbb{T})$*. If we write*

$$S_n(f,t) = \sum_{r=-n}^{n} \hat{f}(r)\exp(irt)$$

and

$$\sigma_n(f,t) = \frac{1}{n}\sum_{j=0}^{n-1} S_j(f,t),$$

then $\sigma_n(f,t) \to f(t)$ *uniformly on* $\mathbb{T}$ *as* $n \to \infty$*.*

The idea of taking averages is due to Cesàro. Exercise 2.8.7 provides a more general context.

We start with a few calculations.

Exercise 2.3.2. *(i) If* $f \in C(\mathbb{T})$*, show that*

$$\sigma_n(f,t) = \sum_{r=-n+1}^{n-1} \frac{n-|r|}{n}\hat{f}(r)\exp(irt).$$

(ii) Show further that

$$\sigma_n(f,t) = \frac{1}{2\pi}\int_{\mathbb{T}} f(s)K_n(t-s)\,ds$$

[1] Indeed, a famous theorem of Carleson shows that (in the language of measure theory) the Fourier sums converge to the correct limit 'almost everywhere'.

(or, more briefly, $\sigma_n(f,t) = f * K_n(t)$) where

$$K_n(t) = \sum_{r=-n+1}^{n-1} \frac{n-|r|}{n} \exp(irt).$$

(iii) Show that

$$\left(\sum_{r=0}^{n-1} \exp\Big(i((-((n-1)/2)+r)t)\Big)\right)^2 = \sum_{r=-n+1}^{n-1} (n-|r|)\exp(irt)$$

and deduce that

$$K_n(t) = \begin{cases} \frac{1}{n}\left(\frac{\sin\left(nt/2\right)}{\sin t/2}\right)^2 & \text{for } t \neq 0 \\ n & \text{for } t = 0. \end{cases}$$

We call K_n the *Fejér kernel*. The next results are easily proved but very useful.

Lemma 2.3.3. *Fejér's kernel K_n has the following properties.*
(i) $\frac{1}{2\pi}\int_{\mathbb{T}} K_n(t)\,dt = 1.$
(ii) $K_n(t) \geq 0$ *for all* t.
(iii) If $\pi > \eta > 0$ *then* $K_n(t) \to 0$ *uniformly for all* $|t| \geq \eta$.

Proof. (i) Observe that

$$\frac{1}{2\pi}\int_{\mathbb{T}} K_n(t)\,dt = \sum_{r=-n+1}^{n-1} \frac{n-|r|}{n} \int_{\mathbb{T}} \exp(irt)\,dt = 1.$$

(ii) Clear from Exercise 2.3.2 (iii).
(iii) From Exercise 2.3.2 (iii) we have

$$0 \leq K_n(t) = \frac{1}{n}\left(\frac{\sin\,(nt/2)}{\sin t/2}\right)^2 \leq \frac{1}{n}\left(\frac{1}{\sin \eta/2}\right)^2 \to 0$$

as $n \to \infty$ for all t with $|t| \geq \eta$. ∎

We now show that the properties obtained in Lemma 2.3.3 suffice to prove Fejér's theorem.

Theorem 2.3.4. *Suppose that* $L_n : \mathbb{T} \to \mathbb{R}$ *is a continuous function with the following properties.*
(i) $\frac{1}{2\pi}\int_{\mathbb{T}} L_n(t)\,dt = 1.$
(ii) $L_n(t) \geq 0$ *for all* t.

(iii) If $\pi > \eta > 0$, then $L_n(t) \to 0$ uniformly for all $|t| \geq \eta$.
Then, if $f : \mathbb{T} \to \mathbb{C}$ is any continuous function,

$$L_n * f(t) = \frac{1}{2\pi}\int_{\mathbb{T}} f(s)L_n(t-s)\,ds \to f(t)$$

uniformly on $\mathbb{T}$.

Proof. Let $\epsilon > 0$. Since any continuous function on $\mathbb{T}$ is uniformly continuous, we can find an $\eta > 0$ such that $|f(s) - f(t)| \leq \epsilon/2$ for all $|s - t| \leq \eta$. By (iii), we can find an N such that $|L_n(t)| \leq (\epsilon/4)\|f\|_\infty$ for all $t \in \mathbb{T}$ and all $n \geq N$.

We now observe that, if $n \geq N$, then

$$\begin{aligned}
\left|\frac{1}{2\pi}\int_{\mathbb{T}} f(s)L_n(t-s)\,ds - f(t)\right| &= \left|\frac{1}{2\pi}\int_{\mathbb{T}} (f(s)-f(t))L_n(t-s)\,ds\right| \qquad \text{by (i)}\\
&\leq \frac{1}{2\pi}\int_{\mathbb{T}} \Big|(f(s)-f(t))L_n(t-s)\Big|\,ds\\
&\leq \frac{1}{2\pi}\int_{\mathbb{T}} |f(s)-f(t)|L_n(t-s)\,ds \qquad \text{by (ii)}\\
&\leq \frac{1}{2\pi}\int_{|t-s|\leq\eta} |f(s)-f(t)|L_n(t-s)\,ds\\
&\qquad + \frac{1}{2\pi}\int_{|t-s|>\eta} |f(s)-f(t)|L_n(t-s)\,ds\\
&\leq \frac{1}{2\pi}\int_{|t-s|\leq\eta} \frac{\epsilon}{2}L_n(t-s)\,ds + \frac{1}{2\pi}\int_{|t-s|>\eta} 2\|f\|_\infty \frac{\epsilon}{4\|f\|_\infty}\,ds\\
&= \frac{\epsilon}{2}\left(\frac{1}{2\pi}\int_{|t-s|\leq\eta} L_n(t-s)\,ds + \frac{1}{2\pi}\int_{|t-s|>\eta} 1\,ds\right)\\
&\leq \frac{\epsilon}{2}\left(\frac{1}{2\pi}\int_{\mathbb{T}} L_n(t-s)\,ds + \frac{1}{2\pi}\int_{\mathbb{T}} 1\,ds\right)\\
&= \epsilon
\end{aligned}$$

for all $t \in \mathbb{T}$ so we are done. ■

The ideas just introduced are very useful. It may help the reader to understand them if she works through Exercises 2.8.8 to 2.8.10.

From our point of view, the most important conclusion to be drawn from Fejér's theorem is the following. Let us say that P is a *trigonometric polynomial of degree n* if

$$P(t) = \sum_{j=-n}^{n} a_j \exp(ijt)$$

with $a_j \in \mathbb{C}$ and at least one of a_n and a_{-n} non-zero. (If we talk about polynomials of degree n or less this will include the zero polynomial.)

Theorem 2.3.5. *If $f \in C(\mathbb{T})$ and $\epsilon > 0$, then we can find a trigonometric polynomial P with $\|f - P\|_\infty < \epsilon$.*

More briefly, the trigonometric polynomials are uniformly dense in $C(\mathbb{T})$.

Proof. Observe that $\sigma_n(f)$ is a trigonometric polynomial and $\sigma_n(f) \to f$ uniformly. ∎

The next exercise gives the a simple case of the Riemann–Lebesgue lemma.

Exercise 2.3.6. *Suppose that $f \in C(\mathbb{T})$ and P is a trigonometric polynomial of degree N such that $\|f - P\|_\infty < \epsilon$. Show that*

$$|\hat{f}(n)| \leq \epsilon$$

for all $|n| \geq N + 1$.

Deduce that, if $g \in C(\mathbb{T})$, $\hat{g}(n) \to 0$ as $|n| \to \infty$.
[*Exercise 2.8.13 shows that, in some sense, this result is best possible.*]

Exercise 2.3.7. *The notion of convolution*

$$f * g(t) = \frac{1}{2\pi} f(t - s)g(s)\, dt$$

will be developed and generalised as these notes proceed. For the moment we suppose that $f, g \in C(\mathbb{T})$.

(i) If $f_n, g_n \in C(\mathbb{T})$ and $f_n \to f$, $g_n \to g$ uniformly, show that $f_n * g_n \to f * g$ uniformly.

(ii) If P and Q are trigonometric polynomials, show that $P * Q$ is a trigonometric polynomial with $\widehat{P * Q}(r) = \hat{P}(r)\hat{Q}(r)$.

(iii) Use Fejér's theorem to show that, if $f, g \in C(\mathbb{T})$ then $f * g \in C(\mathbb{T})$ and $\widehat{f * g}(r) = \hat{f}(r)\hat{g}(r)$.

Once we have the density result of Theorem 2.3.5, it is easy to obtain a uniqueness result.

Theorem 2.3.8. *If $f, g \in C(\mathbb{T})$ and $\hat{f}(n) = \hat{g}(n)$ for all n, then $f = g$.*

Proof. Since $(\widehat{f - g})(n) = \hat{f}(n) - \hat{g}(n)$ for all n, we need only prove the result when $g = 0$, that is to say, we need only show that, if $\hat{f}(n) = 0$ for all n, then $f = 0$.

To this end, observe that we can find a sequence of trigonometric polynomials P_n such that $P_n(t) \to f(t)^*$ uniformly[2]. Since

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}}\left(\sum_{j=-N}^{N} a_j \exp(ijt)\right) f(t)\,dt &= \sum_{j=-N}^{N} a_j \frac{1}{2\pi}\int_{\mathbb{T}} \exp(ijt) f(t)\,dt \\ &= \sum_{j=-N}^{N} a_j \hat{f}(-j) = 0,\end{aligned}$$

we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} P_n(t) f(t)\,dt = 0.$$

However,

$$P_n(t)f(t) \to f(t)^* f(t) = |f(t)|^2$$

uniformly and so

$$\frac{1}{2\pi}\int_{\mathbb{T}} P_n(t) f(t)\,dt \to \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^2\,dt.$$

It follows that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^2\,dt = 0$$

and so, since $|f|^2$ is continuous and positive, it must be the zero function. Thus $f = 0$, as required. ■

We can use the uniqueness theorem to prove a convergence theorem which suffices for many practical purposes.

Lemma 2.3.9. *If $f \in C(\mathbb{T})$ and $\sum_{j=-\infty}^{\infty} |\hat{f}(j)|$ converges, then*

$$\sum_{j=-n}^{n} \hat{f}(j) \exp ijt \to f(t)$$

uniformly on $\mathbb{T}$.

The proof is given by the following exercise.

Exercise 2.3.10. *Suppose that f satisfies the conditions of Lemma 2.3.9.*

(i) By using the Weierstrass M-test, or otherwise, show that $\sum_{j=-n}^{n} \hat{f}(j) \exp ijt$ converges uniformly to $h(t)$ where h is some continuous function.

[2]Here and throughout the book, z^* represents the complex conjugate of z

(ii) By using a result on limits under the integral, to be stated precisely, show that

$$\frac{1}{2\pi}\int_{\mathbb{T}} \exp(-imt)\sum_{j=-n}^{n}\hat{f}(j)\exp ijt\,dt \to \frac{1}{2\pi}\int_{\mathbb{T}}\exp(-imt)h(t)\,dt$$

and deduce that

$$\hat{f}(m) = \hat{h}(m)$$

for all m. Deduce the conclusion of Lemma 2.3.9.

2.4 Mean square convergence

So far we have looked a pointwise convergence and uniform convergence, but there is another mode of convergence which is intimately connected with Fourier series.

I assume that the reader is familiar with the notion of an inner product.

Definition 2.4.1. *If V is a vector space over $\mathbb{C}$ we say that $M : V \times V \to \mathbb{C}$ is an* inner product *if, writing $M(\mathbf{u},\mathbf{v}) = \langle \mathbf{u},\mathbf{v}\rangle$, we have*
(i) $\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{v},\mathbf{u}\rangle^*$.
(ii) $\langle \mathbf{u},\mathbf{u}\rangle \geq 0$.
(iii) $\langle \mathbf{u},\mathbf{u}\rangle = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$.
(iv) $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{w}\rangle + \langle \mathbf{v},\mathbf{w}\rangle$.
(v) $\langle \lambda\mathbf{u},\mathbf{v}\rangle = \lambda\langle \mathbf{u},\mathbf{v}\rangle$.

If V is a vector space over $\mathbb{R}$, condition (i) is replaced by $\langle \mathbf{u},\mathbf{v}\rangle = \langle \mathbf{v},\mathbf{u}\rangle$.

The following two exercises should be revision If not, the reader should not rest until she is thoroughly at home with the results.

Exercise 2.4.2. *Suppose that a vector V space over $\mathbb{C}$ has an inner product $\langle\ ,\ \rangle$.*
(i) If $\langle \mathbf{u},\mathbf{v}\rangle$ is real, verify that

$$\lambda^2\langle \mathbf{u},\mathbf{u}\rangle + 2\lambda\langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{v}\rangle = \langle \lambda\mathbf{u}+\mathbf{v},\lambda\mathbf{u}+\mathbf{v}\rangle \geq 0$$

for all real λ and deduce that

$$\langle \mathbf{u},\mathbf{v}\rangle^2 \leq \langle \mathbf{u},\mathbf{u}\rangle\langle \mathbf{v},\mathbf{v}\rangle.$$

(ii) By considering $e^{i\theta}\mathbf{a}$ for suitable θ, or otherwise, prove the Cauchy–Schwarz inequality

$$\langle \mathbf{a},\mathbf{b}\rangle^2 \leq \langle \mathbf{a},\mathbf{a}\rangle\langle \mathbf{b},\mathbf{b}\rangle.$$

(iii) Show that

$$\|\mathbf{a}\| = \langle \mathbf{a},\mathbf{a}\rangle^{1/2}$$

(with the positive square root) defines a norm.

We now introduce an inner product (and so a norm) on the space of continuous functions.

Exercise 2.4.3. *We work in $C_{\mathbb{C}}(\mathbb{T})$ the space of complex valued continued functions. Show that*

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(t)^* \, dt$$

defines an inner product on $C(\mathbb{T}) = C_{\mathbb{C}}(\mathbb{T})$.

We write

$$\|f\|_2 = \langle f, f\rangle^{1/2} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^2 \, dt\right)^{1/2}.$$

Exercise 2.4.4. *We work in $C(\mathbb{T})$ with the inner product and norm just described.*

(i) Let $e_j = \exp(int)$. Show that

$$\langle e_r, e_s\rangle = \begin{cases} 1 & \text{if } r = s, \\ 0 & \text{otherwise.} \end{cases}$$

(If we use the language of geometry, then the e_r are orthonormal vectors and what follows concerns orthonormal projection and the theorem of Pythagoras.)

Show also that $\hat{f}(j) = \langle f, e_j\rangle$.

(ii) Show that

$$\left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_2^2 = \|f\|_2^2 + \sum_{j=-n}^{n} |a_j - \hat{f}(j)|^2 - \sum_{j=-n}^{n} |\hat{f}(j)|^2.$$

(iv) Deduce that

$$\left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_2 \geq \left\| f - \sum_{j=-n}^{n} \hat{f}(j) e_j \right\|_2$$

with equality if and only if $a_j = \hat{f}(j)$ for $-n \leq j \leq n$.

(v) Suppose that we can find a trigonometric polynomial P of degree at most n such that $\|f - P\|_\infty \leq \epsilon$. By taking $a_j = \hat{P}(j)$ in (iii) show that

$$\left\| f - \sum_{j=-n}^{n} \hat{f}(j) e_j \right\|_2 \leq \epsilon.$$

(vi) By using Fejér's theorem, show that

$$\left\| f - \sum_{j=-n}^{n} \hat{f}(j) e_j \right\|_2 \to 0$$

as $n \to \infty$.

(vii) By using (iii) with $a_j = \hat{f}(j)$*, or otherwise, show that*

$$\sum_{j=-n}^{n} |\hat{f}(j)|^2 \to \|f\|_2^2$$

as $n \to \infty$*. More succinctly we say*

$$\sum_{j=-\infty}^{\infty} |\hat{f}(j)|^2 = \|f\|_2^2.$$

This is called Parseval's identity.

(viii) Of course many, of the ideas carry over to general inner product spaces. Show that if $(X, \langle\ ,\ \rangle)$ *is a a real or complex inner product space and* $\mathbf{e}_j$ *is a collection of orthonormal vectors, then*

$$\left\| \mathbf{f} - \sum_{j=1}^{n} a_j \mathbf{e}_j \right\|_2^2 = \|\mathbf{f}\|_2^2 + \sum_{j=1}^{n} |a_j - \langle \mathbf{f}, \mathbf{e}_j \rangle|^2 - \sum_{j=1}^{n} |\langle \mathbf{f}, \mathbf{e}_j \rangle|^2.$$

Deduce that

$$\|\mathbf{f}\|_2^2 \geq \sum_{j=1}^{n} |\langle \mathbf{f}, \mathbf{e}_j \rangle|^2$$

and so $\sum_{j=1}^{\infty} |\langle \mathbf{f}, \mathbf{e}_j \rangle|^2$ *converges with*

$$\|\mathbf{f}\|_2^2 \geq \sum_{j=1}^{\infty} |\langle \mathbf{f}, \mathbf{e}_j \rangle|^2.$$

This is Bessel's inequality.

Although $(C(\mathbb{T}), \| \ \|_2)$ is an interesting space it is not complete.

Exercise 2.4.5. *Show that* $(C(\mathbb{T}), \| \ \|_2)$ *is not complete.*
[*Look at Exercise 1.2.10 if you need a hint.*]

The following gives an example of a complete infinite dimensional inner product space.

Exercise 2.4.6. *Let us write* $l^2 = l^2(\mathbb{Z})$ *for the space consisting of two-way infinite sequences*

$$\mathbf{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \ldots)$$

with $a_j \in \mathbb{C}$ and $\sum_{j=-\infty}^{\infty} |a_j|^2$ convergent. We write

$$\|\mathbf{a}\|_2 = \left(\sum_{j=-\infty}^{\infty} |a_j|^2 \right)^{1/2}$$

where, as usual, we take the positive square root.

(i) If $\mathbf{a}$, $\mathbf{b} \in l^2$, let us write $\mathbf{a} + \mathbf{b} = \mathbf{c}$ with $c_j = a_j + b_j$. Explain why

$$\left(\sum_{j=-M}^{N} |a_j + b_j|^2 \right)^{1/2} \leq \left(\sum_{j=-M}^{N} |a_j|^2 \right)^{1/2} + \left(\sum_{j=-M}^{N} |b_j|^2 \right)^{1/2} \leq \|\mathbf{a}\|_2 + \|\mathbf{b}\|_2$$

and deduce that $\mathbf{a} + \mathbf{b} \in l^2$ with $\|\mathbf{a} + \mathbf{b}\|_2 \leq \|\mathbf{a}\|_2 + \|\mathbf{b}\|_2$.

(ii) If $\mathbf{a} \in l^2$ and $\lambda \in \mathbb{C}$, let us write $\lambda\mathbf{a} = \mathbf{c}$ with $c_j = \lambda a_j$. Show that, with the operations we have introduced, l^2 is a normed vector space.

(iii) If $\mathbf{a}$, $\mathbf{b} \in l^2$, show (by imitating the ideas of (i), or otherwise) that $\sum_{j=-\infty}^{\infty} a_j b_j^*$ converges absolutely and so converges. If we write

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=-\infty}^{\infty} a_j b_j^*,$$

show that, with the operations we have introduced, l^2 is an inner product space.

(iv) By considering the collection of $\mathbf{a}$ with all but finitely many a_j zero and all $a_j \in \mathbb{Q}$, or otherwise, show that l^2 is separable.

Lemma 2.4.7. *The normed space $(l^2, \| \|_2)$ is complete.*

Proof. By Exercise 1.2.15 (i), it sufficient to show that if $\|\mathbf{a}(n) - \mathbf{a}(m)\|_2 \leq 2^{-n}$ for all $m \geq n$, then $\mathbf{a}(n)$ converges in l^2. We observe that

$$|a_j(n) - a_j(m)| \leq \|\mathbf{a}(n) - \mathbf{a}(m)\|_2 \leq 2^{-n/2}$$

for all $m \geq n$ and so, since $\mathbb{C}$ is complete, $a_j(n) \to a_j$ for some $a_j \in \mathbb{C}$.

Next we note that since the sequence $\mathbf{a}(n)$ is Cauchy it is bounded with $\|\mathbf{a}(n)\|_2 \leq M$ say, so $\sum_{-P}^{Q} |a_j(n)|^2 \leq M^2$ and, allowing $n \to \infty$, this gives $\sum_{-P}^{Q} |a_j|^2 \leq M^2$ for all P, $Q > 0$. Allowing P, $Q \to \infty$, this shows that $\mathbf{a} \in l^2$.

Finally we have

$$\begin{aligned}
\left(\sum_{-N}^{N} |a_j - a_j(n)|^2 \right)^{1/2} &\leq \left(\sum_{-N}^{N} |a_j - a_j(m)|^2 \right)^{1/2} + \left(\sum_{-N}^{N} |a_j(m) - a_j(n)|^2 \right)^{1/2} \\
&\leq \left(\sum_{-N}^{N} |a_j - a_j(m)|^2 \right)^{1/2} + \|\mathbf{a}_n - \mathbf{a}_m\|_2 \\
&\leq \left(\sum_{-N}^{N} |a_j - a_j(m)|^2 \right)^{1/2} + 2^{-n}
\end{aligned}$$

Now, allowing $m \to \infty$, we obtain

$$\sum_{-N}^{N} \left(\sum_{-N}^{N} |a_j - a_j(n)|^2 \right)^{1/2} \leq 2^n.$$

Allowing $N \to \infty$, we obtain $\|\mathbf{a} - \mathbf{a}(n)\|_2 \to 0$ as required. ■

The next exercise provides practice in writing out the kind of argument just used.

Exercise 2.4.8. *Let us write* $l^1 = l^1(\mathbb{Z})$ *for the space consisting of two-way infinite sequences*

$$\mathbf{a} = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \dots)$$

with $a_j \in \mathbb{C}$ *and* $\sum_{j=-\infty}^{\infty} |a_j|$ *convergent. We write*

$$\|\mathbf{a}\|_1 = \sum_{j=-\infty}^{\infty} |a_j|$$

and, if $\mathbf{a}$, $\mathbf{b} \in l^1$, $\lambda \in \mathbb{C}$, *we take*

$$\mathbf{a} + \mathbf{b} = (\dots, a_j + b_j, \dots),$$
$$\lambda\mathbf{a} = (\dots, \lambda a_j, \dots).$$

Show that, with these choices, l^1 *is a well defined, separable, complete normed space.*

In Lemma 4.5.4 we shall see that the spaces l^1 and l^2 give us an example of two *distinct* well behaved infinite dimensional normed spaces.

Returning to l^2, observe that looking at Fourier transforms gives a natural mapping $\theta : (C(\mathbb{T}), \| \ \|_2) \to (l^2, \| \ \|_2)$ defined by $\theta(f)_n = \hat{f}(n)$.

Exercise 2.4.9. *(i) Prove the* polarisation identity *which states that if* $(V, \langle \ , \ \rangle)$ *is a complex inner product vector space with derived norm* $\| \ \|$, *then*

$$4\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 + i\|\mathbf{x} + i\mathbf{y}\|^2 - i\|\mathbf{x} - i\mathbf{y}\|^2$$

for all $\mathbf{x}$, $\mathbf{y} \in V$.

State and prove a corresponding result for real inner product spaces.

(ii) Suppose that $(V_A, \langle \ , \ \rangle_A)$ *and* $(V_B, \langle \ , \ \rangle_B)$ *are complex inner product spaces with derived norms* $\| \ \|_A$ *and* $\| \ \|_B$. *Suppose that the linear map* $\phi : V_A \to V_B$ *is an* isometry *in the sense that* $\|\phi\mathbf{u}\| = \|\mathbf{u}\|$ *for all* $\mathbf{u} \in V_A$. *Show, using (i), that* ϕ *preserves inner products, that is to say*

$$\langle \phi\mathbf{u}, \phi\mathbf{v} \rangle_B = \langle \mathbf{u}, \mathbf{v} \rangle_A.$$

(iii) (A form of Parseval's identity) Show that the map θ in the sentence before the beginning of this exercise is a well defined linear isometry. Conclude that

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)^* = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(t)^*\, dt$$

for all $f,\ g \in C(\mathbb{T})$.

We know that $(l^2, \| \|_2)$ is complete, but $(C(\mathbb{T}), \| \|_2)$ is not, so the two spaces cannot be identified. Can we identify l^2 with some larger space containing $(C(\mathbb{T}), \| \|_2)$ in a natural way?

If we look at the matter abstractly, the answer is yes. Recall from Section 1.7 that we can complete $(C(\mathbb{T}), \| \|_2)$ is an essentially unique way to obtain a complete normed space $(X, \| \|_2)$ with $C(\mathbb{T})$ as a dense subspace.

Exercise 2.4.10. *(i) By using the polarisation identity (see Exercise 2.4.9 (i)), or otherwise, show that X is an inner product space where the inner product restricted to $C(\mathbb{T})$ is the original inner product.*

(ii) If $f_n \in C(\mathbb{T})$, $F \in X$ and $f_n \to F$ in X, show that the $\theta(f_n)$ form a Cauchy sequence and so there exists a $\mathbf{c} \in l^2$ with $\theta(f_n) \to \mathbf{c}$ as $n \to \infty$. If $g_n \in C(\mathbb{T})$ and $g_n \to F$ show that $\theta(g_n) \to \mathbf{c}$. Thus we may define $\tilde{\theta}(F) = \mathbf{c}$.

(iii) Show that $\tilde{\theta}(f) = \theta(f)$ for all $f \in C(\mathbb{T})$.

(iv) Show that $\tilde{\theta} : X \to l^2$ is linear and preserves the norm.

(v) Suppose that $\mathbf{a} \in l^2$. Set

$$P_n(t) = \sum_{j=-n}^{n} a_j \exp ijt.$$

Show that the P_n form a Cauchy sequence in X and so $P_n \to F$ for some $F \in X$. Show that $\theta(F) = \mathbf{a}$.

(vi) Conclude that θ is a surjection and so a linear isometry.

If we look at the matter more concretely and ask what the F in Exercise 2.4.10 'actually looks like' things appear less straight-forward.

Exercise 2.4.11. *Let $a_{2^j} = a_{-2^j} = 1/j$ for $j \geq 0$ and $a_r = 0$ otherwise. Show that, although $\mathbf{a} \in l^2$,*

$$\sum_{j=-n}^{n} a_j \exp ijt \to \infty$$

whenever $t = k2^{-p}\pi$ with $k,\ p \in \mathbb{Z}$ and $p \geq 0$.

We need measure theory to obtain a more direct view of X.

2.5 Jackson's theorems

Once we have the idea of using different kernels such as Dirichlet's kernel and Féjer's kernel, we can try our hand at designing kernels for a particular purpose. The proof of the next theorem provides an excellent example.

Theorem 2.5.1. [Jackson's first theorem] *There exists a constant C with the following property. If $f : \mathbb{T} \to \mathbb{R}$ is once continuously differentiable, then, given $m \geq 1$, we can find a real trigonometric polynomial P of degree at most m such that*

$$\|P - f\|_\infty \leq C m^{-1} \|f'\|_\infty$$

Jackson's theorem provides a quantitative version of the statement that well behaved functions are easier to approximate. (In this connection see Exercises 2.8.12 and 2.8.15.)

Our proof of Theorem 2.5.1 depends on properties of the Jackson kernel J_n defined by

$$J_n(t) = \gamma_n^{-1} K_n(t)^2 = \lambda_n^{-1} \left(\frac{\sin(nt/2)}{\sin t/2} \right)^4$$

for $t \neq 0$, $J_n(0) = \lambda_n^{-1} n^2$ where γ_n and λ_n are chosen so that

$$\frac{1}{2\pi} \int_{\mathbb{T}} J_n(t)\, dt = 1.$$

Lemma 2.5.2. *There exist strictly positive constants A, A' and B such that*

$$A n^3 \geq \lambda_n \geq A' n^3$$

and

$$\frac{1}{2\pi} \int_{\mathbb{T}} |t| J_n(t)\, dt \leq B n^{-1}.$$

The function J_n is positive and has degree $2(n-1)$.

Proof. Using Exercise 2.2.2 (iv), we have

$$\left(\frac{\sin(nt/2)}{t/2} \right)^4 \leq \left(\frac{\sin(nt/2)}{\sin t/2} \right)^4 \leq \left(\frac{\pi}{2} \right)^2 \left(\frac{\sin(nt/2)}{t/2} \right)^4$$

for all $0 < |t| \leq \pi$. Symmetry and simple change of variables give

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{T}} \left(\frac{\sin(nt/2)}{t/2} \right)^4 dt &= 2n^4 \frac{1}{2\pi} \int_0^{\pi} \left(\frac{\sin(nt/2)}{nt/2} \right)^4 dt \\ &= 2n^3 \frac{1}{2\pi} \int_0^{n\pi/2} \left(\frac{\sin u}{u} \right)^4 du \end{aligned}$$

and we note that

$$\int_0^{n\pi/2}\left(\frac{\sin u}{u}\right)^4 \leq \int_0^{\pi/2}\left(\frac{\sin u}{u}\right)^4 du + \int_{\pi/2}^{\infty}\left(\frac{\sin u)}{u}\right)^4 du$$
$$\leq \int_0^{\pi/2}\left(\frac{\sin u}{u}\right)^4 du + \int_{\pi/2}^{\infty}\frac{1}{u^4}\, du = A''$$

with A'' independent of n whilst, trivially

$$\int_0^{n\pi/2}\left(\frac{\sin u}{u}\right)^4 \geq \int_0^{\pi/2}\left(\frac{\sin u}{u}\right)^4 du = A'''$$

where A''' is strictly positive and independent of n.

It follows that there exist positive constants A and A' such that

$$An^3 \geq \lambda_n \geq A'n^3.$$

Similar arguments, which the reader should provide, give the remaining inequality.

Since the degree of K_n is $2(n-1)$ the degree of J_n is $4(n-1)$ ∎

Exercise 2.5.3. *Provide the argument just requested.*

We can now prove a version of Theorem 2.5.1.

Theorem 2.5.4. *There exists a constant C' with the following property. If $f : \mathbb{T} \to \mathbb{R}$ is once continuously differentiable, then, given $n \geq 1$, we can find a real trigonometric polynomial Q of degree at most $4(n-1)$ such that*

$$\|Q - f\|_\infty \leq C'n^{-1}\|f'\|_\infty$$

Proof. Since J_n is a trigonometric polynomial of degree $4(n-1)$, and

$$Q(t) = J_n * f(t) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)J_n(s)\, ds$$

defines a real trigonometric polynomial of degree at most $4(n-1)$.

Now, using the mean value theorem,

$$|f(s) - f(t)| \leq \|f'\|_\infty |t-s|$$

so

$$|Q(t) - f(t)| = \left|\frac{1}{2\pi}\int_{\mathbb{T}}(f(t-s) - f(t))J_n(s)\, ds\right|$$
$$\leq \frac{1}{2\pi}\int_{\mathbb{T}}|f(t-s) - f(t)|J_n(s)\, ds$$
$$\leq \|f'\|_\infty \frac{1}{2\pi}\int_{\mathbb{T}}|s|J_n(s)\, ds \leq Bn^{-1}$$

as required. ∎

Exercise 2.5.5. *Deduce Theorem 2.5.1 from Theorem 2.5.4.*

It is easy to guess the generalisation to higher derivatives.

Theorem 2.5.6. [Jackson's second theorem] *There exists a constant C_k with the following property. If $f : \mathbb{T} \to \mathbb{R}$ is k times continuously differentiable then given $n \geq 1$ we can find a real trigonometric polynomial P of degree at most n such that*

$$\|P - f\|_\infty \leq C_k n^{-k} \|f^{(k)}\|_\infty$$

There is a natural way of proving this theorem by designing new kernels along the lines of the kernel used in proving Jackson's first theorem (see Exercise 2.8.23) but, at least in the form presented in the exercises, that proof requires some knowledge of divided differences (see Exercise 2.8.22).

Another natural way of proving Jackson's second theorem is simply to use his first theorem repeatedly. We need to keep the following simple results in mind.

Exercise 2.5.7. *(i) Suppose that $f \in C(\mathbb{T})$. Show that, if there exists a $g \in C(\mathbb{T})$ such that g is differentiable and $g'(t) = f(t)$, we must have*

$$\frac{1}{2\pi} \int_{\mathbb{T}} f(t)\, dt = 0,$$

that is to say $\hat{f}(0) = 0$.

(ii) Suppose that $f \in C(\mathbb{T})$ and $\hat{f}(0) = 0$. Show that we can find a $g \in C(\mathbb{T})$ with $\hat{g}(0) = 0$ such that g is differentiable and $g'(t) = f(t)$ for all $t \in \mathbb{T}$.

(iii) By applying Exercise 2.3.7 (iii), or otherwise, show that if $f, h \in C(\mathbb{T})$ and $\hat{f}(0) = 0$, then $\widehat{f * h}(0) = 0$.

(iv) By repeated integration by parts, or otherwise, show that if $f \in C(\mathbb{T})$ is r times continuously differentiable and $\hat{f}(0) = 0$ then $\widehat{f^{(r)}}(0) = 0$.

The proof of Theorem 2.5.6 is outlined in the next exercise.

Exercise 2.5.8. *(i) Use the remark of Exercise 2.5.7 (iii) to prove the following modification of Theorem 2.5.1. There exists a constant C with the following property. If $f : \mathbb{T} \to \mathbb{R}$ is once continuously differentiable and $\hat{f}(0) = 0$, then, given $n \geq 1$, we can find a real trigonometric polynomial P of degree at most n with $\hat{P}(0) = 0$ such that*

$$\|P - f\|_\infty \leq C n^{-1} \|f'\|_\infty$$

(ii) Deduce that if $g : \mathbb{T} \to \mathbb{R}$ is once continuously differentiable and Q is a trigonometric polynomial of degree at most n with $\hat{g}(0) = \hat{Q}(0) = 0$, then, given $n \geq 1$, we can find a real trigonometric polynomial R of degree at most n with $\hat{R}(0)$ such that

$$\|g - R\|_\infty \leq C n^{-1} \|g' - Q\|_\infty.$$

(iii) Use induction to show that if $f : \mathbb{T} \to \mathbb{R}$ is k times continuously differentiable, with $\hat{f}(0) = 0$ we can find a real trigonometric polynomial P of degree at most n such that

$$\|f - P\|_\infty \leq C^k n^{-k} \|f^{(k)}\|.$$

(iv) Explain why we can remove the condition $\hat{f}(0) = 0$ in part (iii), thus completing the proof.

2.6 The Weierstrass approximation theorem

In 1875, at the age of seventy, Weierstrass published to following important theorem.

Theorem 2.6.1. *If $f : [a, b] \to \mathbb{R}$ is continuous, then, given any $\epsilon > 0$, we can find a real polynomial P such that $|f(t) - P(t)| \leq \epsilon$ for all $t \in [a, b]$.*

We shall give three different proofs[3] of this result (and refer the reader to Exercises 2.8.17 and 2.8.18 for two more). Each of the proofs is worth studying for its own sake, but we shall later prove the Stone–Weierstrass theorem which includes Theorem 2.6.1 as a special case. Because of this, I shall feel free to leave a large part of the various proofs as exercises for the reader.

Exercise 2.6.2. *Explain why we need only prove Theorem 2.6.1 for one particular choice of $[a, b]$ (for example $[a, b] = [0, 1]$ or $[a, b] = [-1, 1]$).*

Our first proof uses Fejér's theorem. We introduce the Chebychev[4] polynomials T_n.

Exercise 2.6.3. *(i) Explain why $\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n$.*

(ii) By taking real parts of the equation (i) and recalling that $1 = \cos^2\theta + \sin^2\theta$, show that there exists a unique real polynomial T_n of degree n such that

$$\cos n\theta = T_n(\cos\theta)$$

for all real θ.

[3]As a student, Kolmogorov was undecided as to whether to pursue history or mathematics. When presented the results of his first piece of research (on landholding in Novgorod in the 15th–16th century) his teacher told him 'You have supplied one proof of your thesis [based on the observation that certain taxes produced an integer sum of roubles and others did not], and in the mathematics that you study this would perhaps suffice, but we historians prefer to have at least ten proofs.' Kolmogorov decided to choose mathematics.

[4]This has become the standard English transliteration of the name. The French transliteration, used by Chebychev himself, was Tchebycheff, often anglicised as Tchebychev. Hence the 'T' for 'Tchebychev polynomial'.

(iii) By using the binomial theorem, or otherwise, show that, if $n \geq 1$,

$$2^n = (1+1)^n + (1-1)^n = 2\sum_{n \geq n-2r \geq 0} \binom{n}{2r}.$$

Hence show that the coefficient of t^n in $T_n(t)$ is 2^{n-1} for $n \geq 1$. (Note, however, that $T_0(t) = 1$.)

The polynomial T_n is called the nth *Chebychev polynomial*.

Exercise 2.6.4. *Suppose that $f : [-1, 1] \to \mathbb{R}$ is continuous and $\epsilon > 0$.*

(i) Show that, if we set $F(\theta) = f(\cos\theta)$, then $F : \mathbb{T} \to \mathbb{R}$ is well defined and continuous with $F(\theta) = F(-\theta)$.

(ii) Explain why $\sigma_n(F)(\theta) = \sum_{j=0}^{n-1} a_j \cos j\theta$ for some real $a_0, a_1, \dots, a_{n-1}$.

(iii) Use Fejér's theorem and Chebychev polynomials to show that there exists a real polynomial P such that $|f(t) - P(t)| \leq \epsilon$ for all $t \in [0, 1]$.

Exercise 2.6.5. [Jackson's theorem for polynomials] *Suppose that $f : [-1, 1] \to \mathbb{R}$ is a continuously differentiable function with $f(0) = 0$.*

Show that, if we set $F(\theta) = f(\cos\theta)$, then $F : \mathbb{T} \to \mathbb{R}$ is well defined and continuously differentiable with $F(\theta) = F(-\theta)$. (Be careful to check differentiabilty at 0 and π.) Show also that $\|F'\|_\infty \leq 2\|f'\|_\infty$.

By applying Jackson's first theorem (Theorem 2.5.1) show that there exists a constant K independent of f and n such that there exists a polynomial of degree at most n with

$$\|f - P\|_\infty \leq Kn^{-1}\|f'\|_\infty.$$

Explain why we can drop the condition $f(0) = 0$.
[*For the expected extension, see Exercise 2.8.16.*]

Our second proof is close to the original proof of Weierstrass. We start with an echo of the proof of Theorem 2.3.4.

Exercise 2.6.6. *Write $E(t) = (2\pi)^{-1/2}\exp(-t^2/2)$. It is well known (and you may accept, if it is not well known to you) that*

$$\int_{-\infty}^{\infty} E(t)\, dt = 1.$$

(i) Let $K > 0$. If we write $E_K(t) = KE(Kt)$, show that

$$\int_{-\infty}^{\infty} E_K(t)\, dt = 1.$$

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is a bounded uniformly continuous function, show that

$$E_K * f(t) = \int_{-\infty}^{\infty} E_K(t-s) f(s)\, ds \to f(t)$$

uniformly as $K \to \infty$.

Here we use the notation

$$f * g(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\,ds$$

for the convolution of two continuous functions on $\mathbb{R}$. Note that we must check that the integral is defined for these particular functions.

Exercise 2.6.7. *Now suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with $g(t) = 0$ for $|t| \geq 1$.*

(i) Explain why

$$E_K * g(t) = \int_{-1}^{1} E_K(t-s)g(s)\,ds.$$

(ii) By using the Taylor expansion for exp*, or otherwise (be careful), show that, if K is fixed and $\epsilon > 0$ is given, we can find a polynomial P such that*

$$|E_K(x) - P(x)| \leq \epsilon$$

for all $|x| \leq 2$. Show that, for this P and all $t \in [-1, 1]$

$$|P * g(t) - E_K * g(t)| \leq 2\epsilon \sup_{x \in \mathbb{R}} |g(x)|.$$

(iii) By first considering the case $Q(t) = t^m$, or otherwise, show that, if Q is real polynomial, then so is $Q * g$.

(iii) Deduce Theorem 2.6.1 for $[a, b] = [-1, 1]$.

Our third proof is due to Lebesgue. We start with a simple set of observations.

Exercise 2.6.8. *(i) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. If we define $f_n : [0, 1] \to \mathbb{R}$ to be the simplest continuous linear piecewise function with $f_n(r/n) = f(r/n)$, show that $f_n \to f$ uniformly as $n \to \infty$.*

(ii) Let us define $\Delta_{r,n} : [0, 1] \to \mathbb{R}$ by the formula

$$\Delta_{r,n}(t) = \max\{0, 1 - n|t - r/n|\}.$$

Sketch the graph of $\Delta_{r,n}$. If $g : [0, 1] \to \mathbb{R}$ is a continuous function which is linear on each interval $[r/n, (r + 1)/n]$, show that

$$g = \sum_{r=0}^{n} g(r/n)\Delta_{r,n}.$$

(Informally, we say that g is the sum of triangular functions.)

(iii) Show that

$$\Delta_{r,n}(t) = \frac{n}{2}\left(\left|t - \frac{r+1}{n}\right| + \left|t - \frac{r-1}{n}\right| - 2\left|t - \frac{r}{n}\right|\right).$$

(iv) Deduce that the theorem of Weierstrass will follow if we can show that there is a sequence Q_m of real polynomials such that

$$Q_m(t) \to |t|$$

uniformly on $[-2, 2]$.

(v) By considering $2P_n(t/2)$ show that the theorem of Weierstrass will follow if we can show that there is a sequence P_n of real polynomials such that

$$P_n(t) \to |t|$$

uniformly on $[-1, 1]$.

We have reduced the proof of the Weierstrass polynomial approximation theorem to the following lemma.

Lemma 2.6.9. *There is a sequence P_n of real polynomials such that*

$$P_n(t) \to |t|$$

uniformly on $[-1, 1]$.

There are many different proofs of Lemma 2.6.9. Exercise 2.8.20 is particularly direct, but the traditional proof looks at the power series expansion of $(1-x)^{1/2}$. Using the real variable Taylor theorem with remainder, the complex variable Taylor theorem (much easier to apply than the real variable Taylor theorem) or some other method (see for example Exercise 2.8.21) we know that writing

$$a_0 = 1,\ a_1 = -\frac{1}{2},\ \ldots,\ a_r = -\frac{1}{r!}\left(\frac{1}{2}\prod_{k=1}^{r-1}\frac{2k-1}{2}\right),\ \ldots$$

we have

$$\sum_{r=0}^{n} a_r x^r \to (1-x)^{1/2}$$

uniformly for $|x| \leq 1 - \eta$ whenever $1 > \eta > 0$.

Exercise 2.6.10. *(i) Let $\epsilon > 0$. Show that there is an $1 > \delta > 0$ such that*

$$\left||t| - (1 - (1-\delta)(1-t^2))^{1/2}\right| < \epsilon/2$$

for all $|t| \leq 1$.

(ii) Obtain Lemma 2.6.9 by applying the paragraph preceding this exercise.

Here is a simple, but useful, application of the Weierstrass approximation theorem.

Exercise 2.6.11. *If* $f : [a, b] \to \mathbb{R}$ *is continuous and the* moments[5]

$$\int_a^b t^n f(t)\, dt = 0$$

for all n, show that $f = 0$.
[*Hint: Look at the proof of Theorem 2.3.8.*]

Here is another application which is amusing rather than useful

Exercise 2.6.12. *Let* $\mathcal{P}$ *be the vector space of real polynomials on* $[0, 1]$. *Show that, if I is an infinite subset of* $[0, 1]$, *then* $\|p\|_I = \sup_{t\in I} |p(t)|$ *defines a a norm on* $\mathcal{P}$.

Let $I = [0, 1/4]$, $J = [3/4, 1]$. *Show that we can find* $p_n \in \mathcal{P}$ *such that* $p_n \to 0$ *in the norm* $\| \|_I$, *but* $p_n \to 1$ *in the norm* $\| \|_J$.

If the reader makes a mental note each time we use the Wierstrass approximation theorem or its generalisation by Stone she will soon appreciate the central role of the result.

2.7 A cousin and three sisters

In these notes, we mainly use the Baire category theorem to produce negative results, for example to show that there exist continuous functions which are nowhere differentiable.

In this section we shall see how Banach and his colleagues used the Baire category theorem to produce positive results. We shall not make much direct use of these results, but they form an important feature in the landscape of modern analysis.

When we talked about the contraction mapping theorem, I said that Banach replaced many (but not all) versions of a successive approximation argument by a single simple theorem. In the same way he replaced many (but not all) versions of the so called 'method of condensation of singularities' by the following simple theorem.

[5] Recall that in probability $\mathbb{E}X^n$ is call the nth moment of the random variable X.

Theorem 2.7.1. [The principle of uniform boundedness][6] *Suppose that* $(U, \| \|_U)$ *is a complete normed space,* $(V, \| \|_V)$ *a normed space and* $\mathcal{T}$ *is a collection of continuous linear functions* $T : U \to V$*. If, given any* $\mathbf{u} \in U$*, we can find a* $K(\mathbf{u})$ *such that*

$$\|T(\mathbf{u})\| \leq K(\mathbf{u}) \text{ for all } T \in \mathcal{T},$$

then there exists a constant K *such that*

$$\|T\| \leq K \text{ for all } T \in \mathcal{T}.$$

Proof. Consider the closed balls

$$\bar{B}_n = \{\mathbf{v} \in V : \|\mathbf{v}\|_V \leq n\}.$$

Since each $T \in \mathcal{T}$ is continuous, $T^{-1}(\bar{B}_n)$ is closed and so $A_n = \bigcap_{T \in \mathcal{T}} T^{-1}(\bar{B}_n)$ is closed.

Using the hypotheses of the theorem, we see that, if $\mathbf{u} \in U$, then $\mathbf{u} \in A_n$ whenever $n \geq K(\mathbf{u})$. It follows that $\bigcup_{n=1}^{\infty} A_n = V$. The Baire category theorem now tells us that at least one of the A_n must have non-empty interior. Suppose that A_N has non-empty interior. Then we can find $\mathbf{u}_0$ and $\epsilon > 0$ such that

$$\{\mathbf{u} \in U : \|\mathbf{u} - \mathbf{u}_0\|_U \leq \epsilon\} \subseteq A_N,$$

in other words, such that

$$\|\mathbf{u} - \mathbf{u}_0\|_U \leq \epsilon \Rightarrow \|T\mathbf{u}\|_V \leq N$$

for all $T \in \mathcal{T}$. Thus, if $T \in \mathcal{T}$, the linearity of T gives us

$$\begin{aligned} \|\mathbf{u}\|_U \leq \epsilon &\Rightarrow \|(\mathbf{u} + \mathbf{u}_0) - \mathbf{u}_0\|_U,\ \|\mathbf{u}_0 - \mathbf{u}_0\|_U \leq \epsilon \\ &\Rightarrow \|T(\mathbf{u} + \mathbf{u}_0)\|_V,\ \|T\mathbf{u}_0\| \leq M \\ &\Rightarrow \|T\mathbf{u}\|_V = \|T(\mathbf{u} + \mathbf{u}_0) - T\mathbf{u}_0\| \leq \|T(\mathbf{u} + \mathbf{u}_0)\|_V + \|T\mathbf{u}_0\| \leq 2M. \end{aligned}$$

It follows that $\|T\| \leq 2\epsilon^{-1}M$ for all $T \in \mathcal{T}$ and we are done. ∎

To see how this links in with more classical results like Theorem 2.2.3 we note the following corollary.

Exercise 2.7.2. *Suppose that* $(U, \| \|_U)$ *is a complete normed space,* $(V, \| \|_V)$ *a normed space and that we have a sequence of continuous linear functions* $T_n : U \to V$ *such that* $\|T_n\|$ *is unbounded. Show that there exists a* $\mathbf{u} \in U$ *such that* $\|T_n(\mathbf{u})\|$ *is unbounded (and therefore, in particular,* $T_n(\mathbf{u})$ *cannot converge).*

[6] Also called the Banach–Steinhaus theorem after its discoverers. Hahn obtained the result independently at about the same time.

We shall see a very nice use of the principle of uniform boundedness in Exercise 2.8.7. The uniform boundedness theorem has deep roots in classical analysis. The 'three sisters' of the title of this section arise when we seek to answer a more modern question. If U and V are normed spaces and $T : U \to V$ is a continuous linear bijection, when can we deduce that T^{-1} is continuous?

Exercise 2.7.3. *(A reminder to the reader.) If U and V are vector spaces and $T : U \to V$ is both a linear map and a bijection, show that $T^{-1} : V \to U$ is linear.*

The next exercise shows that some condition is needed to ensure continuity of the inverse.

Exercise 2.7.4. *Let $(l^\infty, \| \|_\infty)$ be the vector space of bounded sequences with pointwise addition and scalar multiplication equipped with the norm $\|\mathbf{a}\|_\infty = \sup_n |a_n|$. Let c_{00} be the subspace of sequences all but finitely many of whose entries are non-zero. If $T : (c_{00}, \| \|_\infty) \to (c_{00}, \| \|_\infty)$ is given by*

$$T(a_1, a_2, \dots a_n, \dots) = (a_1, 2^{-1}a_2, \dots n^{-1}a_n, \dots),$$

show that T is a continuous linear bijection whose inverse is not continuous.

However, if we make the two normed spaces *complete*, the desired result is true and gives us our first sister.

Theorem 2.7.5. [The inverse mapping theorem] *Suppose that $(U, \| \|_U)$ and $(V, \| \|_V)$ are complete normed spaces and $T : U \to V$ is a continuous linear bijection. Then $T^{-1} : V \to U$ is also continuous.*

We obtain Theorem 2.7.5 via an apparently more general result

Theorem 2.7.6. [The open mapping theorem][7] *Suppose that $(U, \| \|_U)$ and $(V, \| \|_V)$ are complete normed spaces and $T : U \to V$ is a continuous linear surjection. Then $T : V \to U$ is is an open map (that is to say, T maps open sets to open sets).*

Proof of Theorem 2.7.5 from Theorem 2.7.6. Suppose that f is a bijection from a metric space X to a metric space Y. The statement that f takes open sets to open sets is equivalent to saying that f^{-1} is continuous. ∎

The open mapping theorem for *analytic functions* that appears in complex analysis is rather deep. If we confine ourselves to *linear surjections*, the notion of an open map is not very profound.

[7] Also called the Banach–Schauder theorem.

