Sketch Solutions For The Main Text of
A Stroll Through Completeness and Compactness

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Here are what I believe to be sketch solutions to the bulk of exercises in the main text (i.e. those not in the "Further Exercises"). I have written in haste in the hope that others will help me correct at leisure. I am sure that they are stuffed with errors ranging from the TEXtual through to the arithmetical and not excluding serious mathematical mistakes. I would appreciate the opportunity to correct at least some of these problems. Please tell me of any errors, unbridgeable gaps, misnumberings etc. I welcome suggestions for additions.

ALL COMMENTS GRATEFULLY RECEIVED.

If you can, please use LATEX 2$_\varepsilon$ or its relatives for mathematics. If not, please use plain text. My e-mail is **twk@dpmms.cam.ac.uk**. You may safely assume that I am both lazy and stupid so that a message saying 'Presumably you have already realised the mistake in Exercise *Z*' is less useful than one which says 'I think you have made a mistake in Exercise *Z* because you have have assumed that the sum is necessarily larger than the integral. One way round this problem is to assume that f is decreasing.'

It may be easiest to navigate this document by using the table of contents which follow on the next few pages. To avoid disappointment, observe that those exercises marked ★ have no solution given.

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1. Exercise 1.1.1

(1) To solve

$$\frac{dy}{dx} = x^3y^2$$

rewrite as

$$\int \frac{1}{y^2} = \int x^3\,dx$$

to obtain

$$A - \frac{1}{y} = \frac{x^4}{4}$$

for some constant A, that is to say,

$$y = -\frac{4}{x^4 + B}$$

for some constant B.

(2) We wish to solve

$$\bigstar \qquad x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3.$$

We first seek to solve

$$\bigstar\bigstar \qquad x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

We guess that there might be solutions to $\bigstar\bigstar$ of the form $y = x^\alpha$, obtaining the 'indicial equation'

$$\alpha(\alpha - 1) - 2\alpha + 2 = 0$$

that is to say

$$\alpha^2 - 3\alpha + 2 = 0$$

with roots $\alpha = 1$ and $\alpha = 2$. Thus $y = Ax + Bx^2$ and, waving our hands, we say that, since we have two arbitrary constants A and B, this gives the general solution. (It is not hard to produce a proof for equations of this type but we will discuss more general situations later.)

We now guess that $\bigstar$ might have a solution of the form Cx^3. Substitution gives

$$C(3 \times 2 - 2 \times 3 + 2) = 4$$

so $C = 2$ and we have the complementary solution $y = 2x^3$. By linearity our original equation has the general solution

$$y = Ax + Bx^2 + 2x^3$$

with A and B arbitrary.

Needless say, we shall not sully our hands with this kind of thing in the rest of these notes.

Exercise 1.1.2

(i) We have
$$\frac{1}{x} = \frac{d}{dx}\frac{P(x)}{Q(x)} = \frac{P'(x)}{Q(x)} - \frac{P(x)Q'(x)}{Q(x)^2}$$
and so
$$x(P'(x)Q(x) - Q'(x)P(x)) = Q(x)^2.$$

If $N > M$, then equating coefficients of x^{N+M} gives $Na_Nb_M - Ma_Nb_M = 0$ that is to say
$$(N - M)a_Nb_M = 0$$
contradicting the conditions $a_N,\ b_M \neq 0$.

If $M > N$, then equating coefficients of x^{N+M} gives $b_M^2 = 0$ contradicting the condition $b_M \neq 0$.

If $N = M$, we again obtain $b_M^2 = 0$ and derive a contradiction.

(ii) By the mean value theorem, there exits a $\zeta \in (2^n, 2^{n+1})$ such that
$$l(2^{n+1}) - l(2^n) = (2^{n+1} - 2^n)l'(\zeta) = 2^n\frac{1}{\zeta}$$
Thus
$$1 \geq l(2^{n+1}) - l(2^n) \geq \frac{1}{2}.$$
Thus
$$n \geq \sum_{r=1}^{n} l(2^r) - l(2^{r-1}) \geq n/2$$
so
$$n \geq l(2^n) - l(0) \geq n/2$$
and so $l(2^n) \to \infty$ as $n \to \infty$, but $2^{-n}f(2^n) \to 0$ as $n \to \infty$.

Now suppose $a_N,\ b_M \neq 0$, $P(x) = \sum_{j=0}^{N} a_jx^j$, $Q(x) = \sum_{k=0}^{M} b_kx^k$ and $g(x) = P(x)/Q(x)$. Then $g(x) \to 0$ if $N < M$ and $g(x) \to a_N/b_N$ if $N = M$. Thus, if $g(2^n) \to \infty$ as $n \to \infty$, we must have $N > M$. But if $N = M + 1$, then $xg(x) \to a_N/b_M$ as $x \to \infty$ and if $N \geq M + 2$ $|xg(x)| \to \infty$ as $x \to \infty$. Thus we cannot have $2^{-n}g(2^n) \to 0$ as $n \to \infty$.

Exercise 1.1.3

The first part of the fundamental theorem of analysis tells us that if

$$f(x) = \int_1^x \frac{1}{t}\,dt$$

then $f'(x) = 1/x$.

The uniqueness follows from the mean value theorem.

Exercise 1.1.4

(i) Since $f'(x) = 1/x > 0$ for all $x > 0$, the mean value theorem tells us that f is strictly increasing. Since $f(2^n) \to \infty$ as $n \to \infty$, it follows that $f(x) \to \infty$ as $x \to \infty$.

(ii) We have

$$\frac{d}{dx}(l(xy) - l(x)) = \frac{y}{xy} - \frac{1}{x} = 0$$

so, by the mean value theorem, $l(xy) - l(x) = l(y)$ for all $x > 0$.

Thus $l(x) + l(y) = l(xy)$ for all $x, y > 0$ and, in particular, setting $y = 1$ and noting that $l(1) = 0$, we have $l(x) = -l(1/x)$.

(iii) Since $l(x) \to \infty$ as $x \to \infty$, $l(x) = -l(1/x) \to -\infty$ as $x \to 0+$. Since l is a continuous function, the intermediate value theorem tells us that l is surjective. Since l is strictly increasing, l is injective.

(iv) If $f : (a,b) \to (f(a), f(b))$ is differentiable with $f'(t) > 0$ for all $t \in (a,b)$ then f^{-1} is differentiable with $(f^{-1})'(s) = 1/f'(f^{-1}(s)$. Thus

$$e'(x) = \frac{1}{l'(e(x))} = \frac{1}{1/e(x)} = e(x).$$

(v) We have

$$\frac{d}{dx}\frac{e(x)}{E(x)} = \frac{e'(x)}{E(x)} - \frac{e(x)E'(x)}{E(x)^2} = \frac{e(x)}{E(x)} - \frac{e(x)}{E(x)} = 0.$$

The mean value theorem now gives $e(x)/E(x)$ constant, so considering $x = 0$ we have $e(x)/E(x) = 1$ and so $E(x) = e(x)$ for all x.

Exercise 1.1.5

If we set $c(x) = s'(x)$, then
$$c'(x) = s''(x) = -s(x)$$
so, by induction, c and s are n times differentiable for every n with
$$s^{(2k)}(x) = (-1)^k s(x),\ s^{(2k+1)}(x) = (-1)^k c(x),$$
$$c^{(2k)}(x) = (-1)^k c(x) \text{and } c^{(2k+1)} = (-1)^{k+1} s(x).$$
and
$$c''(x) + c(x) = 0 \text{ for all } x,\ c(0) = s'(0) = 1,\ c'(0) = -s(0) = 0.$$

(i) We have
$$\begin{aligned} f'(x) &= -s'(a-x)c(x) + s(a-x)c'(x) - c'(a-x)s(x) - c(a-x)s'(x) \\ &= c(a-x)c(x) - s(a-x)s(x) + s(a-x)s(x) - c(a-x)s(x) = 0 \end{aligned}$$
for all x so, by the mean value theorem, $f(x) = f(0)$ for all x and
$$s(a-x)c(x) + c(a-x)s(x) = s(a)c(0) + c(a)s(0) = s(a)$$
for all x. Setting $a = u + v$, $x = v$, we obtain
$$s(u+v) = s(u)c(v) + c(u)s(v)$$
for all $u,\ v \in \mathbb{R}$.

(ii) Set
$$g(x) = c(a-x)c(x) - s(a-x)s(x).$$
Then
$$g'(x) = s(a-x)c(x) - c(a-x)s(x) + c(a-x)s(x) - s(a-x)c(x) = 0$$
for all x so, by the mean value theorem, $g(x) = g(0)$ for all x and
$$s(a-x)c(x) + c(a-x)s(x) = s(a)c(0) + c(a)s(0) = c(a)$$
for all x. Setting $a = u + v$, $x = v$, we obtain
$$c(u+v) = c(u)c(v) - s(u)s(v)$$
for all $u,\ v \in \mathbb{R}$.

(iii) If $h(x) = c(x)^2 + s(x)^2$, then
$$h'(x) = -2s(x)c(x) + 2c(x)s(x) = 0$$
for all x so, by the mean value theorem, $h(x) = h(0)$ for all x and
$$c(x)^2 + s(x)^2 = 1.$$
Thus $s(x)^2,\ c(x)^2 \leq 1$ and $|s(x)|, |c(x)| \leq 1$ for all x.

(iv) By the mean value theorem, s is increasing on $[0, b]$. Since c is positive on this intervals and $c^2 = 1 - s^2$, c is decreasing. If $c(x) > 3/5$ on $[0, b]$ the $1 \geq s(b) \geq 3b/5$ so $b \leq 5/3$. Thus, using the intermediate value theorem, there exists a $u \in [0, 5/3]$ such that c is positive on $[0, u]$ and $c(u) = 3/5$. Thus $s(u)$ is positive and so $s(u) = 4/5$. We have $c(2u) = c(u)^2 - s(u)^2 < 0$.

(v) By the intermediate value theorem, since $c(0) = 1$, $c(2u) < 0$, there exists a $w > 0$ such that $c(w) = 0$. Set

$$\omega = \inf\{t \geq 0 \, : \, c(t) = 0\}.$$

We can find t_n with $n^{-1} + t_n \geq \omega$ and $c(t_n) = 0$. Since $t_n \to \omega$, continuity tells us that $c(\omega) = 0$. Since $c(0) = 1$, continuity tells us that $\omega > 0$. By the intermediate value theorem, $c(t) > 0$ on $[0, \omega)$ so s is increasing on $[0, \omega)$, so $s(\omega) > 0$, so, since $s(\omega)^2 + c(\omega)^2 = 1$ we have $s(\omega) = 1$.

(vi) We have

$$\begin{aligned} s(x + \omega) &= s(x)c(\omega) + c(x)s(\omega) = c(x), \\ c(x + \omega) &= c(x)c(\omega) - s(x)s(\omega) = -s(x). \end{aligned}$$

Thus

$$s(x + 2\omega) = c(x + \omega) = -s(x)$$

and so

$$s(x + 4\omega) = -s(x + 2\omega) = s(x).$$

Similarly $c(x + 2\omega) = -c(x)$ and $c(x + 4\omega) = c(x)$.

(vii) Suppose that such a ρ exists with ρ not an integral multiple of ω. Then we can find integers m and n such that, writing $\tau = m\rho - n\omega$, we have $\omega/2 \geq \tau > 0$. Now $s(x + 4\tau) = s(x)$ and $c(x + 4\tau) = c(x)$ for all $x \in \mathbb{R}$. In particular $c(4\tau) = 1$.

But $c(x + \omega) = -c(x)$ so $0 \geq c(x)$ for $x \in [\omega, 2\omega]$, and c is strictly decreasing on $[0, \omega]$, so $1 > c(x)$ for all $x \in (0, 2\omega]$ and in particular $c(4\tau) < 1$. The desired result follows by contradiction.

Exercise 1.1.6

(i) If we write

$$u_r = \left| \frac{1}{(2r+1)!}(-1)^r x^{2r+1} \right|$$

then, if $x \neq 0$,

$$\frac{u_{r+1}}{u_r} = \frac{x^2}{(2r+3)(2r+2)} \to 0$$

as $r \to \infty$. Thus, by the ratio test the given power series converges absolutely for all x and so has radius of convergence ∞.