Lemma 2.7.7. *Suppose that* $(U, \|\ \|_U$ *and* $(V, \|\ \|_V)$ *are complete normed spaces and* $T : U \to V$ *is a continuous linear surjection. Then* T *is open if and only if we can find a* $K > 0$ *such that, if* $\mathbf{v} \in V$*, we can find a* $\mathbf{u} \in U$ *with* $T\mathbf{u} = \mathbf{v}$ *and* $\|\mathbf{u}\|_U \leq K\|\mathbf{v}\|_V$*.*

Proof. Suppose that T is open. Consider the open unit ball

$$B = \{\mathbf{u} \in U : \|\mathbf{u}\|_U < 1\}.$$

Since TB is open and $\mathbf{0} \in TB$ we can find a $\delta > 0$ such that

$$\{\mathbf{w} \in V : \|\mathbf{w}\|_V \leq \delta\} \subseteq TB.$$

We set $K = \delta^{-1}$.

If $\mathbf{v} \in V$ then either $\mathbf{v} = \mathbf{0}$ and we are done, or

$$K^{-1}\|\mathbf{v}\|^{-1}\mathbf{v} \in \{\mathbf{w} \in V : \|\mathbf{w}\|_V \leq \delta\}$$

so we can find a $\mathbf{z} \in B$ such that $T\mathbf{z} = K^{-1}\|\mathbf{v}\|^{-1}\mathbf{v}$. Taking $\mathbf{u} = K\|\mathbf{v}\|\mathbf{z}$ gives the desired result.

Conversely, suppose there exist a $K > 0$ such that, if $\mathbf{v} \in V$, we can find a $\mathbf{u} \in U$ with $T\mathbf{u} = \mathbf{v}$ and $\|\mathbf{u}\|_U \leq K\|\mathbf{v}\|_V$. If A is an open subset of of U and $\mathbf{a} \in A$ then we can find an $\eta > 0$ such that (writing B for the open unit ball as before)

$$A \supseteq \mathbf{a} + \eta B.$$

By linearity,

$$TA \supseteq T\mathbf{a} + \eta TB \supseteq \{\mathbf{v} : \|\mathbf{v} - T\mathbf{a}\| < K^{-1}\eta\}$$

and we have shown that TA is open. ∎

The proof of Theorem 2.7.6 is an ingenious application of the Baire category theorem followed by a successive approximation argument. Throughout we make essential use of the linearity of T.

Proof of Theorem 2.7.6. Let $\bar{B}_n = \{\mathbf{u} \in U : \|\mathbf{u}\| \leq n\}$. Since T is surjective, we know that $\bigcup_{n=1}^{\infty} T(\bar{B}_n) = V$. However, we do not know that $T(\bar{B}_n)$ is closed, so instead we look at $\operatorname{Cl} T(\bar{B}_n)$ the closure of $T(\bar{B}_n)$ in V. We now know that $\bigcup_{n=1}^{\infty} \operatorname{Cl} T(\bar{B}_n) = V$ and $\operatorname{Cl} T(\bar{B}_n)$ is closed, so, by the Baire category theorem, there must be an N such that $\operatorname{Cl} T(B_N)$ has non-empty interior.

Echoing our proof of Theorem 2.7.1 we suppose that A_N has non-empty interior. Then we can find $\mathbf{v}_0$ and $\eta > 0$ such that

$$\{\mathbf{v} \in V : \|\mathbf{v} - \mathbf{v}_0\|_V \leq \eta\} \subseteq \operatorname{Cl} T(B_N).$$

If $\mathbf{w} \in V$ and $\|\mathbf{w}\|_V \le \eta$, then

$$\|\mathbf{v}_0 - \mathbf{v}_0\|_V,\ \|(\mathbf{v}_0 + \mathbf{w}) - \mathbf{v}_0\|_V \le \eta$$

so we can find $\mathbf{x}_n,\ \mathbf{y}_n \in B_N$ with

$$T\mathbf{x}_n \to \mathbf{v}_0 \text{ and } T\mathbf{y}_n \to \mathbf{v}_0 + \mathbf{w}.$$

Thus, setting $\mathbf{z}_n = \mathbf{y}_n - \mathbf{x}_n$, we have $\|\mathbf{z}_n\|_U \in B_{2N}$ and $T\mathbf{z}_n \to \mathbf{w}$.

Rescaling and using linearity, we see that there exists a constant C such that, if $\mathbf{v} \in V$, there exist $\mathbf{u}_n \in U$ with $\|\mathbf{u}_n\|_U \le C\|\mathbf{v}\|_V$ and $\|T\mathbf{u}_n - \mathbf{v}\|_V \to 0$. To apply successive approximation, we only need the apparently weaker result that, whenever $\mathbf{v} \in V$, there exists a $\mathbf{u} \in U$ with $\|\mathbf{u}\|_U \le C\|\mathbf{v}\|_V$ and $\|T\mathbf{u} - \mathbf{v}\|_V \le \|\mathbf{v}\|_V/2$.

If $\mathbf{v} \in V$, we set $\mathbf{x}_0 = \mathbf{0}$ and find inductively $\mathbf{x}_n,\ \mathbf{u}_n \in U$ such that

$$\|\mathbf{x}_{n+1}\|_U \le C\left\|\mathbf{v} - T\left(\sum_{j=0}^{n}\mathbf{x}_j\right)\right\|_V$$
$$\left\|\mathbf{v} - T\left(\sum_{j=0}^{n+1}\mathbf{x}_j\right)\right\|_V \le 2^{-1}\left\|\mathbf{v} - T\left(\sum_{j=0}^{n}\mathbf{x}_j\right)\right\|_V.$$

By induction

$$\|\mathbf{x}_n\|_U \le 2^{-n}\|\mathbf{v}\|$$

and so

$$\left\|\mathbf{v} - T\left(\sum_{j=0}^{n}\mathbf{x}_j\right)\right\|_V \le C2^{-n}\|\mathbf{v}\|_V.$$

Since $\|\ \|_U$ is complete, $\sum_{j=0}^{n}\mathbf{x}_j$ converges to some $\mathbf{u} \in U$ with

$$\|\mathbf{u}\| \le \sum_{j=0}^{n}\|\mathbf{x}_j\|_U \le 2C\|\mathbf{v}\|_V.$$

Thus

$$\begin{aligned}\|\mathbf{v} - T\mathbf{u}\|_V &\le \left\|\mathbf{v} - T\left(\sum_{j=0}^{n}\mathbf{x}_j\right)\right\|_V + \left\|T\left(\sum_{j=0}^{n}\mathbf{x}_j\right) - T\mathbf{u}\right\|_V \\ &\le \left\|\mathbf{v} - T\left(\sum_{j=0}^{n}\mathbf{x}_j\right)\right\|_V + \|T\|\left\|\left(\sum_{j=0}^{n}\mathbf{x}_j\right) - \mathbf{u}\right\|_V \to 0\end{aligned}$$

so $T\mathbf{u} = \mathbf{v}$ and we are done. ∎

Exercise 2.7.8. *Halmos says 'once is a trick, twice is a method and three times a theorem'. State and (if you feel you should) prove a theorem which summarises the last paragraph of the proof just given.*

If we look at the proof of Theorem 2.7.6 we see that adding the hypothesis T injective would not have made much difference to the proof.

Exercise 2.7.9. *Identify the point or points, if any, at which adding the hypothesis T injective would, in your opinion, have made the proof easier*

It is therefore natural to ask if instead of proving Theorem 2.7.5 from Theorem 2.7.6 we could reverse the process and prove Theorem 2.7.6 from Theorem 2.7.5. It is not hard to guess that a quotienting argument might work. Exercise 2.8.27 gives the details.

In general linear maps are neither injective nor surjective. Is there some way of extending our results to such maps? To do this we need some way of transforming a general map to a bijective map and this in turn is provided by the notion of a graph.

Definition 2.7.10. *The* graph $\Gamma(f)$ *of a function* $f : X \to Y$ *is the subset of* $X \times Y$ *given by*

$$\Gamma(f) = \{(x, f(x)) \, : \, x \in X\}.$$

Exercise 2.7.11. *We use the notation of Definition 2.7.10.*

(i) Show that the map $\tilde{f} : X \to \Gamma(f)$ *is bijective.*

(ii) Suppose that (X, d_X) *and* (Y, d_Y) *are metric spaces and we give* $X \times Y$ *a product metric* d *defined by*

$$d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y').$$

If f *is continuous, show that* $\Gamma(f)$ *is closed.*

(iii) Suppose that $X = Y = \mathbb{R}$ *and we use the standard metric. If* $f(x) = 1/x$ *for* $x \neq 0$ *and* $f(0) = 0$ *show* $\Gamma(f)$ *is closed, but* f *is not continuous.*

Exercise 2.7.12. *If* $(U, \| \, \|_U)$ *and* $(V, \| \, \|_V)$ *are normed and we give* $U \times V$ *the standard vector space structure, show that*

$$\|(\mathbf{u}, \mathbf{v})\|_{U\times V} = \|\mathbf{u}\|_U + \|\mathbf{v}\|_V$$

is a norm. (We call this norm the product norm.*) Show that if* $\| \, \|_U$ *and* $(\| \, \|_V)$ *are complete norms so is the product norm* $\| \, \|_{U\times V}$.

We now get our third sister.

Theorem 2.7.13. [The closed graph theorem] *Suppose that* $(U, \| \|_U$ *and* $(V, \| \|_V)$ *are complete normed spaces and* $T : U \to V$ *is a linear map. Then, if* $\Gamma(T)$ *is closed in the product norm,* T *is continuous.*

Proof of Theorem 2.7.13 from Theorem 2.7.5. Observe that $\tilde{T} : U \to \Gamma(T)$ is a linear bijection so $\tilde{T}^{-1} : \Gamma(T) \to U$ is a linear bijection. Moreover if $(\mathbf{u}, \mathbf{v}) \in \Gamma(T)$, then $\mathbf{v} = T\mathbf{u}$, so

$$\|\tilde{T}^{-1}(\mathbf{u}, \mathbf{v})\|_U = \|\mathbf{u}\|_U \leq \|\mathbf{u}\|_U + \|\mathbf{v}\|_V = \|(\mathbf{u}, \mathbf{v})\|_{U\times V}.$$

Thus $\tilde{T}^{-1}$ is a continuous linear bijection.

Since $\Gamma(T)$ is a closed subspace of a complete normed space it is complete under the restriction of the norm. Thus we may apply Theorem 2.7.5 to deduce that $\tilde{T} = (\tilde{T}^{-1})^{-1}$ is continuous. Now

$$\|T\mathbf{u}\|_V \leq \|\mathbf{u}\|_U + \|T\mathbf{u}\|_V = \|\tilde{T}(\mathbf{u})\|_{U\times V} \leq \|\tilde{T}\|\|\mathbf{u}\|_U,$$

so T is continuous. ■

We can reverse the direction of our proofs in a rather elegant manner

Proof of Theorem 2.7.5 from Theorem 2.7.13. Since T is continuous, the graph

$$\Gamma(T) = \{(\mathbf{u}, T\mathbf{u}) : \mathbf{u} \in U\}$$

is closed in $U \times V$. On reflection we see that this means that

$$\begin{aligned}\Gamma(T^{-1}) &= \{(\mathbf{v}, T^{-1}\mathbf{v}) : \mathbf{v} \in V\}\\ &= \{(T\mathbf{u}, \mathbf{u}) : \mathbf{u} \in U\}\end{aligned}$$

is closed in $V \times U$ and so, by the closed graph theorem, T^{-1} is continuous. ■

Exercise 2.7.14. *Consider* $V = C([0, 1])$ *with uniform norm. Let* U *be the subspace of continuously differentiable functions with the subspace norm. Let* $T : U \to V$ *be the map* $Tf = f'$*. Show that*

$$\Gamma(T) = \{(u, Tu) : u \in U\}$$

is closed in $U \times V$ *(hint: this is a standard result on uniform convergence dressed up in abstract clothing) but* T *is not a continuous map. Why does this not contradict the closed graph theorem?*

2.8 Further exercises

Exercise 2.8.1. *(i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is real analytic at every point. Suppose that $x_1, x_2, \dots$ are distinct points with $x_n \to x_0$ as $n \to \infty$. If $f(x_n) = 0$ for all n, show that $f(x) = 0$ for all x.*
[If you have done complex variable theory, you should not need a hint. If not, observe that, if f is not identically zero, we can write $f(x) = (x - x_0)^m h(x)$ with $h(x_0) \neq 0$ in some open interval $(x_0 - \eta, x_0 + \eta)$.]

(ii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is real analytic at every point but is not a polynomial. By using a Baire category argument or otherwise, show that there is a point $y \in \mathbb{R}$ such that $f^{(k)}(y) \neq 0$ for every integer $k \geq 0$.

Exercise 2.8.2. *(i) (Should be revision, otherwise look it up in a text book under 'radius of convergence'.) If $a_n \in \mathbb{R}$, show that either $\sum_{r=0}^{\infty} a_n x^n$ converges absolutely for all x or there exists an $R \geq 0$ such that $\sum_{r=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$ and diverges for $|x| > R$. We call R the radius of convergence and set $R = \infty$ if $\sum_{r=0}^{\infty} a_n x^n$ converges absolutely for all x.*

(ii) Let $\sum_{r=0}^{\infty} a_n x^n$ have radius of convergence R and $|x_0| < R$. If $|x_0| + |h| < R$, show that

$$\sum_{r=0}^{\infty} \sum_{k=0}^{n} \left| \binom{n}{k} a_n x_0^k h^{n-k} \right|$$

converges. Hence, or otherwise, show that we can find b_j such that $\sum_{r=0}^{\infty} b_n x^n$ has radius of convergence at least $R - |x_0|$ and

$$\sum_{r=0}^{\infty} a_n (x_0 + h)^n = \sum_{r=0}^{\infty} b_n h^n$$

for all $|h| < R - |x_0|$.

Exercise 2.8.3. *Here is another metric on $C^\infty([a, b])$ that is sometimes used.*

(i) If $u, v \geq 0$ show, by direct calculation, or otherwise, that

$$u(1 + v)(1 + u + v) + v(1 + u)(1 + u + v) - (u + v)(1 + u)(1 + v) \geq 0$$

and deduce that

$$\frac{u}{1 + u} + \frac{v}{1 + v} \geq \frac{u + v}{1 + u + v}.$$

(ii) If $f, g \in C^\infty([a, b])$, explain why the sum on the left hand side of the definition

$$\rho(f, g) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|f^{(n)} - g^{(n)}\|_\infty}{1 + \|f^{(n)} - g^{(n)}\|_\infty}$$

converges.

(iii) Show that ρ *is a metric on* $C^\infty([a, b])$.

(iv) Let d *be the metric defined in Exercise 2.1.6. Show that* d *and* ρ *are Lipschitz equivalent.*

Exercise 2.8.4. [Local Taylor's theorem] *(i) Let* $\eta > 0$. *Suppose that* $f : (-\eta, \eta)$ *is* n *times differentiable with* $f(0) = f'(0) = \cdots = f^n(0) = 0$. *If* $|h| < \eta$, *show, by using the mean value theorem repeatedly, that*

$$|f(h)| \leq |h|^r |f^{(r)}(\theta_r h)|$$

for some θ_r *with* $0 < \theta_r < 1$ $[1 \leq r \leq n]$.

(ii) If, further, $f^{(n)}$ *is continuous at* 0, *show that*

$$h^{-n} f(h) \to 0$$

as $h \to 0$.

(iii) Deduce that, if $g : (-\eta, \eta)$ *is* n *times differentiable with continuous* n*th derivative at* 0, *then*

$$h^{-n}\left(g(h) - \sum_{r=0}^{n} \frac{g^r(0)}{r!} h^r\right) \to 0$$

as $h \to \infty$. *Conclude that*

$$g(h) = \sum_{r=0}^{n} \frac{g^r(0)}{r!} h^r + \epsilon(h) h^n$$

with $\epsilon(h) \to 0$ *as* $h \to 0$.

Exercise 2.8.5. *If* $x \in \mathbb{R}$, *write*

$$[x] = \{t \in \mathbb{R} : (2\pi)^{-1}(t - x) \in \mathbb{Z}\}.$$

We take $\mathbb{T} = \{[x] : x \in \mathbb{R}\}$.

(i) Show that, if $[x] = [x']$ *and* $[y] = [y']$, *then* $[x + y] = [x' + y']$. *Conclude that we can write*

$$[x] + [y] = [x + y]$$

without ambiguity.

(ii) Explain why, if $x \in \mathbb{R}$,

$$\min\{|x - 2\pi n| : n \in \mathbb{Z}\}$$

is well defined. Show that, if $[x] = [x']$, *then*

$$\min\{|x - 2\pi n| : n \in \mathbb{Z}\} = \min\{|x' - 2\pi n| : n \in \mathbb{Z}\}.$$

Conclude that we can write

$$\|[x]\| = \min\{|x - 2\pi n| \,;\, n \in \mathbb{Z}\}$$

without ambiguity.

(iii) Show that $(\mathbb{T}, +)$ *is an Abelian group.*

(iv) Show that $d([x],[y]) = \|[x-y]\|$ *defines a complete metric on* $\mathbb{T}$.

(v) Show that

$$S^1 = \{z \in \mathbb{C} \,:\, |z| = 1\}$$

is a group under ordinary multiplication. Show that the formula $f([t]) = e^{it}$ *gives a well defined map* $f : \mathbb{T} \to S^1$. *Show that* f *is bijective, that* f *is a group isomorphism and that* f *and* f^{-1} *are continuous if we give* S^1 *the usual metric (i.e.* f *is a homeomorphism).*

(vi) If $a,\ b \in \mathbb{R}$ *and* $0 < b - a \leq 2\pi$ *we write*

$$(a, b] = \{[x] \,:\, a < x \leq b\}$$

and say that $I = (a, b]$ *is a left half open interval of length* $|I| = b - a$. *Show that, if* $0 < c - d \leq 2\pi$, *then* $(c, d] = (a, b]$ *if and only if there exists an integer n with* $c = a + 2\pi n$, $d = b + 2\pi n$. *Explain why this means that (if* $0 < b - a \leq 2\pi$*) we can define the length* $|I|$ *of the interval* $I = (a, b]$ *by* $|I| = b - a$.

(vii) If I, J and K are half open intervals with

$$I \cup J = K \text{ and } I \cap J = \varnothing,$$

show that $|I| + |J| = |K|$.

Exercise 2.8.6. *Use induction to show that*

$$1 + 2\sum_{r=0}^{n} \cos rs = \frac{\sin\left((n + \frac{1}{2})s\right)}{\sin \frac{1}{2}s}$$

when $s \neq 0$.

Exercise 2.8.7. [Summation methods] *We work in the space of bounded sequences* $l^\infty(\mathbb{N})$ *with norm given by* $\|\mathbf{a}\| = \sup_\infty |a_n|$. *(If you do not already know this, check that we have a complete normed space.) Suppose that* $b_{jn} \in \mathbb{F}$ *is such that* $\sum_{j=1}^{\infty} b_{jn}$ *converges absolutely for each n. We write* $T_n\mathbf{a} = \sum_{j=1}^{\infty} b_{jn}a_j$ *for each* $\mathbf{a} \in l^\infty$ *and* $n \geq 1$. *We say that we have a* summation method *if* $T_n\mathbf{a} \to \alpha$ *whenever* $a_j \to \alpha$ *as* $j \to \infty$.

Show that s the set of convergent sequences is a closed subspace of l^∞ *and thus complete. If we write* S_n *for the restriction of* T_n *to s, show that* $S_n : s \to \mathbb{F}$ *is linear and continuous with* $\|S_n\| = \sum_{j=1}^{\infty} |b_{jn}|$. *By using the principle of uniform* *boundedness for the restriction of T_N to s_0 show that, if we have a summation method, then*

(a) There exists a K such that $\sum_{j=1}^{\infty} |b_{jn}| \leq K$ for all j.

Show further, by elementary means, that

(b) $\sum_{j=1}^{\infty} b_{jn} \to 1$ as $n \to \infty$ and

(b) $a_{jn} \to 0$ as $n \to \infty$ for each j.

Show that these three necessary conditions are also sufficient for a summation method. (The necessity and sufficiency was first proved by Toeplitz.)

Check that taking $b_{jn} = 1/n$ for $1 \leq j \leq n$, $b_{jn} = 0$, otherwise, gives a summation method. Deduce that, provided the Fourier coefficients of f are defined, we have, using the notation of Theorem 2.3.1,

$$S_n(f,t) \to f(t) \Rightarrow \sigma_n(f,t) \to f(t)$$

as $n \to \infty$.

Exercise 2.8.8. *(In this question you may interpret the statement that g is an integrable function in any reasonable way which allows for an integrable function not to be continuous. Most people will use 'Riemann integrable' or 'Lebesgue integrable'.) Suppose that $L_n : \mathbb{T} \to \mathbb{R}$ is an integrable function with the following properties.*

(i) $\frac{1}{2\pi}\int_{\mathbb{T}} L_n(t)\, dt = 1$.

(ii) $L_n(t) \geq 0$ for all t.

(iii) If $\eta > 0$, then $L_n(t) \to 0$ uniformly for all $|t| \geq \eta$.

Show that, if $f : \mathbb{T} \to \mathbb{C}$ is a bounded integrable function, then

$$L_n * f(t) \to f(t)$$

at every point t where f is continuous.

Exercise 2.8.9. *Suppose that $L_n \in C(\mathbb{T})$.*

(i) Show that

$$\frac{1}{2\pi}\int_{\mathbb{T}} L_n(t) f(t)\, dt \to f(0)$$

as $n \to \infty$, for each $f \in C(\mathbb{T})$, if and only if the following three conditions hold. (Compare Exercise 2.8.7.)

(a) There exists a constant A such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |L_n(t)|\, dt \leq A \text{ for all } n \geq 1.$$

(b) We have $\frac{1}{2\pi}\int_{\mathbb{T}} L_n(t)\, dt \to 1$ as $n \to \infty$.

(c) If $\eta > 0$, we have

$$\frac{1}{2\pi}\int_{|t|\geq\eta} L_n(t)f(t)\,dt \to 0$$

as $n \to \infty$, for every $f \in C(\mathbb{T})$.

(ii) Show that condition (c) is implied by the following condition.

(c)′ If $\eta > 0$, we have

$$\sup_{|t|\geq\eta} |L_n(t)| \to 0$$

as $n \to \infty$.

Show by means of an example that (a), (b) and (c)′ do not imply

$$\frac{1}{2\pi}\int_{\mathbb{T}} |L_n(t)|\,dt \to 1$$

as $n \to \infty$. (You could consider $L_n = K_n + \lambda_n D_n$ for an appropriate λ_n.)

Exercise 2.8.10. *Although Fejér's theorem is a particularly happy thought, we can use the same idea to show that the trigonometric polynomials are uniformly dense without finding such a neat kernel.*

(i) If $f : \mathbb{T} \to \mathbb{C}$ is continuous and p is a trigonometric polynomial, show that, if

$$p * f(t) = \frac{1}{2\pi}\int_{\mathbb{T}} p(t-s)f(s)\,ds,$$

then $p * f$ is a trigonometric polynomial.

(ii) If $A > 0$ and N is a positive integer, define $K_{A,N} : \mathbb{T} \to \mathbb{C}$ by

$$K_{A,N}(t) = A(2 + \cos t)^N.$$

Show that, given any $\epsilon > 0$, we can find $A > 0$ and N such that

$$K_{A,N}(t) \geq 0 \qquad \text{for all } t,$$
$$|K_{A,N}(t)| \leq \epsilon \qquad \text{whenever } |t| \geq \epsilon,$$

and

$$\frac{1}{2\pi}\int_{\mathbb{T}} K_{A,N}(t)\,dt = 1.$$

(iii) By considering functions of the form

$$P(t) = \frac{1}{2\pi}\int_{\mathbb{T}} K_{A,N}(t-s)f(s)\,ds,$$

show that any continuous function $f : \mathbb{T} \to \mathbb{C}$ can be uniformly approximated by trigonometric polynomials.

Exercise 2.8.11. [Weyl's theorem] *We work on $C(\mathbb{T})$ with the uniform norm (except in part (iv)). The result, which is of importance in itself, also looks forward to the Ergodic Theorem in more advanced work.*

(i) Let $\theta \in \mathbb{T}$ and set

$$W_n(f,t) = \frac{f(t) + f(t+\theta) + f(t+2\theta) + \ldots + f(t+n\theta)}{n+1}.$$

for all $t \in \mathbb{T}$. Explain why W_n maps $C(\mathbb{T})$ to $C(\mathbb{T})$. Show that W_n is a linear map and that, if $f(t) \geq 0$ for all $t \in \mathbb{T}$, then $W_n(f,t) \geq 0$ for all $t \in \mathbb{T}$. (Thus W_n is a positive linear functional.) Show that $\|W_n f\|_\infty \leq \|f\|_\infty$.

(ii) Suppose that $\theta/(2\pi)$ is irrational. By computing the sum of a geometric series, show that, if $e_m(t) = \exp(imt)$, with m an integer, then

$$W_n(e_m,t) \to \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

uniformly in t as $n \to \infty$.

(iii) Now show that

$$W_n(P,t) \to \hat{P}(0)$$

uniformly if P is a trigonometric polynomial and deduce that

$$W_n(f,t) \to \hat{f}(0)$$

uniformly if f is a continuous function.

(iv) We now consider the discontinuous function $\mathbb{I}_{[a,b)}$ with $0 \leq a < b < 2\pi$. By considering appropriate continuous f and g with

$$f(t) \geq \mathbb{I}_{[a,b)} \geq g(t)$$

for all t, or otherwise, show that

$$W_n(\mathbb{I}_{[a,b)},t) \to \frac{b-a}{2\pi}$$

uniformly as $n \to 0$. In particular, if $N(n)$ is the number of integer r with $0 \leq r \leq n$ and $r\theta \in [a,b)$, show that

$$\frac{N(n)}{n+1} \to \frac{|b-a|}{2\pi}$$

as $n \to \infty$.

(v) (This is easy, but provides background.) So far, we have taken $\theta/(2\pi)$ irrational. Show that if $\theta/(2\pi)$ is rational, it will still be true that $W_n f$ converges uniformly to some continuous function g, but that we can always find an f such that g is not constant.

Exercise 2.8.12. *It is easy to obtain a weak quantitative statement of the idea that well behaved functions are easier to approximate. Show, by integrating by parts, that, if $f : \mathbb{T} \to \mathbb{R}$ is k times continuously differentiable, then*

$$|\hat{f}(r)| \leq A_k |r|^{-k} \|f^{(r)}\|_\infty$$

for all $r \neq 0$ and some constant A_k independent of f.

Deduce that, if $k \geq 2$,

$$\|S_n(f) - f\|_\infty \leq B_k |n|^{2-k}$$

for all $n \neq 0$ and some constant B_k independent of f.

Exercise 2.8.13. *By considering $\sum_{j=0}^{\infty} 2^{-j} e^{iN(j)t}$ for suitable $N(j)$, show that the Riemann–Lebesgue lemma of Exercise 2.3.6 cannot be improved. Specifically, show that, if $\kappa(n)$ is a sequence of real numbers with $\kappa(n) \to \infty$ as $n \to \infty$, then we can find an $f \in \mathbb{C}(T)$ such that $\limsup_{n\to\infty} \kappa(n)|\hat{f}(n)| = \infty$.*

Is it true that, given $\kappa(n)$ as above, we can find a once continuously differentiable function g such that $\limsup_{n\to\infty} \kappa(n)|\hat{f}(n)| = \infty$? Give a proof or counter-example.

Exercise 2.8.14. *Let $f \in C(\mathbb{T})$. Prove the following results (a) by direct calculation and (b) by first proving them for $f(t) = \exp(imt)$, then for f a trigonometric polynomial and then, by a limiting argument, for all continuous f.*

(i) If $g(t) = f(-t)$, then $\hat{g}(n) = \hat{f}(-n)$.

(ii) If $f_a(t) = f(t-a)$, then $\hat{f}_a(n) = \exp(ina)\hat{f}(n)$.

(iii) If j is an integer and $f_j(t) = f(jt)$, then

$$\hat{f}_j(n) = \begin{cases} \hat{f}(r) & \text{if } rj = n, \\ 0 & \text{otherwise.} \end{cases}$$

[*It may be found helpful to observe that $\int_0^{2\pi} h(t)\,dt = \sum_{k=1}^{j} \int_{2(k-1)\pi/j}^{2k\pi/j} h(t)\,dt$.*]

Deduce that

$$\frac{1}{2\pi}\int_{\mathbb{T}} e^{ims} f(t - js)\,ds = \begin{cases} e^{irt}\hat{f}(r) & \text{if } rj = m \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.8.15. *(i) If $f \in C(\mathbb{T})$ and P is trigonometric polynomial of degree at most n, show that*

$$\|f - P\|_\infty \geq \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(t) - P(t)|^2\,dt\right)^{1/2} \geq |\hat{f}(m)|$$

for any $|m| > n$.

(ii) We write $\mathcal{P}_n$ *for the set of trigonometric polynomials of degree at most n. Let* $0 < n(1) < n(2) < \dots$. *If*

$$f(t) = \sum_{j=0}^{\infty} 2^{-j} n(j)^{-1} \cos n(j)t$$

show that f is a well defined continuously differentiable function with $\|f'\|_\infty \leq 1$ *and*

$$\inf_{P \in \mathcal{P}_{n(j)-1}} \|f - P\|_\infty \geq 2^{-j} n(j)^{-1}.$$

Deduce that, if κ_n *is a sequence of strictly positive numbers with* $\kappa_n n \to \infty$, *there exists a continuously differentiable function g with*

$$\limsup_{n \to \infty} \kappa_n \inf_{P \in \mathcal{P}_n} \|f - P\|_\infty = \infty.$$

In this sense, Jackson's first theorem is best possible.

(iii) Show that Jackson's second theorem is best possible in a similar sense.

Exercise 2.8.16. *Use induction in the manner of Exercise 2.5.8 (note that you will need to make a fair number of changes) to prove the following version of Jackson's second theorem for polynomials.*

There exists a constant C_k *with the following property. If* $f : [0, 1] \to \mathbb{R}$ *is k times continuously differentiable then, given* $n \geq 1$, *we can find a real trigonometric polynomial P of degree at most n such that*

$$\|P - f\|_\infty \leq C_k n^{-k} \|f^{(k)}\|_\infty.$$

Exercise 2.8.17. [Bernstein's proof of the Weierstrass Approximation Theorem] *(This exercise requires elementary probability theory.) Suppose that* $f : [0, 1] \to \mathbb{R}$ *is continuous. Let* $0 \leq t \leq 1$ *and let* $X_1, X_2, \dots, X_n$ *be independent identically distributed random variables with*

$$\Pr(X_j = 0) = 1 - t, \ \Pr(X_j = 1) = t.$$

Set $Y_n = \sum_{j=1}^n X_j$. *(Thus* Y_n *corresponds to the number of heads obtained in n throws of a coin which has probability t of coming down heads.) Let* $Z_n = n^{-1} Y_n$.

(i) If $\eta > 0$, *justify the following sequence of inequalities.*

$$\begin{aligned} |\mathbb{E} f(Z_n) - f(t)| &\leq \mathbb{E}|f(Z_n) - f(t)| \\ &\leq \Pr(|f(Z_n) - t| \leq \eta) \sup_{|t-s| \leq \eta} |f(s) - f(t)| + 2\|f\|_\infty \Pr(|f(Z_n) - t| > \eta) \\ &\leq \sup_{|t-s| \leq \eta} |f(s) - f(t)| + 2\|f\|_\infty \Pr(|f(Z_n) - t| > \eta). \end{aligned}$$

(ii) By using uniform continuity and Chebychev's inequality, or otherwise, show that, given $\epsilon > 0$, we can find an n_0 such that

$$|\mathbb{E}f(Z_n) - f(t)| < \epsilon$$

for all $t \in [0, 1]$ and all $n \geq n_0$.

(iii) Conclude that, if $n \geq n_0$,

$$\left| \sum_{j=0}^{n} \binom{n}{j} f(j/n) t^j (1-t)^{n-j} - f(t) \right| < \epsilon.$$

Thus

$$\sum_{j=0}^{n} t^j \binom{n}{j} f(j/n)(1-t)^{n-j} \to f(t)$$

uniformly as $n \to \infty$ and we have an explicit formula for polynomials of the type required the Weierstrass theorem.

Exercise 2.8.18. *Here is an alternative proof of Bernstein's theorem using a different set of ideas.*

(i) Let $f \in C([0, 1])$. Show that, given $\epsilon > 0$, we can find an $A > 0$ such that

$$f(x) + A(t - x)^2 + \epsilon/2 \geq f(t) \geq f(x) - A(t - x)^2 - \epsilon/2$$

for all $t,\ x \in [0, 1]$.

(ii) Now show that we can find an N such that, writing

$$h_r(t) = f(r/N) + A(t - r/N)^2,\ g_r(t) = f(r/N) - A(t - r/N)^2,$$

we have

$$g_r(t) + \epsilon \geq f(t) \geq h_r(t) - \epsilon$$

for $|t - r/N| \leq 1/N$. (You may find it helpful to draw diagrams here and in (iii).)

(iii) We say that a linear map $S : C([0, 1]) \to C([0, 1])$ is positive if $F(t) \geq 0$ for all $t \in [0, 1]$ implies $SF(t) \geq 0$ for all $t \in [0, 1]$. Suppose that S is such a positive linear operator. Show that, if $F(t) \geq G(t)$ for all $t \in [0, 1]$, then $(SF)(t) \geq (SG)(t)$ for all $t \in [0, 1]$ $[F,\ G \in C([0, 1])]$. Show also that, if $F \in C([0, 1])$, then $\|SF\|_\infty \leq \|S1\|_\infty \|F\|_\infty$.

(iv) Write $e_r(t) = t^r$. Suppose that S_n is a sequence of positive linear functions such that $\|S_n e_r - e_r\|_\infty \to 0$ as $n \to \infty$ for $r = 0$, $r = 1$ and $r = 2$. Show, using (ii), or otherwise, that $\|S_n f - f\|_\infty \to 0$ as $n \to \infty$ for all $f \in C([0, 1])$.

(v) Let

$$(S_n f)(t) = \sum_{j=0}^{n} \binom{n}{j} f(j/n)(1-t)^j t^{n-j}.$$

Verify that S_n satisfies the hypotheses of part (iv) and deduce Bernstein's theorem.

Exercise 2.8.19. *(i) By consideing the effect of taking imaginary parts in Exercise 2.6.3 (ii), or otherwise, show that there exist polynomials U_{n-1} of degree $n-1$ such that*

$$U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}$$

for $\sin\theta \neq 0$, $U_{n-1}(0) = n$. *(The U_{n-1} are called Chebychev polynomials of the second kind.)*

(ii) Compute the Chebychev polynomials T_n of the first kind and the Chebychev polynomials U_{n-1} of the second kind for $n = 1, 2 \ldots, 5$.

(iii) Explain why we know, without calculation, that the Chebychev polynomials T_n are even when n is even and odd when n is odd. What can you say about the Chebychev polynomials U_n of the second kind? (A function f is called even *if $f(-x) = f(x)$ for all x and* odd *if $f(-x) = -f(x)$ for all x.)*

(iv) Show that

$$T_{n+1}(t) - 2T_n(t) + T_{n-1}(t) = 0.$$

Use this formula to check your calculations of the Chebychev polynomials in (ii) and to give another proof that the coefficient of t^n in $T_n(t)$ is 2^{n-1} for $n \geq 1$.

Exercise 2.8.20. *(i) Let*

$$G_n(t) = \left(\int_1^1 (1-s^2)^n\,ds\right)^{-1} (1-t^2)^n.$$

Show that G_n is a polynomial and that, if $\eta > 0$,

$$G_n(t) \to 0$$

uniformly for $\eta \leq |t| \leq 1$.

(ii) Let $H_n : [-1,1] \to \mathbb{R}$ be given by

$$H_n(t) = \int_{-1}^t G_n(s)\,ds.$$

Show that H_n is a polynomial, that $0 \leq H_n(t) \leq 1$ for all $t \in [-1,1]$ and that, if $\eta > 0$,

$$H_n(t) \to 0$$

uniformly for $-1 \leq t \leq -\eta$, whilst

$$H_n(t) \to 1$$

uniformly for $\eta \leq t \leq 1$.

(iii) Let $P_n : [-1, 1] \to \mathbb{R}$ *be given by*

$$P_n(t) = \int_0^t 2H_n(s) - 1\,ds.$$

Show that P_n *is a polynomial and*

$$P_n(t) \to |t|$$

uniformly on $[-1, 1]$ *as* $n \to \infty$.

Exercise 2.8.21. *(i) Let*

$$a_0 = 1,\ a_1 = -\frac{1}{2},\ \dots,\ a_r = -\frac{1}{r!}\left(\frac{1}{2}\prod_{k=1}^{r-1}\frac{2k-1}{2}\right),\ \dots.$$

Show that the power series $\sum_{r=0}^{\infty} a_r x^r$ *has radius of convergence* 1.

(ii) By multiplying term by term within the circle of convergence, show that

$$\left(\sum_{r=0}^{\infty} a_r x^r\right)^2 = 1 - x$$

for $-1 < x < 1$. *By considering what happens at* $x = 0$ *and using a continuity argument, or otherwise, show that*

$$\sum_{r=0}^{\infty} a_r x^r = (1 - x)^{1/2}$$

for all $-1 < x < 1$.

(iii) (An alternative to (ii).) Write $f(x) = \sum_{r=0}^{\infty} a_r x^r$. *By differentiating term by term within the circle of convergence, show that*

$$-2(1 - x)f'(x) = f(x).$$

Deduce that

$$\frac{d}{dx}\frac{f(x)}{(1 - x)^{1/2}} = 0$$

for $|x| < 1$ *and then that* $f(x) = (1 - x)^{1/2}$ *for* $|x| < 1$.

(iv) (An improvement.) Show that $\sum_{r=0}^{\infty} |a_r|$ *converges and deduce that* $\sum_{r=0}^{\infty} a_r x^r$ *converges uniformly to a continuous function on* $[-1, 1]$. *Deduce that*

$$\sum_{r=0}^{n} a_r x^r \to (1 - x)^{1/2}$$

uniformly on $[-1, 1]$.

(v) Show, using (iv), that

$$\sum_{r=0}^{n} a_r(1 - x^2) \to |x|$$

uniformly on $[-1, 1]$ *and deduce Lemma 2.6.9.*

Exercise 2.8.22. [Divided differences] *(i) Let* $f : \mathbb{R} \to \mathbb{R}$ *be given. Explain why, if* $x_0,\ x_1,\ \dots,\ x_n$ *are distinct real numbers, there is a unique polynomial* P *of degree at most* n *such that* $P(x_j) = f(x_j)$ *for* $0 \le j \le n$.

We define $f[x_0, x_1, \dots, x_n]$ *to be the coefficient of* t^n *in* $P(t)$.

(ii) Continuing with the ideas above, suppose that P_1 *and* P_2 *are polynomials of degree at most* $n - 1$ *with* $P_1(x_j) = f(x_j)$ *for* $0 \le j \le n - 1$ *and* $P_2(x_j) = f(x_j)$ $1 \le j \le n$. *Show that*

$$P(t) = \frac{(x_n - t)P_1(t) + (t - x_1)P_2(t)}{x_n - x_1}$$

defines a polynomial P *of degree at most* n *such that* $P(x_j) = f(x_j)$ *for* $0 \le j \le n$. *Deduce that*

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_1}.$$

(iii) Suppose now that f *is* $n + 1$ *times differentiable and* P *is the polynomial of degree at most* n *such that* $P(x_j) = f(x_j)$ *for* $0 \le j \le n$. *Let*

$$E_x = f(x) - P(x)$$

and consider the function g *defined by*

$$g(t) = E_x \prod_{j=1}^{n+1} (t - x) - \big(f(t) - P(t)\big).$$

If $x,\ x_0\, x_1,\ \dots,\ x_n$ *are distinct points of* $[a, b]$, *show that* g *has at least* $n + 1$ *zeros in* $[a, b]$. *By using Rolle's theorem, show that* g' *has at least* n *zeros in* (a, b). *By repeating the argument many times, show that* $g^{(n)}$ *has at least* 1 *zero in* (a, b). *Deduce that there exists a* $\theta \in (a, b)$ *such that*

$$f^{(n+1)}(\theta) = (n + 1)!E_x.$$

(iv) Continuing with the notation and hypotheses of part (iii), show that

$$|f(t) - P(t)| \le \frac{1}{(n + 1)!} \sup_{s \in [a,b]} |f^{(n+1)}(s)|$$

for all $t \in [a, b]$.

(v) We now look at another similar estimate. Suppose that f *is* n *times differentiable, that* $x_1, \ldots, x_n$ *are distinct points of* $[a, b]$ *and* P *is the polynomial of degree at most* n *such that* $P(x_j) = f(x_j)$ *for* $0 \leq j \leq n$*. By looking at* $f(t) - P(t)$ *and applying Rolle's theorem show that*

$$n!|f[x_0, x_1, \ldots, x_n]| \leq \sup_{s\in[a,b]} |f^{(n)}(s)|.$$

(vi) By specialising to the case when $x_k = x + kh$*, show that, if* f *is* n *times continuously differentiable on* $[a, b]$ *and* $x,\ x + nh \in [a, b]$*, then*

$$\|\triangle_h^n f\|_\infty \leq n!|h|^n\|f^{(n)}\|_\infty.$$

By choosing an appropriate f*, show that this inequality cannot be improved. Note that, as Exercise 2.1.9 shows,* $\|f^{(n)}\|_\infty$ *can increase arbitrarily fast.*

(vii) Using induction, or otherwise, show that, in the case $x_k = x + kh$ *considered in (vi),*

$$\triangle_h^k f(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(x + jh).$$

Exercise 2.8.23. *In the next two exercises we give another proof of Jackson's second theorem using the notion of divided difference introduced in the previous exercise. Here we find a suitable kernel.*

Suppose that we set

$$J_{n,r}(t) = \gamma_n^{-1}K_n(t)^{2r} = \lambda_{n,r}^{-1}\left(\frac{\sin{(nt/2)}}{\sin t/2}\right)^{2r}$$

for $t \neq 0$*,* $J_n(0) = \lambda_n^{-1}n^2$*, where* $\gamma_{n,r}$ *and* $\lambda_{n,r}$ *are chosen so that*

$$\frac{1}{2\pi}\int_{\mathbb{T}} J_{n,r}(t)\,dt = 1.$$

Show that there exist constants $B_{n,r,j}$ *such that*

$$\frac{1}{2\pi}\int_{\mathbb{T}} |t|^j J_n(t)\,dt \leq B_{n,r,j}n^{-j}$$

for all $0 \leq j \leq 2r - 2$.

Exercise 2.8.24. *Suppose that* $f \in C(\mathbb{T})$ *is* k *times continuously differentiable. Continuing with the notation of the previous question, we set*

$$Q_n(t) = \frac{1}{2\pi}\int_{\mathbb{T}} J_{n,k}(s)\sum_{j=1}^{k}(-1)^j\binom{k}{j}f(t - js)\,ds.$$

Show that Q_n is a real trigonometric polynomial with degree at most $2(n-1)k$ and use parts (vi) and (vii) of Exercise 2.8.22 to show that

$$|Q_n(t) - f(t)| \leq C_k n^{-k}$$

for some constant C_k independent of f.

Exercise 2.8.25. *Let $f \in C(\mathbb{T})$ be once continuously differentiable. By integrating by parts, show that $|\hat{f}(n)| \leq |f'(n)||n|^{-1}$ for $n \neq 0$. By using the Cauchy–Schwarz inequality, show that*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = |\hat{f}(0)| + \sum_{n\neq 0} |n(f')\hat{\ }(n)||n|^{-1} \leq |\hat{f}(0)| + 2\|f'\|_2 \left(\sum_{n=1}^{\infty} n^{-2}\right)^{1/2}.$$

Conclude that

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \exp int$$

where the sum converges absolutely.

Exercise 2.8.26. [Morton's fork for uniform boundedness] *(Morton collected taxes for the king using one of two arguments. If a man lived well, he was obviously rich and, if he lived frugally, then he must have savings.) Suppose that $(U, \| \ \|_U)$ is a complete normed space, $(V, \| \ \|_V)$ a normed space and T_n is a sequence of continuous linear functions $T_n : U \to V$. Suppose further that there is a dense subset E of U such that $T_n\mathbf{e}$ converges. Show that either $T_n\mathbf{u}$ converges for all $\mathbf{u} \in U$ or $\|T_n\|$ is unbounded.*

Exercise 2.8.27. *The object of this exercise in to prove Theorem 2.7.6 from Theorem 2.7.5. We do this by looking at quotient norms. The reader should skip any parts of the exercise which are familiar to her.*

(i) Let U be a vector space over $\mathbb{F}$. If K is subspace show that

$$\mathbf{x} \sim \mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} \in K$$

is an equivalence relation. If V/K is the set of equivalence classes

$$[\mathbf{a}] = \{\mathbf{x} : \mathbf{x} \sim \mathbf{a}\},$$

show that

$$[\mathbf{a}] + [\mathbf{b}] = [\mathbf{a} + \mathbf{b}], \ \lambda[\mathbf{a}] = [\lambda\mathbf{a}]$$

give well defined operations and that, with these operations, V/K is a vector space.

(ii) If $(U, \| \|_U)$ is a normed vector space and we write

$$\|[\mathbf{u}]\|_{U/K} = \inf\{\|\mathbf{x}\|_U : \mathbf{x} \in [\mathbf{u}]\},$$

show that

$$|[\mathbf{u}]\|_{U/K} = 0 \Rightarrow [\mathbf{u}] = [\mathbf{0}]$$

if and only if K is closed.

(iii) Continuing with (ii), show that, if K is closed, then $\| \|_{U/K}$ is a norm on U/K.

(iv) Suppose now that K is closed and $\| \|_U$ is complete. If $\|[\mathbf{u}_n]_{U/K}\| \leq 2^{-n}$ show that we can find $\mathbf{x}_n \sim \mathbf{u}_n$ with $\|\mathbf{x}_n\|_U \leq 2^{-n+1}$. Deduce that $\sum_{j=1}^{\infty} \mathbf{u}_j$ converges and thence that $\sum_{j=1}^{\infty} [\mathbf{u}_j]$ converges. Explain why this means that $\| \|_{U/K}$ is complete.

(v) Suppose from now on $(U, \| \|_U$ and $(V, \| \|_V)$ are complete normed spaces and $T : U \to V$ is a continuous linear surjection. Show that K is closed and that the map $\tilde{T} : U/K \to V$ is a continuous linear bijection.

(vi) Theorem 2.7.5 now tells us that $\tilde{T}^{-1}$ is continuous. Deduce that T is open.

Exercise 2.8.28. *(i) Show that the rules*

$$d_m(f, g) = \sum_{n=1}^{\infty} \min\{\sup_{|t|\leq m} |f^{(n)}(t) - g^{(n)}(t)|, 1\} \text{ and } d(f, g) = \sum_{m=1}^{\infty} 2^{-m} d_m(f, g)$$

define a complete metric d on the space $C^\infty(\mathbb{R})$ of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

(ii) Show that, if we use this metric, quasi-all $f \in C^\infty(\mathbb{R})$ fail to be real analytic at every point of $\mathbb{R}$.

Exercise 2.8.29. *(i) We work on $\mathbb{T}$. If*

$$P(t) = \sum_{j=-n}^{n} a_j \exp(ijt)$$

write down $P(u + mt)$ in the same form. (Here $u \in \mathbb{T}$ and $m \in \mathbb{Z}$.)

(ii) If $f \in C(\mathbb{T})$ and $g(t) = f(u + mt)$, use the change of variable formula to find $\hat{g}(k)$. Check that you answer agrees with (i).

Chapter 3

Are there true functions of two variables?

3.1 Hilbert's thirteenth problem

The contents of this chapter allow us to show off some of the ideas we have developed.

Towards the end of the 19th century, table makers devised an ingenious graphical method called nomography by which the values of functions of two variables could be derived by combining tables of functions of one variables in an appropriate manner.

Inspired by this, Hilbert asked whether every function of many variables could be expressed using only functions of two variables.[1] To see what he meant, observe that, for example,

$$xyz = \exp\left(\log x + (\log y + \log z)\right).$$

Here we have expressed a function of three variables $(x, y, z) \mapsto xyz$ using only the functions of one variable $t \mapsto \exp t$, $t \mapsto \log t$ and the rather simple function of two variables $(s, t) \mapsto s + t$.

In the final paragraph of Section 5.5, I shall give some plausible ways in which we might obtain functions of several variables which cannot be expressed using functions of fewer variables. For the moment, the reader is invited to try to find such functions for herself.

Naturally, mathematicians expected that the answer to Hilbert's question would be negative and, equally naturally, no one in the succeeding half century managed

[1] Here, and throughout this chapter, I have preferred to simplify history rather than complicate the mathematical exposition. In particular, Hilbert may have had in mind an *algebraic* version of the problem which has still not been resolved.

to make the slightest progress with the problem. It therefore came as a tremendous surprise when Kolmogorov proved that any continuous function of many variables can be expressed using only continuous functions of three variables and, a year later, his nineteen year old pupil Arnol'd[2] resolved Hilbert's problem completely.

Theorem 3.1.1. *If $n \geq 2$, every continuous function $f : [0,1]^n \to \mathbb{R}$ can be written in terms of continuous functions of two variables.*

Kolmogorov's original proof was extremely abstract, but, as the result and proof passed through various hands, the result became more and more explicit and the proof simplified. It turned out that the only function of two variables required was addition.

Theorem 3.1.2. *If $n \geq 2$, every continuous function $f : [0,1]^n \to \mathbb{R}$ can be written in terms of continuous functions of one variable and addition.*

In other words, there are no true functions of many variables! We shall prove the result for $n = 2$, but only trivial changes are needed to obtain the result for general $n \geq 2$.

Our object is thus to prove the following result.

Theorem 3.1.3. *Every continuous function $f : [0,1]^2 \to \mathbb{R}$ can be written in terms of continuous functions of one variable and addition.*

In fact we shall prove much more.

Theorem 3.1.4. *Let λ be irrational. We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[0 \leq j \leq 4]$ with the following property. Given any continuous function $f : [0,1]^2 \to \mathbb{R}$ we can find a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x,y) = \sum_{j=0}^{4} g(\phi_j(x) + \lambda\phi_j(y)).$$

Exercise 3.1.5. *(i) Explain to a beginning student of mathematics why Theorem 3.1.4 is much stronger than Theorem 3.1.3.*

(ii) Explain to a beginning student of mathematics why Theorem 3.1.4 might be easier to prove than Theorem 3.1.3.

[2]For those who wish to dispense with accents, Arnold.

3.2 Preliminary steps

The argument we use to prove Theorem 3.1.4 is the work of many authors, but took its final form in a paper of Kahane. We start by using one of our standard simple approximation arguments to reduce Theorem 3.1.4 to a less baffling form.

Lemma 3.2.1. *Let λ be irrational. We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[0 \leq j \leq 4]$ with the following property. Given any continuous function $F : [0,1]^2 \to \mathbb{R}$, we can find a continuous function $G : [-1-|\lambda|, 1+|\lambda|] \to \mathbb{R}$ such that $\|G\|_\infty \leq \|F\|_\infty$ and*

$$\sup_{(x,y)\in[0,1]^2} \left| F(x,y) - \sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) \right| \leq \frac{99}{100}\|F\|_\infty.$$

(Of course the choice of $99/100$ is not important.)

Proof of Theorem 3.1.4 from Lemma 3.2.1. This is just a successive approximation argument. Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function and write $f_0 = F$. We now proceed inductively. Once the continuous function $f_{n-1} : [0,1]^2 \to \mathbb{R}$ has been defined, Lemma 3.2.1 tells us that we can find a continuous function $g_n : [-1-|\lambda|, 1+|\lambda|] \to \mathbb{R}$ such that

$(1)_n$ $\|g_n\|_\infty \leq \|f_{n-1}\|_\infty$,

$(2)_n$ $\displaystyle\sup_{(x,y)\in[0,1]^2} \left| f_{n-1}(x,y) - \sum_{j=0}^{4} g_n(\phi_j(x) + \lambda\phi_j(y)) \right| \leq \frac{99}{100}\|f_{n-1}\|_\infty.$

We now set

$(3)_n$ $\displaystyle f_n(x,y) = f_{n-1}(x,y) - \sum_{j=0}^{4} g_n(\phi_j(x) + \lambda\phi_j(y)).$

Using $(2))_n$ and $(3)_n$, we see that

$(4)_n$ $\displaystyle\|f_n\|_\infty \leq \left(\frac{99}{100}\right)^n \|f_0\|_\infty,$

so, by $(1)_n$

$(5)_n$ $\displaystyle\|g_n\|_\infty \leq \left(\frac{99}{100}\right)^{n-1} \|f_0\|_\infty$

and, using the Weierstrass M-test, it follows that $\sum_{n=1}^{\infty} g_n$ converges uniformly to a continuous function G, say.

The definition $(3)_n$ gives us

$$\begin{aligned}\sum_{j=0}^{4}\sum_{n=1}^{N} g_n(\phi_j(x)+\lambda\phi_j(y)) &= \sum_{n=1}^{N}\sum_{j=0}^{4} g_n(\phi_j(x)+\lambda\phi_j(y))\\ &= \sum_{n=1}^{N} f_{n-1}(x,y)-f_n(x,y)\\ &= f_0(x,y)-f_N(x,y).\end{aligned}$$

Allowing $N\to\infty$, we obtain

$$\sum_{j=0}^{4} G(\phi_j(x)+\lambda\phi_j(y)) = f_0(x,y) = f(x,y)$$

as required. ■

Our next simplification requires the following observation.

Lemma 3.2.2. *The space* $(C([0,1]^2), \|\ \|_\infty)$ *is separable.*

There are lots of ways of proving this result. We outline one in the next exercise, but the reader may wish to wait for Exercise 4.4.12 in the next chapter.

Exercise 3.2.3. *(i) Suppose that* $f:[0,1]^2\to\mathbb{R}$ *is continuous. Show that, if we use a two dimensional generalisation of the notion of a piecewise linear function and set*

$$\begin{aligned}&f_n((u+s)2^{-n},(v+t)2^{-n})\\ &\quad=(1-s)(1-t)f(u2^{-n},v2^{-n})+s(1-t)f((u+1)2^{-n},v2^{-n})\\ &\qquad+(1-s)tf(u2^{-n},(v+1)2^{-n})+stf((u+1)2^{-n},(v+1)2^{-n})\end{aligned}$$

for u, v *integers and* s, t *real numbers with* $0\le u,v\le n-1$ *and* $0\le s,t\le 1$, *then* $f_n:[0,1]^2\to\mathbb{R}$ *is a well defined continuous function such that* $f_n\to f$ *uniformly.*

(ii) Let $\mathcal{A}_n$ *consist of those functions such that*

$$\begin{aligned}&g((u+s)2^{-n},(v+t)2^{-n})\\ &\quad=(1-s)(1-t)g(u2^{-n},v2^{-n})+s(1-t)g((u+1)2^{-n},v2^{-n})\\ &\qquad+(1-s)tg(u2^{-n},(v+1)2^{-n})+stg((u+1)2^{-n},(v+1)2^{-n})\end{aligned}$$

for u, v *integers and* s, t *real numbers with* $0\le u,v\le n-1$ *and* $0\le s,t\le 1$ *and satisfying the following further condition*

$$2^n g(u2^{-n},v2^{-n})\in\mathbb{Z}$$

for all $0\le u,v\le n$. *Show that* $\bigcup_{n=1}^{\infty}\mathcal{A}_n$ *is a countable, dense, subset of* $(C([0,1]^2),\|\ \|_\infty)$.

We can now show that Lemma 3.2.1 can be reduced to the following slightly more manageable form.

Lemma 3.2.4. *Let λ be irrational and let $F_1, F_2, \dots$ be a given sequence of non-zero functions in $C([0,1]^2)$. We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[0 \le j \le 4]$ with the following property. There exist continuous functions $G_n : [-1-|\lambda|, 1+|\lambda|] \to \mathbb{R}$ such that $\|G_n\|_\infty \le \|F_n\|_\infty$ and*

$$\sup_{(x,y)\in[0,1]^2} \left| F_n(x,y) - \sum_{j=0}^{4} G_n(\phi_j(x) + \lambda\phi_j(y)) \right| < \frac{9}{10}\|F_n\|_\infty.$$

Proof of Lemma 3.2.1 from Lemma 3.2.4. Choose a dense sequence F_n in $C([0,1]^2, \| \|_\infty$. Now suppose we are given an $F \in C([0,1]^2)$. If $F = 0$, we may take $G = 0$ and there is nothing to prove. If not, then we can find an N such that

$$\|F_N - \tfrac{9}{10}F\|_\infty \le \tfrac{1}{20}\|F\|_\infty.$$

(Note that this implies that $\|F_N\|_\infty \le \|F\|_\infty$.) We now have

$$\begin{aligned}
&\sup_{(x,y)\in[0,1]^2} \left| F(x,y) - \sum_{j=0}^{4} G_n(\phi_j(x) + \lambda\phi_j(y)) \right| \\
&\le \sup_{(x,y)\in[0,1]^2} \left| F_N(x,y) - \sum_{j=0}^{4} G_N(\phi_j(x) + \lambda\phi_j(y)) \right| + \|F_N - \tfrac{9}{10}F\|_\infty \\
&\le \frac{9}{10}\|F_N\|_\infty + \frac{1}{20}\|F\|_\infty \le \frac{9}{10}\|F\|_\infty + \frac{1}{20}\|F\|_\infty \le \frac{98}{100}\|F\|_\infty
\end{aligned}$$

and we are done. (Once again the particular fractions chosen are fairly arbitrary.) ■

3.3 A Baire category argument

In order to apply a Baire category argument, we need an appropriate metric space.

Exercise 3.3.1. *(i) Consider the space Y of continuous functions $\boldsymbol{\phi} : [0,1] \to \mathbb{R}^5$. Show that*

$$d(\boldsymbol{\phi}, \boldsymbol{\psi}) = \max_{0\le j\le 4} \|\phi_j - \psi_j\|_\infty$$

is a well defined metric on Y. Show that d is a complete metric.

(ii) Show that the set X of $\boldsymbol{\phi} \in Y$ such that ϕ_j is increasing (that is to say $\phi_j(s) \le \phi_j(t)$ for $0 \le s \le t \le 1$) for each $0 \le j \le 4$ is closed in X. Conclude that X with metric d is complete.

We can now give a Baire category version of Lemma 3.2.4.

Lemma 3.3.2. *Let (X, d) be as in Exercise 3.3.1. Let λ be irrational and let F_1, F_2, ... be a given sequence of non-zero functions in $C([0, 1]^2)$. Then quasi-all $\boldsymbol{\phi} \in X$ have the property that there exist continuous functions $G_n : [-1-|\lambda|, 1+|\lambda|] \to \mathbb{R}$ such that $\|G_n\|_\infty \leq \|F_n\|_\infty$ and*

$$\sup_{(x,y)\in[0,1]^2} \left| F_n(x, y) - \sum_{j=0}^{4} G_n(\phi_j(x) + \lambda\phi_j(y)) \right| < \frac{9}{10}\|F_n\|_\infty.$$

At first sight, Lemma 3.3.2 looks harder to prove than Lemma 3.2.4, but the fact that the countable union of meagre sets remains meagre means that Lemma 3.3.2 follows from rather simpler lemma.

Lemma 3.3.3. *Let (X, d) be as in Exercise 3.3.1. Let λ be irrational and let $F \in C([0, 1]^2$ be a non-zero function. The set E of functions $\boldsymbol{\phi} \in X$ with the property that there exists a continuous function $G : [-1-|\lambda|, 1+|\lambda|] \to \mathbb{R}$ such that $\|G\|_\infty \leq \|F\|_\infty$ and*

$$\sup_{(x,y)\in[0,1]^2} \left| F(x, y) - \sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) \right| < \frac{9}{10}\|F\|_\infty$$

is open and dense in (X, d).

Part of Lemma 3.3.3 is easy to prove.

Lemma 3.3.4. *The set E described in Lemma 3.3.3 is open.*

Proof. If $\boldsymbol{\phi} \in E$, then, setting

$$\eta = \frac{1}{2}\left(\frac{9}{10}\|F\|_\infty - \sup_{(x,y)\in[0,1]^2} \left| F(x, y) - \sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) \right| \right),$$

we know that $\eta > 0$. Since G is continuous on the closed bounded interval $[-1 - |\lambda|, 1 + |\lambda|]$, it is uniformly continuous and we can find a $\delta > 0$ such that

$$|G(u) - G(v)| < \eta/5$$

whenever $|u - v| < \delta$.

It follows that, if $\boldsymbol{\psi} \in X$ and

$$d(\boldsymbol{\phi}, \boldsymbol{\psi}) < \frac{\delta}{1 + |\lambda|},$$

we have

$$\begin{aligned}\sup_{(x,y)\in[0,1]^2}&\left|F(x,y)-\sum_{j=0}^{4}G(\psi_j(x)+\lambda\psi_j(y))\right|\\ &\leq \sup_{(x,y)\in[0,1]^2}\left|F(x,y)-\sum_{j=0}^{4}G(\phi_j(x)+\lambda\phi_j(y))\right|\\ &\qquad\qquad+\sum_{j=0}^{4}\left|G(\phi_j(x)+\lambda\phi_j(y))-G(\psi_j(x)+\lambda\psi_j(y))\right|\\ &\leq \frac{9}{10}\|F\|_\infty-2\eta+\eta<\frac{9}{10}\|F\|_\infty\end{aligned}$$

and we are done. ■

3.4 The core of the proof

We have reduced the proof of Theorem 3.1.4 to the proof of Lemma 3.3.3 and proved the easy part of that lemma in Lemma 3.3.4. We must now prove the hard part.

Lemma 3.4.1. *The set E described in Lemma 3.3.3 is dense in (X,d).*

By rescaling, we may suppose that $\|F\|_\infty = 1$ and restate Lemma 3.4.1 as follows.

Lemma 3.4.2. *Let (X,d) be as in Exercise 3.3.1. Let λ be irrational and let $F \in C([0,1]^2)$ with $\|F\|_\infty = 1$. Given any $\boldsymbol{\psi} \in X$ and any $\epsilon > 0$, we can find a $\boldsymbol{\phi} \in X$ and a continuous function $G : [-1-|\lambda|, 1+|\lambda|] \to \mathbb{R}$ such that $\|G\|_\infty \leq 1$, $d(\boldsymbol{\psi},\boldsymbol{\phi}) < \epsilon$ and*

$$\sup_{(x,y)\in[0,1]^2}\left|F(x,y)-\sum_{j=0}^{4}G(\phi_j(x)+\lambda\phi_j(y))\right|<\frac{9}{10}.$$

This represents the core of the proof. It is the part where we leave general ideas and consider what F, G and $\boldsymbol{\phi} \in X$ actually look like. The proof of Lemma 3.4.2 occupies the rest of this section. From now on, λ, F, $\boldsymbol{\psi}$ and $\epsilon > 0$ will be fixed and we will try and find appropriate $\boldsymbol{\phi}$ and G.

Since any continuous real valued function on $[0,1]$ or $[0,1]^2$ is uniformly continuous, we can find an integer $N \geq 1$ such that

$$|F(x,y)-F(x',y')| \leq 10^{-2} \text{ whenever } |x-x'|,\ |y-y'| \leq 10N^{-1}$$

and

$$|\psi_j(x) - \psi_j(x')| \leq \epsilon/20 \text{ whenever } |x - x'| \leq 10N^{-1} \text{ and } 0 \leq j \leq 4. \qquad \bigstar$$

(Basically, we want F and ψ_j to be effectively constant over a length scale of about $5N^{-1}$.)

Now choose distinct rational numbers $u_{r,j}$ with $0 < u_{r,j} < u_{r+1,j} < 1$ the property that

$$|u_{r,j} - \psi_j(t)| \leq \epsilon/5$$

when $t \in [0, 1]$, $|t - (5r + j)N^{-1}| \leq 10N^{-1}$, $0 \leq j \leq 4$.

Exercise 3.4.3. *Explain, in as much detail as you think appropriate, why ★ enables us to do this.*

We now define $\phi_j : [0, 1] \to \mathbb{R}$ to be the simplest continuous piecewise linear function such that

$$\phi_j(t) = u_{r,j}$$

for $|t - (5r + j)N^{-1}| \leq 2N^{-1}$ and $t \in [0, 1]$. It is important to have a picture in your mind's eye of what this definition means.

Exercise 3.4.4. *I visualise ϕ_j as a series of (relatively) long flat steps joined by what will (usually) be rather steep slopes. Sketch the form of ϕ_j and see if you agree. Show, that $\|\psi_j - \phi_j\|_\infty < \epsilon$ for $0 \leq j \leq 4$ and so $d(\boldsymbol{\psi}, \boldsymbol{\phi}) < \epsilon$.*

Exercise 3.4.5. *Suppose that μ is irrational. If u, u', v, $v' \in \mathbb{Q}$ and*

$$u + \mu v = u' + \mu v',$$

show that $u = u'$ and $v = v'$.

By Exercise 3.4.5, we know that $u_{r,j} + \lambda u_{s,j} = u_{r',j'} + \lambda u_{s',j'}$ only if $r = r'$, $s = s'$ and $j = j'$. We can therefore define $G : [-1 - |\lambda|, 1 + |\lambda|] \to \mathbb{R}$ unambiguously as the simplest continuous linear function with $|G(t)| \leq 1/8$ for all t and

$$G(u_{r,j} + \lambda u_{s,j}) = \begin{cases} 1/8 & \text{if } F((5r + j)N^{-1}, (5s + j)N^{-1}) \geq 0 \\ -1/8 & \text{otherwise.} \end{cases}$$

Automatically, $\|G\|_\infty = 1/8 \leq 1$, so all that remains to do is to prove the crucial inequality

$$\sup_{(x,y)\in[0,1]^2} \left| F(x, y) - \sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) \right| < \frac{9}{10}.$$

Our proof relies on the following simple observation.

Lemma 3.4.6. *If* $(x, y) \in \mathbb{R}^2$*, then at least* 3 *out of the possible* 5 *values of* j *with* $0 \leq j \leq 4$ *satisfy the following condition. There exist* $r_j,\ s_j \in \mathbb{Z}$ *such that*

$$|x - (5r_j + j)N^{-1}| \leq 2N^{-1} \text{ and } |y - (5s_j + j)N^{-1}| \leq 2N^{-1}.$$

Proof. By periodicity and symmetry, we need only consider the case $x \in [0, N^{-1}]$, $y \in [0, 3N^{-1}]$.

If $x \in [0, N^{-1}]$, $y \in [0, N^{-1}]$, then we can take $j \in \{0, 1, 3, 4\}$,

$$(r_0, s_0) = (r_1, s_1) = (0, 0),\ (r_3, s_3) = (r_4, s_4) = (-1, -1).$$

If $x \in [0, N^{-1}]$, $y \in [N^{-1}, 2N^{-1}]$, then we can take $j \in \{0, 1, 2\}$,

$$(r_0, s_0) = (r_1, s_1) = (r_2, s_2) = (0, 0).$$

If $x \in [0, N^{-1}]$, $y \in [2N^{-1}, 3N^{-1}]$, then we can take $j \in \{1, 2, 4\}$,

$$(r_1, s_1) = (r_2, s_2) = (0, 0),\ (r_4, s_4) = (-1, 0).$$

■

We can now conclude the proof.

Lemma 3.4.7. *With the notation and choices of this section,*

$$\sup_{(x,y)\in[0,1]^2} \left| F(x, y) - \sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) \right| \leq \frac{7}{8}.$$

Proof. The proof is easy but instructive. Suppose that $(x, y) \in [0, 1]^2$. There are three possible cases.

CASE 1 If $|f(x, y)| \leq 1/8$ then, since $\|G\|_\infty = 1/8$, we have

$$\begin{aligned}\left| F(x, y) - \sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) \right| &\leq \left|F(x, y)\right| + \sum_{j=0}^{4} \left|G(\phi_j(x) + \lambda\phi_j(y))\right| \\ &\leq \frac{1}{8} + \frac{5}{8} < \frac{7}{8}.\end{aligned}$$

CASE 2 Now suppose $f(x, y) > 1/8$. By Lemma 3.4.6, at least 3 out of the possible 5 values of j with $0 \leq j \leq 4$ satisfy the following condition. There exist $r_j,\ s_j \in \mathbb{Z}$ such that

$$|x - (5r_j + j)N^{-1}| \leq 2N^{-1} \text{ and } |y - (5s_j + j)N^{-1}| \leq 2N^{-1}.$$

Let A be the set of such j. Recall that we chose N so large that F is effectively constant over a length scale of about $5N^{-1}$. It follows that

$$F(x - (5r_j + j)N^{-1}, y - (5s_j + j)N^{-1}) > 0$$

and so, by definition,

$$G(\phi_j(x) + \lambda\phi_j(y)) = G(u_{r_j,j} + \lambda u_{s_j,j}) = 1/8$$

for $j \in A$.

Since $\|G\|_\infty = 1/8$, we have

$$\begin{aligned}\sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) &= \sum_{j\in A} + \sum_{j\notin A} G(\phi_j(x) + \lambda\phi_j(y)) \\ &\geq \frac{3}{8} - \frac{2}{8} = \frac{1}{8}\end{aligned}$$

and, as before,

$$\sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) \leq \frac{5}{8}$$

so, since $1/8 \leq F(x, y) \leq 1$, it follows that

$$-\frac{1}{2} \leq F(x, y) - \sum_{j=0}^{4} G(\phi_j(x) + \lambda\phi_j(y)) \leq \frac{7}{8}.$$

CASE 3 The case $f(x, y) < -1/8$ is dealt with similarly. ∎

Once we have got over the shock of seeing Kolmogorov's result, we may realise that, in some sense, it is the answer to the wrong question. Most questions in multi-dimensional calculus involve functions of various degrees of smoothness and what we really need to know is whether 'well behaved' functions of many variables can be expressed in terms of 'well behaved' functions of fewer variables.

Of course, there are many interesting choices of what 'well behaved' should mean but one obvious choice is 'continuously differentiable'. Here we have a marvellous theorem of Vistuškin.[3]

Theorem 3.4.8. *If $n > m$ there exist continuously differentiable functions of n variables which are not expressible in terms of continuously differentiable functions of m variables.*

[3] For those who wish to dispense with accents Vitushkin.

To paraphrase Lorentz,'This is the *Fundamental Theorem of the Differential Calculus in Several Variables* since it tells us that interesting examples of the objects studied actually exist'. We shall see later, in Section 5.4, how to prove this result using a quantitative version of uniform equicontinuity and a great deal of ingenuity. For the moment the reader may care to ponder on how she might attack the problem

3.5 Futher Exercises

Exercise 3.5.1. *Let* (X, d) *be as in Exercise 3.3.1. Show that quasi-all* $\boldsymbol{\phi} \in X$ *have the property that* ϕ_j *is strictly increasing (that is to say* $\phi_j(s) < \phi_j(t)$ *for* $0 \leq s < t \leq 1$*) for each* $1 \leq j \leq 5$*. Why does this immediately tell us that we can replace the word 'increasing' by the words 'strictly increasing' in Theorem 3.1.4.*

Exercise 3.5.2. *We say that real numbers* $\lambda_1, \lambda_2, \dots, \lambda_n$ *are independent over* $\mathbb{Q}$ *if the equation*

$$\sum_{j=1}^{n} q_j \lambda_j = 0$$

has no solution with $q_j \in \mathbb{Q}$ $[1 \leq j \leq n]$ *apart from the trivial solution with all* $q_j = 0$.

By using a Baire category argument, or otherwise, show that, if $\lambda_1, \lambda_2, \dots, \lambda_n$ *are independent over* $\mathbb{Q}$*, we can find* λ_{n+1} *such that* $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ *are independent over* $\mathbb{Q}$*. Deduce that we can find* $\lambda_1, \lambda_2, \dots$ *such that* $\lambda_1, \lambda_2, \dots, \lambda_n$ *are independent over* $\mathbb{Q}$ *for all n.*

Exercise 3.5.3. *(i) If* $0 \leq j \leq 4$*, let us say that the square* $[pN^{-1}, (p+1)N^{-1}] \times [qN^{-1}, (q+1)N^{-1}]$ *with p and q integers has colour j if*

$$\begin{aligned}[pN^{-1},&(p+1)N^{-1}] \times [qN^{-1}, (q+1)N^{-1}] \\ &\subseteq [(5r+j-2)N^{-1}, (5r+j+2)N^{-1}] \times [(5s+j-2)N^{-1}, (5s+j+2)N^{-1}]\end{aligned}$$

for some integers r and s.

Take a sheet of squared paper and sketch the resulting colouring. Check that every square has at least three associated colours and no square has five associated colours. Identify the squares which have four associated colours. What is the relevance of this exercise to Lemma 3.4.6.

(ii) We now repeat the exercise with four *colours rather than five. If* $0 \leq j \leq 3$*, let us say that the square* $[pN^{-1}, (p+1)N^{-1}] \times [qN^{-1}, (q+1)N^{-1}]$ *with p and q integers has colour j if*

$$\begin{aligned}[pN^{-1},&(p+1)N^{-1}] \times [qN^{-1}, (q+1)N^{-1}] \\ &\subseteq [(4r+j-1)N^{-1}, (4r+j+2)N^{-1}] \times [(4s+j-1)N^{-1}, (4s+j+2)N^{-1}]\end{aligned}$$

for some integers r and s.

Take a sheet of squared paper and sketch the resulting colouring. Is it true that each square has least three associated colours? Can you verify your answer by a simple counting argument?

(iii) Restate the exercise with n colours. Find, with proof, the greatest integer $\kappa(n)$ such that each square has least $\kappa(n)$ associated colours. Show that $\kappa(n)/n > 1/2$ if and only if $n \geq 5$. Show that $\kappa(n)/n \to 1$ as $n \to \infty$.

Exercise 3.5.4. *Prove that, if λ_1, λ_2, λ_3 are independent over $\mathbb{Q}$, we can find increasing continuous functions $\phi_j : [0, 1] \to \mathbb{R}$ $[0 \leq j \leq 6]$, with the following property. Given any continuous function $f : [0, 1]^3 \to \mathbb{R}$ we can find a function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x_1, x_2, x_3) = \sum_{j=0}^{6} g(\lambda_1\phi_j(x_1) + \lambda_2\phi_j(x_2) + \lambda_3\phi(x_3)).$$

[*If you wish to learn the proof of Theorem 3.1.4 you should probably write out the whole proof. If not, you should identify the only part of the proof which might fail and concentrate on that.*]

Exercise 3.5.5. *Investigate to what extent the proof would be simplified if we replaced Theorem 3.1.4 by the following slightly less demanding result.*

Given any continuous function $f : [0, 1]^2 \to \mathbb{R}$ we can find continuous functions $\phi_r : [0, 1] \to \mathbb{R}$ $[0 \leq r \leq 9]$ and continuous functions $g_j : \mathbb{R} \to \mathbb{R}$ $[0 \leq j \leq 4]$ such that

$$f(x, y) = \sum_{j=0}^{4} g_j(\phi_{2j}(x) + \phi_{2j+1}(y)).$$

Exercise 3.5.6. *Show that the result corresponding to Theorem 3.1.4 is false if $\lambda = 1$ or $\lambda = -1$.*

Chapter 4

Compactness

4.1 Dirichlet's problem treated informally

This section requires a 'mathematical methods' knowledge of vector calculus which we shall use in an informal way. I suggest that those readers without this knowledge and those who shrink from informal arguments skim through this section which will not be used again, but which does provide useful background.

Nineteenth century mathematicians like Gauss, Green and Dirichlet studied Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

or, more concisely[1],

$$\nabla^2\phi = 0,$$

on a region Ω with boundary $\partial\Omega$ subject to the boundary condition

$$\phi(x, y) = f(x, y) \text{ for } (x, y) \in \partial\Omega.$$

(This is now called Dirichlet's problem.) For the moment, we will lose nothing by considering $\Omega = D = \{(x, y) : x^2 + y^2 < 1\}$ with

$$\bar{D} = \operatorname{Cl} D = \{(x, y) : x^2 + y^2 \leq 1\},\ \partial D = \{(x, y) : x^2 + y^2 = 1\}.$$

Plausible Statement 4.1.1. *Let $f : \partial D \to \mathbb{R}$ be continuous. The problem of finding a continuous function $\phi : \bar{D} \to \mathbb{R}$ which is twice continuously differentiable on D and satisfies*

$$\nabla^2\phi = 0 \textit{ on } D,$$
$$\phi = f \textit{ on } \partial D$$

has at most one solution.

[1]More advanced texts write $\nabla^2\phi = \triangle\phi$ and are probably right to do so.

Plausible argument. Suppose that

$$\nabla^2\phi_j = 0 \text{ on } D,$$
$$\phi_j = f \text{ on } \partial D$$

for $j = 1, 2$. If we set $\phi = \phi_1 - \phi_2$, then

$$\nabla^2\phi = 0 \text{ on } D$$
$$\phi = f \text{ on } \partial D$$

Our object is to show that $\phi = 0$.

The key to the matter is the 'energy like' object

$$\iint_D |\nabla\phi|^2\,dx\,dy.$$

Using elementary vector calculus and the divergence theorem, we have

$$\begin{aligned}\iint_D |\nabla\phi|^2\,dx\,dy &= \iint_D \nabla\phi\cdot\nabla\phi\,dx\,dy = \iint_D \nabla\cdot(\phi\nabla\phi) - \phi\nabla^2\phi\,dx\,dy\\ &= \iint_D \nabla\cdot(\phi\nabla\phi)\,dx\,dy = \int_{\partial D}(\phi\nabla\phi)\cdot\mathbf{n}ds = \int_{\partial D}\mathbf{0}\cdot\mathbf{n}ds = 0.\end{aligned}$$

Since $|\nabla\phi|^2 \geq 0$ on D, the only way that this can occur is to have $|\nabla\phi|^2 = 0$ and so $\nabla\phi = \mathbf{0}$ on D. Thus ϕ is constant on $\bar{D}$ and, since $\phi = 0$ on ∂D, $\phi = 0$ on $\bar{D}$. ■

One evident weakness of our argument is that we have not specified the conditions under which the divergence theorem should hold. None-the-less the argument is a very elegant one which mathematicians like Dirichlet took further in a remarkable manner.

Plausible Statement 4.1.2. *Let $f : \partial D \to \mathbb{R}$ be continuous. The problem of finding a continuous function $\phi : \bar{D} \to \mathbb{R}$ which is twice continuously differentiable on D and satisfies*

$$\nabla^2\phi = 0 \text{ on } D,$$
$$\phi = f \text{ on } \partial D$$

always has a solution.

Plausible argument. Let us look at the collection Ψ of functions $\psi : \bar{D} \to \mathbb{R}$ which are twice continuously differentiable on D and satisfy $\psi = f$ on ∂D. We are interested in the function $\phi \in \Psi$ which *minimises* the 'energy like' object $\iint_D |\nabla\psi|^2\,dx\,dy$, that is to say, we look at the $\phi \in \Psi$ which satisfies

$$\iint_D |\nabla\phi|^2\,dx\,dy \leq \iint_D |\nabla\psi|^2\,dx\,dy$$

for all $\psi \in \Psi$.

Suppose now that $\kappa : \bar{D} \to \mathbb{R}$ is twice continuously differentiable on D and satisfies $\kappa = 0$ on ∂D. If h is any real number, we know that $\phi + h\kappa \in \Psi$ and so

$$\begin{aligned}
0 &\leq \iint_D |\nabla(\phi + h\kappa)|^2\,dx\,dy - \iint_D |\nabla\phi|^2\,dx\,dy \\
&= \iint_D \nabla(\phi + h\kappa)\cdot\nabla(\phi + h\kappa)\,dx\,dy - \iint_D \nabla\phi\cdot\nabla\phi\,dx\,dy \\
&= -2h\iint_D \nabla\phi\cdot\nabla\kappa\,dx\,dy + h^2\iint_D \nabla\kappa\cdot\nabla\kappa\,dx\,dy.
\end{aligned}$$

Since this result holds when h is very small and positive and when h is very small and negative, we must have

$$\iint_D \nabla\phi\cdot\nabla\kappa\,dx\,dy = 0.$$

We now use the divergence theorem again to obtain

$$\begin{aligned}
0 &= \iint_D \nabla\phi\cdot\nabla\kappa\,dx\,dy = \iint_D \nabla\cdot(\kappa\nabla\phi)\,dx\,dy - \iint_D \kappa\nabla^2\phi\,dx\,dy \\
&= \int_{\partial D} (\kappa\nabla\phi)\cdot\mathbf{n}ds - \iint_D \kappa\nabla^2\phi\,dx\,dy = -\iint_D \kappa\nabla^2\phi\,dx\,dy.
\end{aligned}$$

Since

$$\iint_D \kappa\nabla^2\phi\,dx\,dy = 0$$

for all appropriate $\kappa : \bar{D} \to \mathbb{R}$, we must have $\nabla^2\phi = 0$ on D so ϕ is a solution of our original problem. ∎

The 'plausible proof' that we have just given is too beautiful to abandon and echos of it appear in many places. However, there are two problems (besides the question the appropriate conditions for applying the divergence theorem) which prevent it from being acceptable as it stands. The first is that we might have

$$\iint_D |\nabla\psi|^2\,dx\,dy = \infty$$

for all $\psi \in \Psi$ (this problem also affects our proposed proof of uniqueness in Plausible Theorem 4.1.1). The second, which was pointed out by Weierstrass, is that even if there exist $\psi \in \Psi$ with

$$\iint_D |\nabla\psi|^2\, dx\, dy < \infty,$$

we cannot be certain that that there exists a $\phi \in \Psi$ which actually minimises this integral.

In the next section we shall see the strength of these objections.

Exercise 4.1.3. *(i) Suppose that $\delta > 0$. Show that if A, $B \in \mathbb{R}$ and*

$$Ah^2 + Bh \geq 0$$

for all $|h| < \delta$, then $B = 0$. (This easy result is often useful.)

(ii) Suppose that if A, B, $C \in \mathbb{R}$ and $Ah^3 + Bh^2 + Ch \geq 0$ for all $h \in \mathbb{R}$. What can you deduce about A, B and C and why?

(iii) Suppose that $\delta > 0$, A, B, $C \in \mathbb{R}$ and $Ah^3 + Bh^2 + Ch \geq 0$ for all $|h| < \delta$. What can you deduce about A, B and C and why?

Exercise 4.1.4. *(i) Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and*

$$\int_a^b f(t)g(t)\, dt = 0$$

for all continuous $g : [a,b] \to \mathbb{R}$. Show that $f = 0$. (Recall Exercise 1.2.9 if necessary.)

(ii) Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and

$$\int_a^b f(t)g(t)\, dt = 0$$

for all continuous $g : [a,b] \to \mathbb{R}$ with $g(a) = g(b) = 0$. Show that $f = 0$.

(iii) Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and

$$\int_a^b f(t)g(t)\, dt = 0$$

for all infinitely differentiable $g : [a,b] \to \mathbb{R}$ with $g(a) = g(b) = 0$. Show that $f = 0$.

Exercise 4.1.5. *Let $f : \partial D \to \mathbb{R}$ be continuous. Suppose that there exists a continuous function $\phi : \bar{D} \to \mathbb{R}$ which is twice continuously differentiable on D and satisfies*

$$\begin{aligned}\nabla^2\phi &= 0 \text{ on } D,\\ \phi &= f \text{ on } \partial D.\end{aligned}$$

If

$$\iint_D |\nabla\phi|^2 \, dx\, dy < \infty,$$

show, assuming that the conditions of the divergence theorem apply, that

$$\iint_D |\nabla\phi|^2 \, dx\, dy \leq \iint_D |\nabla\psi|^2 \, dx\, dy$$

for all continuous $\psi : \bar{D} \to \mathbb{R}$ which are once continuously differentiable on D and satisfy

$$\psi = f \text{ on } \partial D.$$

4.2 Dirichlet's problem treated formally

We now return to our usual standards of proof. Our first task is to prove the uniqueness of the solution of Dirichlet's problem (if such a solution exists) for bounded regions.

We start by establishing a 'maximal principle'.

Lemma 4.2.1. *Let Ω be an open subset of $\mathbb{R}^2$ and $\phi : \Omega \to \mathbb{R}$ a function such that the second partial derivatives*

$$\frac{\partial^2\phi}{\partial x^2}(x, y) \text{ and } \frac{\partial^2\phi}{\partial y^2}(x, y)$$

exist at each point of $(x, y) \in \Omega$. If

$$\frac{\partial^2\phi}{\partial x^2}(x, y) + \frac{\partial^2\phi}{\partial y^2}(x, y) > 0$$

for all $(x, y) \in \Omega$, then ϕ cannot attain a maximum in Ω.

Proof. Suppose that ϕ attains a maximum g, defined by

$$g(x) = \phi(x, y_0)$$

on the open set $\{x : (x, y_0) \in \Omega\}$, has a maximum at x_0 and so, by one dimensional calculus,

$$\frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) = g''(x_0) \leq 0.$$

A precisely similar argument gives

$$\frac{\partial^2 \phi}{\partial y^2}(x_0, y_0) \leq 0$$

so

$$\frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0) \leq 0.$$

The stated result now follows. ■

In the last section we took the notion of the boundary $\partial\Omega$ as given. Since we have reverted to a more precise mode, we must define it.

Definition 4.2.2. *Let (X, d) be a metric space. If E is a subset of X, we define the* boundary *(or* frontier*) ∂E of E by*

$$\partial E = \operatorname{Cl} E \setminus \operatorname{Int} E.$$

The reader should check that this corresponds to her idea of of a boundary when E is a 'well behaved' set in $\mathbb{R}^n$ with the usual Euclidean norm and $1 \leq n \leq 3$. If E is not 'well behaved', we have no 'natural idea' of a boundary, so we may as well use the one we have just given, provided it is technically useful.

Theorem 4.2.3. *Let Ω be a bounded open subset of $\mathbb{R}^2$ and $\phi : \operatorname{Cl}\Omega \to \mathbb{R}$ be a continuous function such that the second partial derivatives*

$$\frac{\partial^2 \phi}{\partial x^2}(x, y) \text{ and } \frac{\partial^2 \phi}{\partial y^2}(x, y)$$

exist at each point of $(x, y) \in \Omega$.

(i) If

$$\frac{\partial^2 \phi}{\partial x^2}(x, y) + \frac{\partial^2 \phi}{\partial y^2}(x, y) \geq 0$$

for all $(x, y) \in \Omega$, then

$$\sup_{(x,y)\in \operatorname{Cl}\Omega} \phi(x, y) = \sup_{(x,y)\in\partial\Omega} \phi(x, y).$$

(ii) If

$$\frac{\partial^2 \phi}{\partial x^2}(x, y) + \frac{\partial^2 \phi}{\partial y^2}(x, y) = 0$$

for all $(x, y) \in \Omega$ and $\phi(x, y) = 0$ for all $(x, y) \in \partial\Omega$, then $\phi(x, y) = 0$ for all $(x, y) \in \operatorname{Cl}\Omega$.

Proof. (i) Consider
$$\phi_\epsilon(x, y) = \phi(x, y) + \epsilon(x^2 + y^2)$$
where $\epsilon > 0$. We observe that $\phi_\epsilon : \operatorname{Cl}\Omega \to \mathbb{R}$ is a continuous function such that the second partial derivatives

$$\frac{\partial^2 \phi_\epsilon}{\partial x^2}(x, y) \text{ and } \frac{\partial^2 \phi_\epsilon}{\partial y^2}(x, y)$$

exist at each point of $(x, y) \in \Omega$ and

$$\frac{\partial^2 \phi_\epsilon}{\partial x^2}(x, y) + \frac{\partial^2 \phi_\epsilon}{\partial y^2}(x, y) \geq 4\epsilon > 0$$

for all $(x, y) \in \Omega$.

Since Ω is bounded, we can find an R such that $x^2 + y^2 \leq R$ for all $(x, y) \in \Omega$ and so $x^2 + y^2 \leq R$ for all $(x, y) \in \operatorname{Cl}\Omega$. We are working in $\mathbb{R}^2$ with its usual metric, so we know that a continuous function on a closed bounded set attains its bounds[2] Thus there exists an $(x_\epsilon, y_\epsilon) \in \operatorname{Cl}\Omega$ such that

$$\phi_\epsilon(x_\epsilon, y_\epsilon) \geq \phi_\epsilon(x, y)$$

for all $(x, y) \in \operatorname{Cl}\Omega$ and so, in particular, for all $(x, y) \in \Omega$. By Lemma 4.2.1 we must have $(x_\epsilon, y_\epsilon) \notin \Omega$ and so $(x_\epsilon, y_\epsilon) \in \partial\Omega$. Thus

$$\sup_{(x,y)\in\operatorname{Cl}\Omega} \phi_\epsilon(x, y) = \phi_\epsilon(x_\epsilon, y_\epsilon) = \sup_{(x,y)\in\partial\Omega} \phi_\epsilon(x, y)$$

and

$$\sup_{(x,y)\in\operatorname{Cl}\Omega} \phi(x, y) \leq \sup_{(x,y)\in\operatorname{Cl}\Omega} \phi_\epsilon(x, y) = \sup_{(x,y)\in\partial\Omega} \phi_\epsilon(x, y) \leq \sup_{(x,y)\in\partial\Omega} \phi(x, y) + \epsilon R^2.$$

Since ϵ was arbitrary

$$\sup_{(x,y)\in\operatorname{Cl}\Omega} \phi(x, y) \leq \sup_{(x,y)\in\partial\Omega} \phi(x, y)$$

and so

$$\sup_{(x,y)\in\operatorname{Cl}\Omega} \phi(x, y) = \sup_{(x,y)\in\partial\Omega} \phi(x, y)$$

(ii) By part (i), $\phi(x, y) \leq 0$ for all $(x, y) \in \operatorname{Cl}\Omega$ and, by part (i) applied to $-\phi$, $\phi(x, y) \geq 0$ for all $(x, y) \in \operatorname{Cl}\Omega$. Thus $\phi(x, y) = 0$ for all $(x, y) \in \operatorname{Cl}\Omega$. ∎

[2] We discuss generalisations of this result in the next section. (Exercise 4.3.12 and Theorem 4.3.15 give the particular results).

Since the appropriate maxima are attained, we have verified the principle that the maximum of a harmonic function ϕ (that is to say, a function satisfying Laplace's equation $\nabla^2\phi = 0$) on a bounded set occurs on the boundary. Our maximum principle immediately gives the uniqueness of the solution of Dirichlet's problem.

Exercise 4.2.4. *Let Ω be a bounded open subset of $\mathbb{R}^2$ and $\phi_j : \mathrm{Cl}\,\Omega \to \mathbb{R}$ be a continuous function such that the second partial derivatives*

$$\frac{\partial^2\phi_j}{\partial x^2}(x,y) \text{ and } \frac{\partial^2\phi_j}{\partial y^2}(x,y)$$

exist at each point of $(x,y) \in \Omega$ and

$$\frac{\partial^2\phi_j}{\partial x^2}(x,y) + \frac{\partial^2\phi_j}{\partial y^2}(x,y) = 0$$

for all $(x,y) \in \Omega$ $[j = 1, 2]$.

If $\phi_1(x,y) = \phi_2(x,y)$ for all $(x,y) \in \partial\Omega$, show that $\phi_1(x,y) = \phi_2(x,y)$ for all $(x,y) \in \Omega$.

Although we can rescue and generalise the uniqueness result for Dirichlet's problem, we cannot do the same for existence. The following beautiful result is due to Zaremba[3].

Example 4.2.5. *We work in $\mathbb{R}^2$. Let*

$$\Omega = \{\mathbf{x} : 0 < \|\mathbf{x}\| < 1\}$$

(that is to say, Ω is the punctured unit disc). If we define $f : \partial\Omega \to \mathbb{R}$ by

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \|\mathbf{x}\| = 1, \\ 1 & \text{if } \mathbf{x} = \mathbf{0}, \end{cases}$$

then f is continuous, but there does not exist a continuous $\phi : \mathrm{Cl}\,\Omega \to \mathbb{R}$ such that ϕ is twice continuously differentiable on Ω and

$$\begin{aligned} \nabla^2\phi &= 0 \text{ on } \Omega, \\ \phi &= f \text{ on } \partial\Omega. \end{aligned}$$

[3]One of the two founders of the Polish school of mathematics and major mathematician in his own right. Lebesgue wrote that Zaremba never published a superfluous paper.

Proof. Suppose that such a solution ϕ exists. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation through an angle θ about $\mathbf{0}$. By symmetry, $\phi \circ R_\theta$ (given by $\phi \circ R_\theta(\mathbf{x}) = \phi(R_\theta \mathbf{x})$) is also a solution. By our uniqueness result, $\phi \circ R_\theta = \phi$ for all θ and so $\phi(\mathbf{x}) = F(\|\mathbf{x}\|)$ for some continuous function $F : [0, 1] \to \mathbb{R}$.

Automatically $F(1) = 0$ and $F(0) = 1$, F is twice differentiable on $(0, 1)$ and

$$\frac{1}{r}\frac{d}{dr}(rF'(r)) = 0.$$

(If this is not clear, do Exercise 4.2.6 below.)

Thus

$$\frac{d}{dr}(rF(r)) = 0,$$

whence

$$rF'(r) = A$$

and

$$F(r) = A \log r + B$$

for some constants A and B and all $0 < r < 1$. Allowing $r \to 0+$ and using continuity, we get $A = 0$ and $B = 1$ so $F(r) = 1$. Allowing $r \to 1-$, we obtain a contradiction. ■

In some sense this is 'physically obvious'. If we have a very thin wire running down the centre of a long earthed cylindrical shell, then if we try to charge the wire 'the field will break down' and we will see electrical discharges. Lebesgue showed that the three dimensional Dirichlet problem may not be soluble if the boundary is strongly cusped.

Exercise 4.2.6. *Let $F : (0, 1) \to \mathbb{R}$ be twice continuously differentiable. If we define*

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : 0 < \|\mathbf{x}\| < 1\}$$

and $\phi : \Omega \to \mathbb{R}$ by

$$\phi(\mathbf{x}) = F(\|\mathbf{x}\|)$$

compute

$$\frac{\partial \phi}{\partial x}(x, y) \text{ and } \frac{\partial^2 \phi}{\partial x^2}(x, y).$$

Show that

$$\nabla^2 \phi(\mathbf{x}) = \frac{1}{r}\frac{d}{dr}(rF'(r))$$

where $r = \|\mathbf{x}\|$.

Show also that

$$|\nabla \phi(\mathbf{x})|^2 = |F'(r)|^2.$$

It is instructive to verify explicitly that the 'energy type' integral

$$\iint_\Omega |\nabla\psi|^2\,dx\,dy$$

does not attain a minimum in Zaremba's example.

Example 4.2.7. *Let Ω and f be as in Example 4.2.5. Then, if $\mathcal{Z}$ is the set of continuous functions $\psi : \operatorname{Cl}\Omega \to \mathbb{R}$ which are twice continuously differentiable on Ω and satisfy $\psi = f$ on $\partial\Omega$, we have*

$$\inf_{\psi\in\mathcal{Z}} \iint_\Omega |\nabla\psi|^2\,dx\,dy = 0$$

but

$$\iint_\Omega |\nabla\psi|^2\,dx\,dy > 0$$

for all $\psi \in \mathcal{Z}$.

Proof. If

$$\iint_\Omega |\nabla\psi|^2\,dx\,dy = 0,$$

then $|\nabla\psi|^2 = 0$ and so $\nabla\psi = 0$ on Ω. Thus ψ is constant on Ω and so on $\operatorname{Cl}\Omega$ which is impossible.

Our main task is thus to show that

$$\inf_{\mathcal{Z}} \iint_\Omega |\nabla\psi|^2\,dx\,dy = 0.$$

To this end, let $1 > \epsilon > 0$ and choose $g : [0,1] \to \mathbb{R}$ a continuously differentiable function such that

$$\begin{aligned} g(s) &= s^{-1} \text{for } 1 \geq s \geq \epsilon,\\ 0 \leq &g(s) \leq 2\epsilon^{-1} \text{ for } \epsilon \geq s \geq \epsilon/2\\ g(s) &= 0 \text{ for } \epsilon \geq s \geq \epsilon/20. \end{aligned}$$

Now set

$$A = \int_0^1 g(s)\,ds,\ F(t) = \frac{1}{A}\left(A - \int_0^t g(s)\,ds\right).$$

We observe that

$$A \geq \int_\epsilon^1 g(s)\,ds = -\log\epsilon$$

that $F(0) = 1$, $F(1) = 0$ and F is twice continuously differentiable with $|F'(t)| = A^{-1}t^{-1}$ for $1 \geq t \geq \epsilon$, $F'(t) = 0$ for $\epsilon/2 \geq t \geq 0$.

Thus, if we set $\phi(\mathbf{x}) = F(\|\mathbf{x}\|)$, we have $\phi \in \mathcal{Z}$ and

$$|\nabla\phi(\mathbf{x})|^2 = F'(\|\mathbf{x}\|)^2.$$

It follows that

$$\begin{aligned}\iint_\Omega |\nabla\phi|^2\,dx\,dy &= 2\pi\int_0^1 F'(r)^2 r\,dr = 2\pi\left(\int_\epsilon^1 F'(r)^2 r\,dr + \int_0^\epsilon F'(r)^2 r\,dr\right)\\ &\leq \frac{2\pi}{A^2}\left(\int_\epsilon^1 r^{-2} r\,dr + \int_0^\epsilon \epsilon^{-2} r\,dr\right)\\ &= \frac{2\pi}{A^2}\left(\int_\epsilon^1 r^{-1}\,dr + \frac{1}{2}\right)\\ &= -\frac{2\pi}{A^2}(2^{-1} + |\log\epsilon|) = \frac{2\pi}{|\log\epsilon|^2}(2^{-1} + |\log\epsilon|) \to 0\end{aligned}$$

as $\epsilon \to 0+$. ■

The reader who wants a slightly simpler example of the same phenomenon can do the following exercise.

Exercise 4.2.8. *If $\mathcal{F}$ is the set of once continuously differentiable functions $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $f(1) = 1$ show that*

$$\inf_{f\in\mathcal{F}} \int_0^1 t^2 f'(t)^2\,dt = 0$$

but

$$\int_0^1 t^2 f'(t)^2\,dt > 0$$

for all $f \in \mathcal{F}$.

A variation on this theme, rather closer to Weierstrass's original example, is given as Exercise 4.10.14.

There is another problem associated with the naive use of minima.

Exercise 4.2.9. [*An example of Hadamard*] *Consider the unit disc $D = \{z : |z| < 1\}$ in $\mathbb{C}$. Recall that, if the power series $\sum_{j=0}^\infty a_n z^n$ converges on D to $g(z)$, then the power series can be differentiated term by term within D.*

(i) Recall (or prove) that if we write

$$g(x+iy) = u(x,y) + iv(x,y)$$

with x, y, u and v real then

$$g'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y}$$

recall (or deduce) that this implies $\nabla^2 u = 0$. *Show also that*

$$\nabla u . \nabla u = |g'(z)|^2.$$

(ii) Suppose $a_n \in \mathbb{R}$ *with* $\sum_{n=0}^{\infty} |a_n| < \infty$. *If we take*

$$u_n(x, y) = r^n \cos n\theta$$

where $r \cos\theta = x$, $r \sin\theta = y$, $r = (x^2 + y^2)^{1/2}$ *and*

$$\psi = \sum_{n=0}^{\infty} a_n u_n,$$

show using (i) (or direct calculation) that ϕ *solves Dirichlet's problem for the boundary condition*

$$\psi(x, y) = \sum_{n=0}^{\infty} a_n u_n(x, y) \text{ whenever } (x, y) \in \partial D.$$

Show also that if we write $\bar{D}(R) = \{z : |z| \leq R\}$

$$\iint_{\bar{D}(R)} \nabla\psi \cdot \nabla\psi \, dx\, dy = \pi \sum_{n=1}^{\infty} n R^{2n} a_n^2$$

for $R < 1$.

If we take $a_n = 2^{-m}$ *when* $n = 2^{2m}$ *and* $a_n = 0$ *otherwise, check that* $\sum_{n=0}^{\infty} a_n$ *converges but*

$$\iint_{\bar{D}(R)} \nabla\psi \cdot \nabla\psi \, dx\, dy \to \infty$$

as $R \to 1-$.