Since a power series can be differentiated term by term within its radius of convergence, we may differentiate term by term twice to obtain

$$S''(x) = \sum_{r=1}^{\infty} \frac{1}{(2r+1)!}(-1)^{r+1} x^{2r+1} = -S(x).$$

Automatically $S(0) = 0$.

Also

$$S'(x) = \sum_{r=0}^{\infty} \frac{1}{(2r)!}(-1)^r x^{2r},$$

so $S(0) = 1$.

(ii) We use the following crude version of Taylor's theorem. If f is infinitely differentiable, then

$$\left| f(x) - \sum_{r=0}^{n} \frac{f^{(r)}(0)}{r!} x^r \right| \leq \frac{\sup_{|t| \leq |x|} |f^{(n+1)}(t)|}{(n+1)!} |x|^{n+1}.$$

We know that $s^{(r)}(t)$ is one of $\pm c(t)$, $\pm s(t)$, so $|f^{(r)}(t)| \leq 1$ and

$$\frac{\sup_{|t| \leq |x|} |s^{(n+1)}(t)|}{(n+1)!} |x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

Thus

$$s(x) = \sum_{r=0}^{\infty} \frac{s^{(r)}(0)}{r!} x^r = S(x)$$

for all x.

Exercise 1.1.7

(i) If $y(t) = t^3$, then $y(0) = 0$ and $y'(t) = 3t^2 = 3y(t)^{2/3}$

If $y(t) = 0$, then $y(0) = 0$ and $y'(t) = 0 = 3y(t)^{2/3}$.

(ii) Direct verification as in (i) shows that y is differentiable except perhaps at a and b and that (except perhaps at these exceptional points) $y'(t) = 3y(t)^{2/3}$. If $a = b = 0$ we have the case of (i) so we may assume $b > a$.

Suppose $b > 0 > a$. Then

$$\frac{y(a+h) - y(a)}{h} = \begin{cases} 0 & \text{if } -a \geq h > 0 \\ h^2 & \text{if } 0 > h \end{cases}$$

so

$$\frac{y(a+h) - y(a)}{h} \to 0$$

as $h \to 0$. Thus y is differentiable at a and $y'(a) = 0 = 3y^2(a)$. The point b may be treated similarly.

If $b = 0 > a$ or $b > 0 = a$ similar arguments hold

(iii) If $a \leq 0$, then

$$y(t) = \begin{cases} (t-a)^3 & \text{for } t \leq a \\ 0 & \text{for } a < t \end{cases}$$

is a solution.

If $b \geq 0$

$$y(t) = \begin{cases} (t-a)^3 & \text{for } t \leq a \\ 0 & \text{for } a < t \end{cases}$$

is a solution.

In both cases the argument follows (ii) very closely.

Exercise 1.1.8

If $y(x) = \tan x$ for $x \in (-\pi/2, \pi/2)$, then $y(0) = 0$ and

$$y'(x) = \frac{1}{(\cos x)^2} = \frac{1}{1 + (\tan x)^2} = \frac{1}{1 + y^2}$$

so we have a solution, but one which is only valid on $(-\pi/2, \pi/2)$.

Exercise 1.2.3

(i) Assumed known.

(ii) Entirely routine.

If $f, g, h \in V^X$, then, if $x \in X$,

$$\begin{aligned}(f + (g + h))(x) &= f(x) + (g + h)(x) = f(x) + \big(g(x) + h(x)\big)\\ &= \big(f(x) + g(x)\big) + h(x) = (f + g)(x) + h(x) = \big((f + g) + h\big)(x)\end{aligned}$$

so $f + (g + h) = (f + g) + h$.

If $f, g, \in V^X$, then, if $x \in X$,

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x)$$

so $f + g = g + f$.

Write $z(x) = 0$ for all $x \in X$. If $f \in V^X$, then,

$$(f + z)(x) = f(x) + z(x) = f(x) + 0 = f(x)$$

so $f + z = f$.

If $f \in V^X$, then, writing $(-f)(x) = -f(x)$ for all $x \in X$ we have

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = z(x)$$

so $f + (-f) = 0$.

If $a,\ b \in \mathbb{F}$, $f \in V^X$, then for all $x \in X$

$$\begin{aligned}(a(bf))(x) &= a\big((bf)(x)\big) = a(bf(x)) = (ab)f(x)\\ (1f)(x) &= 1(f(x)) = f(x)\\ \big((a + b)f\big)(x) &= (a + b)f(x) = af(x) + bf(x) = (af + bf)(x)\end{aligned}$$

so $a(bf) = (ab)f$, $1f = f$, $(a + b)f = af + bf$.

Finally, if $f, g \in V^X$ and $a \in \mathbb{F}$, then, for all $x \in X$,

$$\begin{aligned}\big(a(f + g)\big)(x) &= a\big((f + g)(x)\big) = a(f(x) + g(x))\\ &= af(x) + bf(x) = (af + bf)(x)\end{aligned}$$

so $a(f + g) = af + bg$.

To see that $C_{\mathbb{R}}([0, 1])$ is a vector space, observe first that $\mathbb{R}^{[0,1]}$ is a vector space by (ii) and by standard theorems of elementary analysis

(a) $0 \in C_{\mathbb{R}}([0, 1])$,

(b) $f, g \in\in C_{\mathbb{R}}([0, 1]) \Rightarrow f + g \in C_{\mathbb{R}}([0, 1])$,

(c) $\lambda \in \mathbb{F},\ f \in C_{\mathbb{R}}([0, 1]) \Rightarrow \lambda f \in C_{\mathbb{R}}([0, 1])$,

so (i) applies.

Exercise 1.2.4

(i) $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| \geq 0.$

(ii) $d(\mathbf{u}, \mathbf{u}) = ||\mathbf{0}|| = ||0\mathbf{0}|| = |0|||\mathbf{0}|| = 0.$

Also

$$d(\mathbf{u}, \mathbf{v}) = 0 \Rightarrow ||\mathbf{u} - \mathbf{v}|| = 0 \Rightarrow \mathbf{u} - \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} - \mathbf{v}.$$

(iii) $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \left\|(-1)(\mathbf{v} - \mathbf{u})\right\| = ||\mathbf{v} - \mathbf{u}|| = d(\mathbf{v}, \mathbf{u}).$

(iv) We have

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}) &= ||\mathbf{u} - \mathbf{v}|| + ||\mathbf{v} - \mathbf{w})|| \\ &\leq \left\|(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w})\right\| = ||(\mathbf{u} - \mathbf{w})|| = d(\mathbf{u}, \mathbf{w}). \end{aligned}$$

Exercise 1.2.5★

Exercise 1.2.7

(i) We have

$$0 \leq d(x,y) \leq d(x,x_n) + d(x_n,y) = d(x_n,x) + d(x_n,y) \to 0 + 0 = 0$$

so $d(x,y) = 0$ and $x = y$.

(ii) Given $\epsilon > 0$, we can find an N such that $d(x_n,x) < \epsilon/2$ for $n \geq N$. We then have

$$d(x_n,x_m) \leq d(x_n,x) + d(x,x_m) = d(x_n,x) + d(x_m,x) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n, m \geq N$.

(iii) Given $\epsilon > 0$, we can find an N such that $d(x_n,x_m) < \epsilon/2$ for all $n, m \geq N$. We can also find a J such that $n(j) \geq N$ and $d(x_{n(j)},x) < \epsilon/2$ for all $j \geq J$. We then have

$$d(x,x_n) \leq d(x,x_{n(J)}) + d(x_{nJ},x_n) \leq \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n \geq N$.

(iv) We have

$$\begin{aligned} 0 \leq& \left\|(\mathbf{u}_n + \mathbf{v}_n) - (\mathbf{u} + \mathbf{v})\right\| = \left\|(\mathbf{u}_n - \mathbf{u}) + (\mathbf{v}_n - \mathbf{v}\right\| \\ \leq& \|\mathbf{u}_n - \mathbf{u}\| + \|\mathbf{v}_n - \mathbf{v}\| \to 0 + 0 = 0. \end{aligned}$$

(v) We have

$$\begin{aligned} 0 \leq \|\lambda_n\mathbf{u}_n - \lambda\mathbf{u}\| &= \left\|\lambda_n(\mathbf{u}_n - \mathbf{u}) + (\lambda_n - \lambda)\mathbf{u}\right\| \leq \left\|\lambda_n(\mathbf{u}_n - \mathbf{u})\right\| + \left\|(\lambda_n - \lambda)\mathbf{u}\right\| \\ &= |\lambda_n|\|\mathbf{u}_n - \mathbf{u}\| + |\lambda_n - \lambda|\|\mathbf{u}\| \\ &\leq (|\lambda| + |\lambda_n - \lambda|)\|\mathbf{u}_n - \mathbf{u}\| + |\lambda_n - \lambda|\|\mathbf{u}\| \\ &\to (|\lambda| + 0) \times 0 + 0 \times \|\mathbf{u}\| = 0 \end{aligned}$$

as $n \to 0$.

Exercise 1.2.9

Let $f : [a,b] \to \mathbb{R}$ be continuous with $f(t) \geq 0$ for all $t \in [a,b]$. Suppose that there exists a $c \in [a,b]$ such that $f(c) > 0$. By continuity we can find a $\delta > 0$ such that $|f(t) - f(c)| \leq f(c)/2$ for $t \in [a,b]$, $|t-c| > 0$. Writing $[\alpha,\beta] = [c-\delta, c+\delta] \cap [a,b]$ we have $f(t) \geq f(c)/2$ for $t \in [\alpha,\beta]$. so

$$\int_a^b f(t)\,dt \geq \int_\alpha^\beta f(t)\,dt \geq \int_\alpha^\beta f(c)/2\,dt = (\beta-\alpha)f(c)/2 > 0.$$

If $g(t) = 0$ for $t \in [-1,0) \cup (0,1]$ and $g(0) = 1$, then g is everywhere non-negative and g is not identically zero, but

$$\int_{-1}^1 g(t)\,dt = 0.$$

If $h(t) = t$, then h is continuous and not identically zero, but

$$\int_{-1}^1 h(t)\,dt = 0.$$

Exercise 1.2.10

The collection $[-1,1]^{\mathbb{R}}$ of maps $[-1,1] \to \mathbb{R}$ forms a vector space under pointwise operations. Now

$$f, g \in C([-1,1]) \Rightarrow f + g \in C([-1,1])$$

and

$$\lambda \in \mathbb{R},\ f \in C([-1,1]) \Rightarrow f + g \in C([-1,1])$$

whilst the zero function $0 \in C([-1,1])$. Thus $C([-1,1])$ is a subspace of $[-1,1]^{\mathbb{R}}$ so a vector space.

(i) (a) We have $|f(t)| \geq 0$ for all $t \in [-1,1]$ so

$$\|f\|_1 = \int_{-1}^{1} |f(t)|\,dt \geq 0.$$

(b) By Exercise 1.2.9

$$\|f\|_1 = 0 \Rightarrow \int_{-1}^{1} |f(t)|\,dt = 0 \Rightarrow |f| = 0 \Rightarrow f = 0$$

.

(c) We have

$$\|\lambda f\| = \int_{-1}^{1} |\lambda f(t)|\,dt = \int_{-1}^{1} |\lambda||f(t)|\,dt = |\lambda| \int_{-1}^{1} |f(t)|\,dt = |\lambda|\|f\|_1.$$

(d) Since $|f(t)| + |g(t)| \geq |f(t) + g(t)|$ for all t,

$$\|f\|_1 + \|g\|_1 = \int_{-1}^{1} |f(t)| + |g(t)|\,dt \geq \int_{-1}^{1} |f(t) + g(t)|\,dt = \|f + g\|_1.$$

(ii) We have

$$\|f_n - f\|_1 = \int_{-1}^{1} t^{2n}\,dt = \frac{2}{2n+1} \to 0$$

so $f_n \underset{\| \|_1}{\to} f$.

However $f_n(1) = 1 \nrightarrow 0 = f(1)$, so the sequence does not converge *pointwise* to f.