For this choice the solution to Dirichlet's problem has an infinite energy integral and the plausible arguments of Section 4.1 are doomed to failure.

It is quite easy (although we shall not use the result) t show that, in fact, Dirichet's problem always has a solution for nice regions (see Exercises 4.10.1 and 4.10.2)

4.3 Compactness

In the previous section, we saw that, even in very natural situations, there may be no object which maximises a given function. However, in elementary analysis, we do have a theorem which states that a real valued continuous function on a closed bounded interval is bounded and attains its bounds. In this section we seek suitable generalisations of this theorem and its relatives.

It turns out that a good way to start is to consider the famous Heine–Borel theorem.

Theorem 4.3.1. [Heine–Borel] *We work on $\mathbb{R}$ with the usual metric. The following two conditions on a set E are equivalent.*

(i) E is closed and bounded.

(ii) Suppose that $\mathcal{U}$ is a collection of open sets with

$$\bigcup_{U\in\mathcal{U}} U \supseteq E.$$

Then we can find a finite subset $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup_{V\in\mathcal{V}} V \supseteq E$.

Proof. The reader should know a proof, but the theorem will also follow from the more general results of this section (see Exercise **??**). ∎

Any attempt to extend this result to general metric spaces must bear in mind the following trivial observation.

Exercise 4.3.2. *Show that, if d is a metric on a space X and $\eta > 0$, then*

$$\rho(x, y) = \min\{\eta, d(x, y)\}$$

defines a metric ρ such that

$$x_n \underset{d}{\rightarrow} x \Leftrightarrow x_n \underset{\rho}{\rightarrow} x.$$

(Thus every metric space is homeomorphic to a bounded metric space, that is say to a metric space in which the distance beween any two points is bounded.)

Show further that the two metrics have the same Cauchy sequences.

To get round this difficulty we make the following definition.

Definition 4.3.3. *A metric space (X, d) is* totally bounded *if, given $\epsilon > 0$, we can find a finite collection of open balls $B(x_1, \epsilon)$, $B(x_2, \epsilon)$, …, $B(x_m, \epsilon)$ with*

$$\bigcup_{j=1}^{m} B(x_j, \epsilon) = X.$$

Exercise 4.3.4. *Show that, if we use the usual metric, $\mathbb{R}$ is complete, but not totally bounded and $(0, 1)$ is totally bounded, but not complete.*

Exercise 4.3.5. *If (X, d) is totally bounded and E is a non-empty subset of X, show that (E, d_E) is totally bounded.*
[*Note that this requires two sentences to prove rather than one.*]

We can now state our generalisation of the Heine–Borel theorem.

Theorem 4.3.6. *Let (X, d) be a metric space. Then the following three conditions are equivalent.*

(i) **[The Heine–Borel property]** *Suppose that $\mathcal{U}$ is a collection of open sets with*

$$\bigcup_{U \in \mathcal{U}} U = X.$$

Then we can find a finite subset $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup_{V \in \mathcal{V}} V = X$. (More concisely, any open cover has a finite subcover.)

(ii) **[The Bolzano–Weierstrass property]** *If $x_n \in X$, we can find $n(j) \to \infty$ and $x \in X$ such that $x_{n(j)} \to x$. (More concisely, any sequence in (X, d) has a convergent subsequence.)*

(iii) **[Completeness and totally boundedness]** *The metric space (X, d) is complete and totally bounded.*

Proof. $(i) \Rightarrow (ii)$. Suppose, if possible, that (X, d) has the Heine–Borel property but contains a sequence x_n with no convergent sequence. Then, given $x \in X$, we can find a $\delta(x) > 0$ and an $N(x) \geq 1$ such that

$$d(x_n, x) > \delta(x) \text{ for all } n \geq N(x).$$

The collection of open balls $B(x, \delta(x))$ covers X since

$$\bigcup_{x \in X} B(x, \delta(x)) \supseteq \bigcup_{x \in X} \{x\} = X,$$

so, by the Heine–Borel property, we can find $y_1, y_2, \ldots, y_m$ such that

$$\bigcup_{j=1}^{m} B(y_j, \delta(y_j)) = X.$$

Since $x_n \notin B(y_j, \delta(y_j))$ for $n \geq N(j)$, we have $x_n \notin \bigcup_{j=1}^{m} B(y_j, \delta(y_j)) = X$ for $n \geq \max_{1 \leq j \leq m} N(y_j)$ which is absurd. The required result follows by reductio ad absurdum.

$(ii) \Rightarrow (iii)$. Suppose that (X, d) has the Bolzano–Weierstrass property. Since every sequence has a convergent subsequence, it follows that every Cauchy sequence has a convergent subsequence and so (by Exercise 1.2.7 (iii)) must be convergent. Thus (X, d) is complete.

Suppose, if possible, that (X, d) is not totally bounded. Then there exists an $\epsilon > 0$ such that X cannot be covered by a finite collection of open balls of radius ϵ. It follows that, given $x_1, x_2, \ldots, x_n \in X$, have

$$\bigcup_{j=1}^{n} B(x_j, \epsilon) \neq \varnothing$$

and so we can find $x_{n+1} \in X$ with $x_{n+1} \notin \bigcup_{j=1}^{n} B(x_j, \epsilon)$, that is to say, $d(x_{n+1}, x_j) \geq \epsilon$ for all $n \geq j \geq 1$. Proceeding inductively we can find a sequence $x_1, x_2, \ldots$ with $d(x_r, x_s) \geq \epsilon$ for all $r \neq s$. Such a sequence can contain no convergent subsequence. The required result follows by reductio ad absurdum.

(*iii*) $\Rightarrow$ (*i*). Suppose, if possible, that the metric space (X, d) is complete and totally bounded, but there exists a collection $\mathcal{U}$ of open sets covering X without a finite subcover.

Since (X, d) is totally bounded, we can find a finite set of open balls B_1, B_2, $\ldots B_m$ of radius 1 with

$$\bigcup_{j=1}^{m} B_j = X.$$

If each ball B_j has a finite subcover $\mathcal{U}_j$, that is to say we can find a finite set $\mathcal{U}_j \subseteq \mathcal{U}$ with

$$\bigcup_{U \in \mathcal{U}_j} U \supseteq B_j,$$

then $\mathcal{V} = \bigcup_{j=1}^{m} \mathcal{U}_j$ is a finite subcover of X which is impossible. Thus we can find an open ball Γ_0 of radius 1 which is not covered by any finite subset of $\mathcal{U}$.

Since (X, d) is totally bounded, we can find a finite set of open balls of radius $1/2$ which cover X and so we can find a finite subset of open balls which cover Γ_0 and have non-empty intersection with Γ_0. The argument of the previous paragraph shows that at least one of these balls cannot be covered by any finite subset of $\mathcal{U}$. Thus we have found an open ball Γ_1 of radius $1/2$ which has non-empty intersection with Γ_0 and which cannot be covered by any finite subset of $\mathcal{U}$.

Proceeding inductively, we obtain a sequence of open balls Γ_n of radius 2^{-n} with $\Gamma_{n-1} \cap \Gamma_n \neq \varnothing$ such that Γ_n cannot be covered by any finite subset of $\mathcal{U}$. Let Γ_n have centre x_n. Since $\Gamma_{n-1} \cap \Gamma_n \neq \varnothing$,

$$d(x_{n-1}, x_n) \leq 2^{-n+1} + 2^{-n} < 2^{-n+2}.$$

Thus the x_n form a Cauchy sequence and, by completeness, $x_n \to x$ for some $x \in X$. Since $\mathcal{U}$ covers X, there must be some $U_0 \in \mathcal{U}$ with $x \in U_0$. Since U_0 is open, we can find a $\delta > 0$ with

$$B(x_0, \delta) \subseteq U_0.$$

Since $x_n \to x$, we can find an N such that $2^{-N} < \delta/2$ and $d(x_N, x) < \delta/2$. This tells us that

$$\Gamma_N = B(x_N, 2^{-N}) \subseteq B(x_0, \delta) \subseteq U_0.$$

Thus Γ_N is covered by the finite subset of $\mathcal{U}$ consisting of the single set U_0. This contradicts our definition of Γ_N, so the required result follows by reductio ad absurdum. ■

The following further equivalence is very much more useful than it looks at first sight.

Lemma 4.3.7. [Finite intersection property] *The Heine–Borel property is equivalent to the following 'finite intersection property'. Suppose that $\mathcal{F}$ is a collection of closed sets such that, for every finite non-empty subset $\mathcal{G}$ of $\mathcal{F}$ we have $\bigcap_{F\in\mathcal{G}} F \neq \emptyset$, then $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$. (More concisely, if the intersection of any finite subcollection of closed sets is non-empty the intersection of the entire collection is non-empty.)*

Proof. The Heine Borel property and the finite intersection property are 'complements of each other'. If the Heine-Borel property holds and $\mathcal{F}$ is a collection of closed sets with $\bigcap_{F\in\mathcal{F}} F = \emptyset$, then $\mathcal{U} = \{X \setminus F \ : \ F \in \mathcal{F}\}$ is an open cover of X and so has a finite subcover $\mathcal{V} = \{X \setminus F \ : \ F \in \mathcal{G}\}$ with $\mathcal{G}$ a finite non-empty subset of $\mathcal{F}$. Automatically, $\bigcap_{F\in\mathcal{G}} F = \emptyset$. Thus the Heine Borel property implies the finite intersection property. We leave the converse as a recommended exercise for the reader. ■

Exercise 4.3.8. *Do the exercise just recommended.*

Definition 4.3.9. *We say that a metric space (X, d) is* compact *if it obeys the conditions of Theorem 4.3.6.*

We say that a subset E of a metric space is compact *if the metric space (E, d_E) consisting of E with the restriction metric $d_E(x, y) = d(x, y)$ $[x,\ y \in E]$ is compact.*

Exercise 4.3.10. *(i) Let (X, d) be a metric space and E a subset of X. Show that if U is open in (X, d), then $V = U \cap E$ is open in (E, d_E). By considering balls $B_d(x, \delta) = \{y \in X \ : \ d(x, y) < \delta\}$, in X, balls $B_{d_e}(e, \delta) = \{f \in E \ : \ d_E(e, f) < \delta\}$ in E and sets*

$$U = \bigcup \{B_d(v, \delta) \ : \ v \in E,\ \delta > 0,\ B_d(v, \delta) \subset V\},$$

or otherwise, show that, if V is open in (E, d_e), then $V = U \cap E$ for some U open in (X, d).

(ii) Show that a subset E of a metric space is compact if and only if, given any collection $\mathcal{U}$ of open sets with $\bigcup_{U\in\mathcal{U}} U \supseteq E$, we can find a finite subset $\mathcal{V}$ of $\mathcal{U}$ with $\bigcup_{V\in\mathcal{V}} V \supseteq E$. (We say every open cover $\mathcal{U}$ has a finite subcover.)

Exercise 4.3.11. *(i) Use the Bozano–Weierstrass property to show that a compact subset of metric space is closed.*

Give an example to show that the converse of the result just stated need not be the case.

(iii) Produce four proofs of the next sentence, using each of the three properties discussed in Theorem 4.3.6 (together with Exercise 4.3.10) and Lemma 4.3.7 in turn. A subset of a compact metric space is compact only if it is closed.

Exercise 4.3.12. *Explain why a subset of* $\mathbb{R}^n$ *(with its usual metric) is complete and totally bounded if and only if it is closed and bounded. Conclude that a subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded. Recover Theorem 4.3.1.*

We shall mainly use conditions (i) and (ii) of Theorem 4.3.6 when we wish to establish results about compact metric spaces. Proofs using condition (i) (the Heine–Borel property) have the advantage that they only use the open set structure and so carry over to more general objects called *compact topological spaces*. I recommend that the reader practises using Heine–Borel proofs when she has the choice.

Exercise 4.3.13. *(This exercise occurs very early in many university courses, but there is no harm in briefly revisiting it with greater experience.) Let* $f : X \to Y$ *be a function and let* $\mathcal{A}$ *be a collection of sets in* X*, and* $\mathcal{B}$ *be a collection of sets in* Y*. Show that*

$$f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) = \bigcup_{B\in\mathcal{B}} f^{-1}(B),\ f^{-1}\left(\bigcap_{B\in\mathcal{B}} B\right) = \bigcap_{B\in\mathcal{B}} f^{-1}(B),\ \textit{and}\ f\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} f(A).$$

Show that

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A),$$

but give an example in which X*,* Y *and* $\mathcal{A}$ *are finite and equality does not hold.*

Theorem 4.3.14. *Let* (X, d) *and* (Y, ρ) *be metric spaces and* $f : X \to Y$ *a continuous function. Then, if* E *is a compact subset of* X*, it follows that* $f(E)$ *is a compact subset of* Y*. (More briefly, the continuous image of a compact set is compact.)*

Proof. Suppose that $\mathcal{U}$ is a collection of open sets in (Y, ρ) with $\bigcup_{U\in\mathcal{U}} U \supseteq f(E)$. Then $\tilde{\mathcal{U}} = \{f^{-1}(U) : U \in \mathcal{U}\}$ is a collection of open sets in (X, d) with

$$\bigcup_{W\in\tilde{\mathcal{U}}} W = \bigcup_{U\in\mathcal{U}} f^{-1}(U) = f^{-1}\left(\bigcup_{U\in\mathcal{U}} U\right) \supseteq f^{-1}(f(E)) \supseteq E.$$

Since E is compact we can find a finite collection $W_j = f^{-1}(U_j)$ $[U_j \in \mathcal{U},\ 1 \leq j \leq m]$ with $\bigcup_{j=1}^n W_j \supseteq E$. It follows that

$$\bigcup_{j=1}^n U_j = \bigcup_{j=1}^n f(W_j) = f\left(\bigcup_{j=1}^n W_j\right) \supseteq f(E)$$

and we have shown that every open cover of $f(E)$ has a finite subcover. ■

As an immediate corollary, we obtain a theorem guaranteeing the existence of a maximum under certain circumstances.

Theorem 4.3.15. *Suppose (X, d) is a compact metric space and $f : X \to \mathbb{R}$ a continuous function (for the usual metric on $\mathbb{R}$). Then there exists an $x_0 \in X$ such that $f(x_0) \geq f(x)$ for all $x \in X$. (In other words a real valued continuous function on a compact metric space attains a maximum.)*

Proof. By Theorem 4.3.14, $f(X)$ is a non-empty compact set in $\mathbb{R}$. A compact set in $\mathbb{R}$ is closed and bounded. Since $f(X)$ is non-empty and bounded it has a supremum a, say, and we can find $x_n \in X$ with $f(x_n) \to a$. Since $f(X)$ is closed, we have $a \in f(X)$ and we can find an $x_0 \in X$ with $f(x_0) = a$. ■

If the reader feels that this proof is rather sophisticated, she can find a more direct one in Exercise 4.10.10.

Here is a simple application of Theorem 4.3.15.

Exercise 4.3.16. *If (X, d) is a metric space show that*

$$|d(x, y) - d(x, z)| \leq d(y, z).$$

Deduce that, if E is a non-empty subset of X and $a \in X$, the function $f : E \to \mathbb{R}$ given by $f(e) = d(a, e)$ is continuous. Conclude that, if E is a compact set, there exists an $e_0 \in E$ such that

$$d(a, e_0) \leq d(a, e)$$

for all $e \in E$. (In other words there is a closest point in E to a.)

Is the closest point always unique? Give a proof or counter-example.

Show also that if X is compact then there exist a, $b \in X$ such that $d(a, b) \geq d(x, y)$ for all x, $y \in X$. (We sometimes call $d(a, b)$ the diameter *of X.)*

Exercise 4.10.11 gives some extensions of this result. The reader should have no difficulty in using the Heine–Borel property to prove the following version of a standard theorem.

Exercise 4.3.17. [Uniform continuity] *Suppose (X, d) is a compact metric space (Y, ρ) is a metric space and $f : X \to Y$ is continuous. Show that given any $\epsilon > 0$, we can find a $\delta > 0$ such that*

$$d(x, y) \leq \delta \Rightarrow \rho(f(x), f(y)) < \epsilon.$$

[*Hint: Consider a cover of balls $B(x, \delta_x)$ such that $y \in B(x, 2\delta_x)$ implies that $\rho(f(x), f(y)) < \epsilon/2$.*]

4.4 The Stone–Weierstrass theorem

Theorem 4.3.15 tells us that we can generalise ideas about $C_{\mathbb{F}}([a, b])$ the space of continuous functions $f : [a, b] \to \mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$), equipped with the uniform norm, to $C_{\mathbb{F}}(X)$ where (X, d) is a compact metric space. (The reader should note that this section requires us to distinguish between $C_{\mathbb{R}}$ and $C_{\mathbb{C}}$.)

Exercise 4.4.1. *(Only write the things out that you feel you need to.) If (X, d) is a compact metric space and $C_{\mathbb{F}}(X)$ is the set of continuous functions $f : X \to \mathbb{F}$, check that $C_{\mathbb{F}}(X)$ is a vector space over $\mathbb{F}$. Check also that, if f, $g \in C_{\mathbb{F}}(X)$, the product fg (defined by $(fg)(x) = f(x)g(x)$) also lies in $C_{\mathbb{F}}(X)$.*

Check that

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

gives a well defined norm (the uniform norm) on $C_{\mathbb{F}}(X)$ and that $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$. Show, by means of an example, that the inequality may be strict.

The reader may well feel that it is unnecessary to provide a proof of the next theorem, but I include one as a useful model for similar completeness proofs.

Theorem 4.4.2. *Let (X, d) be a compact metric space. The space $(C_{\mathbb{F}}(X), \| \ \|_\infty)$ is complete.*

Proof. Fix $x \in X$ for the time being and observe that, since

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty,$$

$f_n(x)$ is a Cauchy sequence in $\mathbb{F}$ and so converges to a limit $f(x)$ say.

We note that

$$\begin{aligned} |f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| &\leq \|f_n - f_m\|_\infty + |f_m(x) - f(x)| \\ &\leq \sup_{p,q \geq n} \|f_p - f_q\|_\infty + |f_m(x) - f(x)| \to \sup_{p,q \geq n} \|f_p - f_q\|_\infty \end{aligned}$$

as $m \to \infty$. Thus

$$|f_n(x) - f(x)| \leq \sup_{p,q \geq n} \|f_p - f_q\|_\infty. \qquad \bigstar$$

We now show that $f : X \to \mathbb{F}$ is continuous. Let $x \in X$ and $\epsilon > 0$ be given. We can find an N such that $\|f_p - f_q\|_\infty < \epsilon/3$ for all $p, q \geq N$. Since f_N is continuous, we can find a $\delta > 0$ (depending on x and N) such that

$$d(x, y) < \delta \Rightarrow |f_N(x) - f_N(y)| < \epsilon/3.$$

Thus, using $\bigstar$,

$$|f(x) - f(y)| \leq |f_N(x) - f_N(y)| + |f_N(x) - f(x)| + |f_N(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

whenever $d(x, y) < \delta$. Since x and ϵ were arbitrary, we have shown that f is continuous.

The inequality $\bigstar$ can now be reinterpreted to give

$$\|f - f_n\|_\infty \leq \sup_{p,q \geq n} \|f_p - f_q\|_\infty \to 0$$

as $n \to \infty$ so we are done. ■

Our proof of the completeness of $(C_{\mathbb{F}}(X), \| \ \|_\infty)$ followed a standard pattern. Given a Cauchy sequence with respect to a metric, show that it converges in *some sense* to *something*. Now show that the something lies in the correct space and finally show that the sequence converges in the correct metric to the something.

If the reader wants to practice these ideas, she can do Exercises 4.10.18 and 4.10.19.

In the 1940's the editor of the *Mathematical Magazine* wrote to Marshall Stone requesting an article to help relaunch the journal. In reply Stone sent an account of a generalisation of the Weierstrass polynomial approximation theorem that he had developed.

Definition 4.4.3. *Let (X, d) be a compact metric space. We say that $\mathcal{A}$ is a* subalgebra *of $C_{\mathbb{F}}(X)$ if the following two conditions hold.*

(i) $\mathcal{A}$ is non-empty and

$$f,\ g \in \mathcal{A},\ \lambda,\ \mu \in \mathcal{F} \Rightarrow \lambda f + \mu g \in \mathcal{A}.$$

(In other words, $\mathcal{A}$ is vector subspace of $C_{\mathbb{F}}(X)$.)

(ii) $f,\ g \in \mathcal{A} \Rightarrow f \times g \in \mathcal{A}$. (In other words $\mathcal{A}$ is algebraically closed under multiplication.)

Exercise 4.4.4. *Let (X, d) be a compact metric space. If $\mathcal{A}$ is a subalgebra of $C_{\mathbb{F}}(X)$ and we use the standard uniform norm, show that* $\operatorname{Cl}\mathcal{A}$ *is a subalgebra of $C_{\mathbb{F}}(X)$.*

Exercise 4.4.5. *(i) Verify that, if $a,\ b \in \mathbb{R}$, then*

$$\max\{a, b\} + \min\{a, b\} = a + b,\ \max\{a, b\} - \min\{a, b\} = |a - b|$$

and deduce that

$$\max\{a, b\} = \frac{1}{2}((a + b) + |a - b|),\ \min\{a, b\} = \frac{1}{2}((a + b) - |a - b|).$$

(ii) If $a,\ b,\ c,\ d \in \mathbb{R}$ and $a \neq b$ show that we can find $\lambda,\ \mu \in \mathbb{R}$ such that

$$\lambda + \mu a = c,$$
$$\lambda + \mu b = d.$$

Here is Stone's result.

Theorem 4.4.6. [The Stone–Weierstrass theorem] *Let (X, d) be a compact metric space. Suppose that $\mathcal{A}$ is a subalgebra of $C_{\mathbb{R}}(X)$ and the following two conditions hold.*

(i) $1 \in \mathcal{A}$. (Thus $\mathcal{A}$ contains the constant functions.)

(ii) If $x,\ y \in X$ and $x \neq y$, then there exists an $f \in \mathcal{A}$ with $f(x) \neq f(y)$. (We say that $\mathcal{A}$ separates points.*)*

Then $\mathrm{Cl}\,\mathcal{A} = C(X)$ (that is to say, $\mathcal{A}$ is uniformly dense in $C(X)$).

Proof. By Exercise 4.4.4, it suffices to consider the case when $\mathcal{A}$ is closed and to show that $\mathcal{A} = C_{\mathbb{R}}(X)$.

We first show that, if $f \in \mathcal{A}$, then $|f| \in \mathcal{A}$. Since $|\lambda f| = |\lambda||f|$ we need only consider the case when $\|f\|_\infty \leq 1$. We know from Lemma 2.6.9 that there exist a sequence P_n of polynomials with

$$P_n(t) \to |t|$$

uniformly on $[-1, 1]$ and (since $\mathcal{A}$ is an algebra) $P_n(f) \in \mathcal{A}$ and

$$\left\|P_n(f) - |f|\right\|_\infty \to 0$$

as $n \to \infty$. Thus $|f| \in \mathcal{A}$.

We now observe that, if we set

$$f \vee g(t) = \max\{f(t), g(t)\},\ f \wedge g(t) = \min\{f(t), g(t)\},$$

we have

$$f \vee g = \frac{1}{2}((f + g) + |f - g|),\ f \wedge g = \frac{1}{2}((f + g) - |f - g|),$$

so

$$f, g \in \mathcal{A} \Rightarrow f \vee g, f \wedge g \in \mathcal{A}.$$

We note that $(f \vee g) \vee h = f \vee (g \vee h)$, so we can write $f \vee g \vee h = f \vee (g \vee h)$ in the usual way. By induction

$$f_i \in \mathcal{A} \Rightarrow f_1 \vee f_2 \vee \ldots \vee f_n \in \mathcal{A}$$

and similarly

$$f_i \in \mathcal{A} \Rightarrow f_1 \wedge f_2 \wedge \ldots \wedge f_n \in \mathcal{A}.$$

Now suppose $F \in C_{\mathbb{R}}(X)$. If $x, y \in X$ and $x \neq y$, then we know that we can find $g_{xy} \in \mathcal{A}$ such that $g_{xy}(x) \neq g_{xy}(y)$ and so we can find $\lambda_{xy}, \mu_{xy} \in \mathbb{R}$ such that

$$\lambda_{xy} + \mu_{xy} g_{xy}(x) = F(x),$$
$$\lambda_{xy} + \mu_{xy} g_{xy}(y) = F(y).$$

If we set $f_{xy} = \lambda_{xy} + \mu_{xy} g_{xy}$, we see that

$$f_{xy} \in \mathcal{A},\ f_{xy}(x) = F(x),\ f_{xy}(y) = F(y).$$

If $x = y$, we set $f_{xy}(t) = F(x)$ for all $t \in X$ to obtain the same result.

Combining the results of the last two paragraphs with the compactness of X gives the required result. For suppose $F \in C_{\mathbb{R}}(X)$ and $\epsilon > 0$ are given. Fixing $x \in X$ for the moment, we know that, given any $y \in Y$, we can find an $f_{xy} \in \mathcal{A}$ such that $f_{xy}(x) = F(x)$ and $f_{xy}(y) = F(y)$. Since $f_{xy} - F$ is continuous, we can find a $\delta_{xy} > 0$ such that $|f_{xy}(t) - F(t)| < \epsilon$ for all $t \in B(y, \delta_{xy})$. The balls $B(y, \delta_{xy})$ form an open cover of X, so we can find a finite subcover $B(y(j), \delta_{xy(j)})$ $[1 \leq j \leq n_x]$, say. Set

$$f_x = f_{xy(1)} \wedge f_{xy(2)} \wedge \ldots \wedge f_{xy(n_x)}.$$

Automatically, $f_x \in \mathcal{A}$, $f_x(x) = f(x)$ and

$$f_x(t) \leq f_{xy(j)}(t) < F(t) + \epsilon$$

on each $B(y(j), \delta_{xy(j)})$. We thus have $f_x(t) < F(t) + \epsilon$ for all $t \in X$.

Since $f_x - F$ is continuous, we can find a $\eta_x > 0$ such that $|f_x(t) - F(t)| < \epsilon$ for all $t \in B(x, \eta_x)$. We now allow x to vary. The balls $B(x, \eta_x)$ form an open cover of X, so we can find a finite subcover $B(x(k), \eta_{x(k)})$ $[1 \leq k \leq m]$, say. Set

$$f = f_{x(1)} \vee f_{x(2)} \vee \ldots \vee f_{x(m)}.$$

Automatically, $f \in \mathcal{A}$ and

$$f(t) < F(t) + \epsilon$$

for all $t \in X$. Automatically, also,

$$f(t) \geq f_{x(k)}(t) > F(t) - \epsilon$$

on each $B(x(k), \eta_{x(k)})$, so $f(t) > F(t) - \epsilon$ for all $t \in X$. Thus $f \in \mathcal{A}$ and $\|f - F\|_\infty < \epsilon$. Since ϵ and F were arbitrary, we are done. ∎

(For a slight generalisation see Exercise 4.10.21.)

Exercise 4.4.7. *Find a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that*

$$0 \leq f_n(x) \leq f_{n+1}(x) \leq 1 \text{ for all } n \geq 1 \text{ and all } x \in [0,1]$$

and such that, given $x \in [0,1]$, we can find an $N(x)$ with

$$f_n(x) = f_{N(x)}(x) \text{ for all } n \geq N(x),$$

but the function given by $f(x) = f_{N(x)}(x)$ is not continuous.

Remark The real numbers satisfy the *fundamental axiom of analysis* which takes a number of equivalent forms.

(1) Every bounded sequence has a convergent subsequence.

(2) Every Cauchy sequence converges and, given any $\epsilon > 0$, we can find a positive integer n with $\epsilon > n^{-1}$.

(3) Every increasing sequence bounded above tends to a limit.

Version (1) leads to the idea of compactness. Version (2) leads to the idea of completeness. We shall not pursue version (3) but remark that *latices* involving binary operators like $\wedge$ and $\vee$ provide a possible way of generalising (3).

The Stone–Weierstrass theorem requires modification before we can use it in the context of $C_{\mathbb{C}}(X)$.

Exercise 4.4.8. *(i) Let $\mathcal{P}$ be the subset of $C_{\mathbb{C}}(\mathbb{T})$ consisting of trigonometric polynomials of the form*

$$\sum_{j=0}^{n} a_j e^{2\pi i j t}$$

with $a_j \in \mathbb{C}$. Check that $\mathcal{P}$ is a subalgebra of $C_{\mathbb{C}}(\mathbb{T})$ containing the constant functions and separating points. Show, however, that, if $P \in \mathcal{P}$ and $e_{-1}(t) = e^{-2\pi i t}$, then

$$\int_{\mathbb{T}} |p(t) - e_{-1}(t)|^2 \, dt \geq 1.$$

Deduce that $\operatorname{Cl} \mathcal{P} \neq C_{\mathbb{C}}(\mathbb{T})$.

(ii) If the reader knows complex variable theory, we can look at the ideas of (i) rather differently. Let

$$D = \{z \in \mathbb{C} : |z| < 1\} \text{ and } \bar{D} = \{z \in \mathbb{C} : |z| \le 1\}$$

and let $\mathcal{Q}$ be the subset of $C_{\mathbb{C}}(\bar{D})$ consisting of all polynomials of the form

$$\sum_{j=0}^{n} a_j z^j$$

with $a_j \in \mathbb{C}$. By Morera's theorem, the uniform limit of analytic functions is analytic, so any $f \in \operatorname{Cl}\mathcal{Q}$ must be analytic on D. Deduce that if $g(z) = z^*$ (the complex conjugate of z), then $g \notin \operatorname{Cl}\mathcal{Q}$ and so we have $\operatorname{Cl}\mathcal{Q} \neq C_{\mathbb{C}}(\bar{D})$.

Part (ii) of the example just given suggests a natural modification of Theorem 4.4.6.

Theorem 4.4.9. [The complex Stone–Weierstrass theorem] *Let (X, d) be a compact metric space. Suppose that $\mathcal{A}$ is a subalgebra of $C_{\mathbb{C}}(X)$ and the following three conditions hold.*
(i) $1 \in \mathcal{A}$.
(ii) If $x, y \in X$ and $x \neq y$, then there exists an $f \in \mathcal{A}$ with $f(x) \neq f(y)$.
(iii) $f \in \mathcal{A} \Rightarrow f^* \in \mathcal{A}$.
Then $\operatorname{Cl}\mathcal{A} = C_{\mathbb{C}}(X)$.

The proof is laid out in the next exercise.

Exercise 4.4.10. *We use the notation and hypotheses of Theorem 4.4.9. We write*

$$\Re\mathcal{A} = \{\Re f : f \in \mathcal{A}\}.$$

(i) Show that $\Re\mathcal{A} \subseteq \mathcal{A}$ and that $f \in \mathcal{A} \Rightarrow \Im f \in \Re\mathcal{A}$.
(ii) Use the real Stone–Weierstrass theorem to show that $\Re\mathcal{A}$ is uniformly dense in $C_{\mathbb{R}}(X)$.
(iii) Deduce that $\mathcal{A}$ is uniformly dense in $C_{\mathbb{C}}(X)$.
(iv) At which point (or points) of the argument did you need to use the fact that $f \in \mathcal{A} \Rightarrow f^* \in \mathcal{A}$?

Although it is often possible in a concrete situation to replace the use of the Stone–Weierstrass theorem by ad hoc arguments which may give more information (look at our earlier proofs of Féjer's theorem and the Weierstrass polynomial approximation theorem), the Stone–Weierstrass theorem is a powerful general tool. Here are two examples of its use.

Exercise 4.4.11. *Let (X, d) and (Y, ρ) be compact metric spaces and give $X \times Y$ a product metric $\rho\big((x, y), (x', y')\big) = d(x, x') + d(y, y')$.*

(i) If $f \in C_{\mathbb{F}}(X)$, $g \in C_{\mathbb{F}}(Y)$ show that the equation

$$f \otimes g(x, y) = f(x)g(y)$$

defines an element $f \otimes g$ of $C_{\mathbb{F}}(X \times Y)$.

(ii) By using the Stone–Weierstrass theorem, show that, given $F \in C_{\mathbb{F}}(X \times Y)$ and $\epsilon > 0$, we can find $f_j \in C_{\mathbb{F}}(X)$, $g_j \in C_{\mathbb{F}}(Y)$ $[1 \leq j \leq n]$ such that

$$\left\| F - \sum_{j=1}^{n} f_j \otimes g_j \right\|_\infty < \epsilon.$$

(Prove the result for $\mathbb{F} = \mathbb{R}$ and just state the changes needed for $\mathbb{F} = \mathbb{C}$.)

(iii) The rest of the exercise is an application of (ii) to prove a result on the interchange of order for integration. We use results on the integration of continuous functions on bounded intervals but nothing beyond this.

Show, using uniform continuity that, if $F : [a, b] \times [c, d] \to \mathbb{C}$ is continuous

$$u(y) = \int_a^b F(x, y)\, dx$$

is a continuous function of y and so the repeated integral

$$\int_c^d \left(\int_a^b F(x, y)\, dx \right) dy$$

is well defined.

(iv) By first proving the result for $f \otimes g$ and then using (ii), show that

$$\int_c^d \left(\int_a^b F(x, y)\, dx \right) dy = \int_a^b \left(\int_c^d F(x, y)\, dy \right) dx$$

for all $F \in C([a, b] \times [c, d])$.

The next exercise concerns separability (see Definition 1.5.23).

Exercise 4.4.12. *Let (X, d) be a compact metric space.*

(i) By using total boundedness, or otherwise, show that X is separable.

(ii) Let A be a countable dense subset of X. Show that the equation

$$f_a(x) = \max\{0, 1 - d(a, x)\}$$

defines a continuous function $f_a : X \to \mathbb{R}$ $[a \in A]$*. Let* $\mathcal{P}$ *be the collection of functions of the form*

$$\lambda_0 + \sum_{j=1}^{m} \lambda_j f_{a(j)}$$

with $\lambda_j \in \mathbb{R}$*,* $a(j) \in A$ *and let* $\mathcal{A}$ *be the collection of finite products* $g_1 g_2 \dots g_u$ *with* $g_v \in \mathcal{P}$*.*

Show that, given $F \in C_{\mathbb{R}}(X)$ *and* $\epsilon > 0$*, we can find an* $f \in \mathcal{A}$ *such that* $\|F - f\|_\infty < \epsilon$*.*

By identifying an appropriate countable subset of $\mathcal{A}$*, show that* $(C_{\mathbb{R}}(X), \| \|_\infty)$ *is separable.*

(iii) Show also that $(C_{\mathbb{C}}(X), \| \|_\infty)$ *is separable.*

(iv) Consider $(C_{\mathbb{R}}([0, 1]), \| \|_\infty)$*. Let* $B = \{f \in C_{\mathbb{R}}([0, 1]) : \|f\|_\infty \le 1\}$*. Is* B *totally bounded? Why?*

4.5 Finite dimensions

Most of the vector spaces discussed in this book are infinite dimensional. However, it is natural to try and use intuition based on finite dimensional spaces (often $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$) to understand these larger spaces. This attempt is often very successful[4], but it is important to understand that finite dimensional spaces have several properties that do not carry over to the infinite dimensional case. This section discusses two of these properties.

Our first theorem tells us that 'all normed spaces of a given finite dimension are essentially the same'.

Theorem 4.5.1. *Let* $\| \|$ *and* $\| \|_*$ *be two norms on a finite dimensional space* V *over* $\mathbb{F}$*. Then there exist* $A \ge B > 0$ *such that*

$$A\|\mathbf{v}\|_* \ge \|\mathbf{v}\| \ge B\|\mathbf{v}\|_*$$

for all $\mathbf{v} \in V$*.*

Proof. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be a basis for V. We will write $\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j$ and

$$\|\mathbf{x}\|_2 = \sum_{j=1}^{n} x_j^2$$

so that $\| \|_2$ corresponds to the standard Euclidean norm with the $\mathbf{e}_j$ orthonormal. It suffices to prove the result with $\| \|_* = \| \|_2$.

[4]'I am not afraid to confess that, if I needed to, I would willingly hold up one candle to St. Michael and another to his dragon.' (Montaigne, *Essays*, Book 3, Chapter 1).

One of our desired inequalities follows from the observation that

$$\begin{aligned}\|\mathbf{x}\| = \left\|\sum_{j=1}^{n} x_j\mathbf{e}_j\right\| &\leq \sum_{j=1}^{n} |x_j|\|\mathbf{e}_j\| \leq \max_{1\leq j\leq n}\|\mathbf{e}_j\|\sum_{j=1}^{n}|x_j|\\ &\leq \max_{1\leq j\leq n}\|\mathbf{e}_j\| n^{1/2}\left(\sum_{j=1}^{n}|x_j|^2\right)^{1/2} \leq A\left\|\sum_{j=1}^{n} x_j\mathbf{e}_j\right\|_2\end{aligned}$$

where $A = n^{1/2}\max_{1\leq j\leq n}\|\mathbf{e}_j\|$. (We used the Cauchy–Schwarz inequality to obtain

$$\sum_{j=1}^{n}|x_j| = \sum_{j=1}^{n} 1\times|x_j| \leq \left(\sum_{j=1}^{n}1^2\right)^{1/2}\left(\sum_{j=1}^{n}|x_j|^2\right)^{1/2} = n^{1/2}\left(\sum_{j=1}^{n}|x_j|^2\right)^{1/2},$$

but cruder estimates would also do.)

The other inequality requires[5] a more subtle argument. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = \|\mathbf{x}\|$. The triangle inequality for $\|\ \|$ yields

$$|f(\mathbf{x}) - f(\mathbf{y})| = \big|\|\mathbf{x}\| - \mathbf{y}\|\big| \leq \|\mathbf{x} - \mathbf{y}\|$$

so, by the result established in the previous paragraph,

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq A\|\mathbf{x} - \mathbf{y}\|_2,$$

and f is continuous for the standard norm.

The set

$$S = \left\{\mathbf{x}\in\mathbb{F}^n : \sum_{j=1}^{n}|x_j|^2 = 1\right\}$$

is compact for the standard norm. Thus the continuous function f attains a minimum at some point $\mathbf{a}\in S$. Since $\mathbf{a}\neq\mathbf{0}$, we know that $\sum_{j=1}^{n} a_j\mathbf{e}_j \neq 0$ and so

$$\left\|\sum_{j=1}^{n} x_j\mathbf{e}_j\right\|_* = f(\mathbf{x}) \geq f(\mathbf{a}) = \left\|\sum_{j=1}^{n} a_j\mathbf{e}_j\right\|_* > 0$$

for all $\mathbf{x}\in S$. Writing

$$B = \left\|\sum_{j=1}^{n} a_j\mathbf{e}_j\right\|^{-1},$$

we obtain $\|\mathbf{v}\| \geq B$ for all $\mathbf{v}\in V$ with $\|\mathbf{v}\|_2 = 1$ and so $\|\mathbf{v}\| \geq B\|\mathbf{v}\|_2$ for all $\mathbf{v}\in V$. ■

[5]The word *requires* is chosen deliberately. It can be shown that the result depends on the completeness of $\mathbb{R}$.

Exercise 4.5.2. *Why did it suffice to prove the result when $\| \|_*$ is the standard norm?*

Exercise 4.5.3. *Show that every norm on a finite dimensional vector space is complete.*

We already know that there exist infinite dimensional vector spaces with incomplete norms (see, for example, Exercise 1.2.10), but there are plenty of other examples in this book.

Even if we restrict ourselves to separable complete spaces there are many essentially different kinds of infinite dimensional normed spaces.

Exercise 4.5.4. *In this question we work with real sequences. From Section 2.4 we know that $(l^1(\mathbb{Z}), \| \|_1)$ and $(l^2(\mathbb{Z}), \| \|_2)$ are both complete separable infinite dimensional spaces. In this exercise we show that they are different. More formally, we show that there does not exist a linear map $T : l^1 \to l^2$ and a constant $K > 0$ such at*

$$K\|\mathbf{u}\|_1 \geq \|T\mathbf{u}\|_2 \geq K^{-1}\|\mathbf{u}\|_1$$

for all $\mathbf{u} \in l^1$.

(i) Suppose that $\mathbf{v}_1, \mathbf{v}_2 \in l^2$. Prove the parallelogram law (true for all real inner product spaces)

$$\|\mathbf{v}_1 + \mathbf{v}_2\|_2^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_2^2 = 2\|\mathbf{v}_1\|_2^2 + 2\|\mathbf{v}_2\|_2^2.$$

(ii) Deduce that, if $\mathbf{v}_1, \mathbf{v}_2 \in l^2$, then

$$\min\{\|\mathbf{v}_1 + \mathbf{v}_2\|_2^2, \|\mathbf{v}_1 - \mathbf{v}_2\|_2^2\} \leq \|\mathbf{v}_1\|_2^2 + \|\mathbf{v}_2\|^2.$$

(iii) Show that, if $\mathbf{v}_j \in l^2$ $[1 \leq j \leq 2^n]$ we can find $\zeta_j \in \{-1, 1\}$ $[1 \leq j \leq 2^n]$ such that

$$\left\| \sum_{j=1}^{2^n} \zeta_j \mathbf{v}_j \right\|_2^2 \leq \sum_{j=1}^{2^n} \|\mathbf{v}_j\|_2^2.$$

(iv) Consider l^1 and let $\mathbf{u}_j$ be the infinite sequence with 1 in the jth place and 0 elsewhere. Show that, if $\zeta_j \in \{-1, 1\}$, then

$$\left\| \sum_{j=1}^{N} \zeta_j \mathbf{u}_j \right\|_1 = N.$$

(v) By using these results, or otherwise, prove the result stated in the opening paragraph of this exercise.

Exercise 4.5.5. *Show, however, that* $l^1 \supseteq l^2$ *and* $\|\mathbf{a}\|_1 \geq \|\mathbf{a}\|_2$ *for all* $\mathbf{a} \in l^2$. *Is* l^2 *a subspace of* l^1? *Is* l^2 *a closed subset of of* $(l^1, \| \|_1)$? *Give reasons.*

Notice the following important corollary of Theorem 4.5.1.

Lemma 4.5.6. *Any finite dimensional subspace of a normed space is closed.*

Proof. Let E be a finite dimensional subspace of a normed space $(V, \| \|)$. Let $\| \|_E$ be the restriction of $\| \|$ to E. We know that $(E, \| \|_E)$ is complete (by Theorem 4.5.1) and we know (by Exercise 1.5.10) that any complete subset of a metric space is closed. ■

Exercise 4.5.7. *Consider* $(l^1, \| \|_1)$. *Show that the collection* E *of sequences only finitely many of whose entries are non-zero forms a subspace which is not closed.*

Exercise 4.5.8. *Consider the following statement. 'If* $(V, \| \|_V)$ *and* $(W, \| \|_W)$ *are normed vector spaces with* V *finite dimensional, then every linear map* $T : V \to W$ *is is continuous.'*

(i) Prove the statement using Theorem 4.5.1.

(ii) Prove Theorem 4.5.1 using the statement.

(iii) Let V *be the subspace of* l^1 *consisting of those sequences with only finitely many terms non-zero equipped with the* l^1 *norm. Find a discontinuous linear function* $T : V \to \mathbb{R}$.

Exercise 4.5.9. *Consider* $\mathbb{R}^2$ *with the usual vector space structure. Show that*

$$\|(x, y)\|_{[n]} = n|x| + n^{-1}|y|$$

defines a norm on $\mathbb{R}^2$. *If* $A_n,\ B_n > 0$ *and*

$$A_n\|\mathbf{v}\|_{[n]} \geq \|\mathbf{v}\|_{[1]} \geq B_n\|\mathbf{v}\|_{[n]}$$

for all $\mathbf{v} \in \mathbb{R}^2$ *show that* $A_n \to \infty$ *and* $B_n \to 0$.

We now look at a result of direct relevance to this chapter.

Theorem 4.5.10. *The closed unit ball*

$$\bar{B} = \{\mathbf{v} \in V : \|\mathbf{v}\| \leq 1\}$$

of an infinite dimensional normed space $(V, \| \|)$ *is not compact.*

Our proof will use a simple but very important observation.

Lemma 4.5.11. [Lemma of F. Riesz] *Suppose that E is a subspace of a normed space $(V, \|\ \|)$ such that E is complete under the restriction of the norm $\|\ \|$ to E. If $E \neq V$, then, given any $\epsilon > 0$, we can find $\mathbf{v} \in V$ with $\|\mathbf{v}\| = 1$ such that*

$$\|\mathbf{v} - \mathbf{e}\| > 1 - \epsilon$$

for all $\mathbf{e} \in E$.

It may be worth looking at Exercise 1.5.10 to provide some context.

We obtain Riesz's lemma by proving its contrapositive.

Theorem 4.5.12. [Uniform approximation theorem] *Suppose that E is a subspace of a normed space $(V, \|\ \|)$ such that E is complete for the restriction of the norm $\|\ \|$ to E. If there exists an $\epsilon > 0$ such that, given $\mathbf{v} \in V$ with $\|\mathbf{v}\| = 1$, we can find an $e \in E$ with*

$$\|\mathbf{e} - \mathbf{v}\| < 1 - \epsilon,$$

then $E = V$.

This is yet another appearance of the idea successive approximation.

Proof of Theorem 4.5.12. Suppose the hypothesis holds. Then, using the triangle inequality, we know that, given $\mathbf{v} \in V$ with $\|\mathbf{v}\| = 1$, we can find an $\mathbf{e} \in E$ with

$$\|\mathbf{e} - \mathbf{v}\| < 1 - \epsilon \text{ and } \|\mathbf{e}\| \leq 2.$$

Rescaling, we see that given $\mathbf{v} \in V$, we can find an $\mathbf{e} \in E$ with

$$\|\mathbf{e} - \mathbf{v}\| < (1 - \epsilon)\|\mathbf{v}\| \text{ and } \|\mathbf{e}\| \leq 2\|\mathbf{v}\|.$$

Now suppose that $\mathbf{w} \in V$. We proceed inductively. Set $\mathbf{w}_0 = \mathbf{w}$. If $\mathbf{w}_n \in V$ is defined, we know by the previous paragraph that we can find $\mathbf{e}_{n+1} \in V$ such that

$$\|\mathbf{e}_{n+1} - \mathbf{w}_n\| < (1 - \epsilon)\|\mathbf{w}_n\| \text{ and } \|\mathbf{e}_{n+1}\| \leq 2\|\mathbf{w}_n\|.$$

If we set $\mathbf{w}_{n+1} = \mathbf{w}_n - \mathbf{e}_{n+1}$, a simple induction shows that

$$\|\mathbf{w}_n\| < (1 - \epsilon)^n\|\mathbf{w}\| \text{ and } \|\mathbf{e}_n\| \leq 2\|\mathbf{w}_n\| \leq 2(1 - \epsilon)^n\|\mathbf{w}\|.$$

Since E is a subspace, $\sum_{n=1}^N \mathbf{e}_n \in E$ for all N. By comparison with a geometric series, $\sum_{n=1}^\infty \|\mathbf{e}_n\|$ converges and so, by completeness, there exists an $\mathbf{e} \in E$ such that

$$\left\|\mathbf{e} - \sum_{n=1}^N \mathbf{e}_n\right\| \to 0.$$

On the other hand,

$$\mathbf{w}_N = \mathbf{w} - \sum_{n=1}^N \mathbf{e}_n$$

and $\|\mathbf{w}_N\| \to 0$ as $N \to 0$, so $\|\mathbf{w} - \mathbf{e}\| = 0$ and $\mathbf{w} = \mathbf{e} \in E$. ■

Proof of Theorem 4.5.10. Using Riesz's lemma and the fact that every finite dimensional subspace is complete under the restriction norm, we can find, inductively,

$$\mathbf{e}_n \in \bar{B} = \{\mathbf{v} \in V \,:\, \|\mathbf{v}\| \le 1\}$$

such that

$$\|\mathbf{e}_n - \mathbf{f}\| \ge 1/2 \text{ for all } \mathbf{f} \in \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-1}\}.$$

Automatically $\|\mathbf{e}_r - \mathbf{e}_s\| \ge 1/2$ for all $r \ne s$ so the sequence $\mathbf{e}_n$ has no convergent subsequence. Thus $\bar{B}$ is not compact. ■

Theorem 4.5.10 can be restated in various ways.

Exercise 4.5.13. *Show that the following statements about a normed space V are equivalent.*

(i) V is finite dimensional.
(ii) The closed unit ball of V is compact.
(iii) Every bounded closed subset of V is compact.
(iv) V contains a compact set with non-empty interior.

We give a related example of the use of Riesz's lemma in Exercise 4.10.4.

4.6 Uniform equicontinuity

It is clear that compact metric spaces have very pleasant properties, but Theorem 4.5.10 shows that compact spaces are in some sense 'rather small'.

If we look at $C(X)$ the space of continuous functions on a compact metric space X with the uniform norm, there is a famous theorem which characterises the compact subsets of $C(X)$ exactly.

Definition 4.6.1. *Let (X, d) be a compact metric space. A subset $\mathcal{F} \subseteq C(X)$ is said to be* uniformly equicontinuous[6] *if, given $\epsilon > 0$, we can find a $\delta(\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta(\epsilon)$ and $f \in \mathcal{F}$.*

Thus $\mathcal{F}$ is uniformly equicontinuous if we can choose a $\delta > 0$ *independent of $x \in X$ and of $f \in \mathcal{F}$* such that $d(x, y) < \delta$ forces $|f(x) - f(y)| < \epsilon$.

Our second definition is less novel.

Definition 4.6.2. *Let (X, d) be a compact metric space. A subset $\mathcal{F} \subseteq C(X)$ is said to be* uniformly bounded *if we can find a K such that $\|f\|_\infty \le K$ whenever $f \in \mathcal{F}$.*

[6] Traditionally the word 'equicontinuous' was used instead of the phrase 'uniformly equicontinuous' and the majority of mathematicians follow this older usage. See Exercise 4.10.23.

Theorem 4.6.3. [The Arzelá–Ascoli theorem] *Let (X, d) be a compact metric space. Then $\mathcal{F} \subseteq C(X)$ is compact if and only if $\mathcal{F}$ is closed, uniformly bounded and uniformly equicontinuous.*

We recall that a metric space is compact if and only if it is complete and totally bounded (Theorem 4.3.6). We also remember that, if (Y, ρ) is a complete metric space and $E \subseteq Y$, then (E, ρ_E) is complete if and only if E is closed. Since $(C(X), \| \|_\infty)$ is complete, Theorem 4.6.3 is a consequence of the following lemma.