(iii) If $m \geq n$ we have $f_n(t) - f_m(t) = 0$ for $|t| \geq n^{-1}$ and

$$1 \geq |f_n(t) - f_m(t) \text{ if } |T| \leq n^{-1}$$

. Thus

$$\|f_n - f_m\|_1 \leq \int_{-1/n}^{1/n} 1\,dt = 2/n.$$

Since $2/n \to 0$ as $n \to \infty$, the sequence is Cauchy.

Suppose, if possible, that $f_n \underset{\| \|_1}{\to} f$ for some $f \in C([-1,1])$. If $1 > \delta > 0$, then, whenever $n \geq \delta^{-1}$,

$$0 \leq \int_\delta^1 |f(t) - 1|\, dt = \int_\delta^1 |f(t) - f_n(t)|\, dt \leq \|f - f_n\|_1 \to 0$$

as $n \to \infty$. Thus, by Exercise 1.2.9, $|f(t) - 1| = 0$ for $t \geq \delta$. Since δ was arbitrary, $f(t) = 1$ for $t > 0$. Similarly $f(t) = -1$ for $t < 0$. Thus

$$\lim_{t \to 0+} f(t) = 1 \neq -1 = \lim_{t \to 0-} f(t)$$

and f is not continuous. The desired result follows by reductio ad absurdum.

Exercise 1.2.11

If $x = y$ then

$$\Delta(x,y) = \Delta(x,x) = 0 \geq 0,\ \Delta(x,y) = \Delta(x,x) = \Delta(y,x)$$

and

$$\Delta(x,y) + \Delta(y,z) = \Delta(x,x) + \Delta(x,z) = \Delta(x,z) \geq \Delta(x,z).$$

If $x \neq y$

$$\Delta(x,y) = 1 \geq 0,\ \Delta(x,y) = 1 = \Delta(y,x)$$

and

$$\Delta(x,y) + \Delta(y,z) = 1 + \Delta(y,z) \geq 1 \geq \Delta(x,z) \geq \Delta(x,z).$$

If $\Delta(x,y) = 0$ then $x = y$.

Thus Δ is a metric. If x_n is Cauchy in this metric then we can find an N such that $\Delta(x_n, x_m) < 1/2$ and so $x_n = x_m$ for $n, m \geq N$. Thus $x_n = x_N \to x_N$ in this metric and we have shown that every Cauchy sequence converges.

Suppose X is a normed space and $\mathbf{x} \in X$ with $\mathbf{x} \neq \mathbf{0}$. Then writing $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}|$ we see that $d(2\mathbf{x}, \mathbf{0})$, $d(\mathbf{x}, \mathbf{0}) \neq 0$ so

$$d(2\mathbf{x}, \mathbf{0}) = 2d(\mathbf{x}, \mathbf{0}) \neq d(\mathbf{x}, \mathbf{0})$$

so d cannot be the discrete metric.

Exercise 1.2.12

(i) $|f|$ is continuous on $[a,b]$ and so has a supremum.

(ii) (a) We have $|f(t)| \geq 0$ for all $t \in [-1,1]$ so

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)| \geq 0.$$

(b) We have

$$\|f\|_\infty = 0 \Rightarrow 0 \leq |f(t)| \leq 0 \text{ for all } t \Rightarrow |f| = 0 \Rightarrow f = 0$$

.

(c) We have $\lambda f(t)| = |\lambda||f(t)|$ so

$$\|\lambda f\|_\infty = \sup_{t\in[a,b]} |\lambda f(t)| = \sup_{t\in[a,b]} |\lambda||f(t)| = |\lambda| \sup_{t\in[a,b]} |f(t)| = |\lambda|\|f\|_\infty.$$

(d) Since $|f(t)| + |g(t)| \geq |f(t) + g(t)|$ for all t,

$$\|f+g\|_\infty = \sup_{t\in[a,b]} (|f(t)| + g(t)|) \leq \sup_{t\in[a,b]} |f(t)| + \sup_{t\in[a,b]} |g(t)| = \|f\|_\infty + \|g\|_\infty.$$

Exercise 1.2.14

(i) Given $\epsilon > 0$ we can find an N such that $\sum_{n=N}^\infty d(x_n, x_{n+1}) < \epsilon$. The triangle inequality now gives

$$d(x_n, x_m) \leq \sum_{r=n}^{m-1} d(x_r, x_{r+1}) < \epsilon$$

whenever $N \leq n < m$ and so we have a Cauchy sequence.

(ii) In a complete metric space every Cauchy sequence converges. Now use (i).

(iii) Setting $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{x}_n = \sum_{r=1}^n \mathbf{v}_r$ for $n \geq 1$ we see that $\sum_{r=1}^n \|\mathbf{x}_r - \mathbf{x}_{r-1}\| = \sum_{r=1}^n \|\mathbf{v}_r\|$ converges. Thus, by (ii), $\sum_{r=1}^n \mathbf{v}_r = \mathbf{x}_n$ converges.

(iv) If we take $|\ |$ as the norm, $\mathbb{C}$ is a complete normed vector space over $\mathbb{C}$.

(v) $(C([a,b]), \|\ \|_\infty)$ is complete normed space. Under the stated conditions, $\|f_n\|_\infty \leq M_n$ so, by the comparison test, $\sum_{r=1}^\infty \|f_r\|_\infty$ converges.

Exercise 1.2.15

(i) If the sequence x_n is Cauchy, then stating with $n(0) = 1$ we can find inductively $n(j) > n(j-1)$ such that

$$n,\, m \geq n(j) \Rightarrow d(x_n, x_m) \leq \epsilon)(j).$$

By hypothesis $x_{n(j}$ converges and we know from Exercise 1.2.7 that any Cauchy sequence with a convergent subsequence converges, so the sequence x_n converges. Thus (X, d) is complete.

(ii) If $\mathbf{x}_n$ is a sequence with $\|\mathbf{x}_j - \mathbf{x}_{j-1}\| \leq \epsilon_j$ then, setting $\mathbf{y}_j = \mathbf{x}_j - \mathbf{x}_{j-1}$ we know that there exists a $\mathbf{y}$ such that

$$\mathbf{x}_n - \mathbf{x}_0 = \sum_{j=1}^{n} \mathbf{y}_j \to \mathbf{y}$$

in norms and so $\mathbf{x}_n \to \mathbf{y} - \mathbf{x}_0$.

Part (i) now tells us that the norm is complete.

(iii) Choose $\epsilon(n) = 2^{-n}$. Then, if $d(x_n, x_{n-1}) \leq \epsilon(n)$, we know that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ converges by the comparison test so x_n converges. By part (i), the metric is complete.

Claimed converse Suppose that $(V, \|\ \|)$ is a normed vector space such that whenever $\sum_{r=1}^{\infty} \|\mathbf{v}_r\|$ converges, it follows that $\sum_{r=1}^{n} \mathbf{v}_r$ converges. Then the norm is complete.

To prove this, suppose that the hypothesis of the theorem holds and choose $\epsilon(n) = 2^{-n}$. Then, if $\|x_n\| \leq \epsilon(n)$, we know that $\sum_{n=1}^{\infty} \|\mathbf{x}_n\|$ converges by the comparison test so $\sum_{n=1}^{\infty} \|\mathbf{x}_n\|$ converges. By part (i), the norm is complete.

Exercise 1.2.15

(i) It is easy to check that we have a norm using the fact that $\| \|_\infty$ is a norm on $C([a, b])$.

(a) $\|f\|_{(1)} = \|f\|_\infty + \|f'\|_\infty \geq 0 + 0 = 0$.

(b) $\|f\|_{(1)} = 0 \Rightarrow \|f\|_\infty = 0 \Rightarrow f = 0$.

(c) $\|\lambda f\|_{(1)} = \|\lambda f\|_\infty + \|\lambda f'\|_\infty = |\lambda|\|f\|_\infty + |\lambda|\|f'\|_\infty = |\lambda|\|f\|_1$.

(d) $\|f + g\|_1 = \|f + g\|_\infty + \|f' + g'\|_\infty \leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty = \|f\|_1 + \|g\|_1$.

To see completeness, we us the theorem from elementary analysis which states that if $f_n : [a, b] \to \mathbb{R}$ is continuously differentiable, $f_n(c) \to f(c)$ for some $c \in [a, b]$ and f_n' converges uniformly on $[a, b]$ to h, then f is continuously differentiable on $[a, b]$ with $f' = h$.

Suppose f_n is Cauchy in $(C^1([a, b]), \| \|_1)$. Since

$$\|f_n - f_m\|_1 \geq \|f_n - f_m\|_\infty,\ \|f_n' - f_m'\|_\infty$$

we know that f_n and f_n' are Cauchy in the uniform norm. Thus by the general principle of uniform convergence f_n converges uniformly so some f and f_n' converges uniformly to some h. By the quoted theorem f is continuously differentiable with $f' = h$. Thus $f \in C^1([a, b])$ and

$$\|f - f_n\|_{(1)} = \|f - f_n\|_\infty + \|f_n' - f'\|_\infty \to 0 + 0 = 0$$

The space is complete.

(ii) Repeat the arguments of (i).

(iii) $\| \|_A$ is not a norm since $\|1\|_A = 0$ but $1 \neq 0$.

$\| \|_\infty$ is a norm (viz the uniform norm on a subspace of $C([a, b])$ but not complete. By translation and rescaling we can take $[a, b] = [-1, 1]$. Let $f_n(x) = (x^2 + n^{-2})^{1/2}$ and $f(x) = |x|$. Then $f_n \to f$ uniformly so the f_n form a Cauchy sequence for the uniform norm. However, the uniqueness of the limit tells us that if there were a limit of f_n in $(C^1([-1, 1]), \| \|)$ it would have to be h. Since $h \notin C^1([-1, 1])$ we have a contradiction.

$\| \|_B$ is a norm. The only non-standard part of the checking is the following. If $\|f\|_B = 0$, then $f'(t) = 0$ for all $t \in [a, b]$ and $f(a) = 0$ so by the mean value theorem $f(t) = 0$ for all t i.e. $f = 0$. It is complete by the argument of (i).

Exercise 1.3.1 ★

Exercise 1.3.3

One line proof.

$0 \le d(x_n, a) = d(Tx_{n-1}, Ta) \le Kd(x_{n-1}, a) = Kd(Tx_{n-2}, Ta) \le \ldots \le K^n d(x_0, a) \to 0$ as $n \to \infty$.

Exercise 1.3.4

(i) By the mean value theorem applied to $g(x) = f(x) - x$

$$|Tx - Ty| \le \sup_{t\in\mathbb{R}} |g'(t)||x - y| \le k|x - y|$$

for all $x, y \in [-1, 1]$. Thus T is a contraction mapping on the complete metric space $\mathbb{R}$. By the contraction mapping theorem, there is a unique a such that $Ta = a$ or, equivalently, $f(a) = a$.

(ii) Suppose first that $f(0) \le 0$. Then, since $f'(x) \ge 1 - k$, the mean value theorem tells us that, if $b = (1 - k)^{-1}|f(0)|$

$$f(b) - f(0) \ge b(1 - k)$$

so $f(b) \ge 0$ and, by the intermediate value theorem, there exits an a with $0 \le a \le b$ and $f(a) = 0$. The case $f(0) > 0$ is dealt with similarly.

Exercise 1.3.5

(i) Set $g(x) = -\pi/2 + \tan^{-1} x$. Then $g(x) < 0$ for all x. Further, $g'(x) = (1 + x^2)^{-1}$, so $1 > g'(x) > 0$ for all x.

(ii) By the mean value theorem if $x < y$ we can find a $\zeta \in (x, y)$ such that

$$|Tx - Ty| = |g(x) - g(y)| = |g'(\zeta)(x - y)| < |x - y|.$$

However $Tz = z$ implies $g(z) = 0$ which is impossible.

Exercise 1.4.1★

Exercise 1.4.4

(i) By the mean value inequality

$$|f(u,t) - f(v,t)| \leq \sup_{s\in[\int_0 -\delta, t_0+\delta]} |f,1(s)||u-v| \leq k|u-v|.$$

(ii) We have

$$|f(u,t) - f(v,t)| = \big||u| - |v|\big| \leq |u-v|,$$

so f satisfies a Lipschitz condition. However,

$$\lim_{h\to 0+} h^{-1}(f(h,t) - f(0,t)) = 1 \text{ and } \lim_{h\to 0-} h^{-1}(f(h,t) - f(0,t)) = -1,$$

so the partial derivative $f_{,1}(0,t)$ does not exist.

Exercise 1.4.5

(i) By our local Picard's theorem (Theorem 1.4.3), there exists a unique $y_1 : (b-3\delta, b+\delta)$ such that $y_1(b-\delta) = y(b-\delta)$. If we set $\tilde{y}(t) = y(t)$ for $t \in (a,b)$ and $\tilde{y}(t) = y_1(t)$ for $t \in [b, b{+}\delta)$, then $\tilde{y}(t) = y_1(t)$ for $t \in (b{-}\delta, b{+}\delta)$, so $\tilde{y}$ is differentiable with

$$\tilde{y}'(t) = f(\tilde{y}(t), t)$$

for $t \in (a,b)$ and for $t \in (b-\delta, b+\delta)$, so for $t \in (a, b+\delta)$. The solution is unique since, if

$$z'(t) = f(z(t), t)$$

for $t \in (a, b+\delta)$ and $z(t) = y(t)$ for $t \in (a,b)$, then $z(b-\delta) = y(b-\delta)$ and so $z(t) = \tilde{y}(t)$ for $t \in (b-3\delta, b+\delta)$.

(ii) The same argument as (i) shows that there exists a unique differentiable function $\tilde{y} : (a, b+\delta) \to \mathbb{R}$ satisfying the differential equation

$$\tilde{y}'(t) = f(\tilde{y}(t), t)$$

for all $t \in (a-\delta, b+\delta)$ and the equation

$$\tilde{y}(t) = y(t)$$

for all $t \in (a,b)$.

Thus, using Picard's theorem for the case $n = 0$, we can use induction to find differentiable functions $z_n : (t_0 - (n+2)\delta, t_0 + (n+2)\delta) \to \mathbb{R}$ such that

$$z_n'(t) = f(z_n(t), t)$$

for all $t \in (t_0 - (n+2)\delta, t_0 + (n+2)\delta)$ and $z_n(t) = z_{n-1}(t)$ for all $t \in (n+1)\delta$.

Picard's theorem shows that z_0 is unique and induction, using our previous results, shows that z_n is unique. Setting $x(t) = z_n(t)$
for $t \in (t_0 - (n+2)\delta, t_0 + (n+2)\delta)$ now gives the full result.

EXERCISE 1.4.6 ★

2. Exercise 1.5.3

If $x \in B(a,r)$, then, setting $\delta = r - d(x,a)$, we know that $\delta > 0$ and, by the triangle inequality,

$$y = B(x,\delta) \Rightarrow d(y,a) \leq d(y,x) + d(x,a) < r \Rightarrow y \in B(a,r).$$

i.e. $B(x\delta) \subseteq B(a,r)$. Thus the open ball is open.

Exercise 1.5.4

(i) If $a \in X$, then, trivially, $B(a,1) \subseteq X$. Since $\varnothing$ has no members, every $a \in \varnothing$ satisfies $B(a,1) \subseteq \varnothing$.

(ii) If $a \in \bigcup_{\alpha\in A} U_\alpha$, then $a \in U_\beta$ for some $\beta \in A$. Since U_β is open, we can find a $\delta > 0$ such that $B(a,\delta) \subseteq U_\beta$ and so, automatically, $B(a,\delta) \subseteq \bigcup_{\alpha\in A} U_\alpha$.

(iii) If $a \in \bigcap_{j=1}^n U_j$ then, for each $1 \leq j \leq n$, we have $a \in U_j$, so we can find a $\delta_j > 0$ with $B(a,\delta_j) \subseteq U_j$. Setting $\delta = \min_{1\leq j\leq n} \delta_j$, we have $\delta > 0$ and

$$B(a,\delta) \subseteq \bigcap_{j=1}^{n} B(a,\delta_j) \subseteq \bigcap_{j=1}^{n} U_j.$$

Exercise 1.5.5

We have $(a,b) = B((b+a)/2, (b-a)/2)$ which is an open ball so open.

On the other hand $\bigcap_{n=1}^\infty (-1/n, 1/n) = \{0\}$. Since $0 \in \{0\}$, but $B(0,\delta) \nsubseteq \{0\}$ for all δ, we know that $\{0\}$ is not open.

Exercise 1.5.6

(i)⇒(ii) If $X \setminus E$ is not open we can find $y \notin E$ and $x_n \in E$ such that $d(y,x_n) < 1/n$. Since $x_n \to y$ and $x_n \in E$, (ii) shows that $y \in E$. The result follows by reductio ad absurdum.

(ii)⇒(i) Suppose $x_n \in E$, $x_n \to y$ but $y \notin E$. Since $X \setminus E$ is open we can find a $\delta > 0$ such that $B(y,\delta_n) \subseteq X \setminus E$. Thus $d(x_n,y) \geq \delta$ for all n and $x_n \nrightarrow y$. The result follows by reductio ad absurdum.

Exercise 1.5.8

If $x \notin \bar{B}(a,r)$, then $\delta = d(x,a) - r$. If $F_j \in \mathcal{F}$ for all $1 \leq j \leq n$, and by the triangle inequality $\bar{B}(x,\delta) \cap B(a,r) = \varnothing$. Thus the complement of the closed ball is open and the closed ball is indeed closed.

Exercise 1.5.9

We first use condition (i).

(a) $\varnothing \in \mathcal{F}$ since there is no sequence in $\varnothing$. $X \in \mathcal{F}$ because every $x \in X$.

(b) If $F_\alpha \in \mathcal{F}$ for all $\alpha \in A$, $x_n \in \bigcap_{\alpha \in A} F_\alpha$ and $x_n \to x$, then $x_n \in F_\alpha$ and, since F_α is closed, $x \in F_\alpha$ for each $\alpha \in A$. Thus $x \in \bigcap_{\alpha \in A} F_\alpha$. We have shown that $\bigcap_{\alpha \in A} F_\alpha \in \mathcal{F}$.

(c) If $F_j \in \mathcal{F}$ for all $1 \leq j \leq n$, $x_m \in \bigcup_{1 \leq j \leq n} F_j$ for all m and $x_m \to x$, then there must exist a J with $1 \leq J \leq n$ and $m(r)$ a strictly increasing sequence such that $x_{m(r)} \in F_J$. Since F_j is closed $x \in F_J \subseteq \bigcup_{1 \leq j \leq n} F_j$. We have shown that $\bigcap_{j=1}^n F_j \in \mathcal{F}$.

We now use condition (ii)

(a) X is open so $\varnothing = X^c$ is closed. $\varnothing$ is open so $X = \varnothing^c$ is closed.

(b) F_α is closed so F_α^c is open for each $\alpha \in A$. Thus $\bigcup_{\alpha \in A} F_\alpha^c$ is open and

$$\bigcap_{\alpha \in A} F_\alpha = \left(\bigcup_{\alpha \in A} F_\alpha^c \right)^c \in \mathcal{F}.$$

(c) F_j is closed so F_j^c for open for each $1 \leq j \leq n$. Thus $\bigcap_{1 \leq j \leq n} F_j^c$ is open and

$$\bigcup_{1 \leq j \leq n} F_j = \left(\bigcap_{1 \leq j \leq n} F_j^c \right)^c \in \mathcal{F}.$$

Exercise 1.5.10

(i) If $x,\ y,\ z \in E$

(a) $d_E(x,y) = d(x,y) \geq 0$

(b) $d_E(x,y) = 0 \Rightarrow d(x,y) = 0 \Rightarrow x = y$.

(c) $d_E(x,y) = d(x,y) = d(y,x) = d_E(y,x)$.

(d) $d_E(x,y) + d_E(y,z) = d(x,y) + d(y,z) \geq d(x,z) = d_E(x,z)$.

(ii) If $x_n \in E$ and $d(x_n, x) \to 0$, then the sequence x_n is Cauchy with respect to d and so with respect to d_E. Thus there is a $z \in E$ such that $d_E(x_n, z) \to 0$. Thus

$$d(x,z) \leq d(x_n, x) + d(x_n, x) = d(x_n, x) + d_E(x_n, z) \to 0$$

so $d(x,z) = 0$ and $x = z \in E$.

(iii) If the sequence x_n is Cauchy in (E, d_E), it is Cauchy in (X, d) so converges to x say. Since E is closed, $x \in E$ and $d_E(x_n, x) \to 0$ as $n \to \infty$.

(vi) Observe that V is a vector subspace of $C([a,b])$. Observe further that if $f_n \in V$, $f \in C([a,b])$ and $\|f - f_n\|_\infty \to 0$ then,

$$|f(a) - f(b)| \leq |f(a) - f_n(a)| + |f(b) - f_n(a)| \leq 2\|f_n - f\|_\infty \to 0$$

as $n \to \infty$, so $f(a) = f(b)$ and $f \in V$. Thus V is closed and part (iii) shows that $(V, \|\ \|_\infty)$ is complete.

Exercise 1.5.11

(i)⇒(ii) Suppose that (ii) is false. Then we can find an $x \in X$ and an $\epsilon > 0$, such that for any $\delta > 0$, we can find an $x' \in X$ with $d(x, x') < \delta$, but $\rho(f(x), f(x')) \geq \epsilon$. Choosing $x_n \in X$ with $d(x, x_n) < \delta$, but $\rho(f(x), f(x_n)) \geq 1/n$. we see that (i) is false.

(ii)⇒(iii) Suppose $x \in f^{-1}(U)$. Then $f(x) \in U$, so, since U is open, we can find an $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq U$. By (ii) we can find a $\delta > 0$ such that $f(B_d(x, \delta)) \subseteq B_\rho(f(x), \epsilon)$. Thus $f(B_d(x, \delta)) \subseteq U$, whence $B_d(x, \delta) \subseteq f^{-1}(U)$. We have shown that $f^{-1}(U)$ is open.

(iii)⇒(i) Let $\delta > 0$. Since $B_\rho(f(x), \delta)$ is open, $f^{-1}(B_\rho(f(x), \delta))$ is open. Since $x \in f^{-1}(B_\rho(f(x), \delta))$, we can find an $\epsilon > 0$ such that

$$B_d(x, \epsilon) \subseteq f^{-1}(B_\rho(f(x), \delta))$$

and so

$$f(B_d(x, \epsilon)) \subseteq f^{-1}(B_\rho(f(x), \delta)).$$

Choosing N so large that $x_n \in B_d(x, \epsilon)$ for all $n \geq N$, we have
$\rho(f(x_n), f(x)) < \epsilon$ for all $n \geq N$

Exercise 1.5.13

(i) $d_1(x_n, x) \to 0 \Rightarrow d_2(f(x_n), f(x)) \to 0 \Rightarrow d_3(gf(x_n), gf(x)) \to 0.$

(ii) Let $x \in X_1$ and $\epsilon > 0$. Since g is continuous, we can find an $\eta > 0$ such that

$$d_2(y, f(x)) < \eta \Rightarrow d_3(g(y), gf(x)) < \epsilon.$$

Since f is continuous, we can find a $\delta > 0$ such that

$$d_1(w, x)) < \delta \Rightarrow d_2(f(w), f(x)) < \eta.$$

We now have

$$d_1(w, x)) < \delta \Rightarrow d_3(gf(w), gf(x)) < \epsilon.$$

(iii) If U is open in (X_3, d_3), $g^{-1}(U)$ is open in (X_2, d_2) and so $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in (X_1, d_1).

Exercise 1.5.14

If f is continuous, then

E closed $\Rightarrow E^c$ open $\Rightarrow f^{-1}(E^c)$ open $\Rightarrow f^{-1}(E) = (f^{-1}(E^c))^c$ closed.

If f^{-1} preserves closed sets, then

U open $\Rightarrow U^c$ closed $\Rightarrow f^{-1}(U^c)$ closed $\Rightarrow f^{-1}(U) = (f^{-1}(U^c))^c$ open.