Lemma 4.6.4. *Let (X, d) be a compact metric space. Then $\mathcal{F} \subseteq C(X)$ is totally bounded if and only if $\mathcal{F}$ is uniformly bounded and uniformly equicontinuous.*

Proof. In this proof we need to use open balls $B_{C(X)}(f, \eta)$ in $(C(X), \| \|_\infty)$ and open balls $B_X(x, \eta)$ in (X, d).

First suppose that $\mathcal{F} \subseteq C(X)$ is totally bounded. Then $\mathcal{F}$ can be covered by a finite set of balls $B_{C(X)}(f_1, 1)$, $B_{C(X)}(f_2, 1)$, $\dots B_{C(X)}(f_p, 1)$ with $f_j \in \mathcal{F}$ and the relation

$$\|f\|_\infty \leq 1 + \max_{1 \leq j \leq p} \|f_j\|_\infty$$

for all $f \in \mathcal{F}$ follows at once. Thus $\mathcal{F}$ is uniformly bounded

To see that $\mathcal{F}$ is uniformly equicontinuous, suppose that $\epsilon > 0$ is given. By total boundedness, we can find a finite set of balls

$$B_{C(X)}(g_1, \epsilon/3),\ B_{C(X)}(g_2, \epsilon/3),\ \dots,\ B_{C(X)}(g_N, \epsilon/3)$$

with $g_k \in \mathcal{F}$ which cover $\mathcal{F}$. Since (X, d) is compact, each g_k is uniformly continuous (see Exercise **??**) and we can find a $\delta_k > 0$ such that

$$d(x, y) < \delta_k \Rightarrow |g_k(x) - g_k(y)| < \epsilon/3.$$

Take $\delta = \min_{1 \leq k \leq N} \delta_k$. If $d(x, y) < \delta$ and $f \in \mathcal{F}$, we can find a $1 \leq k \leq N$ such that $\|f - g_k\|_\infty < \epsilon/3$ and,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - g_k(x)| + |g_k(x) - g_k(y)| + |g_k(y) - f(y)| \\ &\leq 2\|f - g_k\|_\infty + |g_k(x) - g_k(y)| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus $\mathcal{F}$ is uniformly equicontinuous.

To prove the converse, we now suppose $\mathcal{F}$ is uniformly equicontinuous and uniformly bounded. Let $\epsilon > 0$ be given. Since $\mathcal{F}$ is uniformly equicontinuous, we can find a $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon/5$$

for all $f \in \mathcal{F}$. The open balls $B_X(x, \delta/2)$ cover X so, by compactness, we can find a finite set of points $x_1, x_2, \ldots, x_N$ such that

$$\bigcup_{r=1}^{N} B_X(x_j, \delta/2) = X.$$

We observe that, if x, $y \in B_X(x_j, \delta/2)$, then, by the triangle inequality, $d(x, y) < \delta$ and so $|f(x) - f(y)| < \epsilon/5$.

Since $\mathcal{F}$ is uniformly bounded, there exists a K such that $|f(t)| \leq K$ for all $t \in X$ and all $f \in \mathcal{F}$ Simple geometry (or compactness) tells us that we can find $M \geq 1$ and λ_1, λ_2 $\ldots$, $\lambda_M \in \mathbb{F}$ such that, given $|\lambda| \leq K$, we can find a $1 \leq u \leq M$ with $|\lambda_u - \lambda| < \epsilon/5$. Let $\mathcal{A}$ be the collection of functions

$$\alpha : \{x_j \,:\, 1 \leq j \leq N\} \to \{\lambda_u \,:\, 1 \leq u \leq M\}$$

and write

$$\mathcal{F}_\alpha = \{f \in \mathcal{F} \,:\, |f(x_j) - \alpha(x_j)| < \epsilon/5 \text{ for all } 1 \leq j \leq N\}$$

for each $\alpha \in \mathcal{A}$. By the choice of the λ_u, we have $\bigcup_{\alpha\in\mathcal{A}} \mathcal{F}_\alpha = \mathcal{F}$ so, if we take

$$\mathcal{B} = \{\alpha \in \mathcal{A} \,:\, \mathcal{F}_\alpha \neq \emptyset\},$$

we have $\bigcup_{\beta\in\mathcal{B}} \mathcal{F}_\beta = \mathcal{F}$.

If $\beta \in \mathcal{B}$, then, by definition, we can find an $f_\beta \in \mathcal{F}_\beta$. We observe that, if $f \in \mathcal{F}_\beta$, then

$$|f(x_j) - f_\beta(x_j)| \leq |f(x_j) - \beta(x_j)| + |f_\beta(x_j) - \beta(x_j)| < \frac{\epsilon}{5} + \frac{\epsilon}{5} = \frac{2\epsilon}{5}$$

and

$$|f(y) - f_\beta(y)| \leq |f(y) - f(x_j)| + |f(x_j) - f_\beta(x_j)| + |f_\beta(y) - f_\beta(x_j)| < \frac{\epsilon}{5} + \frac{2\epsilon}{5} + \frac{\epsilon}{5} = \frac{4\epsilon}{5}$$

for all $y \in B_X(x_j, \delta/2)$ and all $1 \leq j \leq N$. Since the $B_X(x_j, \delta/2)$ cover X, we have $|f(y) - f_\beta(y)| < \epsilon$ for all $y \in X$, and so $\|f - f_\beta\|_\infty \leq 4\epsilon/5 < \epsilon$.

In the previous paragraph we showed that $B_{C(X)}(f_\beta, \epsilon) \supseteq \mathcal{F}_\beta$ for each $\beta \in \mathcal{B}$. It follows that

$$\bigcup_{\beta\in\mathcal{B}} B_{C(X)}(f_\beta, \epsilon) \supseteq \bigcup_{\beta\in\mathcal{B}} \mathcal{F}_\beta = \mathcal{F}$$

and so we can cover $\mathcal{F}$ with a finite set of open balls of radius ϵ. We have shown that $\mathcal{F}$ is totally bounded. ■

It is important that the abstract proof above does not leave the reader with the impression that uniform equicontinuity is a difficult conceit to grasp.

Exercise 4.6.5. *Give an example of a subset of $C(\mathbb{T})$ which is uniformly bounded but not uniformly equicontinuous. Give an example of subset of $C(\mathbb{T})$ which is uniformly equicontinuous but not uniformly bounded.*

Exercise 4.6.6. *Let n and m be strictly positive integers and let*

$$\mathcal{G} = \{f \in C([0,1]) \,:\, \|f\|_\infty \leq 1 \text{ and } |x-y| \leq 2^{-m} \Rightarrow |f(x)-f(y)| \leq 2^{-n}\}.$$

Find a finite subset Γ of $\mathcal{G}$ consisting of piecewise linear functions such that

$$\bigcup_{g\in\Gamma} B(g, 2^{2-n}) \supseteq \mathcal{G}$$

where, as usual,

$$B(g, 2^{1-n}) = \{f \in C([0,1]) \,:\, \|f-g\|_\infty < 2^{2-n}\}.$$

Exercise 4.6.7. *Imagine that you have to lecture on Arzelá–Ascoli theorem in the case when $X = [0,1]$, and $\mathbb{F} = \mathbb{R}$. Sketch the plan of your lecture, taking full advantage of the special features of $[0,1]$ and drawing plenty of diagrams.*

Here is an alternative proof of the sufficiency part of Theorem 4.6.3 which is well worth looking at.

Exercise 4.6.8. *Let (X,d) be a compact metric space.*
(i) Suppose that $\mathcal{F} \subseteq C(X)$ is uniformly equicontinuous and

$$E = \{x_1,\ x_2,\ x_3, \dots\}$$

is a countable dense subset of X. If $f_m \in \mathcal{F}$ and $f_m(x_j)$ tends to a limit as $m \to \infty$ for each j, show, by considering Cauchy sequences or otherwise, that there is a continuous function f such that $\|f_m - f\|_\infty \to 0$ as $m \to \infty$.
(ii) Suppose that $\mathcal{F} \subseteq C(X)$ is uniformly bounded and

$$E = \{x_1,\ x_2,\ x_3, \dots\}$$

is a countable dense subset of X. Show that we can find sequences $N_j(m)$ such that the $N_j(m)$ form a strictly increasing subsequence of the $N_{j-1}(m)$ and $f_{N_j(m)}(x_j)$ tends to a limit as $m \to \infty$. Show that $f_{N_m(m)}(x_j)$ tends to a limit as $m \to \infty$ for each j.
(iii) If $\mathcal{F}$ is closed, uniformly bounded and uniformly equicontinuous, show that any sequence in $\mathcal{F}$ has a convergent subsequence with limit in $\mathcal{F}$.

The following minor variation of the Arzelá–Ascoli theorem often turns out to be more useful than our original version.

Lemma 4.6.9. *Let (X, d) be a compact metric space. If $\mathcal{F} \subseteq C(X)$, then its closure $\operatorname{Cl}\mathcal{F}$ is compact if and only if $\mathcal{F}$ uniformly bounded and uniformly equicontinuous.*

Exercise 4.6.10. *Prove Lemma 4.6.9.*

Exercise 4.6.11 gives a typical example of a compact subset of $C_{\mathbb{R}}(\mathbb{T})$.

Exercise 4.6.11. *We work in $C_{\mathbb{R}}(\mathbb{T})$ with the uniform norm. We write $\mathcal{F}_M$ for the set of continuously differentiable functions f with $\|f\|_\infty + \|f'\|_\infty \leq M$.*

(i) Show that $\operatorname{Cl}\mathcal{F}_M$ is compact.

(ii) Show that the set of continuously differentiable functions f with $\|f\|_\infty \leq 1$ is not uniformly equicontinuous and the set of continuously differentiable functions f with $\|f'\|_\infty \leq 1$ is not uniformly bounded.

(iii) Let $g(x) = |x|$. Sketch g (note that we work on $\mathbb{T}$). Show that $g \in \operatorname{Cl}\mathcal{F}_M$ when M is sufficiently large.

4.7 Countable products of compact metric spaces

This fairly straight-forward topic is developed through a series of exercises.

Exercise 4.7.1. *(We have looked at this kind of thing earlier, see for example Exercise 2.1.6.) Suppose that (X_j, d_j) is a metric space $[j \geq 1]$ and $X = \prod_{j=1}^\infty X_j$. Suppose further that $a_j > 0$ and $\sum_{j=1}^\infty a_j$ converges. (Typically we take $a_j = 2^{-j}$.)*

(i) Show that if $d_j(x, y) \leq 1$ for all $x,\ y \in X_j$ and all $j \geq 1$, then

$$\theta(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^\infty a_j d_j(x_j, y_j)$$

is a well defined metric on X. What can happen if we drop the condition at the beginning of the previous sentence?

(ii) Show that, if $a_j > 0$ and $\sum_{j=1}^\infty a_j$ converges, then

$$\rho(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^\infty a_j \min\{1, d_j(x_j, y_j)\}$$

is a well defined metric on X.

(iii) Show that, if $a_j > 0$ and $\sum_{j=1}^{\infty} a_j$ converges, then

$$\tau(x,y) = \sum_{j=1}^{\infty} \frac{a_j d_j(x_j, y_j)}{1 + d_j(x,y)}$$

is a well defined metric on X.
[*Hint: If you need a hint for part (iii) look at Exercise 2.8.3.*]

Each of the metrics defined in the last exercise seem reasonable and experience shows that ρ and τ are easy to use and produce the kind of theorems we want. However, none of the metrics seems more natural than any of the others. We can make the following observation.

Exercise 4.7.2. *(i) Suppose that we take (X_j, d_j) and (X, d) as in part (i) of Exercise 4.7.1. Show that $\mathbf{x}(n) \underset{d}{\to} \mathbf{x}$ if and only if $x_j(n) \underset{d_j}{\to} x_j$ for each j.*

(ii) Deduce that with the notation and hypotheses of Exercise 4.7.1 of $\mathbf{x}(n) \to \mathbf{x}$ for the metrics ρ and τ if and only if $x_j(n) \underset{d_j}{\to} x_j$ for each j.

(iii) Suppose that we take (X_j, d_j) and (X, d) as in part (i) of Exercise 4.7.1. Show that $\mathbf{x}(n)$ forms a Cauchy sequence for d if and only if $x_j(n)$ forms a Cauchy sequence for each j.

(iv) Show that each of the metrics ρ and τ is complete if and only if each metric d_j is.

Exercise 4.7.3. *We use the hypotheses and notation of Exercise 4.7.1.*

(i) Suppose ρ_1 and ρ_2 are metrics on X such that the following statements about a sequence $\mathbf{x}(n)$ and a point $\mathbf{x}$ in X are equivalent

(a) $x_j(n) \underset{d_j}{\to} x_j$ as $n \to \infty$ for each j.

(b) $\mathbf{x}(n) \underset{\rho_k}{\to} \mathbf{x}$.

Show that identity map $\iota : (X, \rho_1) \to (X, \rho_2)$ is a homeomorphism (that is to say, ι and ι^{-1} are both continuous).

(ii) Suppose ρ is defined as in Exercise 4.7.1. If $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, let us set

$$\rho_\sigma(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{\infty} a_j \min\{1, d_{\sigma(j)}(x_j, y_j)\}.$$

Explain why ρ and rho_{sigma} are homeomorphic, but, by choosing an appropriate a_j and σ that they need not be Lipschitz equivalent (see Definition 1.5.32).

Exercise 4.7.4. *Let $V_j = [0,1]$ with its standard metric d_j. Set $V = \prod_{j=1}^{\infty} V_j$ and let $d(\mathbf{x}, \mathbf{y}) = \sup_{j \ge 1} d_j(x_j, y_j)$. Show that d is a metric but*

$$x_j(n) \to x \text{ as } n \to \infty \text{ for all } j \not\Rightarrow \mathbf{x}_n \underset{d}{\to} \mathbf{x} \text{ as } n \to \infty.$$

Theorem 4.7.5. *Suppose that (X_j, d_j) is a compact metric space for each $j \geq 1$ and $X = \prod_{j=1}^{\infty} X_j$. If d is a metric on X such that*

$$x_j(n) \underset{d_j}{\rightarrow} x_j \text{ as } n \to \infty \text{ for each } j \Leftrightarrow \mathbf{x}(n) \underset{d}{\rightarrow} \mathbf{x} \text{ as } n \to \infty, \qquad \bigstar$$

then (X, d) is compact.

Proof. We show that (X, d) has the Bolzano–Weierstrass property. Suppose $\mathbf{x}(n) \in X$. By using the Bolzano–Weierstrass property of each (X_j, d_j) in turn, we can find sequences $N_j(m)$ such that the $N_j(m)$ form a subsequence of the $N_{j-1}(m)$ with $N_j(m) \geq N_{j-1}(m)$ and $x_j \in X_j$ such that

$$x_j(N_j(m)) \underset{d_j}{\rightarrow} x_j$$

as $m \to \infty$. Automatically

$$x_j(N_m(m)) \underset{d_j}{\rightarrow} x_j$$

so

$$\mathbf{x}(N_m(m)) \underset{d}{\rightarrow} \mathbf{x}$$

and we are done. ∎

When we talk about products of metric spaces in later chapters we will only be interested in products of compact metric spaces. (Exercises 4.10.24 and 4.10.25 consider more general metric spaces and Exercise 4.10.26 shows that we cannot treat norms in the same way.) Our discussion shows that we have a wide choice of explicit metrics, but that (by Exercise 4.7.3) all choices satisfying the condition ★ give homeomorphic metrics. (Further, since a compact metric space is complete, all such metrics have the same Cauchy sequences. Look at the exercise below if you need more detail.)

Exercise 4.7.6. *(i) Prove that if (X, d) and (Y, ρ) are metric spaces with (X, d) complete and $f : X \to Y$ is continuous then, if x_n forms a Cauchy sequence in X, it follows $f(x_n)$ is Cauchy in Y.*

(ii) Let $X = \{1/n : n \geq 1,\ n \in \mathbb{Z}$, d be the usual metric on X, ρ the discrete metric on X and $\iota : (X, d) \to (X, \rho)$ the identity map. Show that ι is a homeomorphism and the sequence $1/n$ is Cauchy in (X, d) but the sequence $1/n = \iota(1/n)$ is not Cauchy in (X, ρ).

4.8 The poor man's circle

Leibniz was extremely proud of his invention of the binary system for representing numbers[7]. He saw it as a symbol of creation and even suggested a medallion with

[7] Cajori writes 'He was strangely partial to a discovery of minor importance'. But Cajori wrote in 1916 (*The Monist*, Volume26) before the electronic computer.

the motto 'Omnibus ex nihilo ducendis sufficit unum' — 'To produce everything out of nothing, one is sufficient'.

This section echos Leibniz's sentiment. I hope that it will provide an interesting and not too hard recap of some of the more important ideas of this chapter. We start with one of the simplest objects available to us — the two point set.

Exercise 4.8.1. *Consider* $\mathbb{D}_2 = \{0, 1\}$.

(i) Show that, if we write $d_*(x, y) = 0$ *if* $x = y$, $d_*(x, y) = 1$ *otherwise, then* $(\mathbb{D}_2, d_*)$ *is a compact metric space.*

(ii) Show that $\mathbb{D}_2$ *is an Abelian group under addition* $+$ *given by* $0+0 = 1+1 = 0$, $0 + 1 = 1 + 0 = 1$.

We write $\mathbb{D}_2^\infty = \prod_{j=1}^\infty X_j$ where $X_j = \mathbb{D}_2$. Thus $\mathbb{D}_2^\infty$ is the collection of sequences

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3 \ldots).$$

In accordance with the ideas of Section 4.7, we give $\mathbb{D}_2^\infty$ a metric

$$d(\boldsymbol{\omega}, \boldsymbol{\omega}') = \sum_{j=1}^\infty 2 \times 3^{-j} d_0(\omega_j, \omega'_j).$$

(The use of 2×3^{-j} rather than 2^{-j} aligns our metric with ideas to be discussed later.) We define addition for $\mathbb{D}_2^\infty$ by

$$\boldsymbol{\omega} + \boldsymbol{\omega}' = (\omega_1 + \omega'_1, \omega_2 + \omega'_2, \omega_3 + \omega'_3 \ldots).$$

Exercise 4.8.2. *(i) Show that* $(\mathbb{D}_2^\infty, d)$ *is a compact metric space with no isolated points.*

(ii) Show that $(\mathbb{D}_2^\infty, +)$ *is an Abelian group.*

The next two exercises tell us that the metric structure of $\mathbb{D}_2^\infty$ is richer than it might at first appear.

Exercise 4.8.3. *We work on* $\mathbb{D}_2^\infty$ *with our standard metric.*

(i) Set $g(\boldsymbol{\omega}) = 1$ *if* $\omega_j = 1$ *for only finitely many* j, $g(\boldsymbol{\omega}) = 0$ *otherwise. Show that* $g : \mathbb{D}_2^\infty \to \mathbb{R}$ *is nowhere continuous.*

(ii) Set $h(\mathbf{0}) = 1$, *set* $h(\boldsymbol{\omega}) = n^{-1}$ *if* $\omega_n = 1$ *but* $\omega_j = 0$ *for* $j \geq n + 1$ *and set* $h(\boldsymbol{\omega}) = 0$ *otherwise. Show that* h *is continuous at exactly those points* $\boldsymbol{\eta}$ *with* $\eta_j = 1$ *infinitely often.*

Exercise 4.8.4. *(A taste of things to come in Section 4.9.5.) We use our standard metrics on* $\mathbb{D}_2^\infty$ *and* $[0, 1]$.

(i) Show that the formula

$$f(\omega) = \sum_{\omega_j \neq 0} 2 \times 3^{-j}$$

gives a well defined function $f : \mathbb{D}_2^\infty \to [0,1]$ *with*

$$K_1 d(\omega, \omega) \geq \tfrac{1}{3}|f(\omega) - f(\omega')| \geq K_2 d(\omega, \omega')$$

for some $K_1 > K_2 > 0$*. Conclude that* f *is continuous and injective.*

(ii) Let $E = f(\mathbb{D}_2^\infty)$*. Conclude from (i) that* $\mathbb{D}_2^\infty$ *and* E *with the standard metrics are Lipschitz equivalent. Show that* E *is a compact uncountable subset of* $[0,1]$*. Use Lemma 4.8.2 to show that* E *has no isolated points.*

(iii) Let $n \geq 1$ *and* $\epsilon_j \in \{0,1\}$ *and* $m = \sum_{j=1}^n \epsilon_j 3^{n-j}$*. Show that*

$$\sum_{j=1}^\infty 2\epsilon_j 3^{-j} \in [2m3^{-n}, (2m+1)3^{-n}].$$

(iv) By using (iii), or otherwise, show that if $e \in E$ *and* $\delta > 0$ *we can find* $x \notin E$ *with* $|x - e| < \delta$*. (So* E *considered as a subset of* $[0,1]$ *has empty interior.)*

Of course, associating a set with two structures is not helpful unless the structures are interlinked.

Exercise 4.8.5. *(i) Consider* $\mathbb{T}$ *with its standard addition and its standard metric. Check that the functions* $A : \mathbb{T}^2 \to \mathbb{T}$ *and* $B : \mathbb{T} \to \mathbb{T}$ *given by* $A(x,y) = x + y$ *and* $Bx = -x$ *are continuous.*

(ii) Show that the function $A : \mathbb{D}_2^\infty \times \mathbb{D}_2^\infty \to \mathbb{D}_2^\infty$ *given by* $A(\boldsymbol{\omega}, \boldsymbol{\zeta}) = \boldsymbol{\omega} + \boldsymbol{\zeta}'$ *is continuous in our standard metric. (Note that* $-\boldsymbol{\omega} = \boldsymbol{\omega}$*.)*

(iii) Check that our standard metric on $\mathbb{D}_2^\infty$ *is* translation invariant *in the sense that*

$$d(\boldsymbol{\omega} + \boldsymbol{\zeta}, \boldsymbol{\zeta}) = d(\boldsymbol{\omega}, \mathbf{0}).$$

When studying $(0,1]$ and $\mathbb{R}$ (particularly when we deal with measure theory) we make much use of dyadic intervals like $(r2^{-n}, (r+1)2^{-n}]$. There is an analogous collection of subsets of $\mathbb{D}_2^\infty$.

Exercise 4.8.6. *Let*

$$H_n = \{\boldsymbol{\omega} \in \mathbb{D}_2^\infty \ : \ \omega_r = 0 \text{ for all } r \leq n\}.$$

(i) Show that H_n *is a subgroup of* $\mathbb{D}_2^\infty$*.*

(ii) Show that

$$H_n = \{\boldsymbol{\omega} \in \mathbb{D}_2^\infty \ : \ d(\mathbf{0}, \boldsymbol{\omega}) < 2 \times 3^{-n)}\} = \{\boldsymbol{\omega} \in \mathbb{D}_2^\infty \ : \ d(\mathbf{0}, \boldsymbol{\omega}) \leq 3^{-n}\}$$

Deduce that H_n is both open and closed.

(iii) Check that H_n has 2^n disjoint cosets each of the form

$$\{\omega \in \mathbb{D}_2^\infty : \omega_r = \eta_r \text{ for all } r \leq n\}$$

where $\eta_j \in \mathbb{D}_2$. Show that each coset is open and closed. Show that, if $d(\omega, \omega') \leq 3^{-n}$, ω and ω' belong to the same coset of H_n.

(iv) By thinking about uniform continuity, or otherwise, show that if $f : \mathbb{D}_2^\infty \to \mathbb{C}$ is continuous and $\epsilon > 0$, then there exists an n such that

$$|f(\omega) - f(\omega')| < \epsilon$$

whenever ω and ω' belong to the same coset of H_n.

We shall write $\mathcal{H}_n$ for the collection of the 2^n cosets of H_n.

We can now define a 'Riemann integral' for functions on $\mathbb{D}_2^\infty$ in much the same way as we defined a 'Riemann integral' [8] for functions on an interval $[a, b] \subseteq \mathbb{R}$.

Exercise 4.8.7. *Suppose that $f : \mathbb{D}_2^\infty \to \mathbb{R}$ is a bounded function. If $n \geq 1$, let*

$$\mathcal{I}_n^L(f) = \sup\left\{2^{-n} \sum_{J \in \mathcal{H}_n} a_J : \sum_{J \in \mathcal{H}_n} a_J \mathbb{I}_J(\omega) \leq f(\omega) \text{ for all } \omega \in \mathbb{D}_2^\infty\right\}$$

and

$$\mathcal{I}_n^U(f) = \inf\left\{2^{-n} \sum_{J \in \mathcal{H}_n} a_J : \sum_{J \in \mathcal{H}_n} a_J \mathbb{I}_J(\omega) \geq f(\omega) \text{ for all } \omega \in \mathbb{D}_2^\infty\right\}.$$

(i) Explain why $\mathcal{I}_n^L(f)$ and $\mathcal{I}_n^U(f)$ exist.

(ii) Check that

$$\mathcal{I}_n^L(f) \leq \mathcal{I}_{n+1}^L(f) \leq \mathcal{I}_{n+1}^U(f) \leq \mathcal{I}_n^U(f)$$

for all $n \geq 1$.

(iii) Deduce that there exist $I^L(f)$ and $I^U(f)$ such that

$$\mathcal{I}_n^L(f) \to \mathcal{I}^L(f) \text{ and } \mathcal{I}_n^U(f) \to \mathcal{I}^U(f)$$

as $n \to \infty$. Check that $\mathcal{I}^L(f) \leq \mathcal{I}^U(f)$.

If $\mathcal{I}^L(f) = \mathcal{I}^U(f)$ we say that f is 'Riemann integrable' and write $f \in \mathcal{R}$.

(iv) By using uniform continuity, or otherwise, show that, if f is continuous, then $f \in \mathcal{R}$.

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[8] If you have seen the formal definition of an integral before, just check that what we do is plausible. If you have not met such a definition, convince yourself that what we do is sensible. Since measure theory provides a much more powerful integral, the discussion is just background information.

If $f \in \mathcal{R}$ we shall write

$$\int_{\mathbb{D}_2^\infty} f(\omega)\,d\omega = \mathcal{I}^L(f).$$

As the next exercise shows, although our integral does work for some functions which are not continuous it fails for others.

Exercise 4.8.8. *(i) If $\boldsymbol{\eta} \in \mathbb{D}_2^\infty$, show that $\mathbb{I}_{\{\boldsymbol{\eta}\}}$ is not continuous, but $\mathbb{I}_{\{\boldsymbol{\eta}\}} \in \mathcal{R}$.*

(ii) Let E consist of those $\boldsymbol{\eta} \in \mathbb{D}_2^\infty$ such that there exists an $N(\boldsymbol{\eta})$ with $\eta_j = 0$ for all $j \geq N(\boldsymbol{\eta})$. Show that $\mathbb{I}_E \notin \mathcal{R}$.

Our integral behaves well 'algebraicly'.

Exercise 4.8.9. *Suppose that $f, g \in \mathcal{R}$ that λ, $\mu \in \mathbb{R}$ and $\boldsymbol{\eta} \in \mathbb{D}_2^\infty$.*

(i) Show that $\lambda f, g + f \in \mathcal{R}$ and

$$\int_{\mathbb{D}_2^\infty} \lambda f(\boldsymbol{\omega}) + \mu g(\boldsymbol{\omega})\,d\boldsymbol{\omega} = \lambda \int_{\mathbb{D}_2^\infty} f(\omega)\,d\omega + \mu \int_{\mathbb{D}_2^\infty} f(\omega)\,d\omega.$$

(ii) Show that, if $f(\boldsymbol{\omega}) \geq 0$ for all $\boldsymbol{\omega} \in \mathbb{D}_2^\infty$, then

$$\int_{\mathbb{D}_2^\infty} f(\omega)\,d\omega \geq 0.$$

(iii) Check that

$$\int_{\mathbb{D}_2^\infty} 1\,d\omega = 1.$$

(iv) Show that if we write $f_{\boldsymbol{\eta}}(\boldsymbol{\omega}) = f(\boldsymbol{\omega} + \boldsymbol{\eta})$ then $f_{\boldsymbol{\eta}} \in \mathcal{R}$

$$\int_{\mathbb{D}_2^\infty} f(\boldsymbol{\omega} + \boldsymbol{\eta})\,d\boldsymbol{\omega} = \int_{\mathbb{D}_2^\infty} f(\boldsymbol{\omega})\,d\boldsymbol{\omega}.$$

(v) Explain in as much detail as you require (which may be very little indeed) how to extend the integral to complex valued functions.

A further interlinking of algebraic and metric structures occurs when we ask about continuous homomorphisms. (Recall that if $(G, \times)$ and $(H, .)$ are groups we say that $f : G \to H$ is a homomorphism if $f(x \times y) = f(x).f(y)$ for all x, $y \in G$.)

Lemma 4.8.10. *Consider $\mathbb{T}$ with standard addition and metric and*

$$S^1 = \{z \in \mathbb{C} \,:\, |z| = 1\}$$

with the usual metric and the group structure given by multiplication. The continuous homomorphisms $\chi : \mathbb{T} \to S^1$ have the form $\chi(t) = \exp int$ where $n \in \mathbb{Z}$.

Proof. It is easy to check that $\chi(t) = \exp(int)$ with $n \in \mathbb{Z}$ defines a continuous homomorphism. We need to show that these are the only possible continuous homomorphisms.

To this end, suppose that χ is a continuous homomorphism. Since χ is a homomorphism, $\chi(0) = 1$. Further, since χ is continuous, we can find a $\delta > 0$ such that

$$|1 - \chi(t)| = |\chi(0) - \chi(t)| < 1/2$$

for all $|t| < \delta$.

Now suppose that $0 < s < \delta$ so $|1 - \chi(s)| < 1/2$ and we can find a unique $\theta \in (0, \pi/2)$ with $\chi(s) = \exp(i\theta)$ and $|\theta| < \pi/2$. Since χ is a homomorphism,

$$\big(\chi(s/2)\big)^2 = \chi(s/2 + s/2) = \chi(s) = \exp(i\theta).$$

It follows that $\chi(s/2) = \pm\exp(i\theta/2)$ and so, since $|s/2| < \delta$ and thus $|1 - \chi(s/2)| < 1/2$, it follows that $\chi(s/2) = \exp(i\theta/2)$. Induction now gives $\chi(2^{-n}s) = \exp(i2^{-n}\theta)$ for all integer $n \geq 1$.

We now fix on some particular $s = 2^{-M}$ with $0 < s < \delta$ and M an integer. This will, in turn, give us an associated θ with $\theta \in (0, \pi/2)$ and $\chi(s) = \exp(i\theta)$. Since χ is a homomorphism, we have $\chi(q2^{-n}s) = \exp(iq2^{-n}\theta)$ for any integer q and any positive integer n. Taking $q = 2^M r$ and $\lambda = 2^M\theta$ we have

$$\chi(r2^{-n}) = \exp(ir2^{-n}\lambda)$$

for any integer r and any positive integer n.

If $x \in \mathbb{T}$ we can find integers $r(m)$ together with positive integers $n(m) \to \infty$ such that $r(m)2^{-n(m)}s \to x$ as $m \to \infty$ and so, by continuity,

$$\exp(ir(m)2^{-n(m)}\lambda) = \chi(r(m)2^{-n(m)}) \to \chi(x)$$

whilst, again by continuity,

$$\exp(ir(m)2^{-n(m)}\lambda) \to \exp(i\lambda x).$$

Thus $\chi(x) = \exp(i\lambda x)$ for all $x \in \mathbb{T}$.

Finally we observe that, taking $x = 2\pi$ we have

$$1 = \chi(0) = \chi(2\pi) = \exp(2\pi\lambda)$$

and so λ must be an integer. ∎

The reader may feel that this is a lot of fuss about nothing, but one of the answers to why Fourier methods are so powerful is that they deal with continuous group homomrphisms.

Exercise 4.8.11. *(i) Show that the continuous homomorphisms* $\chi : \mathbb{R} \to S^1$ *are given by* $\chi(t) = \exp(i\lambda t)$ *with* $\lambda \in \mathbb{R}$.

(ii) Show that the continuous homomorphisms $\chi : \mathbb{T}^2 \to S^1$ *are given by* $\chi(x, y) = \exp\left(i(nx + my)\right)$ *with* n, $m \in \mathbb{Z}$.

What are the continuous homomorphisms $\chi : \mathbb{D}_2^\infty \to S^1$? A little experimentation suggests the following answer.

Lemma 4.8.12. *The function* $\chi : \mathbb{D}_2^\infty \to S^1$ *is a continuous homomorphism if and only if we can find an* N *and* $\zeta_j \in \{-1, 1\}$ *for* $1 \leq j \leq N$ *such that*

$$\chi(\omega) = \prod_{j=1}^{N} \zeta_j^{\omega_j}.$$

Exercise 4.8.13. *(i) Let* $\chi_j : \mathbb{D}_2^\infty \to S^1$ *be given by the conditions*

$$\chi_j(\omega) = \begin{cases} 1 & \text{if } \omega_j = 0, \\ -1 & \text{if } \omega_j = 1. \end{cases}$$

Show that χ_j *is a continuous homomorphism.*

(ii) Suppose that $N \geq$ *and* $\zeta_j \in \{-1, 1\}$ *for* $1 \leq j \leq N$ *and*

$$\chi(\omega) = \prod_{j=1}^{N} \zeta_j^{\omega_j}.$$

Check that

$$\chi(\omega) = \prod_{j=1}^{N} \chi_j(\omega_j)$$

for $\boldsymbol{\omega} \in \mathbb{D}_2^\infty$ *and that* χ *is a continuous homomorphism.*

Proof of Lemma 4.8.12. In view of Exercise 4.8.13 we need only show that any continuous homomorphism has the form stated.

To this end, let $\chi : \mathbb{D}_2^\infty \to S^1$ be a continuous homomorphism. Observe that since χ is a homomorphism

$$\chi(\omega)^2 = \chi(\omega + \omega) = \chi(\mathbf{0}) = 1$$

and so $\chi(\omega) = \pm 1$ for each $\boldsymbol{\omega} \in \mathbb{D}_2^\infty$. Echoing the initial arguments of the proof of Lemma 4.8.10 we see that, since χ is continuous, we can find a $\delta > 0$ such that

$$|1 - \chi(\omega)| = |\chi(\mathbf{0}) - \chi(\omega)| < 1/2$$

for all $d(\boldsymbol{\omega}, \mathbf{0}) < \delta$. Putting our two observations together, we see that we can find an $N \geq 1$ such that $\chi(\boldsymbol{\omega}) = 1$ for all $d(\boldsymbol{\omega}, \mathbf{0}) < \delta$. It follows that there exists an N such that

$$\chi(\boldsymbol{\omega}) = 1 \text{ whenever } \omega_j = 0 \text{ for all } j \leq N.$$

Let $\boldsymbol{\theta}(j) \in \mathbb{D}_2^\infty$ take the value 1 in the jth place and the value 0 elsewhere (more briefly $\theta_i(j) = \delta_{ij}$). If we set $\zeta_j = \chi(\boldsymbol{\theta}(j))$, then $\zeta_j \in \{-1, 1\}$ and

$$\begin{aligned} \chi(\boldsymbol{\omega}) &= \chi\left(\sum_{j=1}^{N} \omega_j \boldsymbol{\theta}(j) + (0, 0, \dots, 0, \omega_{N+1}, \omega_{N+2}, \dots)\right) \\ &= \left(\prod_{j=1}^{N} \chi(\omega_j \boldsymbol{\theta}(j))\right) \times \chi(0, 0, \dots, 0, \omega_{N+1}, \omega_{N+2}, \dots) \\ &= \prod_{j=1}^{N} \zeta_j^{\omega_j} \end{aligned}$$

for all $\boldsymbol{\omega} \in \mathbb{D}_2^\infty$. ■

We now develop the analogies between the collection $\hat{\mathbb{T}}$ of continuous homomorphisms $e_n : \mathbb{T} \to S^1$ given by $e_n(t) = \exp int$ and the collection $\widehat{\mathbb{D}_2^\infty}$ of continuous homomorphisms $\chi : \mathbb{D}_2^\infty \to S^1$. We sometimes refer to the elements of $\widehat{\mathbb{D}_2^\infty}$ as *characters*.

Exercise 4.8.14. *(i) Show that $\hat{\mathbb{T}}$ forms an Abelian group under the following multiplication rule: $e_n \cdot e_m(t) = e_n(t)e_m(t)$ for all $t \in \mathbb{T}$.*

(ii) Show that $\widehat{\mathbb{D}_2^\infty}$ forms an Abelian group under the following multiplication rule: $\phi \cdot \psi(\boldsymbol{\omega}) = \phi(\boldsymbol{\omega})\psi(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \mathbb{D}_2^\infty$. Check that $\chi(\boldsymbol{\omega}) = \chi(-\boldsymbol{\omega}) = \chi(\boldsymbol{\omega})^{-1}$.

We shall sometimes write χ_0 for the 'unit character' given by $\chi_0(\boldsymbol{\omega}) = 1$ for all $\boldsymbol{\omega} \in \mathbb{D}_2^\infty$.

Exercise 4.8.15. *(i) If $\chi \in \widehat{\mathbb{D}_2^\infty}$ and $\chi \neq \chi_0$, show that*

$$\int_{\mathbb{D}_2^\infty} \chi(\boldsymbol{\omega})\, d\boldsymbol{\omega} = 0.$$

What is the value of $\int_{\mathbb{D}_2^\infty} \chi_0(\boldsymbol{\omega})\, d\boldsymbol{\omega}$?

(ii) Show that the formula

$$\langle f, g\rangle = \int_{\mathbb{D}_2^\infty} f(\boldsymbol{\omega})g(\boldsymbol{\omega})^*\, d\boldsymbol{\omega}$$

defines an inner product on the space $C(\mathbb{D}_2^\infty)$ of continuous functions $f : \mathbb{D}_2^\infty \to \mathbb{C}$.

(iii) Show that the characters $\chi \in \widehat{\mathbb{D}_2^\infty}$ form an orthonormal set for this inner product.

Exercise 4.8.16. *Recall the definition of H_n in Exercise 4.8.6 and write G_n for the set of $\chi \in \widehat{\mathbb{D}_2^\infty}$ such that $\chi(\boldsymbol{\omega}) = 1$ whenever $\boldsymbol{\omega} \in H_n$.*

(i) Show that G_n is a subgroup of $\widehat{\mathbb{D}_2^\infty}$.

(ii) Show that $\chi \in G_n$ if and only if we can find $\zeta_j \in \{-1, 1\}$ for $1 \le j \le n$ such that

$$\chi(\boldsymbol{\omega}) = \prod_{j=1}^{n} \zeta_j^{\omega_j}.$$

Deduce that G_n has precisely 2^n elements.

(iii) Check that G_n consists of those $\chi \in \widehat{\mathbb{D}_2^\infty}$ which are constant on each coset $\boldsymbol{\omega} + H_n$.

(iv) Explain why $\widehat{\mathbb{D}_2^\infty}$ is countable.

Analogy with the relation of $\mathbb{Z}$ (via $n \leftrightarrow \exp(int)$) to $\mathbb{T}$ suggests that we give $\widehat{\mathbb{D}_2^\infty}$ the discrete metric.

More importantly the developing analogy suggests that we define Fourier coefficients for $f \in C(\mathbb{D}_2^\infty)$ by

$$\hat{f}(\chi) = \langle f, \chi \rangle.$$

If $f \in C(\mathbb{D}_2^\infty)$ and A is a finite subset of $\widehat{\mathbb{D}_2^\infty}$, we write

$$S_A(f, \boldsymbol{\omega}) = \sum_{\chi \in A} \hat{f}(\chi)\chi(\boldsymbol{\omega}).$$

We get particularly simple results if $A = G_n$. Parts (i) and (ii) of the next exercise parallel Lemma 2.2.1 and Exercise 2.2.2, but the remaining parts are a very pleasant surprise.

Exercise 4.8.17. *(i) If $f \in C(\mathbb{D}_2^\infty)$, show that*

$$S_A(f, \boldsymbol{\omega}) = \int_{\mathbb{D}_2^\infty} f(\boldsymbol{\eta}) D_A(\boldsymbol{\omega} - \boldsymbol{\eta})\, d\boldsymbol{\eta},$$

where

$$D_A(\boldsymbol{\eta}) = \sum_{\chi \in A} \chi(\boldsymbol{\eta}).$$

(ii) Show that $D_{G_n} = 2^n \mathbb{I}_{H_n}$, in other words, that

$$D_{G_n}(\boldsymbol{\eta}) = \begin{cases} 2^n & \text{if } \boldsymbol{\eta} \in H_n, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Let $\mathcal{H}_n$ consist of the cosets of H_n. If $f \in C(\mathbb{D}_2^\infty)$, show that

$$S_{G_n}(f, \boldsymbol{\eta}) = \sum_{J \in \mathcal{H}_n} c_J \mathbb{I}_J(\boldsymbol{\eta})$$

where

$$C_J = 2^n \int_{\mathbb{D}_2^\infty} f(\boldsymbol{\eta})\mathbb{I}_J(\boldsymbol{\eta})\, d\boldsymbol{\eta}.$$

(iv) Deduce, using uniform continuity, that

$$S_{G_n}(f, \boldsymbol{\eta}) \to f(\boldsymbol{\eta})$$

uniformly on $\mathbb{D}_2^\infty$ as $n \to \infty$.
[The analogy in (iv) is not quite as close to the study of partial sums $\sum_{r=-n}^{n} \hat{f}(r) \exp irt$ as might at first appear. The correct analogy here is the study of partial sums $\sum_{r=-2^n}^{2^n} \hat{f}(r) \exp irt$.]

We give another easy proof of Exercise 4.8.17 (iv) as Exercise 4.10.27. Exercise 4.8.17 (iv) makes it very easy to obtain results corresponding to Exercise 2.4.4

Exercise 4.8.18. *We use the norm $\| \|_2$ induced by our inner product on $\mathbb{D}_2^\infty$. Show that, if $A(j)$ is a finite subset of $\widehat{\mathbb{D}_2^\infty}$ with*

$$A(1) \subseteq A(2) \subseteq A(3) \subseteq \dots \text{ and } \bigcup_{j=1}^{\infty} A(j) = \widehat{\mathbb{D}_2^\infty},$$

then

$$\left\| f - \sum_{\chi \in A(j)} \hat{f}(\chi)\chi \right\|_2 \to 0$$

as $j \to \infty$.

4.9 Perfect sets and the Hausdorff metric

What does a typical compact set look like? The question is too vague to have an exact answer, but not too vague to lead to interesting ideas. We need to introduce the ideas of a perfect set and a totally disconnected set.

Definition 4.9.1. *Let (X, d) be a metric space and $E \subset X$. We say that $e \in E$ is an* isolated point *of E if we can find a $\delta > 0$ with $d(y, e) > \delta$ for all $y \in E$ with $y \neq e$. If $x \in X$ is an isolated point of X, we just say that x is an isolated point.*

Definition 4.9.2. *A non-empty compact subset E of a metric space (X, d) is said to be* perfect *if it has following properties.*

(i) If $e \in E$ and $\delta > 0$, we can find $x \notin E$ with $d(x, e) < \delta$. (So E has empty interior.)

(ii) If $e \in E$ and $\delta > 0$, we can find $e' \in E$ with $d(e, e') < \delta$. (So E has no isolated pints.)

Definition 4.9.3. *A non-empty subset E of a metric space* (X, d) *is said to be* totally disconnected *if, whenever* $x, y \in E$, $x \neq y$, *we can find U and V open such that* $x \in U$, $y \in V$ $U \cup V \supseteq E$ *and* $U \cap V = \emptyset$

Exercise 4.9.4. *We work with the usual norms on* $[0, 1]$ *and* $[0, 1]^2$.

(i) Give a simple example of a closed subset of $[0, 1]$ *which is totally disconnected, but not perfect.*

(ii) Check that $\{(x, y) : x \in [1/4, 3/4], y = 1/2\}$ *is perfect, but not totally disconnected.*

(iii) If E is a perfect subset of $[0, 1]$ *and* $s, t \in E$ *with* $s < t$ *show that we can find an* $a \in (s, t)$ *with* $a \notin E$. *Deduce that E is totally disconnected.*

The first person to produce totally disconnected perfect sets was Henry Smith, a mathematician of world standard in 19th century Oxford. Since this was something like being a phoenix in a flock of hens, we may wonder how he was viewed by his colleagues. Fortunately he was a good College and University man, 'a thorough man of the world, quite free from shyness' wise, witty and master of entire spectrum of human knowledge (as known in Oxford).

He was a tutor at Balliol[9], ran the new teaching laboratory (learning chemistry in order to teach it) and helped organise a new system of joint college lectures. Among the positions he held (in addition to the Savilian Professorship of Geometry) were membership of the Hebdomadal Council, the secretaryship of the Ashmolean Society, Keeper of the Oxford University Museum, Mathematical Examiner for the University of London, member of a Royal Commission to review scientific education practice, member of the commission to reform University of Oxford governance, and chairman of the committee of scientists overseeing the Meteorological Office. The Times Obituary said 'It is probable that of the thousands of Englishmen who knew Henry Smith scarcely one in a hundred ever thought of him as a mathematician at all.'

Smith's main work was in the theory of elliptic geometry and number theory, where he produced work of permanent value. His sets occur in a beautiful paper entitled 'On the Integration of Discontinuous Functions' (see Volume II of his collected works).

We shall give several proofs that such sets exist. Our first proof[10] is a direct construction. It is part of the tool box of many analysts and so well worth mastering.

Theorem 4.9.5. *There exist perfect subsets of* $[0, 1]$.

[9]He even taught on Sunday afternoon, telling his students that 'It was lawful on the Sabbath day to pull an ass out of the ditch'.

[10]Hard working readers in possession of a good memory will recall Exercise 4.8.4 and murmur 'second proof'.

Proof. Let $1 > \lambda_j > 0$. We start with two disjoint closed intervals

$$I_0 = [0, \lambda_1/2] \text{ and } I_0 = [1 - \lambda_1/2, 1]$$

of equal length $\lambda_1/2$ and proceed inductively as follows.

At the beginning of the nth step we have 2^{n-1} disjoint closed intervals $I_{\omega(1),\omega(2),\ldots,\omega(n-1)}$ with $\omega(j) \in \{0, 1\}$, each of the same length $2^{-n+1}\lambda_1\lambda_2 \ldots \lambda_{n-1}$. We now remove an open interval of length $2^{-n+1}\lambda_1\lambda_2 \ldots \lambda_{n-1}(1-\lambda_n)$ from the centre of $I_{\omega(1),\omega(2),\ldots,\omega(n-1)}$ forming two new closed intervals $I_{\omega(1),\omega(2),\ldots,\omega(n-1),0}$ and $I_{\omega(1),\omega(2),\ldots,\omega(n-1),1}$.

More formally, if $I_{\omega(1),\omega(2),\ldots,\omega(n-1)} = [a, b]$ then

$$I_{\omega(1),\omega(2),\ldots,\omega(n-1),0} = \left[a, a + \lambda_n \frac{b-a}{2}\right] \text{ and } I_{\omega(1),\omega(2),\ldots,\omega(n-1),1} = \left[b - \lambda_n \frac{b-a}{2}, b\right]$$

We now have 2^n disjoint closed intervals $I_{\omega(1),\omega(2),\ldots,\omega(n-1),\omega(n)}$ with $\omega(j) \in \{0, 1\}$, each of the same length and total length $\lambda_1\lambda_2 \ldots \lambda_n$ and move on to the next inductive step.

We write

$$E_n = \bigcup \{I_{\omega(1),\omega(2),\ldots,\omega(n)} \ : \ \omega(j) \in \{0, 1\},\ 1 \le j \le n\}$$

and observe that

$$E_1 \supseteq E_2 \supseteq E_3 \ldots.$$

Finally we set $E = \bigcap_{n=1}^{\infty} E_n$ and claim that E satisfies the conditions of Theorem 4.9.5.

The set E_n is composed of disjoint closed sets of length less than 2^{-n}. Thus, if $x \in E$, and so $x \in E_n$, we can find a $y \notin E_n$, (so $y \notin E$, automatically) such that $|y-x| > 2^{-n-1}$. We observe immediately that the finite intersection characterisation of compactness (see Lemma 4.3.7) shows that E is bounded, compact and non-empty and return to the task of showing that E has property (ii).

Suppose $x \in E$. Automatically $x \in E_n$ and so $x \in I = I_{\omega(1),\omega(2),\ldots,\omega(n-1),\omega(n)}$ for some particular $\omega(j) \in \{0, 1\}$. Let $\omega'(n) = 1 - \omega(n)$ and set

$$J = I_{\omega(1),\omega(2),\ldots,\omega(n-1),\omega'(n)}.$$

By construction, the set $K_m = E_m \cap J$ is non-empty and

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots.$$

Thus $K = \bigcap_{j=1}^{\infty} K_j = J \cap E$ is non-empty. Choose $y \in J \cap E$. Since $I \cap J = \varnothing$, we have $y \neq x$. Since

$$x,\ y \in I \cup J \subseteq I_{\omega(1),\omega(2),\ldots,\omega(n-1)}$$

we have $|x - y| \le 2^{-n+1}$. We have shown that our set E is perfect. ■

Exercise 4.9.6. *It is important that the reader should eventually have an 'intuitive' picture of E. Start this process by sketching E_1, E_2 and E_3.*

Most mathematicians wold call the set E we have constructed 'a Cantor type set', but the notion is rarely defined exactly.

Our set E has a further property which is important for the study of integration. Recall that the intervals composing E_n have total length $\lambda_1\lambda_2\dots\lambda_n$. A decreasing sequence of positive terms tends to a limit so

$$\lambda_1\lambda_2\dots\lambda_n \to \lambda$$

for some λ.

Exercise 4.9.7. *(i) Explain why $1 > \lambda \geq 0$.*

(ii) If $\lambda_j = 2/3$ for each j (in this case E is called the Cantor set *or Cantor middle third set), show that $\lambda = 0$.*

(iii) If $1 > \alpha > 0$, show that we can choose λ_j so that $\lambda = alpha$.

Thus λ can take any value in the interval $[0, 1)$.

If (X, d) is a metric space, then we know that for any three points x, y, $z \in X$,

$$d(x,z) \leq d(x,y) + d(y,z) \text{ and } d(x,y) \leq d(x,z) + d(y,z)$$

so that

$$|d(x,y) - d(x,z)| \leq d(x,y)$$

and the function $y \mapsto d(x,y)$ is continuous. In particular, if E is a non-empty compact set in X, the function $y \mapsto d(x,y)$ attains its minimum on E.

Definition 4.9.8. *If E is a non-empty compact set in a metric space (X, d) we write $d(x, E) = \sup_{e\in E} d(x, e)$.*

Exercise 4.9.9. *Suppose we work in $\mathbb{R}$ with the usual metric d. Give an example of a non-empty compact set E a point $x \notin E$ and two distinct points u, $v \in E$ such that*

$$d(x,u) = d(x,v) = d(x,E).$$

Exercise 4.9.10. *Let (X, d) be a metric space.*

(i) If E is a non-empty compact set, show that the map $x \mapsto d(x, E)$ is continuous.

(ii) Deduce that, if E and F are non-empty compact sets then there exist points $e \in E$ and $f \in F$ such that

$$d(e, f) = \inf_{y\in E} d(y, F).$$

(iii) Are the points e and f necessarily unique?

If E and F are as in Exercise 4.9.10 (ii) we write $\tau(E,F) = \inf_{\mathbf{y}\in E} d(\mathbf{y},F)$.

Exercise 4.9.11. *Let (X,d) be a metric space. Show that if we consider the space $\mathcal{K}$ of non-empty compact sets, then the following results hold.*
(i) $\tau(E,F) \geq 0$ for all $E,\ F \in \mathcal{K}$.
(ii) If $E \cap F \neq \emptyset$ then $\tau(E,F) = 0$.
(iii) $\tau(E,F) = \tau(F,E)$ for all $E,\ F \in \mathcal{K}$.
(iv) We can give examples with $\tau(E,F) + \tau(F,G) < \tau(E,G)$.

Looking at Definition 1.2.1, we see that τ has some, but not all, of the properties required by a metric.

Two other variations on the same theme are even less promising.

Exercise 4.9.12. *If $E,\ F \in \mathcal{K}$ set $\tau_1(E,F) = \inf_{e\in E}\sup_{f\in F} d(e,f)$ and $\tau_2(E,F) = \sup_{e\in E}\sup_{f\in F} d(e,f)$. Show that if $a,\ b \in E$ then*

$$\tau_2(E,E) \geq \tau_1(E,E) \geq d(a,b)/2.$$

We now try the remaining variation on this theme. If $E,\ F \in \mathcal{K}$ let us set

$$\sigma(E,F) = \sup_{x\in E} d(e,F).$$

Exercise 4.9.13. *Show that, with the notation just introduced, there exists an $e \in E$ with $d(e,F) = \sigma(E,F)$.*

Exercise 4.9.14. *Let (X,d) be a metric space. Show that, if we consider the space $\mathcal{K}$ of non-empty compact sets, then the following results hold.*
(i) $\sigma(E,F) \geq 0$ for all $E,\ F \in \mathcal{K}$.
(ii) $\sigma(E,F) = 0 \Leftrightarrow E \subseteq F$
(iii) We can give examples with $\sigma(E,F) \neq \sigma(F,E)$.

This still does not look very hopeful, but we now observe that the key triangle inequality holds.

Lemma 4.9.15. *With the notation we have established*

$$\sigma(E,G) \leq \sigma(E,F) + \sigma(F,G)$$

for all $E,\ F,\ G \in \mathcal{K}$.

Proof. Given $e \in E$, we can find $f \in F$ such that $d(e,f) = d(e,F)$. If $g \in G$, then

$$d(e,G) \leq d(e,g) \leq d(e,f) + d(f,g) = d(e,F) + d(f,g)$$

Since $g \in G$ was arbitrary,

$$d(e, G) \le d(e, F) + d(f, G) \le \sigma(E, F) + \sigma(F, G)$$

and so

$$\sigma(E, G) \le \sigma(E, F) + \sigma(F, G).$$

■

This enables us to define the Hausdorff metric ρ.

Definition 4.9.16. *Let (X, d) be a metric space and consider the space $\mathcal{K}$ of non-empty compact sets. If $E,\ F \in \mathcal{K}$, we set*

$$\rho(E, F) = \sigma(E, F) + \sigma(F, E),$$

that is to say,

$$\rho(E, F) = \sup_{e \in E} \inf_{f \in F} d(e, f) + \sup_{f \in F} \inf_{e \in E} d(f, e).$$

Exercise 4.9.17. *Check that Hausdorff metric ρ is indeed a metric on the space $\mathcal{K}$ of non-empty compact subsets of (X, d).*

If we work with compact metric spaces then, as we shall proceed to demonstrate, $(\mathcal{K}, \rho)$ is complete. (Exercise 4.10.28 shows that this is not true in general.) It will be helpful to recall that the compact subsets of a compact metric space are precisely the closed subsets (Exercise 4.3.11 (ii)).

We need two preliminary lemmas.

Lemma 4.9.18. *Let (X, d) be a compact metric space. Suppose that we have a sequence F_n of non-empty compact sets F_n with $F_n \supseteq F_{n+1}$. Then there exists a non-empty compact set with $\rho(F, F_n) \to 0$.*

Proof. By Lemma 4.3.7, $F = \bigcap_{n=1}^{\infty} F_n$ is non-empty compact set. We claim that $\rho(F, F_n) \to 0$ as $n \to 0$.

Suppose not. Since $F \subseteq F_{n+1} \subseteq F_n$, this means that there is an $\eta > 0$ and a sequence $x_n \in F_n$ with $d(x_n, F) \ge \eta$. Since $x_n \in F_1$ and F_1 is compact, we can find $n(j) \ge j$ and $x \in F_1$ such that $d(x_{n(j)}, x) \to 0$ as $j \to \infty$. For each fixed m we know that $x_{n(j)} \in F_m$ whenever $j \ge m$ so, since F_m is closed, $x \in F_m$. It follows that $x \in F$ and $d(x_{n(j)}, F) \le d(x_{n(j)}, x) \to 0$ contradicting our initial hypothesis. The result follows by reductio ad absurdum. ■

Lemma 4.9.19. *Let (X, d) be a compact metric space and let $\eta > 0$. If E is compact, then so is*

$$E(\eta) = \{x \in X \ : \ \text{there exists a } y \in E \text{ with } d(x, y) \le \eta\}.$$

Proof. Suppose that $x_n \in E(\eta)$ and $x_n \to x$. We wish to show that $x \in E(\eta)$.

By definition, we can find $y_n \in E$ with $d(x_n, y_n) \le \eta$. By the compactness of E, we can find $n(j) \to \infty$ and $y \in E$ with $y_{n(j)} \to y$. Since

$$d(x, y) \le d(x, x_{n(j)}) + d(y, y_{n(j)}) + d(x_{n(j)}, y_{n(j)}) \le d(x, x_{n(j)}) + d(y, y_{n(j)}) + \eta \to \eta,$$

we have $d(x, y) \le \eta$ and $x \in E(\eta)$ as required. ∎

Theorem 4.9.20. *If (X, d) is a compact space, the associated Hausdorff metric is complete.*

Proof. By Exercise 1.2.15 (i), it is sufficient to show that, if we have a sequence E_n of non-empty compact sets with $\rho(E_n, E_m) \le 2^{-2n}$ for all $m \ge n \ge 1$, the sequence converges.

To this end, consider

$$F_n = \{x \in X : \text{there exists a } y \in E_n \text{ with } d(x, y) \le 2^{-n}\}.$$

We observe that if $x \in F_m$ with $m \ge n + 1$, then then there exists a $y \in E_m$ with $d(x, y) \le 2^{-m}$ and, since $\rho(E_n, E_m) \le 2^{-2n}$, there exists a $z \in E_n$ with $d(z, y) \le 2^{-2n}$. It follows that $d(z, x) \le 2^{-m} + 2^{-2n} \le 2^n$ and $z \in F_n$.

We now have a sequence F_n of non-empty compact sets with $F_n \supseteq F_{n+1}$. By Lemma 4.9.18, we know that there exists a non-empty compact set F with $\rho(F_n, F) \to 0$. It follows that

$$\rho(E_n, F) \le \rho(E_n, F_n) + \rho(F_n, F) \le 2^{-n} + \rho(F_n, F) \to 0$$

as $n \to \infty$ so we are done. ∎

If we use the Hausdorff metric then (in the appropriate sense) we see that the ‘typical’ compact subset of $[0, 1]$ is perfect. In fact we shall prove a slightly stronger theorem (although the reader will loose very little if she takes $X = [0, 1]$).

Theorem 4.9.21. *Let (X, d) be a compact metric space with no isolated points. Consider the collection $\mathcal{K}$ of non-empty closed subsets of X with the Hausdorff metric ρ. Quasi-all (see Definition 1.6.4) elements of $\mathcal{K}$ are perfect and totally disconnected.*

We recall that if quasi-all objects have property $\mathcal{P}_1$ and quasi-all objects have property $\mathcal{P}_2$, then quasi-all objects have both property $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus Theorem 4.9.21 follows by combining the results of the next three lemmas.

Lemma 4.9.22. *We work with the notation and hypotheses of Theorem 4.9.21. Let $\mathcal{E}_k$ be the collection of compact sets E such that there exists an $x \in E$ with $B(x, 1/k) \cap E = \{x\}$.*

(i) The set $\mathcal{E}_k$ is closed in the Hausdorff metric.

(ii) The set $\mathcal{E}_k$ is nowhere dense in the Hausdorff metric.

(iii) The set $\mathcal{E}$ of compact sets with an isolated point is meagre with respect to the Hausdorff metric.

Proof. (i) Suppose that $E_n \in \mathcal{E}_k$ and $E_n \to E$ in the Hausdorff metric. By definition, we can find $x_n \in E_n$ with $B(x, 1/k) \cap E = \{x_n\}$. By compactness, we can find $n(j) \to \infty$ and $x \in X$ such that $d(x_n(j), x) \to 0$. We check that $x \in E$.

Suppose, if possible, that $B(x, 1/k) \cap E \neq \{x\}$. Then we can find a $y \in E$ such that $d(x, y) < 1/k$. Set $\delta = (1/k - d(x, y))/2$. Since $E_n \to E$ and $n(j) \to \infty$, we can find a J such that the Hausdorff distance $\rho(E_{n(J)}, E) < \delta$ and so there exists a $y' \in E_{n(J)}$ with $d(y, y') < \delta$ and so with $d(x_{n(J)}, y') < 1/k$, contrary to our hypothesis.

Thus $E \in \mathcal{E}_k$ and $\mathcal{E}_k$ is closed.

(ii) Let $G \in \mathcal{K}$ and let $\epsilon > 0$. Choose $\eta < \min\{\epsilon/3, 1/k\}$. By compactness we can find a finite set of open balls $B(x_j, \eta)$ $[1 \leq j \leq N]$ with $x_j \in E$ and $\bigcup_{j=1}^{N} B(x_j, \eta) \supseteq G$. Now choose $y_j \neq x_j$ with $y_j \in B(y_j, \eta)$. If we set

$$F = \{x_j \,:\, 1 \leq j \leq N\} \cup \{y_j \,:\, 1 \leq j \leq N\},$$

then

$$\rho(G, F) \leq \max_{1 \leq j \leq N} d(x_j, y_j) + \eta \leq 2\eta < \epsilon$$

and $G \notin \mathcal{E}_k$.

(iii) Observe that $\mathcal{E} = \bigcup_{k=1}^{\infty} \mathcal{E}_k$. ■

Lemma 4.9.23. *We work with the notation and hypotheses of Theorem 4.9.21. Let $\mathcal{F}_k$ be the collection of compact sets F such that there exists a ball $B(x, 1/k) \subseteq F$.*

(i) The set $\mathcal{F}_k$ is closed in the Hausdorff metric.

(ii) The set $\mathcal{F}_k$ is nowhere dense in the Hausdorff metric.

(iii) The set $\mathcal{F}$ of compact sets with non-empty interior is meagre.

Proof. (i) Suppose that $F_n \in \mathcal{F}_k$ and $F_n \to F$ in the Hausdorff metric. Then we can find an $x_n \in X$ such that $B(x_n, 1/k) \subseteq F_n$. By compactness, we can find an $x \in X$ and a strictly increasing sequence $n(r)$ such that $x_{n(r)} \to x$. By restricting ourselves to this sequence we may suppose $x_n \to x$.

Given $1/k > \epsilon > 0$ we can find an $N(\epsilon)$ such that

$$d(x, x_n) < \epsilon/2 \text{ and } \rho(F, F_n) < \epsilon/2$$

for $n \geq N$ and so

$$B(x, (1/k) - \epsilon) \supseteq F.$$

Since ϵ was arbitrary $B(x, 1/k) \subseteq F$ and we are done.

(ii) Let $G \in \mathcal{K}$ and let $\epsilon > 0$. By compactness we can find a finite collection of open balls $B(y_m, 1/(4k))$ $[1 \leq m \leq M]$ such that $X = \bigcup_{m=1}^{M} B(y_m, 1/(4k))$. Choose $\eta > 0$ such that $\eta < \epsilon/2$ and $\eta < 1/4k$ and set $E = F \setminus \bigcup_{m=1}^{M} B(y_m, \eta)$. Then $\rho(E, G) \leq 2\eta < \epsilon$ and E can contain no open ball of the form $B(y, 1/k)$.

(iii) Observe that $\mathcal{F} = \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{k} \mathcal{F}_{j,k}$, so $\mathcal{F}$ is the countable union of closed nowhere dense sets. ■

Lemma 4.9.24. *We work with the notation and hypotheses of Theorem 4.9.21. Let $\mathcal{E}_{Mk}$ be the collection of compact sets E such that there does not exist a δ with $1/k > \delta > 0$ and points $x_m \in E$ $[1 \leq m \leq M]$ such that $d(x_m, x_n) > 5\delta$ for all $1 \leq m < n \leq M$, but $\bigcup_{m=1}^{M} B(x_m, \delta) \supseteq E$.*

(i) The set $\mathcal{E}_k$ is closed in the Hausdorff metric.

(ii) The set $\mathcal{E}_k$ is nowhere dense in the Hausdorff metric.

(iii) The collection $\mathcal{E}$ of compact sets such that there exists a $\epsilon > 0$ such that, if $\epsilon > \delta > 0$, there does not exist a finite collection of points $x_m \in E$ $[1 \leq m \leq M]$ such that $d(x_m, x_n) \geq 5\delta$ for all $m \neq n$, but $\bigcup_{m=1}^{M} B(x_m, \delta) \supseteq E$ is meagre.

(iv) Quasi-all compact subsets are totally disconnected.

Proof. (i) We show that $\mathcal{E}_k^c$ is open. Suppose that F is a compact set with $F \notin E_k$. Then we can find $1/k > \delta > 0$ and points $x_m \in F$ $[1 \leq m \leq M]$ such that $d(x_m, x_n) > 5\delta$ for all $1 \leq m < n \leq M$, but $\bigcup_{m=1}^{M} B(x_m, \delta) \supseteq E$.

$$\left(\bigcup_{m=1}^{M} B(x_m, \delta)\right)^c \text{ is compact and disjoint from the compact set } E$$

we have $\bigcup_{m=1}^{M} B(x_m, \delta - \eta) \supseteq E$ for some $\eta > 0$.

Now suppose $\rho(F, G) < \epsilon$. Then we can find $y_m \in G$ such that $d(x_m, y_m) < \epsilon$. Provided only that ϵ is sufficiently small $d(y_m, y_n) > 5\delta$ for all $1 \leq m < n \leq M$ and $B(y_m, \delta) \supseteq B(x_m, \delta - \eta)$ so $G \not\mathcal{E}_k^c$.

(ii) Let $\epsilon > 0$ Since E is a non-empty compact set, we can find a finite collection of open balls $B(x_m, \epsilon/2)$ with $x_m \in E$ $[1 \leq m \leq M]$ and $x_m \neq x_n$ $[1 \leq m \leq n \leq M]$. Choose $0 < \delta < \min_{m \neq n} d(x_m, x_n)$ with $\delta < \epsilon/2$ and set $G = \bigcup_{m=1}^{M} \bar{B}(x_m, \delta/2)$. Then G is compact,

$$\rho(G, E) \leq \delta/2 + \epsilon/2 < \epsilon,$$

$d(x_m, x_n) > 5\delta$ for all $1 \leq m < n \leq M$ and $G =\subseteq \bigcup_{m=1}^{M} \bar{B}(x_m, \delta)$.

(iii) Observe that $\mathcal{E} = \bigcup_{k=1}^{\infty} \mathcal{E}_k$.

(iv) We show that if $F \notin \mathcal{E}$ then F is totally disconnected. For suppose x and y are distinct points of F. Choose an integer $k > 10d(x,y)$. Since $F \notin \mathcal{E}_k$ we can find a $1/k > \delta > 0$ and $x_m \in E$ $[1 \leq m \leq M]$ such that $d(x_m, x_n) > 5\delta$ for all $1 \leq m < n \leq M$, but $\bigcup_{m=1}^{M} B(x_m, \delta) \supseteq E$. Suppose $x \in B(x_r, \delta)$. Then $y \in U = \bigcup_{m \neq r} B(x_m, \delta)$, U and $B(x_r, \delta)$ are open and disjoint with $U \cup B(x_r, \delta) \supseteq E$ as required. ■

This completes the proof of Theorem 4.9.21.

We complete our discussion by showing that all Cantor type sets have the same structure.

Theorem 4.9.25. *All compact totally disconnected perfect sets (in metric spaces) are homeomorphic.*

It will be helpful to restate Theorem 4.9.25

Theorem 4.9.26. *If (X, d) is compact and totally disconnected with no isolated points then (X, d) is homeomorphic to $(\mathbb{D}_2^\infty, \kappa)$ where κ is a standard metric, and, in particular,*

$$\kappa(\omega(n)), \omega) \to 0 \Leftrightarrow \omega_j(n) = \omega_j \textit{ for n sufficiently large, depending on } j.$$

Exercise 4.9.27. *Quickly check that Theorem 4.9.25 and Theorem 4.9.26 are equivalent.*

The key to the proof is in the next simple lemma.

Lemma 4.9.28. *If (X, d) is compact and totally disconnected containing at least two points then, given $x \in X$ and $\delta > 0$ find disjoint non-empty open sets U and V with $U \cup V = X$ and $x \in V \subseteq B(x, \delta)$.*

Proof. Consider the closed (so compact) set $E = X \setminus B(x, \delta)$. If $e \in E$, then since X is totally disconnected, we can find U_e and V_e open disjoint sets such that $e \in U_e$, $x \in V_e$, and $U_e \cup V_e = X$. Since E is compact, we can find a finite collection of points $e(j) \in E$ $[1 \leq j \leq N]$ such that $\bigcup_{j=1}^{N} U_{e(j)} \supseteq E$. If we now take $U = \bigcup_{j=1}^{N} U_{e(j)}$, $V = \bigcup_{j=1}^{N} V_{e(j)}$, then, by inspection, U and V have the desired properties. ■

Notice that U and V in Lemma 4.9.28 are both open and closed.

Exercise 4.9.29. *Explain in one sentence why this is the case.*

For the rest of the section we use the portmanteau word *clopen* to mean that a set is both open and closed.

Lemma 4.9.30. *If (X, d) is compact and totally disconnected containing at least two points then we can find a finite collection of disjoint non-empty clopen sets W_j $[1 \leq j \leq N]$ with $\bigcup_{j=1}^{N} W_j = X$ such that there exists x_j with $W_j \subseteq B(x_j, \delta)$ $[1 \leq j \leq N]$*

Proof. By the previous result we can find clopen sets V_y such that $V_y \subseteq B(y, \delta)$ for each $y \in X$. Since X is compact we can find a finite collection of points $y(m)$ $[1 \leq m \leq M]$ with $\bigcup_{m=1}^{M} V_{y(m)} = X$. Since $A \cup B$ and $A \setminus B$ are clopen whenever A and B are, the sets

$$V_1,\ V_2 \setminus V_1, V_3 \setminus (V_1 \cup V_2), V_4 \setminus (V_1 \cup V_2 \cup V_3), \ldots$$

are clopen and the non-empty sets among these provide the W_j that we require. ■

Exercise 4.9.31. *(This is more about understanding what is going on than any thing else.) Now suppose that X in Lemma 4.9.30 is also perfect.*

(i) Explain why the (W_j, d) are perfect and totally disconnected.

(ii) Use (i) to show that, (using the notation of Lemma 4.9.30, if $M \geq N$ can find a finite collection of disjoint non-empty clopen sets U_k $[1 \leq k \leq M]$ with $\bigcup_{k=1}^{M} U_k = X$ such that there exist y_k with $U_k \subseteq B(y_k, \delta)$ $[1 \leq j \leq M]$.

(iii) Hence show that if $2^m \geq N$ we can find non-empty clopen sets $I_{\mathbf{w}}(r)$ with $\mathbf{w} \in \mathbb{D}_2^r$ $[1 \leq r \leq m]$ with the following properties.

(a) The $I_{\mathbf{w}}(r)$ are disjoint for each r.

(b) $I_{(0)}(1) \cup I_{(1)}(1) = X$ and $I_{(\mathbf{u},0)}(r+1) \cup I_{(\mathbf{u},1)}(r+1) = I_{\mathbf{u}}(r)$ for all $\mathbf{u} \in \mathbb{D}_2^r$ and all $1 \leq r \leq m-1$

(c) For each $\mathbf{w} \in \mathbb{D}_w^n$ there exists a $y_{\mathbf{w}}$ with $I_{\mathbf{w}}(n) \subseteq B(y_{\mathbf{w}}, \delta)$

Proof of Theorem 4.9.26. By repeated application of Exercise 4.9.31 we can find $m(q) \geq 1$ and disjoint clopen $I_{\mathbf{w}}(r)$ with $\mathbf{w} \in \mathbb{D}_2^r$ such that, writing $n(q) = m(1) + m(2) + \ldots + m(q)$:–

(a) The $I_{\mathbf{w}}(r)$ are disjoint for each r.

(b) $I_{(0)}(1) \cup I_{(1)}(1) = X$ and $I_{(\mathbf{u},0)}(r+1) \cup I_{(\mathbf{u},1)}(r+1) = I_{\mathbf{u}}(r)$ for all $\mathbf{u} \in \mathbb{D}_2^r$ and all $1 \leq r$.

(c) For each $\mathbf{w} \in \mathbb{D}_w^{n(q)}$ there exists a a $y_{\mathbf{w}}$ with $I_{\mathbf{w}} \subseteq B(y_{\mathbf{w}}, 2^{-q})$.

Now suppose $x \in X$. By (a) and (b), there exists a unique $\mathbf{w}_r(x) \in\in \mathbb{D}_2^r$ with $x \in I_{\mathbf{w}_r(x)}(r)$. By (b),

$$\mathbf{w}_{r+1}(x) = (\mathbf{w}_r(x), 0) \text{ or } \mathbf{w}_{r+1}(x) = (\mathbf{w}_r(x), 1).$$

Thus there exists a unique $\boldsymbol{\zeta}(x) \in \mathbb{D}_2^{\infty}$ such that

$$\mathbf{w}_r(x) = (\zeta_1(z), \zeta_2(x), \ldots \zeta_r(x))$$

On the other hand, if $\omega \in \mathbb{D}_2^\infty$ we know that the $I_{\omega_1,\omega_2,\dots,\omega_r}(r)$ are clopen, so closed and so compact. Since

$$I_{\omega_1,\omega_2,\dots,\omega_r}(r) \supseteq I_{\omega_1,\omega_2,\dots,\omega_{r+1}}(r+1)$$

the set $\bigcap_{r=1}^\infty I_{\omega_1,\omega_2,\dots,\omega_r}(r)$ is non-empty and condition (c) tells that it consists of one point which we denote by $Z(\omega)$. By inspection, $Z \circ \zeta$ is the identity map on X and $\zeta \circ Z$ is the identity map on $\mathbb{D}_2^\infty$ so $\zeta : \mathbb{D}_2^\infty \to X$ and $Z : X \to \mathbb{D}_2^\infty$ are inverse bijective maps. By inspection, using condition (c), both maps are continuous so X and $\mathbb{D}_2^\infty$ are homeomorpic. ■

4.10 Further exercises

Exercise 4.10.1. *(i) Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and consider polar coordinates (r,θ) with $x + iy = r\exp(i\theta)$. If $f : \mathbb{T} \to \mathbb{R}$ is a continuous real function, check that $\hat{f}(-n) = hatf(n)^*$. Now set*

$$\phi(r,\theta) = \hat{f}(0) + \sum_{j=1}^\infty \hat{f}(n)r^n e^{in\theta n} + \sum_{n=1}^\infty \hat{f}(n)^* r^n e^{-in\theta}$$

so that

$$\phi(r,\theta) = \hat{f}(0) + \sum_{j=1}^\infty \hat{f}(n)r^n e^{in\theta n} + \left(\sum_{n=1}^\infty \hat{f}(n)^* r^n e^{in\theta}\right)^* .$$

By considering an appropriate g in Exercise 4.2.9 (i), or otherwise, show that ϕ satisfies Laplace's equation on D.

(ii) Observe that the sum $S_N(t, theta) \sum_{n=-N}^N f(t)r^{|n|}e^{in(\theta-t)}$ converges uniformly in θ for r and deduce that

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2pi} f(t)\sum_{n=-\infty}^\infty r^{|n|}e^{in(\theta-t)}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)P_r(\theta-t)\,dt = f * P(\theta)$$

where

$$P_r(\theta) = \sum_{=-\infty}^\infty r^{-|n|}e^{i|n|\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

for $1 > r > 0$ (and $P(0) = 1$) is the Poisson kernel.
[*The formula $\phi = f * P_r$ contains no reference to complex variable or Fourier analysis. We could, for example, simply have set $\phi = f * P_r$ and used (careful) differentiation under the integral sign to verify that $\nabla^2\phi = 0$. Mathematicians familiar with two dimensional potential theory would have other reasons to consider the result as 'evident'.*]

Exercise 4.10.2. *We continue with the ideas and notation of the previous exercise.*

(i) Show that, if $0 < r < 1$,

(a) $P_r(\theta) \geq 0$ for all θ

(b) $\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$

(c) If $0 < \eta < \pi$, then $P_r(\theta) \to 0$ uniformly for $|\theta| \in [\eta, \pi]$ as $r \to 1-$.

(ii) By using the ideas of Theorem 2.3.4 show that, if f is continuous $f * P_r \to f$ uniformly as $r \to 1-$.

(iii) (Note that we use polar coordinates.) Show that if f is a real valued continuous function on $\partial D = \{(1, \theta) : \theta \in \mathbb{R}\}$ and we define $\phi : \mathrm{Cl}\,\Omega \to \mathbb{R}$ by

$$\phi(r, \theta) = \begin{cases} f * P_r(\theta) & \text{if } 0 \leq r < 1 \\ f(1, \theta) & \text{if } r = 1 \end{cases}$$

Conclude that we can always solve Dirichlet's problem for the disc with continuous boundary conditions.

(iv) Suppose that $\Gamma \subseteq \mathbb{C}$ is open and there exists a bijective map $T : Cl\Gamma \to ClD$ such that $\partial\Gamma = \partial D$, $T : Cl\Gamma \to ClD$, $T^{-1} : Cl\Gamma \to ClD$ are continuous and $T : \Gamma \to D$, $T^{-1} : \Gamma \to D$ are analytic[11]. Show (or recall from a first course in complex analysis) that we can always solve Dirichlet's problem with continuous boundary conditions for Γ.

[*Note, however, that our demands on T impose rather strong constraints on the boundary of Γ.*]

Exercise 4.10.3. *(i) Suppose that (X, d) is a compact metric space and (Y, ρ) is a metric space. If $f : X \to Y$ is continuous show that $f(E)$ is closed whenever E is closed*

(ii) Give an example of a continuous surjective map $g : [0, 1] \to [0, 1]$ (with the usual metric) which does not take open sets to open sets.

(iii) If (X, d) and (Y, ρ) are compact metric spaces and $f : X \to Y$ is continuous and injective, show that, whenever E is closed in X, then $f(E)$ is closed in Y. Deduce that whenever U is open in X, then $f(U)$ is open in Y. (This is an 'open mapping' theorem).

(iv) Suppose that (X, d) is a compact metric space and (Y, ρ) is a metric space. Show that if $f : X \to Y$ is continuous and bijective, then f^{-1} is continuous. (Thus f is a homeomorphism.)

(v) Suppose that (X, d) and (Y, ρ) are compact metric spaces, Show that

$$\Gamma(f) = \{(x, f(x)) : x \in X\}$$

is closed under the product metric if and only if f is continuous.

[11] Some of these conditions are implied by others

Exercise 4.10.4. *(i) Let E be an infinite dimensional normed space. Use Riesz's lemma to show that, given* $\mathbf{b}$, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in E$ *and* $\eta > 0$*, we can find a* $\mathbf{c} \in E$ *such that* $\|\mathbf{c} - \mathbf{b}\| \leq 1$ *and* $\|\mathbf{c} - \mathbf{a}_j\| > 1 - \eta$ *for* $n \geq 1$.

(ii) Suppose that $1 > \epsilon > 0$ *and that* $n(j)$ *is a strictly increasing sequence of positive integers. If* $\mathbf{a}_j \in E$ *show that we can find* $\mathbf{b}_r$ *such that*

$$\|\mathbf{b}_1\| < 1 - \epsilon/4 \text{ and } \|\mathbf{b}_1 - \mathbf{a}_j\| \geq 1 - \epsilon/2 \text{ for } 1 \leq j \leq n(1)$$

whilst

$$\|\mathbf{b}_r - \mathbf{b}_{r-1}\| \leq 2^{-4r}\epsilon \text{ and } \|\mathbf{b}_r - \mathbf{a}_j\| \geq 2^{-4r+1} \text{ for } n(r) + 1 \leq j \leq n(r+1)$$

for $r \geq 2$

(iii) If the norm on E is complete, show that, with the notation of (ii), $\mathbf{b}_r$ *converges to a point* $\mathbf{b}$ *with* $\|\mathbf{b}\| < 1$,

$$\|\mathbf{b} - \mathbf{a}_j\| > 1 - \epsilon \text{ for } 1 \leq j \leq n(1),$$

and

$$\|\mathbf{b} - \mathbf{a}_j\| > 1 - 2^{-4r-2}\epsilon \text{ for } n(r) + 1 \leq j \leq n(r+1).$$

(iv) Conclude that the closed unit ball in an infinite dimensional complete normed space cannot be covered by a sequence of balls $B(\mathbf{a}_j, r_j)$ *centre* $\mathbf{a}_j$ *and radius* r_j *such that* $r_j < 1$ *and* $r_j \to 0$.

(v) Explain why a separable normed space can be covered by a sequence of balls of any given fixed radius.

(vi) Show that any finite dimensional normed space can be covered by some sequence of balls $B(\mathbf{a}_j, r_j)$ *such that* $r_j \to 0$.

(vii) (This part uses the ideas of Exercise 1.8.6.) Show that any normed vector space of countable dimension can be covered by some sequence of balls $B(\mathbf{a}_j, r_j)$ *such that* $r_j \to 0$.

Exercise 4.10.5. *Suppose that a sequence of functions* $f_n : [0, 1] \to \mathbb{R}$ *is uniformly equicontinuous and that* $f_n(x) \to f(x)$ *as* $n \to \infty$ *for each* $x \in [0, 1]$*. Show that* $f_n \to f$ *uniformly.*

Exercise 4.10.6. *Use Theorem 4.2.3 to prove the following version of the maximum principle for analytic functions. Let* Ω *be a non-empty bounded open subset of* $\mathbb{C}$ *and* $f : \mathrm{Cl}\,\Omega \to \mathbb{C}$ *be a continuous function such that* f *is analytic on* Ω*. Then* $|f(z)|$ *attains its maximum on* $\partial\Omega$.
[*Hint: Consider* $\phi(x, y) = \Re e^{i\theta} f(x + iy)$.]

Exercise 4.10.7. *(i) Show that, given $\epsilon > 0$ we can find an infinitely differentiable function f, with $f^{(r)}(0) = 0$ for $r \geq 1$, $f(0) = 1$ and $f(1) = 0$, such that*

$$\int_0^1 tf'(t)^2\,dt < \epsilon.$$

(ii) Show that, given $\epsilon > 0$ we can find a polynomial P, with $P(0) = 1$ and $P(1) = 0$, such that

$$\int_0^1 tP'(t)^2\,dt < \epsilon.$$

[*Both parts just require ad hoc adjustments to the ideas of Exercise 4.2.8.*]

Exercise 4.10.8. *4.10.14 Let $\mathcal{F}$ be the set of infinitely differentiable functions $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$ and let $\mathcal{G}$ be the set of functions $g : [0,1] \to \mathbb{R}$ with $g(0) = g(1) = 0$ which are continuous and have bounded continuous derivative except at a finite number of points.*

(i) Show, by considering appropriate piecewise linear functions, that

$$\inf_{g\in\mathcal{G}} \int_0^1 g(t)^2 + (1 - g'(t)^2)^2\,dt = 0.$$

Show that

$$\int_0^1 g(t)^2 + (1 - g'(t)^2)^2\,dt > 0$$

for all $g \in \mathcal{G}$.

(ii) By modifying the example you used in (i), or otherwise, show that

$$\inf_{f\in\mathcal{F}} \int_0^1 f(t)^2 + (1 - f'(t)^2)^2\,dt = 0.$$

although

$$\int_0^1 f(t)^2 + (1 - f'(t)^2)^2\,dt > 0$$

for all $f \in \mathcal{F}$.

Exercise 4.10.9. *Let E be a non-empty subset of $\mathbb{R}^n$ with the usual metric. Show that, if every continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and attains its bounds, then E is closed and bounded (that is to say, compact).*

Exercise 4.10.10. *Suppose that (X, d) is a compact metric space and $f : X \to \mathbb{R}$ a continuous function (for the usual metric on $\mathbb{R}$).*

(i) Show that X can be covered by open balls $B(x, \delta_x)$ such that

$$y \in B(x, \delta_x) \Rightarrow |f(y) - f(x)| \leq 1.$$

Use the Heine–Borel property to show that $f(X)$ is bounded.

(ii) Let $M = \sup_{x\in X} f(x)$ and

$$E = \{x \in X \,:\, f(x) < M.\}$$

Show that E can be covered by open balls $B(e, \eta_e)$ with $e \in E$ such that

$$y \in B(e, \eta_e) \Rightarrow |f(y)| \leq (M + f(e))/2.$$

Use the Heine–Borel property to show that $E \neq X$ and deduce that there exists an $x_0 \in X$ with $f(x_0) = M$.

Exercise 4.10.11. *Let (X, d) be a metric space.*

(i) If A is a non-empty compact subset of X and $y \in X$, show, by considering a subsequence of some sequence of $a_n \in A$ with $d(a_n, y) \to \inf_{a\in A} d(a, y)$, that there exists an $a_0 \in A$ with $d(a_0, y) \geq d(a, y)$ for all $a \in A$.

(ii) If A is a non-empty compact subset of X and B is a non-empty closed subset of X show that there exists an $a_0 \in A$ and a $b_0 \in B$ with $d(a_0, b_0) \geq d(a, b)$ for all $a \in A$, $b \in B$.

(iii) Let $X = \mathbb{R}^2$, d the usual metric and

$$A = \{(x, y) \,:\, xy = 1,\ x > 0\},\ B = \{(x, y) \,:\, xy = -1,\ x < 0\}.$$

Show that A and B are closed, but there do not exist $\mathbf{a}_0 \in A$ and $\mathbf{b}_0 \in B$ with $d(\mathbf{a}_0, \mathbf{b}_0) \geq d(\mathbf{a}, \mathbf{b})$ for all $\mathbf{a} \in A$, $\mathbf{b} \in B$.

Exercise 4.10.12. *(i) Let (X, d) be a compact metric space. Suppose that $T : X \to X$ is a map such that*

$$d(Tx, Ty) < d(x, y)$$

for all $x \neq y$. Show that the map $f : X \to \mathbb{R}$ given by $f(x) = d(Tx, x)$ is continuous and deduce that there exists an $x_0 \in X$ with $Tx_0 = x_0$. Show that x_0 is unique.

(ii) Show, by means of an example, that the conclusion of (i) may be false if we replace the condition $d(Tx, Ty) < d(x, y)$ for $x \neq y$ by $d(Tx, Ty) \leq d(x, y)$. Show, by means of an example, that the conclusion of (i) may be false if we replace the condition (X, d) compact by (X, d) complete.

(iii) Let (X, d) be a compact metric space. Suppose that $T : X \to X$ is a map such that find a polynomial P, with $P(0) = 1$ and $P(1) = 0$, such that

$$\int_0^1 tP'(t)^2\, dt < \epsilon.$$

[*Both parts just require ad hoc adjustments to the ideas of Exercise 4.2.8.*]

Exercise 4.10.13. *(i) Let $\mathcal{G}$ be the collection of continuous functions $g : [0,1] \to \infty$ which are continuous and piecewise continuously differentiable. Show that*

$$\inf_{g\in\mathcal{G}} \int_0^1 (g(t)^2 + (1 - g'(t)^2)^2)\, dt = 0,$$

but that

$$\int_0^1 (g(t)^2 + (1 - g'(t)^2)^2)\, dt > 0$$

for all $g \in \mathcal{G}$.
[*Hint: Think about $g \in \mathcal{G}$ such that $(1-g'(t)^2)^2 = 0$ except at a finite set of points.*]

(ii) Let $\mathcal{F}$ be the collection of continuously differentiable functions $f : [0,1] \to \infty$. Show that

$$\inf_{f\in\mathcal{F}} \int_0^1 (f(t)^2 + (1 - f'(t)^2)^2)\, dt = 0$$

but

$$\int_0^1 (f(t)^2 + (1 - f'(t)^2)^2)\, dt > 0$$

for all $f \in \mathcal{F}$.

Exercise 4.10.14. *Let $\mathcal{F}$ be the set of infinitely differentiable functions $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$ and let $\mathcal{G}$ be the set of functions $g : [0,1] \to \mathbb{R}$ with $g(0) = g(1) = 0$ which have bounded continuous derivative except at a finite set of points.*

(i) Show, by considering appropriate piece wise linear functions, that

$$\inf_{g\in\mathcal{G}} \int_0^1 g(t)^2 + (1 - g'(t)^2)^2\, dt = 0.$$

Show that

$$\int_0^1 g(t)^2 + (1 - g'(t)^2)^2\, dt > 0$$

for all $g \in \mathcal{G}$.

(ii) By modifying the example you used in (i), or otherwise, show that

$$\inf_{f\in\mathcal{F}} \int_0^1 f(t)^2 + (1 - f'(t)^2)^2\, dt = 0.$$

although

$$\int_0^1 f(t)^2 + (1 - f'(t)^2)^2\, dt > 0$$

for all $f \in \mathcal{G}$.

Exercise 4.10.15. *Let E be a non-empty subset of $\mathbb{R}^n$ with the usual metric. Show that, if every continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and attains its bounds, then E is closed and bounded (that is to say, compact).*

Exercise 4.10.16. *(i) Let (X, d) be a compact metric space. Suppose that $T : X \to X$ is a map such that*

$$d(Tx, Ty) < d(x, y)$$

for all $x \neq y$. Show that the map $f : X \to \mathbb{R}$ given by $f(x) = d(Tx, x)$ is continuous and deduce that there exists an $x_0 \in X$ with $Tx_0 = x_0$. Show that x_0 is unique.

(ii) Show, by means of an example, that the conclusion of (i) may be false if we replace the condition $d(Tx, Ty) < d(x, y)$ for $x \neq y$ by $d(Tx, Ty) \leq d(x, y)$. Show, by means of an example, that the conclusion of (i) may be false if we replace the condition (X, d) compact by (X, d) complete.

(iii) Let (X, d) be a compact metric space. Suppose that $T : X \to X$ is a map such that there exists an $N \geq 1$ with

$$d(T^N x, T^N y) < d(x, y)$$

for all $x \neq y$. Show that there exists a unique $x_0 \in X$ such that $Tx_0 = x_0$.
[*Hint: $T^N(Tx) = T(T^N x)$.*]

(iv) In the theory of Markov chains, we are interested in the space

$$X = \left\{ \mathbf{q} \in \mathbb{R}^m \ : \ \sum_{j=1}^{m} q_j = 1,\ q_i \geq 0 \text{ for } 1 \leq i \leq m \right\}.$$

Suppose that $P = (p_{ij})$ is an $m \times m$ matrix with $p_{ij} \geq 0$ $[1 \leq i, j \leq m]$ and $\sum_{j=1}^{m} p_{ij} = 1$ for $1 \leq i \leq m$. Verify that, if $\mathbf{q} \in X$, then $\mathbf{q}P \in X$.

Suppose further that there exists an $N \geq 1$ such that the matrix P^N has no zero entries. By considering the metric

$$d(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{m} |u_i - v_i|$$

on X, show that there exists a unique $\boldsymbol{\pi} \in X$ with $\boldsymbol{\pi} P = \boldsymbol{\pi}$. (In the language of Markov chains, every irreducible, aperiodic finite Markov chain has a unique associated invariant probability.)

Exercise 4.10.17. *Here is another approach to finite Markov chains. Suppose that we wish to rank web sites. One way of ranking web sites is to consider Simon, a rather simple minded web user. If Simon is on web page, he jumps to any linked*[12]

[12] Note that links run in only one direction on the web.

web page. Thus if web site j is one of r web sites linked to web site i, Simon jumps to j with probability $p_{ij} = 1/r$. If web site j is not linked, $p_{ij} = 0$. (In order to make sure that $r \neq 0$, we adopt the convention that every web page is linked to itself.) Web pages which Simon visits frequently are highly ranked (because we suspect that they must have many links from other highly ranked web pages) and those that he visits infrequently have low ranks.

Unfortunately, this simple procedure will not always give the desired result, since Simon might get trapped on a page with no links to any other or (more difficult to prevent) in a small set of pages with links to each other but no links beyond those. To remedy, this we consider Simon's very slightly more sophisticated sister Susan. When Susan is at site i she behaves like Simon with probability $1-\alpha$ but jumps to a completely random page with probability $\alpha > 0$. If we suppose that there are m pages in all, this gives the probability that Susan jumps from page i to page j as

$$p_{ij}(\alpha) = (1-\alpha)p_{ij} + \alpha m^{-1}.$$

Verify that $p_{ij}(\alpha) > 0$ $[1 \leq i,\ j \leq m]$ and $\sum_{j=1}^m p_{ij}(\alpha) = 1$ for $1 \leq i \leq m$.

As in Exercise 4.10.16, we consider the space

$$X = \left\{ \mathbf{q} \in \mathbb{R}^m \ :\ \sum_{j=1}^m q_j = 1,\ q_i \geq 0 \text{ for } 1 \leq i \leq m \right\}$$

with the metric

$$d(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^m |u_i - v_i|.$$

If $\mathbf{q} \in X$, let $T_\alpha(\mathbf{q}) = \tilde{\mathbf{q}}$ with

$$\tilde{q}_j = \sum_{i=1}^n q_i p_{ij}(\alpha).$$

(i) Show that T_α is a contraction mapping and deduce that there is a unique $\boldsymbol{\pi}(\alpha) \in X$ with

$$T_\alpha \boldsymbol{\pi}(\alpha) = \boldsymbol{\pi}(\alpha).$$

and that, given any $\mathbf{q} \in X$, we have $T_\alpha^n \mathbf{q} \underset{d}{\to} \boldsymbol{\pi}(\alpha)$.

(ii) In practice, we are interested in how fast $T_\alpha^n \mathbf{q}$ approaches $\boldsymbol{\pi}(\alpha)$. If we take $\alpha = 3/4$, how large must n be for you to guarantee that $d(T_\alpha^n \mathbf{q}, \boldsymbol{\pi}(\alpha)) \leq 10^{-16}$? (In real life, we would start from $\mathbf{q}$ representing last week's result and, since things do not change very much in a week, a single iteration would suffice.)

(iii) We now forget the ranking problem and consider any p_{ij} with $p_{ij} \geq 0$, $\sum_{j=1}^m p_{ij} = 1$. Since we made no use of the way we obtained the p_{ij} in the ranking

problem, the result of (i) still applies. Use compactness to show that there exists a $\boldsymbol{\pi} \in X$ *and a sequence* $\alpha_n \to 0$ *such that* $\boldsymbol{\pi}(\alpha_n) \underset{d}{\to} \boldsymbol{\pi}$*. Show that* $\boldsymbol{\pi}P = \boldsymbol{\pi}$*. (In the language of Markov chains, every finite Markov chain has an associated invariant probability.)*

(iv) Suppose that $m = 2$ *and*

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that there is a unique $\boldsymbol{\pi} \in X$ *with* $\boldsymbol{\pi}P = \boldsymbol{\pi}$*, but that, if* $\mathbf{q} \neq \boldsymbol{\pi}$*,*

$$d(T^n\mathbf{q}, \boldsymbol{\pi}) \nrightarrow 0$$

as $n \to \infty$*.*

(v) Suppose that $m = 4$ *and*

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Show that, if $\mathbf{q} \in X$*, then there exists a* $\boldsymbol{\pi}_\mathbf{q} \in X$ *with* $\boldsymbol{\pi}_\mathbf{q}P = \boldsymbol{\pi}_\mathbf{q}$ *and* $d(T^n\mathbf{q}, \boldsymbol{\pi}_\mathbf{q}) \to 0$ *as* $n \to \infty$*. By finding* $\boldsymbol{\pi}_\mathbf{q}$ *explicitly, show that* $\boldsymbol{\pi}_\mathbf{q}$ *depends on* $\mathbf{q}$*.*

(vi) Explain why (iv) and (v) do not contradict the result obtained in part (iii).

Exercise 4.10.18. *Let* (X, d) *be a compact metric space and* (Y, ρ) *a metric space. Let* Z *be the space of all continuous functions* $f : X \to Y$*. Show that*

$$\rho_\infty(f, g) = \sup_{x \in X} \rho(f(x), g(x))$$

is a well defined metric on Z*. Show that* (Z, ρ_∞) *is complete if and only if* (Y, ρ) *is complete.*

Exercise 4.10.19. *A function* $f : [0, 1] \to \mathbb{R}$ *is said to be of* bounded variation *if there exists an* M *such that, whenever* $0 = x_0 < x_1 < \ldots < x_n = 1$*, we have* $\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq M$*. We write*

$$V(f) = \inf \left\{ \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \,;\, 0 = x_0 < x_1 < \ldots < x_n = 1,\ n \geq 1 \right\}.$$

(i) Show that the space $BV([0, 1])$ *of functions of bounded variation is a vector space over* $\mathbb{R}$ *and that*

$$\|f\|_{BV} = |f(0)| + V(f)$$

is a complete norm on $BV([0,1])$.

(ii) Show that, if we use this norm, the subspace of continuous functions in $BV([0,1])$ *is closed with a dense complement. Thus quasi-all functions in* $BV([0,1])$ *are discontinuous.*

(iii) Show that the function $g : [0,1] \to \mathbb{R}$ *given by* $g(x) = x \sin x^2$ *for* $x \neq 0$, $g(0) = 0$ *is continuous but not of bounded variation.*

Exercise 4.10.20. [Dini's theorem] *(i) Let* (X,d) *be a compact metric space. Suppose that* $f_n : X \to \mathbb{R}$ *is continuous, that* $f : X \to \mathbb{R}$ *and that, for each* $x \in X$, $f_n(x)$ *is an increasing sequence with* $f_n(x) \to f(x)$. *Show that, given* $\epsilon > 0$ *and* $x \in X$, *we can find an integer* $N(x)$ *and an open ball* $B(x, \delta_x)$ *such that* $f(y) - f_n(y) < \epsilon$ *for all* $y \in B(x, \delta_x)$. *Deduce that there exists an* $N(\epsilon)$ *such that* $|f(y) - f_n(y)| < \epsilon$ *for all* $n \geq N(\epsilon)$. *This is Dini's theorem:- an increasing sequence of continuous functions on a compact metric space tending to a continuous limit converges uniformly.*

(ii) Show that Dini's theorem fails if we replace the condition (X,d) *compact by* (X,d) *complete. Show that Dini's theorem fails if we remove the condition* f_n *increasing. Show that Dini's theorem fails if we remove the condition* f_n *continuous, but keep the condition* f *continuous. Show that Dini's theorem fails if we remove the condition* f *continuous, but keep the condition* f_n *continuous.*

(iii) Define $p_n : [-1,1] \to \mathbb{R}$ *inductively by* $p_0(x) = 0$ *and*

$$p_{n+1}(x) = p_n(x) + \frac{x^2}{2} - \frac{p_n(x)^2}{2}.$$

If x is fixed, show, by induction, that

$$0 \leq p_n(x) \leq |x| \text{ and } p_{n+1} - p_n(x) \geq 0.$$

Conclude that $p_n(x)$ *tends to a limit* $p(x)$ *with* $0 \leq p(x) \leq |x|$. *Show that in fact,* $p(x) = |x|$.

Show that $p_n(x) \to |x|$ *uniformly on* $[0,1]$ *and use this fact to give another proof of Lemma 2.6.9.*

Exercise 4.10.21. *In this exercise we give a slightly more general form of the Stone–Weierstrass theorem. We work over* $\mathbb{R}$.

(i) Show that we can find a sequence of polynomials P_n *with* $P_n(0) = 0$ *such that* $P_n(t) \to |t|$ *uniformly on* $[-1,1]$ *as* $n \to \infty$.

(ii) If a, b, c, $d \in \mathbb{R}$ *and* $a \neq b$, a, $b \neq 0$, *show that we can find* λ, $\mu \in \mathbb{R}$ *such that*

$$\lambda a + \mu a^2 = c,$$
$$\lambda b + \mu b^2 = d.$$

(iii) Let (X, d) *be a compact metric space and suppose that* $\mathcal{A}$ *is a subalgebra of* $C_{\mathbb{R}}(X)$. *We say that* $\mathcal{A}$ *is* strongly separating *if, whenever* $x, y \in X$ *and* $x \neq y$, *there exists an* $f \in \mathcal{A}$ *with* $f(x) \neq f(y)$ *and* $f(x), f(y) \neq 0$. *Show that* $Cl\mathcal{A} = C_{\mathbb{R}}(X)$ *if and only if* $\mathcal{A}$ *is strongly separating.*

(iv) If $0 \notin [a, b]$ *show that there is a sequence of polynomials* Q_n *with zero constant term such that* $Q_n(t) \to 1$ *uniformly on* $[a, b]$.