Exercise 1.5.16

(i) The intersection of closed sets is closed so ClE is closed. Since $\operatorname{Cl} E$ is the intersection of sets all of which contain E, we have $\operatorname{Cl} E \supseteq E$. If F is closed and $F \supseteq E$, then by construction $F \supseteq \operatorname{Cl} E$. (Thus '$\operatorname{Cl} E$ is the smallest closed set set containing E.')

(ii) '$\int E$ is open and $\operatorname{Int} E \subseteq E$. If U is open and $U \subseteq E$ then $U \subseteq \operatorname{Int} E$.'

The union of open sets is open so $\operatorname{Int} E$ is open. Since $\operatorname{Int} E$ is the union of sets all of which are contained in E, we have $\operatorname{Int} E \supseteq E$. If U is open and $U \subseteq E$, then by construction $U \supseteq \operatorname{Int} E$. (Thus '$\operatorname{Int} E$ is the largest open set contained in E.')

(iii) Since $\operatorname{Int} E \subseteq E$ and $\operatorname{Int} E$ is open, it follows that $X \setminus \operatorname{Int} E \supseteq X \setminus E$ and $X \setminus \operatorname{Int} E$ is closed. If F is closed and $F \supseteq E$ then F^c is open and $F^c \subseteq E$. Thus $F^c \subseteq \operatorname{Int} E$ and $F \supseteq X \setminus \operatorname{Int} E$. We have shown that $\operatorname{Cl}(X \setminus E) = X \setminus \operatorname{Int} E$

Exercise1.5.17

(i) If there exist $x_n \in E$ with $x_n \to x$ as $n \to \infty$ the if F is closed and $F \supseteq E$ we have $x_n \in F$ and $x_n \to x$. Thus $x \in F$ by Exercise 1.5.6. We have shown that x lies in every closed set containing E so $x \in \operatorname{Cl} E$.

If $x \in ClE$ and $x_n \in E$ with $x_n \to x$ as $n \to \infty$, then $x_n \in ClE$ (since $E \subseteq \operatorname{Cl} E$) and, since $\operatorname{Cl} E$ is closed, Exercise 1.5.6 tells us that $x \in \operatorname{Cl} E$.

(ii) If there exists a $\delta > 0$ such that $B(x, \delta) \subset E$ then since $B(x, \delta)$ is open $x \in B(x, \delta) \subseteq \operatorname{Int} E$

If there does not exist a $\delta > 0$ such that $B(x, \delta) \subset E$ then we can find $x_n \notin E$, but with $x_n \in B(x, 1/n)$. We have $x_n \to X$ and $x_n in E^c$ so, by (i), $x = \operatorname{Cl}(E^c) = X \setminus \operatorname{Int} E$.

Exercise1.5.20

(i) $B(1/2, 1/4) \cap \mathbb{Z} = \emptyset$.

(ii) We need to show that, if $x \in \mathbb{Q}$ there exist $x_n \in \mathbb{Q}$ with $x_n \to x$. Since $x_n \to x \Leftrightarrow x_n \to -x$ and $0 \to 0$, we may suppose $x > 0$. Sine $1/n \to 0$ we can find an integer m with $1/x > 1/m$ and so $m > x$. Now x belongs to one of the intervals $(r_n 2^{-n}, (r_n + 1)2^n]$ with $0 \leq r_n \leq 2^n m - 1$ ($r_n \in \mathbb{Z}$). Since $0 < x - r_n 2^{-n} \leq 2^{-n} \to 0$ as $n \to \infty$, $r_n 2^{-n} \to x$ and we are done.

[Do not spend too much time worrying about this.]

Exercise1.5.21

(i) If $x \neq 0$ then $x \in E \subseteq \operatorname{Cl} E$. Since $1/n \in E$ and $1/n \to 0$ we have $0 \in E$. Thus $\operatorname{Cl} E = \mathbb{R}$.

If $x \in E$, then $|x - y| < |x| \Rightarrow |f(x) - f(y)| = 0$. Thus f is continuous on E.

Suppose a $\tilde{f}$ exists with the properties described. Then $\tilde{f}(1/n) = f(1/n) = 1 \to 1$ as $n \to \infty$ so by continuity $\tilde{f}(0) = 1$. But $\tilde{f}(-1/n) = f(1 - /n) = 1 \to 1$ as $n \to \infty$ so by continuity $\tilde{f}(0) = -1$ which is impossible.

(ii) If there does not exist a $\delta > 0$ with $B(x, \delta) \subseteq E$, then we can find $x_n in E^c$ with $x_n \in B(x, 1/n)$. Since $x_n \to x$ $x \in \operatorname{Cl}(E^c) = X \setminus \operatorname{Int} E$ and $x \notin IntE$. If there does exist a $\delta > 0$ with $B(x, \delta) \subset E$, then, since $B(x, \delta)$ is open $x \in B(x, \delta) \subseteq \operatorname{Int} E$.

Exercise 1.5.18

By translation, we may take $\mathbf{a} = \mathbf{0}$. If $\mathbf{x} \notin B(\mathbf{0}, r)$, then $\|\mathbf{x}\| \geq r$ so, whenever $\delta > 0$,

$$(1 + r2^{-1}\delta\|\mathbf{x}\|^{-1}) \in B(\mathbf{x}, \delta) \setminus \bar{B}(\mathbf{0}, r)$$

and so $\mathbf{x} \notin \operatorname{Int} \bar{B}(\mathbf{0}, r)$. Thus $\operatorname{Int} \bar{B}(\mathbf{0}, r) \subseteq B(\mathbf{0}, r)$. Since $B(\mathbf{0}, r)$ is open and $B(\mathbf{0}, r) \subseteq \bar{B}\mathbf{0}, r)$, it follows that

$$B(\mathbf{0}, r) = \operatorname{Int} \bar{B}(\mathbf{0}, r).$$

If $\mathbf{x} \in \bar{B}(\mathbf{0}, r)$, then $(1 - n^{-1})\mathbf{x} \in B(\mathbf{0}, r)$ so $(1 - n^{-1})\mathbf{x} \in \operatorname{Cl} B(\mathbf{0}, r)$. Since $(1 - n^{-1})\mathbf{x} \to \mathbf{x}$ and $\operatorname{Cl} B(\mathbf{0}, r)$ is closed, we have $\mathbf{x} \in \operatorname{Cl} B(\mathbf{0}, r)$. Thus

$$B(\mathbf{0}, r) \subseteq \bar{B}(\mathbf{0}, r) \subseteq \operatorname{Cl} B(\mathbf{0}, r).$$

Since $\bar{B}(\mathbf{0}, r)$ is closed, it follows that

$$\bar{B}(\mathbf{0}, r) = \operatorname{Cl} B(\mathbf{0}, r).$$

In the discrete metric (X, d) every set is open so every set is closed. Thus $\operatorname{Cl} A = A$ and $\operatorname{Int} A = A$ for every A. Provided that X has at least two points

$$\operatorname{Cl} B(a, 1) = B(a, 1) = \{a\} \neq X = \bar{B}(a, 1) = \operatorname{Int} \bar{B}(a, 1).$$

Exercise 1.5.24

(i) $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ and countable.

(ii) If $E \subseteq X$ and $a \notin E$ then $\Delta(a, e) = 1$ for all $e \in E$ so $a \notin \operatorname{Cl} E$. Thus E is dense in X if and only if $E = X$. Thus X is separable if and only if it is countable.

EXERCISE 1.5.25

(i) d_1 is the Euclidean metric.

(ii) (a) Since $|x_1 - y_1|, |x_2 - y_2| \geq 0$ we have

$$d_2(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \geq 0.$$

(b) We have

$$d_2(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow |x_1 - y_1|, |x_2 - y_2| = 0 \Rightarrow x_1 - y_1 = 0,\ x_2 - y_2 = 0 \Rightarrow \mathbf{x} = \mathbf{y}.$$

(c) We have

$$d_2(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\} = \max\{|x_2 - y_2|, |x_1 - y_1|\} = d_2(\mathbf{y}, \mathbf{x}).$$

(d) We have

$$\begin{aligned} d_2(\mathbf{x}, \mathbf{y}) + d_2(\mathbf{y}, \mathbf{z}) &= \max\{|x_1 - y_1|, |x_2 - y_2|\} + \max\{|y_1 - z_1|, |y_2 - z_2|\} \\ &\geq \max\{|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|\} \\ &\geq \max\{|x_1 - z_1|, |x_2 - z_2|\} = d_2(\mathbf{x}, \mathbf{z}). \end{aligned}$$

(iii) (a) We have

$$d_3(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| \geq 0.$$

(b) We have

$$d_3(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow |x_1 - y_1|, |x_2 - y_2| = 0 \Rightarrow x_1 - y_1 = 0,\ x_2 - y_2 = 0 \Rightarrow \mathbf{x} = \mathbf{y}.$$

(c) We have

$$d_3(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| = |x_2 - y_2| + |x_1 - y_1| = d_3(\mathbf{y}, \mathbf{x}).$$

(d) We have

$$\begin{aligned} d_3(\mathbf{x}, \mathbf{y}) + d_2(\mathbf{y}, \mathbf{z}) &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &\geq |x_1 - z_1| + |x_2 - z_2| = d_3(\mathbf{x}, \mathbf{z}). \end{aligned}$$

(iv) (a) We have

$$d_4(\mathbf{x}, \mathbf{y}) \geq \max\{1, d_1(\mathbf{x}, \mathbf{y})\} \geq 0$$

(b) We have

$$d_4(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow d_1(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow \mathbf{x} = \mathbf{y}.$$

(c) We have

$$d_4(\mathbf{x}, \mathbf{y}) = \max\{1, d_1(\mathbf{x}, \mathbf{y})\} = \max\{1, d_1(\mathbf{y}, \mathbf{x})\} = |x_2 - y_2| + |x_1 - y_1| = d_4(\mathbf{y}, \mathbf{x}).$$

(d) We have

$$\begin{aligned} d_4(\mathbf{x},\mathbf{y}) + d_4(\mathbf{y},\mathbf{z}) &= \max\{1, d_1(\mathbf{x},\mathbf{y})\} + \max\{1, d_1(\mathbf{y},\mathbf{z})\} \\ &\geq \max\{1, d_1(\mathbf{x},\mathbf{y}) + d_1(\mathbf{y},\mathbf{z})\} \end{aligned}$$

$$\geq \max\{1, d_1(\mathbf{x},\mathbf{z})\} = d_4(\mathbf{x},\mathbf{z}).$$

(v) d_5 is the discrete metric.

Exercise 1.5.26

This is just Definition 1.5.12 with $f = \iota$.

Exercise 1.5.27

Just apply Exercise 1.5.26 to ι and ι^{-1}.

EXERCISE 1.5.29

Write $(X,d) \sim (Y,\rho)$ if (X,d) and (Y,ρ) are homeomorphic.

(i) The identity map $\iota : (X,d) \to (X,d)$ is a bijection with ι and $\iota^{-1} = \iota$ continuous. Thus $(X,d) \sim (X,d)$.

(ii) $(X,d) \sim (Y,\rho)$, then there exists a bijection $f : (X,d) \to (Y,\rho)$ such that f and f^{-1} are both continuous. It follows that $f^{-1} : (Y,\rho) \to (X,d)$ is a bijection such that f^{-1} and $(f^{-1})^{-1} = f$ are both continuous. Thus $(Y,\rho) \sim (X,d)$.

(iii) $(X,d) \sim (Y,\rho)$ and $(Y,\rho) \sim (Z,\tau)$, then there exist bijections $f : (X,d) \to (Y,\rho)$ and $g : (Y,\rho) \to (Z,\tau)$ such that f, f^{-1}, g and g^{-1} are continuous. If we consider the composition map $h = g \circ f$ given by $h(x) = g(f(x))$, then $h : (X,d) \to (Z,\tau)$ is a bijection whilst h and $h^{-1} = f^{-1} \circ g^{-1}$ are continuous. Thus $(X,d) \sim (Z,\tau)$.

Conditions (i), (ii) and (iii) together show that $\sim$ is an equivalence relation.

EXERCISE 1.5.30

Let $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$.