Exercise 4.10.22. *Here is a rather direct proof of the Stone–Weierstrass theorem. In this exercise* (X, d) *and* $\mathcal{A}$ *satisfy the hypotheses of Theorem 4.4.6 and* E *and* F *are disjoint non-empty closed subsets of* X.

(i) Show that, given $x \in E$, *we can find a* $g \in \mathcal{A}$ *such that* $g(x) = 0$, $0 \leq g(t) \leq 1$ *for all* $t \in X$ *and* $g(t) > 0$ *for all* $t \in F$. *Explain why we can find an integer* $M \geq 1$ *such that* $g(t) \geq 2/M$ *for all* $t \in F$.

(ii) Show that we can find open sets $U_1, U_2, \ldots, U_m$ $g_1, g_2, \ldots, g_m \in \mathcal{A}$ *and strictly positive integers* $M_1, M_2, \ldots, M_m$ *such that* $\bigcup_{r=1}^{m} U_r \supseteq E$ *whilst*

$$\begin{aligned} 0 \leq g_r(t) &\leq 1 \text{ for all } t \in X \\ g_r(t) &\geq \frac{2}{M_r} \text{ for all } t \in F \\ g_r(t) &\leq \frac{1}{2M_r} \text{ for all } t \in U_r \end{aligned}$$

for each r *with* $1 \leq r \leq m$

(iii) By using elementary calculus, or otherwise, show that

$$1 - Nt \leq (1-t)^N \leq \frac{1}{Nt}$$

for all integers $N \geq 1$ *and all* t *with* $0 < t \leq 1$.

(iv) By using (iii) and considering $h_r(t) = 1 - (1 - g_r^{n_r}(t))^{M_r^{n_r}}$ *with* n_r *sufficiently large show that we can find* $h_r \in \mathcal{A}$ *such that*

$$\begin{aligned} 0 \leq h_r(t) &\leq 1 \text{ for all } t \in X \\ g_r(t) &\geq \left(\frac{3}{4}\right)^{1/m} \text{ for all } t \in F \\ g_r(t) &\leq \frac{1}{4} \text{ for all } t \in U_r \end{aligned}$$

for each r *with* $1 \leq r \leq m$.

(iv) Now set $p(t) = h_1(t)h_2(t)\dots h_m(t) - 1/2$. *Show that* $p \in \mathcal{A}$ *and*

$$-\frac{1}{2} \le p(t) \le \frac{1}{2} \text{ for all } t \in X$$
$$p(t) \ge \frac{1}{4} \text{ for all } t \in F$$
$$p(t) \le -\frac{1}{4} \text{ for all } t \in E.$$

(v) Suppose that $f \in C(X)$ *with* $\|f\|_\infty \le 1$. *By setting*

$$E = \{x \in X \,:\, f(x) \le -1/4\},\ F = \{x \in X \,:\, f(x) \ge 1/4\}$$

show that we can find a $p \in \mathcal{A}$ *such that* $\|p\|_\infty \le 1/2$ *and* $\|f - p\|_\infty \le 3/4$.

(vi) Use a successive approximation argument to show that $\mathcal{A}$ *is uniformly dense in* $C(X)$.

Exercise 4.10.23. *Let* (X, d) *be a metric space and* $C(X)$ *the space of continuous functions* $f : X \to \mathbb{F}$. *A subset* $\mathcal{F} \subseteq C(X)$ *is said to be* equicontinuous at the point x *if, given* $\epsilon > 0$, *we can find a* $\delta(\epsilon) > 0$ *such that* $|f(x) - f(y)| < \epsilon$ *whenever* $d(x, y) < \delta(\epsilon)$ *and* $f \in \mathcal{F}$. *Show that, if* X *is compact and* $\mathcal{F}$ *is equicontinuous at every point of* X, *then* $\mathcal{F}$ *is uniformly equicontinuous.*

Give an example of a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ *which are are uniformly continuous and such that the set*

$$\mathcal{F} = \{f_n \,:\, n \ge 1\}$$

is equicontinuous at each point, but not uniformly equicontinuous.

Exercise 4.10.24. *Suppose that* (X_j, d_j) *is a metric space* $[j \ge 1]$ *and* $X = \prod_{j=1}^{\infty} X_j$. *Suppose further that* d *is a metric on* X *such that*

$$x_j(n) \underset{d_j}{\to} x_j \text{ as } n \to \infty \text{ for each } j \Leftrightarrow \mathbf{x}(n) \underset{d}{\to} \mathbf{x}.$$

If (X_j, d_j) *is separable for each* j, *show that* (X, d) *is separable.*

Exercise 4.10.25. *We know (see Exercise 1.5.31) that there exists a space* Y *and two metrics* ρ_1 *and* ρ_2 *on* Y *such that*

$$y(n) \underset{\rho_1}{\to} y \Leftrightarrow y(n) \underset{\rho_2}{\to} y$$

but (Y, ρ_1) *is complete and* (Y, ρ_2) *is not. Use this fact to find complete metric spaces* (X_j, d_j) $[j \ge 1]$ *and a metric* d *on* $X = \prod_{j=1}^{\infty} X_j$ *such that*

$$x_j(n) \underset{d_j}{\to} x_j \text{ as } n \to \infty \text{ for each } j \Leftrightarrow \mathbf{x}(n) \underset{d}{\to} \mathbf{x}$$

but (X, d) *is not complete.*

Exercise 4.10.26. *By Exercise 4.7.1 we know that if we have a collection of non-trivial normed spaces $(V_j, \| \|_j)$, we can find a metric d on the product $V = \prod_{j=1}^{\infty} V_j$ such that the following two statements are equivalent.*

(a) $\|\mathbf{v}_j(n) - \mathbf{v}_j\|_j \to 0$ as $n \to \infty$ for each j.

(b) Setting $\mathbf{v}_j = (\mathbf{v}_j(1), \mathbf{v}_j(2), \dots)$ and $\mathbf{v}_j = (\mathbf{v}(1), \mathbf{v}(2), \dots)$ we have $d(\mathbf{v}_j, \mathbf{v}) \to 0$ as $n \to \infty$

We know that V can be made into a vector space in a standard manner. The object of this exercise, which take longer to discuss than to do, is to show that d cannot be derived from a norm on V. To this end, let $\| \|$ be a norm on V.

(i) Let $\mathbf{w}_j(n) = \mathbf{0}$ for $j \neq n$ and $\mathbf{w}_n(n) \neq 0$. Explain why $\|\mathbf{w}(n)\| \neq 0$ and deduce that we can find a $\mathbf{v}(n) \in V$ such that $\mathbf{v}_j(n) = \mathbf{0}$ for $j \neq n$ and $\|\mathbf{v}(n)\| = n$.

(ii) Show that $d(\mathbf{v}_j, \mathbf{0}) \to 0$ but $\|\mathbf{v}_j\| \to \infty$ as $j \to \infty$.

Exercise 4.10.27. *Here is another way of looking at the result of Exercise 4.8.17 (iv). Recall the definition of H_n in Exercise 4.8.6 and write G_n for the set of $\chi \in \widehat{\mathbb{D}^\infty}$ such that*

$$\chi(\boldsymbol{\omega}) = 1 \text{ whenever } \boldsymbol{\omega} \in H_n.$$

(i) Let C_n be the collection of functions $f : \mathbb{D}^\infty \to \mathbb{C}$ such that f is constant on each coset of H_n. Show that C_n is a vector subspace of $C(\mathbb{D}^\infty)$ with dimension 2^n. By observing that there are 2^n elements of G_n show that G_n is a basis for C_n.

(ii) Consider the map $T : C(\mathbb{D}^\infty) \to C_n$ given by

$$Tf(\boldsymbol{\omega}) = 2^n \int_{\mathbb{D}^\infty} \mathbb{I}_{\omega + H_n}(\boldsymbol{\eta})\, d\boldsymbol{\eta}.$$

By (i), we can write

$$Tf = \sum_{\chi \in G_n} a_\chi \chi.$$

Show that $a_\chi = \hat{f}(\chi)$ and deduce that

$$\sum_{\chi \in G_n} \hat{f}(\chi)\chi = \sum_{J \in \mathcal{H}_n} c_J \mathbb{I}_J$$

where $\mathcal{H}_n$ is the collection of cosets of H_n and

$$C_J = 2^n \int_{\mathbb{D}^\infty} f(\boldsymbol{\eta}) \mathbb{I}_J(\boldsymbol{\eta})\, d\boldsymbol{\eta}.$$

Exercise 4.10.28. *Consider the metric space consisting of open interval $(-1, 1)$ with the usual metric. Check that $I_n = [-1 + 1/n, 1 - 1/n]$ is a compact set and that the sequence I_n is Cauchy with respect to the Hausdorff metric, but does not converge.*

Exercise 4.10.29. *(i) Describe a procedure analogous to that given on page 146 to give a set in* $\mathbb{R}^n$ *which is perfect and totally disconnected showing (in as much detail as you consider desirable) that it has the stated properties.*

(ii) If E_j *is a perfect totally disconnected set in* $\mathbb{R}^{n(j)}$ *show that* $E_1 \times E_2$ *is perfect and totally disconnected in* $\mathbb{R}^{n(1)+n(2)}$.

Chapter 5

Three applications

In this chapter we prove three classical theorems. Although each proof involves other ideas specific to the subject discussed, each proof involves compactness (and uniform equicontinuity) at a crucial moment.

5.1 More on differential equations

(The reader who only wants to see uniform equicontinuity in action can skip to Theorem 5.1.13. The rest of the section simply sets the scene.)

What happens if we try to relax the conditions or try to improve the conclusions of the version of Picard's theorem given in Theorem 1.4.3? Recall that, in the absence of of the Lipschitz condition, uniqueness may break down.

Exercise 5.1.1. *Reread Exercise1.1.7. Why is it consistent with the result of Theorem 1.4.3?*

In Exercise 1.4.5 we showed that, under certain circumstances, we can extend our result on the existence of local solutions to one on global solutions. (If the reader has not already done this exercise, she should do it now.)

In Exercise 1.1.8 we saw that it may not always be possible to obtain global solutions. We remind ourselves of that example by carefully reworking it.

Exercise 5.1.2. *Define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by* $f(x, y) = (1 + x^2)$*. Check that* $f(x, y)$ *is bounded when* $|x| \leq R$ *for each* $R > 0$*. Suppose that*

$$g'(t) = f(g(t), t)$$

for $|t| < \pi/2$ *and* $g(0) = 0$*. Write* $G(t) = g(\tan t)$*. By computing* $G'(t)$*, and using the mean value theorem, show that*

$$g(t) = \tan^{-1} t$$

for $|t| < \pi/2$. Deduce that there does not exist any once differentiable function $h : \mathbb{R} \to \mathbb{R}$ with

$$h'(t) = f(g(t), t)$$

and $h(0) = 0$.

Exercise 5.1.2 is rather disturbing so we seek conditions which prevent this kind of 'explosion'. We start with a definition.

Definition 5.1.3. *Let us say that a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is* locally Lipschitz *if, given x_0, $t_0 \in \mathbb{R}$, we can find δ_0, $\eta_0 > 0$ and $K_0 > 0$ depending on (x_0, t_0) such that*

$$|f(u,t) - f(v,t)| \leq K_0|u - v|$$

for all $t \in [t_0 - \delta_0, t_0 + \delta_0]$ and all u, $v \in [x_0 - \eta_0, x_0 + \eta_0]$.

Exercise 5.1.4. *(i) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous with continuous first partial derivative in the first variable, show that f is locally Lipschitz.*

(ii) Show that a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz if and only if, given $a > 0$, we can find a $K_a > 0$ such that

$$|f(u,t) - f(v,t)| \leq K_a|u - v|$$

for all $|t|$, $|u|$, $|v| \leq a$.

We also need a couple of preliminary remarks.

Exercise 5.1.5. *(Easy when you see what is going on.) Suppose that $a, , b > 0$ $h : [-a, a] \times [-b, b] \to \mathbb{R}$ is a continuous function and a, $b > 0$. Let us define $g : [-a, a] times \mathbb{R} \to \mathbb{R}$ by*

$$g(x,y) = \begin{cases} h(x,-b) & \text{for } |x| \leq b,\ y < -b \\ h(x,y) & \text{for } |x| \leq b,\ |y| \leq b \\ h(x,b) & \text{for } |x| \leq b,\ b < y \end{cases}$$

and $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} g(-b,y) & \text{for } x < -b \\ h(x,y) & \text{for } |x| \leq b \\ g(b,y) & \text{for } x < b. \end{cases}$$

(i) Show that f is continuous and bounded.

(ii) If $|f(u,t) - f(v,t)| \leq K|u - v|$, for all $u, v \in [-a, a]$, $t \in [-b, b]b$ show that

$$|f_b(u,t) - f_b(v,t)| \leq K|u - v|$$

for all t, u, $v \in \mathbb{R}$.

Exercise 5.1.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that*

$$|f(u,t)| \le K(1 + |u|)$$

for all u, $t \in \mathbb{R}$ and some $K > 0$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a once differentiable function with

$$g'(t) = f(g(t),t),\ g(0) = x_0$$

By showing that $e^{-Kt}(|g(t)| + 1)$ is a decreasing function, or otherwise, show that $|g(t)| \le (e^{+Kt} + 1)(|x_0| + 1)$ for all $t \ge 0$

Show that if

$$G'(t) = f(G(t),t),\ G(t_0) = x_0.$$

then

$$|G(t - t_0) - x_0| \le e^{K|t-t_0|}(|x_0| + 1)$$

for all t.

Lemma 5.1.7. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous locally Lipschitz function such that*

$$|f(u,t)| \le K(1 + |u|)$$

for all u, $t \in \mathbb{R}$ and some $K > 0$. Then we can find a unique once differentiable function $g : \mathbb{R} \to \mathbb{R}$ with

$$g'(t) = f(g(t),t),\ g(t_0) = x_0.$$

Proof. Without loss of generality, let $t_0 = 0$. Let $T > 0$ and $S > (e^{+Kt} + 1)(|x_0| + 1)$. By Exercise 5.1.4 (ii), there is a $C > 0$ such that

$$|f(u,t) - f(v,t)| \le C|u - v| \text{ for all } t \in [-T,T] \text{ and all } u,v \in [-S,S].$$

By Exercise 5.1.5 we can find a continuous function $F : \mathbb{R}^2 \to \mathbb{R}$ with $F(u,t) = f((u,t)$ for $(u,t) = [-S,S] \times [-T,T]$ such that $F(u,t) - F(v,t)| \le C|u - v|$ for all $u, v, t \in mathbbR$. By the global Picard theorem proved in Exercise1.4.5 there is one and only one differentiable function x $x : \mathbb{R} \to \mathbb{R}$ such that

$$x'(t) = F(x(t),t) \text{ for all } t \in \mathbb{R} \text{ and } x(0) = x_0.$$

By Exercise 5.1.6 $x(t) \in [-S,S]$ for all $t \in [-T,T]$ so

$$x'(t) = f(x(t),t) \text{ for all } t \in [-T,T]$$

Since T was arbitrary, the full result follows. ■

Exercise 5.1.8. *Arnol'd's charming book* Yesterday and Long Ago *contains the following passage.*

> *Once M. L. Lidov explained to me that mathematical theorems, like the uniqueness theorem in the theory of differential equations, contradict physical reality. For example two integral curves* [*graphs of solutions*] *of the equation $\dot{x} = -x$ (with different initial data $x(0) = 0$ and $x(0) = 1$) will not intersect. For this reason a ship cannot moor smoothly using its engine alone; either the process would take an infinite time or the ship would strike the moorage. That is why in the final stage of mooring a sailor performs the task manually by throwing a mooring rope over a bitt*[1]*. For the same reason, spaceships landing on the moon or Mars should have legs which dampen the motion.*

Think about explaining the argument just given to some one who has done a first course in calculus. Can you see any way of explaining it to some one who has not done any calculus but is mathematically able with an open mind?

Exercise 5.1.9. *Explain why the proof of Lemma 5.1.7 fails when we try to apply it in Exercise 5.1.2.*

Exercise 5.1.10. . *If $\beta > 1$, show that we can find a locally Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$|f(u,t)| \leq (1 + |u|)^\beta$$

for all $u,\ t \in \mathbb{R}$, but there does not exist any once differentiable function $g : \mathbb{R} \to \mathbb{R}$ with

$$g'(t) = f(g(t), t) \text{ and } g(0) \neq 0.$$

A very similar argument to those we have already looked at gives the following standard result.

Lemma 5.1.11. *Suppose that $x_0, t_0 \in \mathbb{R}$, $\delta, \eta > 0$, $k > 0$, $M > 0$ and $M\delta < \eta$. If the continuous function $f : [x_0 - \eta, x_0 + \eta] \times [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ satisfies the Lipschitz condition*

$$|f(u,t) - f(v,t)| \leq k|s - r|$$

for all $t \in [t_0 - \delta, t_0 + \delta]$ and $u,\ v \in [x_0 - \eta, x_0 + \eta]$ together with the condition

$$|f(u,t)| \leq M$$

for all $t \in [t_0 - \delta, t_0 + \delta]$ and $u \in [x_0 - \eta, x_0 + \eta]$, then there is a unique differentiable function $g : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ such that

$$g'(t) = f(g(t), t) \text{ and } g(t_0) = x_0.$$

[1] One of the strong posts firmly fastened in pairs in the deck of a ship for fastening cables and belaying ropes.

Exercise 5.1.12. *(i) Explain informally why we need the condition $M\delta < \eta$ in Lemma 5.1.11.*
(ii) Prove Lemma 5.1.11 in as much detail as you consider appropriate.

We now prove a theorem of Peano which shows that the differential equations of the type we have been discussing always have a local solution (so the Lipschitz condition is irrelevant to existence).

Theorem 5.1.13. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then, given $(x_0, t_0) \in \mathbb{R}^2$, we can find a $\delta > 0$ and differentiable function $x : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ such that*

$$x'(t) = f(x(t), t)$$

for all $t \in [t_0 - \delta, t_0 + \delta]$ and $x(t_0) = x_0$.

Proof. Without loss of generality, we may suppose that that $x_0 = t_0 = 0$. Since f is continuous on $[-1, 1]^2$, it is uniformly continuous and we can find an M such that

$$|f(u, t)| \leq M$$

for all $(u, t) \in [-1, 1]^2$. Now choose δ, $\eta > 0$ such that δ, $\eta < 1$ and $M\delta < \eta$. From now on we work on $[-\eta, \eta] \times [\delta, \delta]$.

By the Stone–Weierstrass theorem or direct construction, we can find a sequence of continuously differentiable functions $f_n : [-\eta, \eta] \times [\delta, \delta] \to \mathbb{R}$ such that

$$|f_n(u, t)| \leq M$$

for all $(u, t) \in [-\eta, \eta] \times [\delta, \delta]$ and $f_n \to f$ uniformly on $[-\eta, \eta] \times [\delta, \delta]$. Since continuously differentiable functions automatically satisfy the Lipschitz condition required by Lemma 5.1.11, we can find once differentiable functions $g_n : [\delta, \delta] \to \mathbb{R}$ with

$$g_n'(t) = f_n(g_n(t), t) \text{ and } g_n(0) = 0.$$

We now use the power of equicontinuity arguments. Observe that by the mean value theorem

$$|g_n(t) - g_n(s)| \leq M|t - s|,$$

so the collection of the g_n is uniformly equicontinuous and so has compact closure in $C([-\delta, \delta])$. It follows that some subsequence of the g_n converges uniformly. By extracting this subsequence, we may suppose that $g_n \to g$ uniformly on $C([-\delta, \delta])$. We observe that, since $|g_n(t)| \leq \eta$ for all $t \in [-\delta, \delta]$, we have $|g(t)| \leq \eta$ for all $t \in [-\delta, \delta]$. Since f is uniformly continuous on $[-\eta, \eta] \times [\delta, \delta]$, we have

$$|f_n(g_n(t), t) - f(g(t), t)| \leq |f_n(g_n(t), t) - f(g_n(t), t)| + |f(g_n(t), t) - f(g(t), t)| \to 0 + 0 = 0$$

uniformly on $[\delta, \delta]$. Thus, by elementary theorems on integrals,

$$g_n(t) = g_n(t) - g_n(0) = \int_0^t g_n'(s)\,ds = \int_0^t f_n(g_n(s), s)\,ds \to \int_0^t f(g(s), s)\,ds.$$

as $n \to \infty$. We thus have

$$g(t) = \int_0^t f(g(s), s)\,ds,$$

so g is differentiable and

$$g'(t) = f(g(t), t)$$

for all $t \in [-\delta, \delta]$. Since $g(0) = 0$, we are done. ∎

Note, once again, that Theorem 5.1.13 gives existence, but not uniqueness. Contraction mapping arguments give existence and uniqueness. Compactness arguments give existence, but not uniqueness.

Exercise 5.1.14. *(Just to emphasise the point.)*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by by $f(u, t) = \frac{3}{2}|u|^{1/2}$ and consider a sequence of functions $f_n : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f_{2n-1}(u, t) = \begin{cases} 3n^{1/2} & \text{for } |u| \le n^{-1} \\ 3u^{2/3} & \text{for } |u| > n^{-1} \end{cases}$$

and

$$f_{2n}(u, t) = \begin{cases} 0 & \text{for } |u| \le n^{-1} \\ |u| - n^{-1} & \text{for } n^1 < |u| \le 2n^{-1} \\ 3|u|^{2/3} & \text{for } |u| > n^{-1}. \end{cases}$$

Check that the functions f_n are continuous and we can find a $K_n > 0$ such that

$$|f_n(u, t) - f_n(v, t)| \le K_n|u - v|$$

for all u, v, $t \in \mathbb{R}$. We thus know that there is a unique differentiable function $g_n : \mathbb{R} \to \mathbb{R}$ with

$$g_n'(t) = f_n(g_n(t), t) \text{ and } g_n(0) = 0.$$

Find g_n.

Check that there are two differentiable functions $G_1, G_2 : \mathbb{R} \to \mathbb{R}$ such that $g_{2n-1} \to G_1$ and $g_{2n} \to G_2$ uniformly on $\mathbb{R}$. Check that $f_n \to f$ uniformly on $\mathbb{R}^2$ and that

$$G_q'(t) = f(G_q(t), t),\ G_q(0) = 0$$

for $q = 1, 2$, but $G_1(t) \ne G_2(t)$ for $t \ne 0$.

5.2 The Riemann mapping theorem

The next two sections require a first course in complex variable including Morera's theorem and Rouché's theorem. We work in $\mathbb{C}$ unless otherwise stated.

The reader should recall the following definition.

Definition 5.2.1. *If Ω_1 and Ω_2 are open subsets of $\mathbb{C}$, we say that f is a* conformal mapping *of Ω_1 to Ω_2 if $f : \Omega_1 \to \Omega_2$ is a bijective analytic map.*

If the reader may have seen the following corollary of Rouché's theorem. If not, we give a proof in Exercise 5.6.5.

Lemma 5.2.2. *If Ω_1 and Ω_2 are open and f is a* conformal mapping *of Ω_1 to Ω_2, then the inverse map f^{-1} is analytic and so a conformal map of Ω_2 to Ω_1.*

Exercise 5.2.3. *Let us say that two open subsets Ω_1 and Ω_2 of $\mathbb{C}$ are* conformally equivalent *if there is a conformal mapping of Ω_1 to Ω_2. Check that conformal equivalence is indeed an equivalence relation.*

Riemann claimed that any reasonable open subset of $\mathbb{C}$ could be conformally mapped to the open unit disc

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

His argument depended on ideas very similar to those which we discussed in Section 4.1 and the objections to those ideas which we discussed there were first raised by Weierstrass in connection with Riemann's argument. Of course, mathematicians like Weierstrass did not doubt that Riemann's claim was correct for sufficiently well behaved sets, but it took a long time to state and prove a precise version. Remarkably, it turned out that the strongest possible version of Riemann mapping theorem was, in fact, correct.

We know (by Liouville's theorem) that any analytic map of $\mathbb{C}$ to D is a constant map. It is also clear that 'no region with holes' can be mapped by a bijective function with continuous inverse to a region like D 'without holes'[2]. The Riemann mapping theorem states that these are the only constraints. Any *proper* (that is to say neither $\varnothing$ nor $\mathbb{C}$) open region without holes can be mapped conformally to D.

Our first task is to say what we mean by a 'region without holes'.

Definition 5.2.4. *We say that an open subset Ω of $\mathbb{C}$ is* pathwise connected *if, whenever $z_1, z_2 \in \Omega$, we can find a continuous function $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = z_1$, $\gamma(1) = z_2$.*

[2]More concisely a region of the plane with holes cannot be homeomorphic to one without.

In what follows we will need γ to be 'well behaved'. Exercise 5.6.6 shows that we can replace the condition 'γ continuous' in Definition 5.2.4 by 'γ well behaved' (for example piecewise smooth). Exercise 5.6.7 recalls some of the properties of analytic functions on open pathwise connected sets.

Definition 5.2.5. *We say that on open subset Ω of $\mathbb{C}$ is* simply connected *if it is pathwise connected and, whenever $g : \Omega \to \mathbb{C}$ is analytic and C is a closed contour lying within Ω, we have*

$$\int_C g(z)\, dz = 0.$$

In more advanced work, the notion of simply connected is defined in a more general context and it is shown that, when Ω is an open subset of $\mathbb{C}$, the more general notion coincides with the one given here.

The reader should not accept Definition 5.2.5 without question but should convince herself that it is indeed a reasonable interpretation[3] of 'without holes'. She should observe that if Ω 'has a hole' and the point a 'lies in the hole', then the equation $g(z) = (z - a)^{-1}$ defines an analytic function on Ω such that

$$\int_C g(z)\, dz = 2\pi i$$

whenever C is a closed contour lying in Ω 'which goes round a once in an anticlockwise direction'.

Exercise 5.2.6. *Show, using Definition 5.2.5, that, if Ω_1 and Ω_2 are conformally equivalent, then, if Ω_1 is simply connected, so is Ω_2.*

The results we need on simply connected open sets should be more or less familiar (though, possibly, in a slightly different form) and will therefore be presented as an exercise.

Exercise 5.2.7. *Let Ω be a simply connected open subset of $\mathbb{C}$ such that $0 \notin \Omega$.*

(i) Let $z_0, z \in \Omega$. Explain why, if C_1 and C_2 are well behaved paths from z_0 to z, we have

$$\int_{C_1} \frac{dw}{w} = \int_{C_2} \frac{dw}{w}.$$

Conclude that, if α_0 is a constant chosen so that $\exp(\alpha_0) = z_0$, we can define a function $L : \Omega \to \Gamma$ by

$$L(z) = \alpha_0 + \int_\Gamma \frac{dw}{w},$$

[3] I do not claim that it is the most natural. This is not a book about complex variable theory or algebraic topology.

where Γ *is any well behaved path from* z_0 *to* z.

(ii) Suppose that the straight line path $\Gamma(h)$ *from* z *to* $z+h$ *lies within* Ω. *Show that*

$$L(z+h) - L(z) - \frac{h}{z} = \int_{\Gamma(h)} \left(\frac{1}{w} - \frac{1}{z}\right) dw$$

and, by bounding the size of the right hand side, show that L *is analytic on* Ω *with*

$$L'(z) = \frac{1}{z}.$$

(iii) Show that

$$\frac{d}{dz}\frac{\exp L(z)}{z} = 0$$

for all $z \in \Omega$ *and deduce that* $\exp L(z) = z$ *for all* $z \in \Omega$. *Show also that* $L(\exp z) = z$ *for all* $z \in \Omega$. *Informally, we say that* L *is a logarithm function on* Ω.

(iv) If we set $S(z) = \exp(L(z)/2)$, *show that* $S : \Omega \to \mathbb{C}$ *is an analytic function with* $S(z)^2 = z$. *Informally, we say that* S *is a square root function on* Ω. *Show that*

$$S'(z) = \frac{S(z)}{2z}.$$

(v) Suppose that $S_1, S_2 : \Omega \to \mathbb{C}$ *are analytic functions with* $S_1(z)^2 = z = S_2(z)^2$ *for all* $z \in \Omega$. *Show that* $S_1 = S_2$ *or* $S_1 = -S_2$.

(vi) Suppose that $L_1, L_2 : \Omega \to \mathbb{C}$ *are analytic functions with* $\exp L_1(z) = z = \exp L_2(z)$ *for all* $z \in \Omega$. *Show that we can find an* $n \in \mathbb{Z}$ *with* $L_1(z) = L_2(z) + 2n\pi i$ *for all* $z \in \Omega$.

We can now state the Riemann mapping theorem.

Theorem 5.2.8. *If* Ω *is a proper open simply connected subset of* $\mathbb{C}$, *then* Ω *is conformally equivalent to the open unit disc* D.

The existence of a square root function is used twice in our proof of the Riemann mapping theorem. The first application is not very hard but provides a useful preliminary simplification.

Lemma 5.2.9. *If* Ω *is a proper open simply connected subset of* $\mathbb{C}$, *then* Ω *is conformally equivalent to an open set* $U \subseteq D$ *with* $0 \in U$.

Proof. Since $\Omega \neq \mathbb{C}$, we can find a point $a \notin \Omega$. If we set $T_1 z = z - a$, then T_1 maps Ω conformally to a set Ω_1 with $0 \notin \Omega_1$. Since Ω and thus Ω_1 is non-empty, we can find a point $b \in \Omega_1$. If we set $T_2 z = b^{-1} z$, then T_2 maps Ω_1 conformally to a set Ω_2 with $0 \notin \Omega_2$ and $1 \in \Omega_1$. By Exercise 5.2.7, we can find a square root function S on Ω_2. By replacing S by $-S$ if necessary, we may take $S1 = 1$.

Since Ω_2 is open, we can find a $0 < \delta < 1$ such that the open disc $D(1,\delta) \subset \Omega_2$. Since the square root function on $D(1,\delta)$ is unique (once we fix its value at 1), we know that

$$S(re^{i\theta}) = r^{1/2}e^{i\theta/2}$$

for all $-\pi < \theta < \pi$, $r > 0$ and $re^{i\theta} \in D(1,\delta)$. In particular, we can find an $\eta > 0$ such that

$$D(1,\eta) \subseteq S(D(1,\delta)) \subseteq S\Omega_2$$

(we could have used a more abstract argument to reach this conclusion). Since the equation $z^2 = w^2$ has the roots $z = w$ and $z = -w$, we deduce that

$$D(-1,\eta) \cap S\Omega_2 = \emptyset.$$

Thus S is a conformal map of Ω_2 to an open set Ω_3 disjoint from $D(-1,\eta)$. If we set $T_3z = 4\eta^{-1}(z+1)$, then T_3 maps Ω_3 conformally to a set Ω_4 disjoint from $D(0,4)$. Setting $T_4z = 1/z$, we see that T_4 maps Ω_4 to an open subset Ω_5 of $D(0,1/2)$. Take some $c \in \Omega_5$ and let $T_5z = z - c$ so that T_5 maps Ω_5 to an open subset U of D with $0 \in U$.

The map $R = T_5T_4T_3ST_2T_1$ is a conformal map of Ω to U. ■

Exercise 5.2.10. *Draw diagrams to illustrate the proof just given.*

Thus Riemann's theorem reduces to a slightly simpler form.

Theorem 5.2.11. *If Ω is an open simply connected subset of D with $0 \in \Omega$, then Ω is conformally equivalent to D.*

Before trying to find a conformal map from Ω to D, it is natural to ask to what extent the answer is unique. The next remark is trivial but central to our discussion.

Lemma 5.2.12. *Suppose that Ω is an open subset of $\mathbb{C}$. If $T, S : \Omega \to D$ are conformal, then TS^{-1} is a conformal map of D to itself. Conversely, if $T : \Omega \to D$ and $U : D \to D$ are conformal maps, then so is $UT : \Omega \to D$.*

Proof. Immediate. ■

Exercise 5.2.13. *Suppose that Ω_1 and Ω_2 are open subsets of $\mathbb{C}$. If $T_j : \Omega_j \to D$ is a conformal map $[j = 1, 2]$, write down a conformal map $S : \Omega_1 \to \Omega_2$. Show how to write every conformal map $P : \Omega_1 \to \Omega_2$ in terms of T_1, T_2 and conformal maps $U : D \to D$.*

We thus need to investigate conformal maps from the open unit disc to itself and, for this, the key tool is Schwarz's lemma.

Lemma 5.2.14. [Schwarz's lemma] *If* $f : D \to \mathbb{C}$ *is analytic,* $|f(z)| \le 1$ *for all* $z \in D$ *and* $f(0) = 0$*, then*

$$|f(z)| \le |z| \text{ for all } 0 < |z| < 1 \text{ and we have } |f'(0)| \le 1.$$

Further, if $|f(w)| = |w|$ *for some* w *with* $0 < |w| < 1$ *or if* $f'(0) = 1$*, then* $f(z) = e^{i\theta}z$ *for all* $z \in D$ *and some real* θ *(thus* f *is a rotation about* 0 *through an angle* θ*).*

Proof. Define $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{for } 0 < |z| < 1 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Then g is analytic on $D \setminus \{0\}$ and continuous on D. But we know that, if an analytic function is bounded in the neighbourhood of an isolated singularity, that singularity is removable. Thus $g : D \to \mathbb{C}$ is analytic.

If $0 < \epsilon < 1$, we know that

$$|g(z)| = \frac{|f(z)|}{|z|} \le \frac{1}{|z|} \le (1-\epsilon)^{-1}$$

whenever $|z| = (1-\epsilon)$. It follows, by the maximum principle for analytic functions[4], that

$$|g(z)| \le (1-\epsilon)^{-1}$$

for all $|z| \le 1 - \epsilon$. Since ϵ was arbitrary, we have

$$|g(z)| \le 1,$$

whence $|f(z)| \le |z|$ for all $0 < |z| < 1$ and we have $|f'(0)| \le 1$.

If $|f(w)| = |w|$ for some w with $0 < |w| < 1$ or if $|f'(0)| = 1$, then $|g(w)| = 1$ for some w with $|w| < 1$. Another version of the maximum principle tells us that if the modulus of an analytic function on a pathwise connected open set attains a global maximum at some point in that set, then the analytic function must be constant. Choosing θ so that $g(w) = e^{i\theta}$ we obtain

$$g(z) = g(w) = e^{i\theta}$$

and so $f(z) = e^{i\theta}z$ for all $z \in D$. ■

Exercise 5.6.4 (which uses ideas from the rest of this section) gives a simple but important generalisation of Schwarz's inequality called Pick's inequality.

The reader will probably have met the next exercise before.

[4] Closely related to the maximum principle for harmonic functions, see Exercise 4.10.6

Exercise 5.2.15. *We work in $\mathbb{C}$. If $|a| < 1$ show that*

$$\left|\frac{a - z}{a^* z - 1}\right| = 1$$

whenever $|z| = 1$. Hence, using standard facts about Möbius maps, show that the map $T_a : D \to \mathbb{C}$ given by

$$T_a z = \frac{z - a}{a^* z - 1}$$

is a conformal map of D to itself with $T_a(a) = 0$ and $T_a(0) = a$. Show also that

$$T_a'(0) = |a|^2 - 1.$$

(The remainder of the exercise will not be used later.) Suppose we replace the condition $|a| < 1$ by the condition $|a| > 1$. Show that T_a maps $D \setminus \{1/a^*\}$ conformally to an open set to be identified. What happens if $|a| = 1$?

We can now completely identify the conformal maps of D to itself.

Lemma 5.2.16. *The conformal maps from D to itself are precisely the maps $T_{a,\theta} : D \to D$ given by*

$$T_{a,\theta}(z) = e^{i\theta} \frac{z - a}{a^* z - 1}$$

with $|a| < 1$ and θ real.

Proof. A simple argument using Exercise 5.2.15 shows that the $T_{a,\theta}$ are conformal maps of D to itself.

Suppose now that S is a conformal map of D to itself. If $S^{-1}0 = a$ then, automatically, $|a| < 1$ and $f = ST_a^{-1}$ is a conformal map of D to itself with $f(0) = 0$. Since $f, f^{-1} : D \to D$ satisfy the conditions of Schwarz's lemma, we must have $|f'(0)|, |(f^{-1})'(0)| \leq 1$. However, $z = f^{-1}(f(z))$ for all $|z| < 1$, so differentiation gives

$$1 = f'(z)(f^{-1})'(f(z)).$$

Setting $z = 0$, we get

$$1 = f'(0)(f^{-1})'(0)$$

and the equation is only consistent with the inequalities $|f'(0)|, |(f^{-1})'(0)| \leq 1$ if $|f'(0)| = |(f^{-1})'(0)| = 1$. Since $|f'(0)| = 1$, Schwarz's lemma shows that there exists a real θ such that $f = R_\theta$ where

$$R_\theta(z) = e^{i\theta} z$$

for all $z \in D$. Thus $ST_a^{-1} = R_\theta$ and $S = R_\theta T_a$ as required. ∎

Exercise 5.2.17. *Show, using the notation of the previous lemma, that* $T_{a,\theta} = T_{b,\phi}$ *if and only if* $a = b$ *and* $\theta - \phi \equiv 0 \pmod{2\pi}$ *or if* $a = b = 0$.

Exercise 5.2.18. *Let* Ω *be a proper open simply connected subset of* $\mathbb{C}$ *containing the point* a, U *an open subset of* D *and suppose that there exists a conformal map* $S : \Omega \to D$.

(i) Show that there exists an open subset V *of* D *and a conformal map* $T : \Omega \to V$ *such that* $T(a) = 0$ *and* $T'(a)$ *is real and positive.*

(ii) Suppose that V *is an open subset of* D *and* $T_1, T_2 : \Omega \to V$ *are conformal maps such that* $T_j(a) = 0$ *and* $T_j'(a)$ *is real and positive for* $j = 1, 2$. *Show ,using Schwarz's lemma, that* $T_1 = T_2$.

(iii) Let T *be as in (i). Suppose that* $R : \Omega \to C$ *is an analytic function with* $R(\Omega) \subseteq V$, $R(a) = 0$ *and* $R'(a)$ *real and positive. Show, that* $R'(a) \leq T'(a)$ *and that, if* $R'(0) = T'(0)$, *then* $R = T$.

It is a useful heuristic principle that it is easier to resolve a problem with exactly one solution than one with many solutions. Lemma 5.2.16 and Exercise 5.2.18 enable us to replace 'find a conformal map' by 'find the unique conformal map with the following additional properties'.

We shall prove the following strengthened form of Theorem 5.2.11.

Theorem 5.2.19. *If* Ω *is an open simply connected subset of* D *and* $0 \in \Omega$, *then there exists a conformal map* $f : \Omega \to D$ *with* $f(0) = 0$ *and* $f'(0)$ *real and positive.*

Before moving on to the next section, I would like to add one remark. In elementary complex variable theory we use the heuristic 'see how the boundaries transform'. This worked in Exercise 5.2.15 because Möbius maps are defined on the whole plane $\mathbb{C}$ (apart, perhaps, from one point). If we look at the more general conformal maps considered in Riemann's theorem, we run into two linked problems. The first is that the boundary of an open simply connected set may be rather complicated and the second is that the conformal maps may not extend to 'nice' maps on the closure of the sets considered. I do not wish to spend time showing precisely how nasty things can become, but the following example should convince readers that it would be ill advised to try to include boundaries in our discussion. The exercise requires readers to have met conformal mappings in an elementary setting.

Exercise 5.2.20. *Find an explicit conformal map* T *taking*

$$\Omega = D \setminus \{x \in \mathbb{R} : 0 \leq x < 1\}$$

to D. *Show that there is no continuous function* $S : \mathrm{Cl}\,\Omega \to \mathrm{Cl}\,D$ *with* $Sz = Tz$ *for all* $z \in \Omega$.

Explain why, if $\tilde{T}$ *is any conformal map of* Ω *to* D, *there is no continuous function* $\tilde{S} : \mathrm{Cl}\,\Omega \to \mathrm{Cl}\,D$ *with* $\tilde{S}z = \tilde{T}z$ *for all* $z \in \Omega$.

5.3 Equicontinuity for analytic functions

If we wish to use the ideas introduced in earlier chapters to study the space $A(\Omega)$ of analytic functions on an open set Ω, then we need to find an appropriate metric. One choice is suggested by the way that we handle power series $\sum_{j=0}^{\infty} a_j z^j$ with radius of convergence 1. We know that the behaviour of such series becomes worse as $|z|$ approaches 1 (that is to say, as z approaches the boundary of the open unit disc D). We often deal with this problem by proving results for $|z| \leq r$ with $r < 1$ (so we only deal with a region of 'uniformly good behaviour') and then remarking that, since r is arbitrary, the results hold for all $|z| < 1$. Such arguments can be restated in terms of a metric on $A(D)$ given by

$$d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \max\{1, \sup_{z \in K_m} |f(z) - g(z)|\},$$

where $K_m = \{z : |z| \leq 1 - 2^{-m}\}$. (You may wish to compare Section 4.7.)

The next few exercises show how to produce a similar metric on $A(\Omega)$ when Ω is a general open subset of $\mathbb{C}$. They are closely related to similar ideas in Section 4.7

Exercise 5.3.1. *(i) Let Ω be an open set in $\mathbb{C}$. Let $\mathcal{B}$ be the collection of open discs $D(u + iv, r)$ with centre $u + iv$ and radius r, where $u, v, r \in \mathbb{Q}$, $0 < r < 1$ and $D(u + iv, r) \subseteq \Omega$. Show that $\mathcal{B}$ is countable and $\bigcup_{B \in \mathcal{B}} B = \Omega$. Deduce that we can find compact sets $K_1, K_2, \ldots$ such that $K_m \subseteq K_{n+1} \subseteq \Omega$ for all n and $\bigcup_{j=1}^{\infty} K_j = \Omega$.*

(ii) Let Ω be an open set in $\mathbb{C}$ and suppose that $K_1, K_2, \ldots$ are compact sets such that $K_m \subseteq K_{n+1} \subseteq \Omega$ and $\bigcup_{j=1}^{\infty} \operatorname{Int} K_j = \Omega$. Show that, if L is a compact subset of Ω, we can find an N with $K_N \supseteq L$.

Exercise 5.3.2. *Let Ω be a non-empty open set in $\mathbb{C}$ and suppose that $K_1, K_2, \ldots$ are compact sets such that $K_n \subseteq K_{n+1} \subset \Omega$ and $\bigcup_{m=1}^{\infty} \operatorname{Int} K_j = \Omega$. Show that the equation*

$$d_{\mathcal{K}}(f, g) = \sum_{m=1}^{\infty} 2^{-m} \max\{1, \sup_{z \in K_m} |f(z) - g(z)|\}$$

defines a metric on $A(\Omega)$, the space of analytic functions $f : \Omega \to \mathbb{C}$. Use Morera's theorem (which tells us that the uniform limit of analytic functions is analytic) to show that d_K is complete.

Exercise 5.3.3. *We use the notation and hypotheses of Exercise 5.3.2. Let $f_n, f \in A(\Omega)$. Show that the following three statements are equivalent.*

(i) $d_{\mathcal{K}}(f_n, f) \to 0$ as $n \to \infty$.

(ii) $f_n \to f$ uniformly on each K_m.

(iii) $f_n \to f$ uniformly on each compact subset L of Ω. (We say that f converges uniformly on compacta

Exercise 5.3.4. *Let Ω be an open set in $\mathbb{C}$. Suppose that K_1, K_2, ... are compact sets such that $K_m \subseteq \Omega$ and $\bigcup_{m=1}^{\infty} \operatorname{Int} K_n = \Omega$ and that L_1, L_2, ... are compact sets such that $L_m \subseteq L_n subseteq \Omega$ and $\bigcup_{m=1}^{\infty} \operatorname{Int} L_j = \Omega$. If $f_n, f \in A(\Omega)$ show that, using the notation introduced above,*

$$d_{\mathcal{K}}(f_n, f) \to 0 \Leftrightarrow d_L(f_n, f)$$

(Conclude thatidentity map is a homeomorphism of $(A(\Omega), d_{\mathcal{K}})$ and $(A(\Omega), d_L)$.

If $g_n \in A(\Omega)$, show that

$$d_{\mathcal{K}}(g_n, g_m) \to 0 \Leftrightarrow d_L(g_n, g_m) \to 0$$

as m, $n \to \infty$. (Thus $(A(\Omega), d_{\mathcal{K}})$ and $(A(\Omega), d_L)$ have the same Cauchy sequences.)

Exercise 5.3.4 tells us that, for many purposes, we can switch from one metric $d_{\mathcal{K}}$ to another d_L without any problems. We shall refer to the metrics so defined as 'uniform on compacta metrics' f_n, $f \in A(\Omega)$.

Exercise 5.3.5. *We use the notation and hypotheses of Exercise 5.3.2. Let $a \in \Omega$ By using Cauchy's integral formula for the derivative, show that, if we use a uniform on compacta metric the map $J_a : A(\Omega) \to \mathbb{C}$ given by $J_a f = f'(a)$ is continuous.*

Exercise 5.2.18 suggests (as does the link with Dirichlet's problem) that our proof of the Riemann mapping theorem should use the existence of an appropriate maximum. We expect to use compactness to establish the existence of such a maximum, so it is appropriate to seek conditions for a subset of $A(\Omega)$ to have compact closure with respect to the metrics we are considering. These conditions turn out to be very simple.

Exercise 5.3.6. *Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ a set of analytic functions $f : \Omega \to \mathbb{C}$. Show that the following conditions are equivalent.*

(i) Given any compact set $L \subseteq \Omega$ we can find a $M(L) > 0$ such that $|f(z)| \leq M(L)$ for all $z \in L$ and $f \in \mathcal{F}$.

(ii) Given any $w \in \Omega$ we can find an $M_w > 0$ and a $\delta_w > 0$ such that the open disc $D(w, \delta_w) \subseteq \Omega$ and $|f(z)| \leq M_w$ whenever $z \in D(w, \delta_w)$ and $f \in \mathcal{F}$.

Definition 5.3.7. *Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ a set of analytic function $f : \Omega \to \mathbb{C}$. We say that $\mathcal{F}$ is* uniformly bounded on compacta *(or* locally bounded*) if it satisfies the conditions of Exercise 5.3.6.*

Theorem 5.3.8. [Montel's theorem][5] *Let Ω be an open set in $\mathbb{C}$ and $\mathcal{F}$ a set of analytic function $f : \Omega \to \mathbb{C}$. Then every sequence of functions in $\mathcal{F}$ contains*

[5]This is the standard name for this theorem, but there is another results of a similar, but deeper, nature which also called 'Montel's theorem'.

a subsequence which is uniformly convergent on compacta (in other words the closure of $\mathcal{F}$ is compact in any uniform on compacta metric) if and only if $\mathcal{F}$ is uniformly bounded on compacta.

Proof. Necessity is easy to prove. If $\mathcal{F}$ is not bounded on compacta, then there must be a compact subset L of Ω such that

$$\sup_{f\in\mathcal{F}}\sup_{z\in L}|f(z)| = \infty.$$

If we chose $f_n \in \mathcal{F}$ so that

$$\sup_{z\in L}|f_{n+1}(z)| \geq 1 + \sup_{z\in L}|f_n(z)|,$$

no subsequence of the f_n can converge uniformly on L and so no subsequence of the f_n can converge uniformly on compacta.

Sufficiency requires a more delicate argument. Suppose that $\mathcal{F}$ is bounded on compacta. By Exercise 5.3.1 we can find a countable sequence of open discs $D_m = D(z_m, r_m)$, such that $D(z_m, 3r_m) \subseteq \Omega$ and $\bigcup_{n=1}^{\infty} D_m = \Omega$. Write

$$\mathcal{F}_m = \{f|_{\mathrm{Cl}\,D_m} : f \in \mathcal{F}\},$$

so that $\mathcal{F}_m$ is a subset of $C(\mathrm{Cl}\,D_m)$. We shall show that $\mathcal{F}_m$ is uniformly equicontinuous.

To this end, observe that, since $\mathrm{Cl}\,D(z_m, 2r_m)$ is a compact subset of Ω, we can find an $M(m)$ such that $|f(z)| \leq M(m)$ for all $|z-z_m| \leq 2r_m$. If $w_1,\ w_2 \in \mathrm{Cl}\,D(z_m, r_m)$ and C_m is the contour described by $z_m + r\exp(i\theta)$ as θ runs from 0 to 2π, Cauchy's formula gives

$$\begin{aligned}
|f(w_1) - f(w_2)| &= \left|\frac{1}{2\pi}\int_C \frac{f(z)}{z-w_1}\,dz - \frac{1}{2\pi}\int_C \frac{f(z)}{z-w_2}\,dz\right| \\
&= |w_1 - w_2|\left|\frac{1}{2\pi}\int_C \frac{f(z)}{(z-w_1)(z-w_2)}\,dz\right| \\
&\leq |w_1 - w_2|\frac{1}{2\pi}\,\mathrm{length}(C)\sup_{z\in C}\left|\frac{f(z)}{(z-w_1)(z-w_2)}\right| \\
&\leq |w_1 - w_2|\frac{2r_m M(m)}{r_m^2} = 2r_m^{-1}M(m)|w_1 - w_2|.
\end{aligned}$$

Thus $\mathcal{F}_m$ is uniformly equicontinuous and its uniform closure must be compact in $(C(\mathrm{Cl}\,D_m), \|\ \|_{\mathrm{Cl}\,D_m})$ (where we write $\|g\|_{\mathrm{Cl}\,D_m} = \sup_{z\in\mathrm{Cl}\,D_m}|g(z)|$).

Let $f_n \in \mathcal{F}$. Exactly the same argument as we used in Theorem 4.7.5, shows that we can find a sequence $n(j) \to \infty$ and a function $f : \Omega \to \mathbb{C}$ such that

$$\sup_{z\in\mathrm{Cl}\,D_m}|f_{n(j)}(z) - f(z)| \to 0$$

as $j \to \infty$ for all $m \geq 1$. Since the uniform limit of analytic functions is analytic, f is analytic on D_m for each m and so f is analytic on Ω. We have $d_D(f_{n(j)}, f) \to 0$ and so we are done. ■

Exercise 5.3.9. *(Short and not very relevant.) Consider the space X of bounded analytic functions on the open unit disk. Show that X is complete, but not compact, under the norm* $\| \ \|_X$ *given by*

$$\|f\|_X = \sup_{|z|<1} |f(z)|.$$

If $Y = \{f \in X : \|f\|_X \leq 1\}$ *show that Y is closed subsete of X but not compact.*

Of course, the closure of $\mathcal{F}$ need not coincide with $\mathcal{F}$. We shall be dealing with the limit of of injective analytic functions and will make use of the following result.