Suppose that $f : V \to X$ is bijective. Choose $\mathbf{u} \in V$ with $\mathbf{u} \neq \mathbf{0}$ and set $\mathbf{u}_n = n^{-1}\mathbf{u}$. Then $\mathbf{u}_n \neq \mathbf{0}$, so $f(\mathbf{u}_n) \neq f(\mathbf{0})$ and

$$\Delta(f(\mathbf{u}_n), f(\mathbf{0}) = 1 \nrightarrow 0$$

although $d(\mathbf{u}_n,\mathbf{0}) = n^{-1}\|\mathbf{u}\| \to 0$. Thus f is not continuous.

Exercise 1.5.31

(i) Let I be an interval. We know from elementary analysis that a surjective function $g : I \to \mathbb{R}$ which is continuously differentiable and has $g'(x) > 0$ for all $x \in I$ is bijective and has continuously differentiable inverse g^{-1}. Setting $f = g$ we see that f is a homeomorphism.

Observe that $x_n = 1 - 2^{-n}$ is a Cauchy sequence in $(-1, 1)$, but, if $y \in (-1, 1)$, then we can find an N such that $x_N \geq y$ so $|x_n - y| \geq 2^{-N-1}$ for all $n \geq N + 1$ and $x_n \nrightarrow y$. Thus $(-1, 1)$ is not complete.

(ii) Set $d(x, y) = |f^{-1}(x) - f^{-1}(y)|$ with f as in (i). We have $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$ and, since f^{-1} is bijective, $d(x, y) = 0$ if and only if $x = y$. Automatically $d(x, y) = d(y, x)$ and

$$\begin{aligned} d(x, y) + d(y, z) &= |f^{-1}(x) - f^{-1}(y)| + |f^{-1}(y) - f^{-1}(z)| \\ &\geq |f^{-1}(x) - f^{-1}(z)| = d(x, z). \end{aligned}$$

If we set $z_n = f(1 - 2^{-n})$, then $|z_n - z_m| \leq 2^{-p}$ for n, $m \geq p$, so we have a Cauchy sequence, but the argument of (i) (with $y = g(z)$) shows that $z_n \nrightarrow z$ for any $z \in \mathbb{R}$.

EXERCISE 1.5.33

(i) Observe that

$$\rho(f(x_n), f(x)) \le Kd(x_n, x) \to 0$$

as $d(x_n, x) \to 0$, so f is continuous and that

$$d(f^{-1}(z_n), f^{-1}(z)) \le K\rho(z_n, z) \to 0$$

as $\rho(z_n, z) \to 0$, so f^{-1} is continuous.

(ii) Write $(X, d) \sim (Y, \rho)$ if (X, d) and (Y, ρ) are Lipschitz equivalent.

(a) The identity map $\iota : (X, d) \to (X, d)$ is a bijection with

$$d(x, y) \ge d\big(\iota(x), \iota(y)\big) \ge d(x, y).$$

(b) If $(X, d) \sim (Y, \rho)$, then there exists a bijection $f : (X, d) \to (Y, \rho)$ and a constant $K > 0$ such that

$$Kd(x, y) \ge \rho\big(f(x), f(y)\big) \ge K^{-1}d(x, y).$$

We have

$$\begin{aligned} K\rho(a, b) &= K\rho(f(f^{-1}(a)), f(f^{-1}(b))) \\ &\ge KK^{-1}d(f^{-1}(a), f^{-1}(b)) = d(f^{-1}(a), f^{-1}(b)) \end{aligned}$$

and

$$d(f^{-1}(a), f^{-1}(b)) = KK^{-1}d(f^{-1}(a), f^{-1}(b)) \ge K^{-1}\rho(f(f^{-1}(a)), f(f^{-1}(b))) = K^{-1}\rho(a, b)$$

for all $a,\ b \in Y$, so $(Y, \rho) \sim (X, d)$.

(c) If $(X, d) \sim (Y, \rho)$ and $(Y, \rho) \sim (Z, \tau)$, then there exist bijections $f : (X, d) \to (Y, \rho)$ and $g : (Y, \rho) \to (Z, \tau)$ together with $K,\ L > 0$ such that

$$Kd(x, y) \ge \rho(f(x), f(y)) \ge K^{-1}d(x, y) \text{ for all } x,\ y \in X$$

and

$$L\rho(a, b) \ge \tau(g(a), g(b)) \ge L^{-1}\rho(a, b)\text{for all } a,\ b \in y.$$

If we consider the composition map $h = g \circ f$ given by $h(x) = g(f(x))$, then $h : (X, d) \to (Z, \tau)$ is a bijection and

$$KLd(x, y) \ge \tau\big(h(x), h(y)\big) \ge (KL)^{-1}d(x, y).$$

for all $x,\ y \in X$. Thus $(X, d) \sim (Z, \tau)$.

Conditions (a), (b) and (c) together show that $\sim$ is an equivalence relation.

(iii) Just repeat (ii) with $X = Y = Z$.

(iv) By definition, t here exists a bijection $f : (X, d) \to (Y, \rho)$ and a constant $K > 0$ such that

$$Kd(x, y) \ge \rho\big(f(x), f(y)\big) \ge K^{-1}d(x, y).$$

Let (a_n) be a Cauchy sequence in (Y, ρ). Then given $\epsilon > 0$ we can find an N such that $\rho(a_m, a_n) < K^{-1}\epsilon$ for m, $n \geq N$. Thus $d(f^{-1}(a_m), f^{-1}(a_n)) < K^{-1}\epsilon$ for m, $n \geq N$ and the $f^{-1}(a_n)$ form a Cauchy sequence in (X, d). It follows that there exists a $z \in X$ such that $d(f^{-1}(a_n), z) \to 0$. We now have $\rho(a_n, f(z)) = \rho(f(f^{-1}(a_n)), f(z)) \to 0$ as $n \to \infty$. Thus (Y, ρ) is complete.

(v) $\rho(x, y) \geq 0$. If $\rho(x, y) = 0$, then $x^3 = y^3$, so $x = y$. Further, $\rho(x, y) = \rho(y, x)$ since $|a| = |-a|$. Finally

$$\rho(x, y) + \rho(y, z) = |x^3 - y^3| + |y^3 - z^3| \geq |x^3 - z^3| = \rho(x, z).$$

Thus ρ is a metric.

We know that $f(x) = x^{1/3}$ defines a continuous function on the closed bounded interval $[-1, 1]$. Thus f is uniformly continuous. If (x_n) is a Cauchy sequence for ρ then, given $\epsilon > 0$ we can find a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for $|x - y| < \delta$ and an N such that $|x_n^3 - x_m^3| = \rho(x_m, x_n) < \delta$ for n, $m > N$. Thus

$$|x_n - x_m| = |f(x_n^3) - f(x_m^3)| < \epsilon$$

for n, $m \geq N$ and the x_n form a Cauchy sequence for the usual metric. We can thus find $z \in [-1, 1]$ such that $|x_n - z| \to 0$ and so, by continuity,

$$\rho(x_n, z) = |x_n^3 - z^3| \to 0$$

as $n \to \infty$. Thus ρ is complete.

If d is our usual metric

$$\frac{d(x, 0)}{\rho(x, 0)} = |x|^{2/3} \to 0$$

as $d(x, 0) \to 0$ so d and ρ are not Lipschitz equivalent.

Exercise 1.5.34

(i) Since T is continuous at $\mathbf{0}$, there must exist a $\delta > 0$ such that

$$||T\mathbf{u}||_V = ||T\mathbf{u} - T\mathbf{0}|| \leq 1 \text{ for all } ||\mathbf{u}||_U \leq \delta.$$

Taking $K = \delta^{-1}$ we obtain

$$||T\mathbf{u}||_V = K||\delta T\mathbf{u}||_V = K||T(\delta\mathbf{u})||_V \leq K$$

whenever $||\mathbf{u}||_U \leq 1$.

In particular, the set $E = \{||T\mathbf{u}||_V : ||\mathbf{u}||_U \leq 1\}$ is a non empty bounded subset of $\mathbb{R}$ and so has a supremum $||T||$. Since E consists of positive numbers, we have $||T|| \geq 0$.

We observe that if $\mathbf{w} \neq \mathbf{0}$, then $||(||\mathbf{w}||_U\mathbf{w}||_U = 1$ so

$$||T\mathbf{w}||_V = ||\mathbf{w}||_U||T(||\mathbf{w}||_U^{-1}\mathbf{w})||_V \leq ||T||||\mathbf{w}||_U.$$

Noting that

$$||T\mathbf{0}||_V = ||\mathbf{0}||_V = 0 = ||T||||\mathbf{0}||_U$$

completes the proof.

If $T : U \to V$ is a linear map, with $||T\mathbf{u}||_V \leq K||\mathbf{u}||_U$ for all $\mathbf{u} \in U$, then

$$||T\mathbf{u} - T\mathbf{w}||_V = ||T(\mathbf{u} - \mathbf{w})||_V \leq K||\mathbf{u} - \mathbf{w})||_U,$$

so T is continuous.

(ii) If $T,\ S \in \mathcal{L}_C(U, V)$ then

$$\begin{aligned} ||(\lambda T + \mu S)\mathbf{u}||_V &= ||\lambda T(\mathbf{u}) + \mu S(\mathbf{u})||_V \\ &\leq |\lambda|||T(\mathbf{u})||_V + |\mu|||T(\mathbf{u})||_V \\ &\leq (|\lambda|||T|| + \mu||S||)||\mathbf{u}||_U \end{aligned}$$

so $\lambda T + \mu S$ is continuous.

(iii) We saw in (ii) that $||T|| \geq 0$.

$||\lambda T\mathbf{u}||_V = |\lambda|||T\mathbf{u}||_V$ so taking suprema gives $||\lambda T|| = |\lambda|||T||$.

If $T \neq 0$, we can find a $\mathbf{w} \in U$ such that $T\mathbf{w} \neq \mathbf{0}$. Since $||T\mathbf{w}||_V > 0$ we must have $||T|| > 0$. Thus $||T|| = 0$ implies $T = 0$.

A calculation along the lines of (ii), shows that

$$||(T + S)\mathbf{u}||_V \leq (||T|| + ||S||)||\mathbf{u}||$$

for all $\mathbf{u} \in U$. Taking suprema yields $||T + S|| \leq ||T|| + ||S||$. Thus we have a norm.

If $T \in \mathcal{L}_C(U, V)$ and $S \in \mathcal{L}_C(V, W)$, then, by elementary linear algebra, $ST \in \mathcal{L}(U, W)$, and, by earlier results of this question,

$$||ST\mathbf{u}||_W \leq ||S||||T\mathbf{u}||_V \leq ||S||||T||||\mathbf{u}||$$

for all $\mathbf{u} \in U$. Thus $ST \in \mathcal{L}_C(U, W)$ and $||ST|| \leq ||S||||T||$.

(iv) Observe that

$$\begin{aligned}
&\|T(\lambda\mathbf{u}+\mu\mathbf{w})-\lambda T\mathbf{u}-\mu T\mathbf{w}\|_V\\
&\qquad=\|T(\lambda\mathbf{u}+\mu\mathbf{w})-\lambda T\mathbf{u}-\mu T\mathbf{w}-(T_n(\lambda\mathbf{u}+\mu T_n\mathbf{w})-\lambda T_n\mathbf{u}-\mu T_n\mathbf{w})\|_V\\
&\qquad\leq\|T(\lambda\mathbf{u}+\mu\mathbf{w})-T_n(\lambda\mathbf{u}+\mu\mathbf{w})\|_V+|\lambda|\|T\mathbf{u}-T_n\mathbf{u}\|_V+|\mu|\|T\mathbf{w}-T_n\mathbf{w}\|_V\to 0.
\end{aligned}$$

Thus

$$\|T(\lambda\mathbf{u}+\mu\mathbf{w})-\lambda T\mathbf{u}-\mu T\mathbf{w}\|_V=0$$

and

$$T(\lambda\mathbf{u}+\mu\mathbf{w})=\lambda T\mathbf{u}-\mu T\mathbf{w}.$$

Thus T is linear.

Under the further stated conditions $\|T_n\mathbf{u}\|_V\leq K\|\mathbf{u}\|_U$ for each n so $\|T\mathbf{u}\|\leq K\|\mathbf{u}\|_U$ for each $\mathbf{u}\in U$, so T is continuous.

(v) Just observe that

$$\|T_n\mathbf{u}-T_m\mathbf{u}\|_V=\|(T_n-T_m)\mathbf{u}\|\leq\|T_n-T_m\|\|\mathbf{u}\|_U.$$

(vi) Suppose that the sequence T_n is Cauchy with respect to the operator norm. By (v) and the completeness of $\|\ \|_V$ we can find, for each $\mathbf{u}\in U$, a $T\mathbf{u}\in V$ such that $\|T_n\mathbf{u}-T\mathbf{u}\|_V\to 0$ as $n\to\infty$. Since any Cauchy sequence is bounded we can find a K such that $\|T_n\|\leq K$ for all n. Thus $\|T\mathbf{u}\|\leq K\mathbf{u}$ for all $\mathbf{u}\in U$ and (iv) tells us that $T\in\mathcal{L}_C(U,V)$.

Finally, we observe that, given any $\epsilon>0$, we can find an N such that $\|T_n-T_m\|\leq\epsilon$ for $n,m\geq N$. Thus, if $N\leq n<m$,

$$\begin{aligned}
\|T_n\mathbf{u}-T\mathbf{u}\|_V&\leq\|T_n\mathbf{u}-T_m\mathbf{u}\|_V+\|T_m\mathbf{u}-T\mathbf{u}\|_V\\
&\leq\epsilon\|\mathbf{u}\|_U+\|T_m\mathbf{u}-T\mathbf{u}\|_V\\
&\to\epsilon\|\mathbf{u}\|_U
\end{aligned}$$

as $m\to\infty$. Thus

$$\|(T_n-T)\mathbf{u}\|_V\leq\epsilon\|\mathbf{u}\|$$

for all $\mathbf{u}\in U$ and

$$\|T_n-T\|\leq\epsilon$$

for all $n\geq N$.

Exercise 1.5.36

Since $|a_n| \geq 0$, we have $\|\mathbf{a}\|_\infty \geq 0$.

If $\|\mathbf{a}\|_\infty = 0$, then $|a_n| = 0$ and so $a_n = 0$ for all n that is to say $\mathbf{a} = \mathbf{0}$.

$\|\lambda\mathbf{a}\|_\infty = \sup|\lambda a_n| = \sup|\lambda||a_n| = |\lambda|\sup|a_n| = |\lambda|\|\mathbf{a}\|_\infty$.

We have $|a_n + b_n| \leq |a_n| + |b_n|$ for each n so

$$\|\mathbf{a} + \mathbf{b}\|_\infty \leq \|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty.$$

Thus $\| \ \|_\infty$ is a norm.

(It is not complete, since, if we define

$$\mathbf{a}(n) = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, 0, \ldots)$$

we have $\|\mathbf{a}(n) - \mathbf{a}(m)\|_\infty \leq 1/N$ for all n, $m \geq N$, so we have a Cauchy sequence. However, if $\mathbf{c} \in C_{00}$, then we can find an M such that $c_n = 0$ for all $n \geq M$ and so $\|\mathbf{a}(n) - \mathbf{c}\|_\infty \geq 1/M$ for all $n \geq M$. Thus $\|\mathbf{a}(n) - \mathbf{c}\| \nrightarrow 0$ as $n \to \infty$.)

T is well defined since only finitely many terms in the appropriate sum are non-zero. We observe that

$$T(\lambda\mathbf{a} + \mu\mathbf{b}) = \sum_{j=1}^{\infty} j(\lambda a_j + \mu b_j) = \mu\sum_{j=1}^{\infty} ja_j + \lambda\sum_{j=1}^{\infty} jb_j = \lambda T\mathbf{a} + \mu T\mathbf{b}$$

so T is linear.

However, if if $\mathbf{e}_n$ is the sequence with 1 in the nth place and 0 elsewhere, $\|\mathbf{e}_n\|_\infty = 1$, but $|T\mathbf{e}_n| = n$. Exercise 1.5.34 (i) tells us that T cannot be continuous.

Exercise 1.5.38

(i) Observe that

$$T(T^{-1}(\lambda \mathbf{v})) = \lambda \mathbf{v} = \lambda T(T^{-1}\mathbf{v}) = T(\lambda T^{-1}\mathbf{v})$$

so, since T is injective, $T^{-1}(\lambda \mathbf{v}) = \lambda T^{-1}\mathbf{v}$ for all $\lambda \in \mathbb{F}$ and all $\mathbf{v} \in V$.

Similarly

$$\begin{aligned} T(T^{-1}(\mathbf{v}+\mathbf{w})) &= \mathbf{v}+\mathbf{w} = T(T^{-1}\mathbf{v}) + T(T^{-1}\mathbf{w}) \\ &= T(T^{-1}\mathbf{v} + T^{-1}\mathbf{w}) \end{aligned}$$

so, since T is injective, $T^{-1}(\lambda \mathbf{v}) = T^{-1}\mathbf{v} + T^{-1}\mathbf{w}$ for all $\mathbf{v}$, $\mathbf{w} \in V$.

(ii) We know that T is continuous if and only if there exists a $K_1 > 0$ such that

$$K_1 \|\mathbf{u}\|_U \geq \|T\mathbf{u}\|_V$$

for all $\mathbf{u} \in U$. Thus all we need to do is to prove that T^{-1} is continuous if and only if there exists a K_2 such that

$$\|T\mathbf{u}\|_V \geq K_2 \|\mathbf{u}\|$$

for all $\mathbf{u} \in U$.

To this end, observe that if T^{-1} is continuous then

$$\|\mathbf{u}\|_U = \|T^{-1}(T\mathbf{u})\| \leq \|T^{-1}\| \|T\mathbf{u}\|$$

so $\|T\mathbf{u}\| \geq \|T^{-1}\|^{-1}\|\mathbf{u}\|$ for all $\mathbf{u} \in U$.

On the other hand, if $\|T\mathbf{u}\|_V \geq K_2\|\mathbf{u}\|_U$, then, if $\mathbf{v} \in V$ we can write $\mathbf{u} = T^{-1}(\mathbf{v})$ to obtain

$$\|T^{-1}(\mathbf{v})\|_U = \|T^{-1}(T\mathbf{u})\|_U = \|\mathbf{u}\|_U \leq K_2^{-1}\|T\mathbf{u}\|_V = K_2^{-1}\|\mathbf{v}\|.$$

Thus T^{-1} is continuous.

(iii) Let I be the identity map $I : U \to U$. Then $1 = \|I\| = \|TT^{-1}\| \leq \|T\|\|T^{-1}\|$.

(iv) Let $T(x, y) = (x, 2y)$ so T is linear and bijective as is T^{-1} given by $T^{-1}(x, y) = (x, y/2)$. $T(0, 1) = (0, 2)$ so $\|T\| \geq 2$. $T^{-1}(1, 0) = (1, 0)$ so $\|T^{-1}\| \geq 1$ and $\|T\|\|T\|^{-1} \geq 2 > 1$.

(v) Part (ii) shows that isomorphic normed vector spaces are Lipschitz equivalent.

Exercise 1.6.2

Suppose E has empty interior. If $x \in X$, then $B(x, 1/n) \cap (X \setminus E) \neq \emptyset$ so we can find $x_n \in (X \setminus E)$ with $d(x_n, x) < 1/n$ for all $n \geq 1$. Thus $X \setminus E$ is dense.

Suppose $X \setminus E$ is dense. If $x \in X$ then, given any $\delta > 0$, we can find $y \in X \setminus E$ with $d(x, y) < \delta$. Thus $B(x, \delta) \cap (X \setminus E) \neq \emptyset$. Thus E has empty interior.

Exercise 1.6.5

Observe that $[0, 1/2]$ with the standard metric is a complete metric space so cannot be meagre by Baire's theorem. If $(1/2, 1]$ were meagre, then $[3/4, 1]$ would be meagre and again this is impossible.

Exercise 1.6.6

The countable union of countable sets is countable.

Exercise 1.6.8

(i) If x is isolated we can find a $\delta > 0$ such that $B(x, \delta) = \{x\}$. But any open ball is open.

If $\{x\}$ is open, then since $x \in \{x\}$ we can find a $\delta > 0$ such that $B(x, \delta) \subseteq \{x\}$ so x is isolated.

(ii) Let (X, d) be a complete non-empty metric space without isolated points. If $x \in X$ then $E = \{x\}$ is closed and we can find $y_n \notin E_n$ such that $d(y_n, x) \leq 1/n$ so $X \setminus E$ is dense. Thus the countable union of one point sets $\{x_n\}$ cannot be X and X must be uncountable.

(iii) $E = \{0\} \cup \{1, 1/2, 1/3, \ldots\}$ is a closed subset of $\mathbb{R}$ with the usual metric so complete with that metric yet countable and infinite.

(iv) $\mathbb{R}$ with the discrete metric is an uncountable complete metric space with every point isolated.

Exercise 1.7.3

(i) We first show that d_Z is well defined

Let $\epsilon > 0$. Then we can find an N with $d(x_n, x_m)$, $d(y_n, y_m) < \epsilon/2$ for all n, $m \geq N$. Thus

$$\begin{aligned}|d(x_n, y_n) - d(x_m, y_m)|& \\ &= |(d(x_n, y_n) - d(x_m, y_n)) + (d(x_m, y_n) - d(x_m, y_m))| \\ &\leq |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m| \\ &\leq d(x_n, x_m) + d(y_n, y_m) < \epsilon\end{aligned}$$

for n, $m \geq N$. Thus $d(x_n, y_n)$ forms a Cauchy sequence in $\mathbb{R}$ with the usual metric and

$$\rho_Z(\mathbf{x}, \mathbf{y}) = \lim_{n\to\infty} d(x_n, y_n)$$

is well defined.

(ii) The results claimed in the first sentence follow by taking limits in the relations

$$d(x_n, y_n) \geq 0,\ d(x_n, y_n) = d(y_n, x_n) \text{ and } d(x_n, y_n) + d(y_n, z_n) \leq d(x_n, z_n).$$

Consider $X = \mathbb{R}$ with the usual metric. If $x_n = 1/n$, $y_n = 0$ then $\mathbf{x}$, $\mathbf{y} \in Z$, $\mathbf{x} \neq \mathbf{y}$, but $\rho_Z(\mathbf{x}, \mathbf{y}) = 0$

(iii) By results in (ii), $\rho(\mathbf{x}, \mathbf{x}) = 0$ so $\mathbf{x} \sim \mathbf{x}$. We have

$$\mathbf{x} \sim \mathbf{y} \Rightarrow \rho(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow \rho(\mathbf{y}, \mathbf{x}) = 0 \Rightarrow \mathbf{y} \sim \mathbf{x}).$$

If $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$, then

$$0 \leq \rho(\mathbf{x}, \mathbf{z}) \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}) = 0 + 0 = 0$$

so $\rho(\mathbf{x}, \mathbf{z}) = 0$ and $\mathbf{x} \sim \mathbf{z}$. Thus $\sim$ is an equivalence relation on Z.

(iv) If $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{y}$, $\mathbf{y}' \in Z$ and $\mathbf{x} \sim \mathbf{x}'$, $\mathbf{y} \sim \mathbf{y}'$, then

$$\rho(\mathbf{x}, \mathbf{x}') = \rho(\mathbf{y}, \mathbf{y}') = 0$$

so

$$\rho(\mathbf{x}, \mathbf{y}) \leq \rho(\mathbf{x}', \mathbf{y}') + \rho(\mathbf{y}, \mathbf{y}') + \rho(\mathbf{x}, \mathbf{x}') = \rho(\mathbf{x}', \mathbf{y}')$$

and, similarly,

$$\rho(\mathbf{x}', \mathbf{y}') \leq \rho(\mathbf{x}, \mathbf{y})$$

so $\rho(\mathbf{x}, \mathbf{x}') = \rho(\mathbf{y}, \mathbf{y}')$.

(v) Since $\mathbf{x}$ is a Cauchy sequence, we can find an N such that $d(x_n, x_m) < \epsilon$ for n, $m \geq N$. Set $x'_n = x_N$ for $n \leq N$ and $x'_n = x_n$ for $n > N$.