Theorem 5.3.10. [Hurwitz's theorem] *Suppose that* Ω *is a pathwise connected non-empty open set. If* $f_n : \Omega \to \mathbb{C}$ *is an injective analytic function and* $f_n \to f$ *uniformly on compacta, then either* f *is a constant function or* f *is injective.*

Proof. Suppose that f is not constant and not injective. Then we can find $z_1, z_2 \in \Omega$ such that $z_1 \neq z_2$ but $f(z_1) = f(z_2)$. By subtracting a constant, we may suppose $f(z_1) = f(z_2) = 0$. Since the zeros of a non-constant analytic function on a pathwise connected open set are isolated (see Exercise 5.6.5 (i) and Exercise 5.6.7 (vi)), we can find a $\delta > 0$ such that the open discs $D(z_1, 2\delta)$ and $D(z_2, 2\delta)$ are disjoint, $D(z_j, 2\delta) \subseteq \Omega$ and $f(z) \neq 0$ for $0 < |z - z_j| < 2\delta$ $[j = 1, 2]$. Since a continuous function on a compact set is bounded and attains its bounds, we can find an $\eta > 0$ such that $|f(z)| \geq \eta$ for $|z - z_j| < \delta$. Since $f_n \to f$ uniformly on compact sets, we can find an N such that $|f_N(z) - f(z)| \leq \eta/2$ and so $|f(z)| > |f_N(z) - f(z)|$ for all $|z - z_j| = \delta$ $[j = 1, 2]$. Rouché's theorem tells us that $f_N(z) = 0$ has a solution in each of the discs $D(z_1, \delta)$ and $D(z_2, \delta)$ which is incompatible with the hypothesis that f_N is injective. The theorem follows by reductio ad absurdum. ■

We are now in a position to embark on a proof of the Riemann mapping theorem.

Lemma 5.3.11. *If* Ω *is an open simply connected subset of D and* $0 \in \Omega$*, then there exists an injective analytic function* $f : \Omega \to D$ *with* $f(0) = 0$*,* $f'(0)$ *real and positive such that, if* $g : \Omega \to D$ *is an injective analytic function with* $g(0) = 0$ *and* $g'(0)$ *real and positive, then* $f'(0) \geq g'(0)$.

Proof. We work in $A(\Omega)$ with a uniform on compacta metric $d_{\mathcal{K}}$. Let $\mathcal{G}$ consist of those injective analytic functions $g : \Omega \to D$ with $g(0) = 0$, $g'(0)$ real and

$g'(0) \geq 1$. If we set $h(z) = z$, then $h \in \mathcal{G}$, so $\mathcal{G}$ is non-empty. Since $g \in \mathcal{G}$ implies $|g(z)| < 1$ for all $z \in \Omega$, $\mathcal{G}$ is uniformly bounded on compacta. We shall show that $\mathcal{G}$ is closed and so, by Montel's theorem, compact.

To this end, suppose that $h_n \in \mathcal{G}$, $h \in A(\Omega)$ and $d_{\mathcal{K}}(h_n, h) \to 0$. By the continuity of the maps $g \mapsto g(0)$ and $g \mapsto g'(0)$ (see Exercise 5.3.5), we know that $h(0) = 0$, that $h'(0)$ real and that $h'(0) \geq 1$. Since $h'(0) \neq 0$, Hurwitz's theorem (Theorem 5.3.10) tells us that h is injective. We know that $|h(z)| \leq 1$ for all $z \in \Omega$ and that the modulus a non-constant analytic function on a pathwise connected open set cannot attain a maximum, so $|h(z)| < 1$ for all $z \in \Omega$ (see for example Exercise 5.6.7). Thus $h \in \mathcal{G}$.

Since $\mathcal{G}$, is compact, the continuous map $g \mapsto g'(0)$ must attain a maximum on $\mathcal{G}$ and this gives us the result. ■

Lemma 5.3.12. *Suppose that Ω is an open simply connected subset of D with $0 \in \Omega$ and $f : \Omega \to D$ is an injective analytic function with $f(0) = 0$ and $f'(0)$ real and positive. If f is not surjective, we can find an injective analytic function $g : \Omega \to D$ with $g(0) = 0$, $g'(0)$ real and $g'(0) > f'(0)$.*

The proof of lemma 5.3.12 is based on the rather vague idea that if E is a subset of D on which a square root function S is defined, then in some sense, SE must be bigger than E.

Proof. Since f is not surjective, $D \setminus f(\Omega) \neq \emptyset$ and we can choose $a \in D \setminus f(\Omega)$. We now know (see Exercise 5.2.15) that there exists a Möbius map T_a with $T_a(a) = 0$ and $T_a(0) = a$ which maps D conformally to itself. Thus $T_a f : \Omega \to D$ is an injective analytic map with $0 \notin T_a f(\Omega)$. Since Ω and $T_a f(\Omega)$ are conformally equivalent we know (see Exercise 5.2.6) that $T_a f(\Omega)$ is pathwise connected. Lemma 5.2.7 tells us that there exists an analytic square root function $S : T_a f(\Omega) \to \mathbb{C}$. We observe that $S T_a f(\Omega) \subseteq D$.

Let $G : \Omega \to \mathbb{C}$ be defined by $G = T_{S(a)}^{-1} S T_a f$. Automatically, G is injective analytic function and $G(\Omega) \subseteq D$. We have

$$G(0) = (T_{S(a)}^{-1} S T_a f)(0) = (T_{S(a)}^{-1} S T_a)(0) = (T_{S(a)}^{-1} S)(a) = T_{S(a)} S(a) = 0$$

and, using the function of a function rule,

$$\begin{aligned} G'(0) &= (T_{S(a)}^{-1} S T_a f)'(0) = f'(0)(T_{S(a)}^{-1} S T_a)'(f(0)) = f'(0)(T_{S(a)}^{-1} S T_a)'(0) \\ &= f'(0) T_a'(0)(S'(T_a(f(0)))((T_{S(a)}^{-1})'(S T_{(a)}(f(0))) \\ &= f'(0) \times T_a'(0) \times \frac{S(T_a(0))}{2T_a(0)} \times \frac{1}{T_{S(a)}'(T_{S(a)}^{-1} S T_a(0))} = \frac{T_a'(0)}{S(a) \times T_{S(a)}'(0))} f'(0) \\ &= \frac{1 - |a|^2}{2S(a)(1 - |S(a)|^2)} f'(0) = \frac{1 + |a|}{2S(a)} f'(0). \end{aligned}$$

Since $0 < |a| < 1$, we have

$$\left|\frac{1 + |a|}{2S(a)}\right|^2 = \frac{1 + 2|a| + |a|^2}{4|a|} = \frac{|a| + |a|^{-1}}{4} + \frac{1}{2} > 1,$$

so $|G'(0)| > f'(0)$. If we choose θ so that $e^{i\theta}G'(0) > 0$ and set $g(z) = e^{i\theta}G(z)$, we have found a function g of the kind required. ■

Exercise 5.3.13. *In the last paragraph of the previous proof we showed by direct calculation that*

$$|(T_{S(a)}^{-1} S T_a)'(0)| > 1.$$

However, we can also see this without calculation. Let P, Q be Möbius maps of D to itself, and let $W(z) = z^2$. Show that $|PWQ^{-1}(z)| < 1$ for all $|z| < 1$, but PWQ^{-1} is not an injection (so not a bijection). Deduce that, if $PWQ^{-1}(0) = 0$, then $|(PWQ^{-1})'(0)| < 1$.

Suppose that Ω is an open subset of D such that there exist an analytic function $S : \Omega \to \mathbb{C}$ with $WS(z) = z$ for all $z \in \Omega$. Consider the map

$$QSP^{-1} : P(\Omega) \to QS(\Omega).$$

Show that $(QSP^{-1})^{-1} = PWQ^{-1}$ and deduce that, if $0 \in P(\Omega) \cap QS(\Omega)$ and $QSP^{-1}(0) = 0$, then $|(QSP^{-1})'(0)| > 1$. Obtain the result given in the first sentence of the exercise.

Lemmas 5.3.11 and 5.3.12 yield Theorem 5.2.19 and, by the discussion of the previous section, this gives Theorem 5.2.8.

5.4 There are functions of many variables

In Chapter 3 we showed that continuous functions of two variables could be expressed using only continuous functions of one variable and addition. In the next two sections, we prove a result of Vistuškin which shows that, if we demand continuous differentiability, there are genuine functions of several variables. The proof is quite hard, so most readers will be happy just to absorb the general idea of what is going on. However the general idea is rather interesting.

It is slightly easier to work with the circle $\mathbb{T}$, rather than an interval, because it it gives us direct access to results like Jackson's theorems. Throughout when we refer to $C(\mathbb{T}^m)$ we will mean the space of continuous functions $f : \mathbb{T}^m \to \mathbb{R}$ where $\mathbb{T}^m$ is given the standard metric

$$d(\approx, \sim) = \left(\sum_{j=1}^{m} |t_j - s_j|^2\right)^{1/2}$$

and $\mathbb{R}$ has its standard metric.

First we need to be clear about what it means for a function of n variables to be written in terms of functions of m variables.

Definition 5.4.1. *Let $n > m \geq 1$. If E is a subset of $C(\mathbb{T}^m)$, define $\mathcal{E}_r(E)$, inductively by setting $\mathcal{E}_0$ to be the set of all functions $f \in C(\mathbb{T})$ given by*

$$f(x_1, x_2, \ldots, x_n) = g(x_{q(1)}, x_{q(2)}, \ldots, x_{q(m)})$$

with $1 \leq q(j) \leq m$ and $g \in E$ and by taking $\mathcal{E}_r(E)$ to be the set of all functions $f \in C(\mathbb{T}^n)$ given by

$$f(\mathbf{t}) = g(u_1(\mathbf{t}), u_2(\mathbf{t}), \ldots, u_m(\mathbf{t}))$$

with $u_l \in \mathcal{E}_{r-1}(E)$ $[1 \leq l \leq m]$ and $g \in E$. We say that an $f \in C(\mathbb{T}^n)$ is written in terms of functions in E if $f \in \mathcal{E}_r(E)$ for some $r \geq 1$.

We shall write $C^1(\mathbb{T}^n)$ for the space of continuously differentiable functions $f : \mathbb{T}^n \to \mathbb{R}$.

We shall prove the following theorem.

Theorem 5.4.2. [A theorem of Vistuškin] *If $n > m \geq 1$ there exists an $f \in C^1(\mathbb{T}^n)$ which cannot be written in terms of functions in $C^1(\mathbb{T}^m)$.*

Our proof depends on the notion of ϵ-entropy introduced by Kolmogorov

Definition 5.4.3. *We work in $C(\mathbb{T}^n)$ equipped with the uniform norm. Let E be a subset of $C(\mathbb{T}^n)$ and $\epsilon > 0$. If E cannot be covered by a finite set of closed balls*

$$\tilde{B}(f, \epsilon) = \{g \in C(\mathbb{T}^n) \,:\, \|f - g\|_\infty \leq \epsilon\},$$

we take $H(\epsilon, E) = \infty$. If E can be covered by a finite set of such balls, we write $N(\epsilon, E)$ for the least number of balls required and define $H(\epsilon, E)$, the ϵ-entropy of *E by*

$$H(\epsilon, E) = \log N(\epsilon, E).$$

Suppose that we are using the functions $f \in E$ as messages, but we cannot distinguish two messages f_1 and f_2 if their uniform distance is less than about ϵ. Then *very roughly speaking* we can only distinguish about $N(\epsilon, E)$ messages and the amount of information we can send (defined *roughly speaking* as the logarithm of the number of possible distinct messages) is about $H(\epsilon, K)$.

We need the following very simple observation. (Here $\mathrm{Cl}_\infty E$ denotes the closure in the uniform norm.)

Exercise 5.4.4. *Let E be a subset of $C(\mathbb{T}^n)$ and $\epsilon > 0$. Then*

$$H(\epsilon, E) = H(\epsilon, \mathrm{Cl}_\infty E).$$

Since we are interested in the behaviour of $H(\epsilon, E)$ when it is finite, we shall only be interested in those E whose uniform closure is compact in $(C(\mathbb{T}^n), \| \|_\infty)$, that is to say, those E which are bounded and uniformly equicontinuous.

The sets E we shall consider are balls in $C^1(\mathbb{T}^n)$ with an appropriate norm which we denote by $\| \|_*$.

Exercise 5.4.5. *If $f \in C^1(\mathbb{T}^n)$ we write*

$$Df(\mathbf{t}) = \left(\frac{\partial f}{\partial t_1}(\mathbf{t}), \frac{\partial f}{\partial t_2}(\mathbf{t}), \dots, \frac{\partial f}{\partial t_n}(\mathbf{t})\right)$$

(so Df is a contious fuction from $\mathbb{T}^n$ to $\mathbb{R}^n$).

(i) Show that

$$\|f\|_* = \|f\|_\infty + \|Df\|_\infty$$

defines a complete norm on $C^1(\mathbb{T}^n)$.

(ii) Using the mean value theorem, show that

$$|f(\mathbf{t}) - f(\mathbf{s})| \leq \|f\|_* |d(\sim, \approx)$$

There is nothing particularly special about the norm we have chosen.

Exercise 5.4.6. *If $\| \|_A$ is a norm on $\mathbb{R}^n$, check quickly that*

$$\|f\|_{**} = \|f\|_\infty + \sup_{\mathbf{t} \in \mathbb{T}^n} \|Df(\mathbf{t})\|_A$$

defines a norm on $C^1(\mathbb{T}^n$. Use Theorem 4.5.1 to show that there are constants $C_1, C_2 \geq 1$ such that

$$C_1 \|f\|_* \geq \|f_{**}\| \geq C_2 \|f\|_*.$$

[*Thus using $\| \|_{**}$ in place of $\| \|_*$ merely changes various constants in various estimates that we make.*]

The key inequality is given by the next theorem.

Theorem 5.4.7. *Let $\bar{B}_n$ be the closed unit ball in $(C^1(\mathbb{T}^n), \| \|_*)$. Then there exist constants C_n and C_n' such that*

$$C_n \epsilon^{-n} \leq H(\epsilon, B_n) \leq C_n' \epsilon^{-n} \log \epsilon^{-1}$$

for $0 < \epsilon < 1/2$.

Notice that this theorem is a *quantitative* version of the much simpler observation that B_n is uniformly equicontinuous. Notice also that we will be considering two sorts of balls:- *uniform balls*, that is to say balls in $(C(\mathbb{T}^n), \| \|_\infty)$, and C^1-*balls*, that is to say balls in $(C^1(\mathbb{T}^n), \| \|_*)$.

The proof of Theorem 5.4.7 is quite hard and we shall delay it until the next section. For the moment, we shall content ourselves with showing how it implies Vistuškin's theorem.

Exercise 5.4.8. *(Easy, but may be helpful in thinking about definitions.) By translation and rescaling, show that Theorem 5.4.7 implies the following result. if $\bar{B}$ is a closed ball of radius $\rho > 0$ in $(C^1(\mathbb{T}^n), \| \|_*)$, with uniform closure $\mathrm{Cl}_\infty \bar{B}$ then*

$$C_n \rho^n \epsilon^{-n} \leq H(\epsilon, \bar{B})$$

for $0 < \rho\epsilon < 1/2$.

Lemma 5.4.9. *Let $n > m \geq 1$. Let $\bar{B}_m$ be the closed unit ball in $(C^1(\mathbb{T}^m), \| \|_*)$. If r and q are strictly positive integers then, using the notation of Definitions 5.4.1 and 5.4.3, we know that there is a constant $C(r, q, m, n)$ depending on r, q, m and n such that*

$$H(\epsilon, \mathcal{E}_r(q\bar{B}_m)) \leq C(r, q, m, n)\epsilon^{-m} \log \epsilon^{-1}$$

for all $0 < \epsilon < 1/2$.

Proof. We fix q and m and prove the result by induction on r.

Observe first that Theorem 5.4.7 tells us that, if $1 \leq q(j) \leq m$, the set of all functions $f \in C(\mathbb{T})$ given by

$$f(x_1, x_2, \ldots, x_n) = g(x_{q(1)}, x_{q(2)}, \ldots, x_{q(m)})$$

$g \in E$ can be covered by P uniform balls of radius ϵ where $P \leq C'\epsilon^{-m} \log \epsilon^{-1}$ for some constant C' (depending on m and q). Since $\mathcal{E}_0$ is the union of m^n such sets the required result holds for $r = 0$ with $C(r, q, m, n) = m^n C'$.

Suppose that $r \geq 1$ and

$$H(\epsilon, \mathcal{E}_{r-1}(q\bar{B}_m)) \leq C(r-1, q, m, n)\epsilon^{-m} \log(\epsilon^{-1})$$

for all $0 < \epsilon < 1/2$. Then we know that $\mathcal{E}_{r-1}(q\bar{B}_m)$ can be covered by M uniform balls of radius $\epsilon/2$ with

$$\begin{aligned}\log M \leq H(\epsilon/2, \mathcal{E}_{r-1}(q\bar{B}_m)) &\leq C(r-1, q, m, n)(\epsilon/2)^{-m} \log\left((\epsilon/2)^{-1}\right) \\ &\leq 2^{m+1} C(r-1, q, m, n)\epsilon^{-m} \log \epsilon^{-1},\end{aligned}$$

provided that $0 < \epsilon < 1/4$. Thus we can find $V \subseteq C(\mathbb{T}^m)$ such that V contains M elements and, if $u \in \mathcal{E}_{r-1}(q\bar{B}_m)$, then we can find a $v \in V$ with $\|u - v\|_\infty \leq \epsilon/2$

By Theorem 5.4.7, we know that $\bar{B}_m$ can be covered N uniform balls of radius $\epsilon/(2q)$ with

$$\log N \leq H(\epsilon/(2q), \bar{B}_m)) = C'_m (2q)^m \epsilon^{-m} (\log \epsilon^{-1} + \log 2q) < C'_m 2^{m+1} q^m \epsilon^{-m} \log \epsilon^{-1},$$

provided that $0 < \epsilon < (2q)^{-1}$. By rescaling we know that $q\bar{B}_m$ can be covered by N balls of radius $\epsilon/2$. Thus we can find $\Gamma \subseteq C(\mathbb{T}^m)$ with the property that, if $g \in C^1(\mathbb{T}^m)$ and $\|g\|_* \leq q$, then we can find a $\gamma \in \Gamma$ with $\|g - \gamma\|_\infty \leq \epsilon/2$.

Now suppose that $f \in \mathcal{E}_r(q\bar{B}_m)$. By definition, we can find $g \in q\bar{B}_m$ and $u_l \in \mathcal{E}_{r-1}(E)$ $[1 \le l \le m]$ such that

$$f(\mathbf{t}) = g(u_1(\mathbf{t}), u_2(\mathbf{t}), \dots, u_m(\mathbf{t})).$$

The two previous paragraphs tell us that we can find $\gamma \in \Gamma$ and $v_l \in V$ such that

$$\|g - \gamma\|_\infty \le \epsilon/2 \text{ and } \|u_l - v_l\|_\infty \le \epsilon/2 \text{ for } 1 \le l \le m.$$

Set

$$F(\mathbf{t}) = F_{\gamma,\mathbf{v}}(\mathbf{t}) = \gamma(v_1(\mathbf{t}), v_2(\mathbf{t}), \dots, v_m(\mathbf{t})).$$

Using Exercise 5.4.5, we see that

$$\begin{aligned}
|F(\mathbf{t}) - f(\mathbf{t})| &\le \left|\gamma(v_1(\mathbf{t}), v_2(\mathbf{t}), \dots, v_m(\mathbf{t})) - g(u_1(\mathbf{t}), u_2(\mathbf{t}), \dots, u_m(\mathbf{t}))\right| \\
&\le \left|\gamma(v_1(\mathbf{t}), v_2(\mathbf{t}), \dots, v_m(\mathbf{t})) - g(v_1(\mathbf{t}), v_2(\mathbf{t}), \dots, v_m(\mathbf{t}))\right| \\
&\qquad + \left|g(v_1(\mathbf{t}), v_2(\mathbf{t}), \dots, v_m(\mathbf{t})) - g(u_1(\mathbf{t}), u_2(\mathbf{t}), \dots, u_m(\mathbf{t}))\right| \\
&\le \|\gamma - g\|_\infty + q \max_{1 \le l \le m} |v_l(\mathbf{t}) - u_l(\mathbf{t})| \\
&\le \|\gamma - g\|_\infty + q \max_{1 \le l \le m} \|v_l - u_l\|_\infty \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

We have shown that $\mathcal{E}_r(q\bar{B}_m)$ can be covered by the uniform balls of radius ϵ centres the $F_{\gamma,\mathbf{v}}$ with $\gamma \in \Gamma$ and $\mathbf{v} \in V^M$. There are $M \times N^m$ such balls, so, by the definition of ϵ-entropy,

$$\begin{aligned}
H(\epsilon, \mathcal{E}_r(q\bar{B}_m)) &\le \log N^m M = m \log N + \log M \\
&\le mC'_m 2^{m+1} q^m \epsilon^{-m} \log \epsilon^{-1} + 2^{m+1} C(r-1, q, m, n) \epsilon^{-m} \log \epsilon^{-1} \\
&= (mC'_m 2^{m+1} q^m + 2^{m+1} C(r-1, q, m, n)) \epsilon^{-m} \log \epsilon^{-1}
\end{aligned}$$

for all $0 < \epsilon < (2q)^{-1}$. It follows that there is a constant $C(r, q, m, n)$ such that

$$H(\epsilon, \mathcal{E}_r(q\bar{B}_m) \le C(r, q, m, n) \epsilon^{-m} \log \epsilon^{-1}$$

for all $0 < \epsilon < 1/2$. This completes the induction. ■

The point of Exercise 5.4.8 and Lemma 5.4.9 emerges when we compare the two estimates.

Lemma 5.4.10. *Let $n > m \ge 1$ and let r and q be strictly positive integers. Let B be a ball in $(C^1(\mathbb{T}^n), \| \|_*)$ with uniform closure $\mathrm{Cl}_\infty B$ and let $\mathrm{Cl}_\infty \mathcal{E}_r(q\bar{B}_m)$ be the uniform closure of $\mathcal{E}_r(q\bar{B}_m)$.*

(i) $\mathrm{Cl}_\infty B \setminus Cl_\infty \mathcal{E}_r(q\bar{B}_m) \ne \emptyset$.

(ii) We can find an $f \in B$ and an $\eta > 0$ such that

$$\|f - F\|_* < \eta \Rightarrow F \notin \mathcal{E}_r(q\bar{B}_m)$$

for all $F \in C^1(\mathbb{T}^n)$.

Proof. (i) Let B have radius ρ. Using the notation and results of Exercise 5.4.4 Exercise 5.4.8 and Lemma 5.4.9, we know that, if $0 < \epsilon < 1/2$, $\mathrm{Cl}_\infty B$ requires at least

$$\exp\left(C_n \rho^n \epsilon^{-n}\right)$$

uniform balls of radius ϵ to cover it but that

$$\exp\left(C(r, q, m, n)\epsilon^{-m} \log \epsilon^{-1}\right)$$

uniform balls of radius ϵ will cover $\mathcal{E}_r(q\bar{B}_m)$.

Now

$$\frac{C_n \rho^n \epsilon^{-n}}{C(r, q, m, n)\epsilon^{-m} \log \epsilon^{-1}} = \frac{C_n \rho^n}{C(r, q, m, n)} \times \frac{\epsilon^{-n+m}}{\log \epsilon^{-1}} \to \infty$$

as $\epsilon \to 0+$. It follows that, when ϵ is small, we require strictly more uniform balls of radius ϵ to cover $\mathrm{Cl}_\infty B$ than to cover $\mathcal{E}_r(q\bar{B}_m)$. The result follows.

(ii) By part (i), we can find an $f_0 \in \mathrm{Cl}_\infty B$ with $f_0 \notin Cl_\infty \mathcal{E}_r(q\bar{B}_m)$. By elementary properties of closed sets, we can find an $\eta > 0$ such that

$$\|f - F\|_\infty < 2\eta \Rightarrow F \notin \mathrm{Cl}_\infty \mathcal{E}_r(q\bar{B}_m).$$

for all $F \in C(\mathbb{T}^n)$. Choose $f \in B$ so that $\|f - f_1\|_\infty < \eta$. Automatically

$$\|f - F\|_\infty < \eta \Rightarrow F \notin \mathcal{E}_r(q\bar{B}_m).$$

for all $F \in C(\mathbb{T}^n)$.

If $F \in C^1(\mathbb{T}^n)$ then, by definition,

$$\|f - F\|_\infty \leq \|f - F\|_*$$

and the result follows. ∎

We can now prove a Baire category version of Theorem 5.4.2.

Theorem 5.4.11. *If $n > m \geq 1$ and we work in $(C^1(\mathbb{T}^n), \|\ \|_*)$, then quasi-all functions in $(C^1(\mathbb{T}^n), \|\ \|_*)$ cannot be written in terms of functions in $C^1(\mathbb{T}^m)$.*

Proof. Let $\mathcal{F}_{q,r,m}$ be the closure of $\mathcal{E}_r(q\bar{B}_m)$ in $(C^1(\mathbb{T}^n), \|\ \|_*)$. Lemma 5.4.10 (ii) tells us that $\mathcal{F}_{q,r,m}$ has dense complement and so is meagre. It follows that $\bigcup_{q=1}^\infty \bigcup_{r=1}^\infty \mathcal{F}_{q,r,m}$ is meagre. Trivially

$$\bigcup_{q=1}^\infty \bigcup_{r=1}^\infty \mathcal{F}_{q,r,m} \supseteq \bigcup_{q=1}^\infty \bigcup_{r=1}^\infty \mathcal{E}_r(q\bar{B}_m)$$

and $\bigcup_{q=1}^\infty \bigcup_{r=1}^\infty \mathcal{E}_r(q\bar{B}_m)$ is the set of all functions in $C^1(\mathbb{T}^n)$ which can be written in terms of functions in $C^1(\mathbb{T}^m)$. The required result follows. ∎

5.5 Estimates of ϵ-entropy

In this section we prove Theorem 5.4.7 and so complete the proof of our Vistuškin's theorem Theorem 5.4.2. We need to prove two inequalities. The first inequality says that we need at least a certain number of ϵ uniform balls to cover B_n. The proof of this inequality is very direct and simple enough to be presented as an exercise. The reader should think of the case $n = 2$ as asking 'How many different messages can we write in a page of Braille?'

Exercise 5.5.1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that $0 \leq F(\mathbf{t}) \leq 2$ for all $\mathbf{t} \in \mathbb{R}^n$, $F(\mathbf{t}) = 0$ if $\max_{1\leq j\leq n} |t_j| \geq 1/2$ and $F(\mathbf{0}) = 2$. Explain why we can define*

$$R = \sup_{\mathbf{t}\in\mathbb{R}^n} |F(t)| + 2\pi q \sum_{j=1}^{n} \sup_{\mathbf{t}\in\mathbb{R}^n} \left(\sum_{j=1}^{n} \frac{\partial F}{\partial t_j}(\mathbf{t})^2 \right)^{1/2}.$$

If q is a strictly positive integer and $\delta > 0$, we define $F_q : \mathbb{R}^n \to \mathbb{R}$ by

$$F_q(\mathbf{t}) = \begin{cases} F(2\pi q\mathbf{t}) & \text{if } 0 \leq t_j \leq 1/(2q\pi) \text{ for all } 1 \leq j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch F_q in the case $n = 1$.

We take B to be the closed unit ball in $(C^1(\mathbb{T}^n), \| \|_*)$.

(i) Show that $F_q \subseteq q\delta RB$ (the closed ball centre $\mathbf{0}$ radius $q\delta R$ in $(C^1(\mathbb{T}^n), \| \|_*)$).

(ii) Consider Θ_q the space of functions $\theta : \{1, 2, \ldots, q\}^n \to \{1, -1\}$. Set

$$F_{q,\theta} = \sum_{\mathbf{k}\in\{1,2,\ldots,q\}^n} \theta(\mathbf{k}) F_q(\mathbf{t} + 2\pi q^{-1}\mathbf{k}).$$

Sketch some typical $F_{q,\theta}$ in the case $n = 1$. Explain why $F_{q,\theta} \in \delta RB$.

(iii) Explain why, given $\theta \in \Theta_q$, we have

$$\|F_{q,\theta} - F_{q,\theta'}\|_\infty = 4\delta.$$

Conclude that δRB cannot be covered by 2^{q^n} uniform balls of radius δ.

(iv) By rescaling show that B cannot be covered by 2^{q^n} uniform balls of radius $1/(qR\delta)$.

(v) Deduce that there exists an $\eta > 0$ such that, if $0 < \epsilon < 1/2$ and B can be covered by balls in $(C(\mathbb{T}^n, \| \|_\infty)$ of radius ϵ, then $\log N \geq \eta\epsilon^{-n}$. Check that we have proved one of the inequalities in Theorem 5.4.7.

The second inequality we need to establish says that we need at most a certain number of uniform ϵ balls to cover B_n. Our proof uses a multi-dimensional extension of Jackson's first theorem (Theorem 2.5.1).

Theorem 5.5.2. *There exists constants A_m with the following property. If $f : \mathbb{T}^m \to \mathbb{R}$ is once continuously differentiable then, given $N \geq 1$, we can find a real trigonometric polynomial*

$$P_N(\mathbf{t}) = \sum_{|j(u)| \leq N} a_{j(1)j(2)\dots j(n)} \exp\left(i \sum_{v=1}^{n} j(v) t_v \right)$$

such that

$$\|P_N - f\|_\infty \leq A_m N^{-1}.$$

Notice that we are working with trigonometric polynomials in several dimensions (the reader may prefer to call them 'multinomials') and that although the $a_{j(1)j(2)\dots j(n)}$ may be complex the resulting trigonometric polynomial is real.

We give the proof in the exercise that follows.

Exercise 5.5.3. *By Lemma 2.5.2, there exists a J_n a positive trigoneometric polynomial of degree at most $4(n-1)$ such that*

$$\frac{1}{2\pi} \int_{\mathbb{T}} J_n(t)\, dt = 1 \text{ and } \frac{1}{2\pi} \int_{\mathbb{T}} |t| J_n(t)\, dt \leq B n^{-1}.$$

We define $H_n : \mathbb{T}^m \to \mathbb{R}$ by $H_n(\mathbf{t}) = \prod_{k=1}^{m} J_n(t_k)$

(i) Show, by using the inequalty $\|\mathbf{t}| \leq \sum_{k=1}^{m} |t_k|$ or otherwise, that

$$\frac{1}{(2\pi)}^{m} \int_{\mathbb{T}}^{m} J_n(\mathbf{t})\, d\mathbf{t} = 1$$

and

$$\frac{1}{(2\pi)}^{m} \int_{\mathbb{T}}^{m} \|\|\mathbf{t}\| J_n(\mathbf{t})\, d\mathbf{t} \leq B^m n^{-1}.$$

(ii) Adapt the argument of the proof of Theorem 2.5.4 to show that there is a constant B such that, given $n \geq 1$, we can find a real trigonometric polynomial Q of degree at most $4(n-1)$ such that

$$\|Q - f\|_\infty \leq n^{-1} \|f'\|_\infty.$$

(iii) Deduce Theorem 5.5.2.

It will be helpful to make the following rather simple remark.

Exercise 5.5.4. *(i) Let us work in a normed space $(V, \|\ \|)$. If $\mathbf{x}, \mathbf{y} \in V$, $\|\mathbf{x}\| \leq 1$, and $\|\mathbf{x} - \mathbf{y}\| \leq \epsilon$ for some $\epsilon > 0$, show that*

$$\left\| \mathbf{x} - \frac{1}{1+\epsilon} \mathbf{y} \right\| \leq 2\epsilon, \text{ and } \left\| \frac{1}{1+\epsilon} \mathbf{y} \right\| \leq 1.$$

(ii) Conclude that, if $\|f\|_\infty \leq 1$, we can strengthen Theorem 5.5.2 (replacing $A(m)$ by a new constant $A'(m)$) to include the extra conclusion $\|P_N\|_\infty \leq 1$. Observe that this implies that $|\hat{P}(\mathbf{j})| \leq 1$ for all $\mathbf{j} \in \mathbb{Z}^n$.

Once we have the multidimensional version of Jackson's first theorem, matters are again so straight-forward that the remaining steps can again be given as an exercise.

Exercise 5.5.5. *We take B to be the closed unit ball in $(C^1(\mathbb{T}^n), \|\ \|_*)$.*

(i) Let Γ_M be the set of real trigonometric polynomials in $C(\mathbb{T}^n)$ of the form

$$P(\mathbf{t}) = \sum b_{\mathbf{j}} \exp(2\pi i \mathbf{j} \cdot \mathbf{t})$$

with degree at most $M - 1$ and with $|b_{\mathbf{j}}| \leq 1$ for all $\mathbf{j}$.

Use Theorem 5.5.2 and Exercise 5.5.4 to show that there exists a constant A, independent of N, such that, given $f \in B$, we can find a $P \in \Gamma_N$ with

$$\|P - f\|_\infty \leq AM^{-1}.$$

(ii) Let Γ'_M be the set of real trigonometric polynomials in $C(\mathbb{T}^n)$ of the form

$$Q(\mathbf{t}) = \sum a_{\mathbf{j}} \exp(2\pi i \mathbf{j} \cdot \mathbf{t})$$

degree at most $M - 1$ and such that $|a_{\mathbf{j}}| \leq 1$ and $M^{n+1}\Re a_{\mathbf{j}}$, $M^{n+1}\Im a_{\mathbf{j}} \in \mathbb{Z}$. (Thus the $a_{\mathbf{j}}$ lie on the vertices of a square grid in $\mathbb{C}$.)

Observe that given any P as described in (i) we can find a Q as described in the paragraph above such

$$|a_{\mathbf{j}} - b_{\mathbf{j}}| \leq 2M^{-n-1}$$

for all $\mathbf{j}$ and explain why this implies

$$\|P - Q\|_\infty \leq (2M - 1)^n \times 2M^{-n-1} \leq 2^{n+1}M^{-1}.$$

Conclude that there exists a constant A' independent of M such that, given $f \in B$, we can find a $Q \in \Gamma'_M$ with

$$\|Q - f\|_\infty \leq A'M^{-1}.$$

(iii) Show that Γ'_M contains at most

$$((2M^{2n+1} + 1)^2)^{(2M-1)^n}$$

elements. Deuce that there exists a constant A'' such that B can be covered by at most N' uniform balls of radius A/M where

$$\log N' \leq A''M^n \log M.$$

(iv) Deduce that there is a constant A''' such that, if $0 < \epsilon < 1/2$, then B can be covered by at most N uniform balls of radius ϵ where

$$\log N \leq A'''\epsilon^{-n} \log \epsilon^{-1}.$$

We have proved the second inequality in Theorem 5.4.7.

If the reader is interested and is prepared to work fairly hard, the ideas of the last two sections can be used to prove a more general version of Vistuškin's theorem. (The theorem actually proved by Vistuškin is still more general.) We write $C^p(\mathbb{T}^n)$ for the space of real continuously p times differentiable functions on $\mathbb{T}^n$.

Theorem 5.5.6. *Suppose $n > n' \geq 1$, $p,\ p' \geq 1$ and $n/p > n'/p'$. Then there exists an $f \in C^p(\mathbb{T}^n)$ which cannot be written in terms of functions in $C^{p'}(\mathbb{T}^{m'})$*

The proof is set out in a series of exercises starting with Exercise 5.6.9.

Hilbert's 13th problem is associated with a great deal of beautiful mathematics discovered by Vistuškin, Kolmogorov and others. If you wish to go further, you should consult the account in [**?**] which I have used or, if you cannot read French and are prepared to work hard, go directly to [**?**].

Vistuškin's theorem tells us that there exist genuine functions of n variables, but does not tell us how to obtain them. I do not know how to write down such a function, but, if $g : \mathbb{R}^4 \to \mathbb{R}$ is a well behaved (but not too well behaved) function such that there exists an *implicit function* defined by

$$g(x_1, x_2, x_3, u(x_1, x_2, x_3)) = 0$$

for $(x_1, x_2, x_3) \in [0, 1]^3$, then u must be a reasonable candidate. Other candidates include the solutions of appropriate partial differential equations.

5.6 Further exercises

Exercise 5.6.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a locally Lipschitz function with the property that, given any $a > 0$, we can find we can find a constant $K(a)$ such that*

$$|f(u, t)| \leq K(a)(1 + |u|)$$

for all $u \in \mathbb{R}$ and all $|t| \leq a$. Show that, given any $(x_0, t_0) \in \mathbb{R}$, there exists a once differentiable function $g : \mathbb{R} \to \mathbb{R}$ with

$$g'(t) = f(g(t), t),\ g(t_0) = x_0.$$

Exercise 5.6.2. *Here is another, more direct, way of obtaining Peano's theorem on the existence of solutions of differential equations. For simplicity, consider a continuous function $f : \mathbb{R} \times [0, 1] \to \mathbb{R}$ with $|f(u, t)| \leq K$ for all $(u, t) \in \mathbb{R} \times [0, 1]$. We wish to show that the system*

$$g'(t) = f(g(t), t),\ g(0) = 0$$

has a solution. Following a very natural idea which goes back at least as far as Euler, define $g_n : [0,1] \to \mathbb{R}$ to be the simplest piecewise linear function with $g_n(0) = 0$ and

$$g_n((r+1)/n) = g_n(r/n) + n^{-1} f(g_n(r/n), r/n)$$

for $0 \le r \le n-1$.

(i) Show that $g_n(t) \in [-K, K]$ for all $t \in [0,1]$.

(ii) By using uniform continuity, or otherwise, show that that, given $\epsilon > 0$, we can find an $N(\epsilon)$ and a $\delta(\epsilon)$ such that

$$\Big| g_n(s) - g_n(t) - (s-t) f\big(g_n(t), t\big) \Big| \le \epsilon |s-t|$$

for all $n \ge N(\epsilon)$, $s, t \in [0,1]$, $|s-t| \le \delta(\epsilon)$.

(iii) Show that there exists a sequence $n(j) \to \infty$ and a continuous function $g : [0,1] \to \mathbb{R}$ such that $g_{n(j)} \to g$ uniformly.

(iv) Show that g is differentiable with

$$g'(t) = f(g(t), t), \ g(0) = 0.$$

Exercise 5.6.3. *If Ω is an open set such that*

$$|z| < \delta \Rightarrow z \in \Omega$$

and $f : \Omega \to \mathbb{C}$ is an analytic function with $f(0) = 0$ and $|f(z)| < R$ for all $z \in \Omega$, show that $|f'(0)| \le R\delta^{-1}$. What can we say if $|f'(0)| = R\delta^{-1}$?

Exercise 5.6.4. [Pick's inequality] *Let $a, b \in \mathbb{C}$ and let $R, S > 0$. Set*

$$D_1 = \{z \in \mathbb{C} : |z-a| < R\}, \text{ and } D_2 = \{z \in \mathbb{C} : |z-b| < S\}.$$

If $f : D_1 \to \mathbb{C}$ is analytic and $f(D_1) \subseteq D_2$, show that

$$\left| S \frac{f(z) - f(w)}{S^2 - f(z)^* f(w)} \right| \le \left| R \frac{z-w}{R^2 - z^* w} \right|$$

for $z \ne w$, $z, w \in D_1$ and

$$|f'(w)| \le \frac{R}{S} \frac{S^2 - |f(w)|^2}{R^2 - |w|^2}$$

for $w \in D_1$ Show that if we have equality in the first inequality for any z and w or in the second for any w, then f is a Möbius map.

Exercise 5.6.5. *(i) (Principle of isolated zeros) Let* Ω *be an open subset of* $\mathbb{C}$*, and let* $f : \Omega \to \mathbb{C}$ *be analytic. Show, by using Taylor's theorem, or otherwise, that, if* $a \in \Omega$ *and* $f(a) = 0$*, then either there exists a* $\delta > 0$ *with* $D(a, \delta) \subseteq \Omega$ *such that* $f(z) = 0$ *for all* $|z - a| < \delta$ *or there exists a* $\delta > 0$ *with* $D(a, \delta) \subseteq \Omega$ *such that* $f(z) \neq 0$ *for all* $0 < |z - a| < \delta$*. (We continue the discussion in Exercise 5.6.7.)*

(ii) Let Ω *be an open subset of* $\mathbb{C}$ *with* $0 \in \Omega$ *and let* $g : \Omega \to \mathbb{C}$ *be an analytic function with* $g(0) = f'(0) = 0$*. Suppose that there exists a* $\delta_1 > 0$ *with* $D(a, \delta_1) \subseteq \Omega$ *such that* $g(z) \neq 0$ *for all* $0 < |z| < \delta_1$*. Explain why there exists a* δ_2 *with* $\delta_1 \geq \delta_2 > 0$ *such that* $g(z) \neq 0$ *for all* $0 < |z| < \delta_2$*. Set* $\delta_3 = \delta_2/2$*. Explain, using compactness, why there exists an* $\eta > 0$ *such that* $|g(z)| \geq \eta$ *for all* $|z| = \delta_3$*. Use Rouché's theorem to show that if* $0 < |a| < |\eta|$ *the equation* $z = a$ *has at least two roots with* $|z| < \delta_3$*. Why must these roots be distinct?*

(iii) Use (ii) to show that if Ω_1 *and* Ω_2 *are open subsets of* $\mathbb{C}$ *and* $f : \Omega_1 \to \Omega_2$ *is a bijective analytic map, then* $f'(z) \neq 0$ *for all* $z \in \Omega_1$*. Deduce that* f^{-1} *is analytic.*

Exercise 5.6.6. *Suppose that* Ω *is an open set and* $\gamma : [0, 1] \to \Omega$ *is continuous. Show, using a compactness argument, that there exists a* $\delta > 0$ *such that*

$$|z - \gamma(t)| < 3\delta, \ t \in [0, 1] \Rightarrow z \in \Omega.$$

Explain why we can find $0 = t_0 < t_1 < \cdots < t_n = 1$ *such that*

$$|\gamma(t_j) - \gamma(t_{j-1})| < \delta$$

for all $1 \leq j \leq n$ *and deduce that we can find a continuous piecewise linear map* $\tilde{\gamma} : [0, 1] \to \Omega$ *with* $\tilde{\gamma}(0) = \gamma(0)$ *and* $\tilde{\gamma}(1) = \gamma(1)$*.*

Exercise 5.6.7. *(i) A 'well known theorem' states that a non-constant function analytic in an open region* Ω *and continuous on the closure of that region can only attain its maximum modulus on the boundary. Consider* $\Omega = D(0, 1) \cup D(2, 1)$ *and* $f : \Omega \to \mathbb{C}$ *given by*

$$f(z) = \begin{cases} z & \text{if } |z| \leq 1 \\ 1 & \text{if } |z - 2| \leq 1. \end{cases}$$

Show that the theorem, as stated, does not apply.

In this question we show that the theorem is true if Ω *is pathwise connected. We shall use parts of Exercise 5.6.6 in the discussion.*

(ii) Let Ω *be an open set and* $f : \Omega \to \mathbb{C}$ *an analytic function. Suppose that* $a \in \Omega$*,* $R \geq \delta > 0$*,* $D(a, R) \subseteq \Omega$ *and* $f(z) = 0$ *for all* $z \in D(a, \delta)$*. Show, using Taylor's theorem, that* $f(z) = 0$ *for all* $z \in D(a, R)$*.*

(iii) We use the notation of Exercise 5.6.6. Suppose that there exists an η *with* $3\delta \geq \eta > 0$ *and* $f(z) = 0$ *for all* $z \in D(\gamma(0), \eta)$*. Show that* $f(z) = 0$ *for all* $z \in D(\gamma(1), 3\delta)$*.*

(iv) Suppose that Ω *is a pathwise connected open set and* $f : \Omega \to \mathbb{C}$ *is an analytic function. Show, using (iii), or otherwise, that, if* $f = 0$ *on some open disc in* Ω*, then* $f = 0$ *on* Ω*.*

(v) Let Ω *be an open set and* $f : \Omega \to \mathbb{C}$ *be an analytic function. Suppose that* $a \in \Omega$*,* $\delta > 0$ *and* $D(a, \delta) \subseteq \Omega$*. Show, by considering Taylor series, that either* f *is constant on* Ω *or we can find an* $A \neq 0$ *an* $N \geq 1$ *and an* η *with* $\delta \geq \eta > 0$ *such that*

$$|f(z) - f(a) - Az^N| \leq |Az^N|/2.$$

Deduce that, if f *is not constant on* Ω*,* $|f|$ *does not have a maximum at* a*.*

(vi) Use Exercise 5.6.5 (i) to show that, if f *is a non-constant analytic function on a pathwise connected open set, then the zeros of* f *are isolated, that is to say, if* $a \in \Omega$ *and* $f(a) = 0$*, we can find a* $\delta > 0$ *such that* $D(a, \delta) \subseteq \Omega$ *and* $f(z) \neq 0$ *for* $0 < |z - a| < \eta$*.*

Exercise 5.6.8. *Let* d *be a uniform on compacta metric (see page* **??***) for the open unit disc*

$$D = \{z \in \mathbb{C} \, : \, |z| < 1\}.$$

(i) Show that the polynomials are dense in $(A(D), d)$*.*

(ii) Let $g_n(z) = nz^n$ *for* $z \in D$*. Show that* $d(g_n, 0) \to 0$ *as* $n \to \infty$*.*

(iii) Show that, if $\epsilon > 0$*,* $K > 0$ *and* P *is a polynomial then we can find a polynomial* Q *and an* R *with* $0 < R < 1$ *such that* $d(P, Q) < \epsilon$ *but* $|Q(z)| > K$ *for* $R < |z| < 1$*.*

(iv) If $\epsilon > 0$*,* $K > 0$ *and* $f \in A(D)$*, then we can find a polynomial* Q *and an* R *with* $0 < R < 1$ *such that* $d(f, Q) < \epsilon$ *but* $|Q(z)| > K$ *for* $R < |z| < 1$*.*

(v) If $K > 0$*, show that quasi-all* $f \in A(D)$ *have the property that there exists an* R *with* $0 < R < 1$ *such that such that* $|f(z)| > K$ *for* $R < |z| < 1$*.*

(vi) Show that quasi-all $f \in A(D)$ *have the property that* $|f(z)| \to \infty$ *uniformly as* $|z| \to 1$*.*

(vii) If f *has the property described in (vi), show that if* Ω *is an open disc in* $\mathbb{C}$*,* $\Omega \cap D \neq \emptyset$ *and* $g : \Omega \to \mathbb{C}$ *is analytic with* $g(z) = f(z)$ *for* $z \in \Omega \cap D$*, then* $\Omega \subseteq D$*. (If the reader is familiar with the term 'natural boundary' she will be able to restate (vii) more briefly.)*

Exercise 5.6.9. *(Our final set of exercises give a proof of Theorem 5.5.6. The first exercise is easy and instructive.) We use the norm* $\| \, \|_{(p)}$ *on* $C^p(\mathbb{T}^n)$ *given by*

$$\|f\|_{(p)} = \|f\| + \frac{1}{n} \sum_{1 \leq p_1 + p_2 + \ldots + p_n \leq p} \sup_{\mathbf{t} \in \mathbb{T}^n} \left| \frac{\partial^{p_1 + p_2 + \ldots + p_n} f}{\partial^{p_1} x_1 \partial^{p_2} x_2 \ldots \partial^{p_n} x_n}(\mathbf{t}) \right|.$$

By adapting Exercise 5.5.1, prove the following result. Let $B_{p,n}$ *be the closed unit ball in* $(C^p(\mathbb{T}^n), \| \, \|_{(p)})$ *and let its uniform closure, that is to say, its closure in*

$(C(\mathbb{T}^n), \|\ \|_\infty)$, be B_n. Then there exists a constant $C_{p,n}$ such that

$$C_{p,n}\epsilon^{-n/p} \leq H(\epsilon, B_{p,n})$$

for $0 < \epsilon < 1/2$.

Exercise 5.6.10. *(If you choose to skip this exercise you should not feel too many pangs of conscience.) By adapting the proof of the second Jackson Theorem (Theorem 2.5.6) in the same way as we adapted the proof of the first Jackson Theorem to prove Theorem 5.5.2), prove the following result. There exist constants $A_{p,n}$ with the following property. If $f \in C^p(\mathbb{T}^n)$, then, given $N \geq 1$, we can find a real trigonometric polynomial*

$$P_N(\mathbf{t}) = \sum_{|j(u)| \leq N} a_{j(1)j(2)\dots j(n)} \exp\left(i \sum_{v=1}^{n} j(v)t_v\right)$$

such that

$$\|P_N - f\|_\infty \leq A_{p,n}N^{-p}\|f\|_p.$$

Exercise 5.6.11. *By adapting Exercise 5.5.5, prove the following result. Let $B_{p,n}$ be the closed unit ball in $(C^p(\mathbb{T}^n), \|\ \|_{(p)})$. Then there exists a constant $C'_{p,n}$ such that*

$$H(\epsilon, B_{p,n}) \leq C'_{p,n}\epsilon^{-n/p} \log \epsilon^{-1}$$

for $0 < \epsilon < 1/2$.

Exercise 5.6.12. *By adapting the proof of Lemma 5.4.9, prove the following result. Let $n > m \geq 1$. Let $B_{p,n}$ be the unit ball in $(C^p(\mathbb{T}^n), \|\ \|_{(p)})$. If r and q are strictly positive integers, then there is a constant $C(r, q, m, n)$, depending on r, q, m and n, such that*

$$H(\epsilon, \mathcal{E}_r(qB_{p,n})) \leq C(r, q, p, m, n)\epsilon^{-m} \log \epsilon^{-1}$$

for all $0 < \epsilon < 1/2$.

Exercise 5.6.13. *Prove Theorem 5.5.6.*

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