(vi) The fact that $\tilde{d}$ is well defined follows from the last part of (ii). The fact that $\tilde{d}([\mathbf{x}], [\mathbf{y}]) = 0$ implies $[\mathbf{x}] = [\mathbf{y}]$ follows from the definition of $\sim$. The remaining conditions for a metric follow from the corresponding conditions on ρ.

(vii) Observe that, if $n,\ m,\ p \geq N$, then

$$d(y_n, y_m) = d(x_n(n), x_m(m)) \leq d(x_n(n), x_p(n))) + d(x_p(n), x_p(m))) + d(x_p(m), x_m(m))$$
$$\leq 2^{-N+1} + d(x_p(n), x_p(m))$$

Allowing $p \to \infty$ we obtain

$$d(y_n, y_m) \leq 2^{-N+1} + \rho(\mathbf{x}(n), \mathbf{x}(m)) \leq 2^{-N+2}$$

for all $n,\ m \geq N$.Thus $\mathbf{y} \in Z$. In much the same way, if $n,\ m,\ p \geq N$, then

$$\begin{aligned} d_(y_n, x_n(m)) &= d(x_n(n), x_n(m)) \\ &\leq d(x_n(n), x_p(n)) + d(x_p(n), x_p(m)) + d(x_p(m), x_n(m)) \\ &\leq 2^{-N+1} + d(x_p(n), x_p(m)) \end{aligned}$$

and, allowing $p \to \infty$,

$$d(y_n, x_n(m)) \leq 2^{-N+1} + \rho(\mathbf{x}(n), \mathbf{x}(m)) \leq 2^{-N+2}$$

for all $n,\ m \geq N$. Thus $\rho(\mathbf{x}(m), \mathbf{y}) \leq 2^{-N+2}$ for $m \geq N$ and $\rho(\mathbf{x}(m), \mathbf{y}) \to 0$ as $m \to \infty$.

(viii) By Exercise 1.2.15 it is sufficient to show that, working in $(\tilde{X}, \tilde{d})$, if $[\mathbf{x}(n)]$ satisfies

$$\tilde{d}([\mathbf{x}(n)], [\mathbf{x}(m)]) < 2^{-N} \text{ for all } n,\ m \geq N,$$

then there exists a $[\mathbf{y}] \in \tilde{X}$ with $\tilde{d}([\mathbf{x}(n)], [\mathbf{y}]) \to 0$ as $n \to \infty$.

By part (v) of this exercise, we may suppose that

$$d(x_j(n), x_k(n)) < 2^{-N} \text{ for all } j,\ k \text{ and all } n \geq N.$$

The required result now follows from part (vii).

(ix) Every constant sequence is a Cauchy sequence, so θ is well defined. Automatically

$$\tilde{d}(\theta(x), \theta(y)) = \lim_{n\to\infty} d(x, y) = d(x, y)$$

so θ is an isometry. If $\mathbf{x} \in Z$ then

$$\tilde{d}(\mathbf{x}, \theta(x_n)) = \lim_{j\to\infty} d(x_j, x_n) \to 0$$

as $n \to \infty$.

Exercise 2.1.1

(i) The stated result is true for $n = 0$ with $P_0(x) = 1$.

Suppose it is true for $n = m$. Then, by standard theorems on differentiation, $E^{(m)}$ is differentiable at $x \neq 0$ with

$$E^{(m+1)}(x) = -\frac{1}{x^2}P'_m(1/x)E(x) + \frac{2}{x^3}P_{m(1/x)}E(x) = P_{m+1}(1/x)$$

where P_{m+1} is the polynomial given by $P_{m+1}(t) = -t^2P'_m(t) + 2t^3P_m(t)$.

The result now follows by induction.

(ii) The stated result is true for $n = 0$ with $P_0(0) = 0$. Suppose it is true for $n = m$. Then

$$\frac{E^{(m)}(h) - E^m(0)}{h} = \frac{1}{h}P_m(h)\exp(-1/h^2) \to 0$$

as $h \to 0$. Thus $E^{(m)}$ is differentiable at 0 with derivative 0.

The result now follows by induction.

(iii) Thus E is infinitely differentiable, but

$$E(h) \neq 0 = \sum_{n=0}^{\infty} 0 \times h^n = \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!}h^n$$

for all $h \neq 0$.

Exercise 2.1.2

Observe that

$$f^{(r)}(x) = \sum_{n=r}^{\infty} n(n-1)\dots(n-r+1)a_nx^{n-r}$$

so

$$f^{(r)}(0) = \frac{a_r}{r!}.$$

Exercise 2.1.5

(i) Let

$$A = \left(\int_{-\infty}^{\infty} E(x-\eta)E(x+\eta)\,dx\right)^{-1}.$$

The required results can now be read off.

(ii) Let $\eta = \delta/2$, take h as in part (i) and set

$$k(x) = \int_{-\infty}^{x} h(s-c-2\eta) - h(s-d+2\eta)\,ds.$$

The required results can now be read off.

(iii) Set $\tilde{f}(x) = k(x)f(x)$.

Exercise 2.1.6

(i) Since $2^{-n}\min\{1,\|f^{(n)}-g^{(n)}\|_\infty\}\leq 2^{-n}$ the comparison test tells that the sum converges.

(ii) Since $2^{-n}\min\{1,\|f^{(n)}-g^{(n)}\|_\infty\}\geq 0$, $d(f,g)\geq 0$. Further, if $d(f,g)=0$, we must have $\min\{1,\|f-g\|_\infty\}=0$ so $\|f-g\|_\infty=0$ and $f=g$.

Since

$$2^{-n}\min\{1,\|f^{(n)}-g^{(n)}\|_\infty\}=2^{-n}\min\{1,\|g^{(n)}-f^{(n)}\|_\infty\}$$

we have $d(f,g)=d(g,f)$. Since

$$\begin{aligned}2^{-n}\min\{1,\|f^{(n)}-g^{(n)}\|_\infty+2^{-n}\min\{1,\|g^{(n)}-h^{(n)}\|_\infty\\ \geq 2^{-n}\min\{1,\|f^{(n)}-h^{(n)}\|_\infty,\end{aligned}$$

we have $d(f,g)+d(g,h)\geq d(f,h)$.

(iii) Suppose that f_j is a Cauchy sequence for $(C^\infty([a,b],d)$ and that $n\geq 0$. Then, given $\epsilon>0$ with $1>\epsilon$, we can find a J such that $d(f_j,f_k)\leq 2^{-n}\epsilon$ and so, automatically,

$$\|f_j^{(n)}-f_k^{(n)}\|_\infty\leq\epsilon$$

for all $j,\ k\geq J$. Thus $f_j^{(n)}$ is a Cauchy sequence for $(C([a,b],\|\ \|_\infty)$ and, since $C([a,b])$ is complete under the uniform norm we can find $F_n\in C([a,b])$ such that $f_j^{(n)}\to F_n$ uniformly on $[a,b]$ as $j\to\infty$ for each n.

It is a theorem of elementary analysis that, if u_n is a continuously differentiable function on $[a,b]$ and $u_n\to u$, $v_n'\to v$ uniformly on $[a,b]$, then u is continuously differentiable with $u'=v$. Thus, by induction, $F=F_0$ is infinitely differentiable with $F^{(n)}=F_n$.

Given $\epsilon>0$, we can find an N such that $2^{-N}\leq\epsilon$. Thus

$$\begin{aligned}d(f_j,F)&\leq\sum_{n=0}^{N}2^{-n}\min\{1,\|f_j^{(n)}-F^{(n)}\|_\infty\}+\sum_{n=N+1}^{\infty}2^{-n}\\&\leq\epsilon+\sum_{n=0}^{N}2^{-n}\min\{1,\|f_j^{(n)}-F^{(n)}\|_\infty\}\to\epsilon\end{aligned}$$

as $j\to\infty$. Since ϵ was arbitrary, $d(f_j,F)\to 0$ and we have shown that d is a complete metric.

Exercise 2.1.9

Just replace $(n!)^2$ by M_n in Lemma 2.1.7and 2.1.8.

Exercise 2.2.2

(i) Observe that

$$\frac{1}{2\pi}\int_{\mathbb{T}} D_n(s)\,ds = \sum_{r=-n}^{n}\frac{1}{2\pi}\int_{\mathbb{T}}\exp(irs)\,ds$$

and

$$\frac{1}{2\pi}\int_{\mathbb{T}}\exp(irs)\,ds = \begin{cases}0 & \text{if } r\neq 0,\\ 1 & \text{if } r=0.\end{cases}$$

(ii) We have $D_n(0) = \sum_{r=-n}^{r=n} 1 = 2n+1$. If $s \neq 0$, then $\exp is \neq 0$ and

$$\begin{aligned}D_n(s) &= \exp(-ins)\sum_{r=0}^{2n}(\exp(is))^r\\ &= \exp(-ins)\frac{\exp(i(2n+1)s)-1}{\exp(is)-1}\\ &= \exp(-ins)\frac{\exp(i(n+1/2)s)-\exp(-i(n+1/2)s)}{\exp(is/2)-\exp(-is/2)}\\ &= \frac{\sin\left((n+\frac{1}{2})s\right)}{\sin\frac{1}{2}s}.\end{aligned}$$

(iii) Left to reader.

(iv) Observe if $f(t) = t - \sin t$, then $f'(t) = 1 - \cos t \geq 0$ for $t \geq 0$ so $f(t) \leq f(0)$, that is to say, $\sin t \leq t$ for $t \geq 0$. On the other hand, if $g(t) = (2t/\pi) - \sin t)$, then $g''(t) \geq 0$ for $0 \leq t \leq \pi/2$ and $g(0) = g(\pi/2) = 0$ so $g(t) \leq 0$ for $0 \leq t \leq \pi/2$ and

$$\frac{2t}{\pi} \leq \sin t \leq t$$

for $0 \leq t \leq \pi/2$. (Or just say convexity.)

(v) Using (iv),

$$|D_n(t)| = \left|\frac{\sin\left((n+\frac{1}{2})t\right)}{\sin\frac{1}{2}t}\right| \geq \frac{\left|\sin\left((n+\frac{1}{2})t\right)\right|}{|t|/2} \geq 2\pi\left|\sin\left((n+\tfrac{1}{2})t\right)\right|\frac{n+\frac{1}{2}}{r+1}$$

for $r\pi/(n+\frac{1}{2}) \leq t \leq (r+1)\pi/(n+\frac{1}{2})$ so

$$\begin{aligned}\frac{1}{2\pi}\int_{r\pi/(n+\frac{1}{2})}^{(r+1)\pi/(r+\frac{1}{2})}|D_n(t)|\,dt &\geq \frac{1}{2r+1}\int_{r\pi/(n+\frac{1}{2})}^{(r+1)\pi(r+\frac{1}{2})}\left|\sin\left((n+\tfrac{1}{2})t\right)\right|dt\\ &= \frac{1}{2r+1}\int_0^{\pi/(n+\frac{1}{2})}\left|\sin\left((n+\tfrac{1}{2})t\right)\right|dt\frac{\pi}{n+\frac{1}{2}}\\ &= B\frac{1}{2r+1} \geq Ar+1\end{aligned}$$

for appropriate $A,\ B > 0$.

(vi) Thus

$$\frac{1}{2\pi}\int_{\mathbb{T}} |D_n(s)|\,ds \geq \sum_{r=2}^{n} \frac{1}{2\pi}\int_{r\pi/(n+\frac{1}{2})}^{(r+1)\pi/(n+\frac{1}{2})} |D_n(t)|\,dt \geq A\sum_{r=2}^{n}\frac{1}{r} \to \infty$$

as $n \to \infty$.

(vii) If $\eta > 0$ let g_η be the simplest continuous linear function such that

$$g_\eta(t) = \begin{cases} 1 & \text{if } D_n(-t) \geq \eta \\ -1 & \text{if } D_n(-t) \leq -\eta. \end{cases}$$

Then $g_\eta \in C(\mathbb{T})$, $-1 \leq g_\eta(s) \leq 1$ for all $s \in \mathbb{T}$, and

$$S_n(g_\eta, 0) = \frac{1}{2\pi}\int_{\mathbb{T}} g_\eta(s) D_n(-s)\,ds \to \left(\frac{1}{2\pi}\int_{\mathbb{T}} |D_n(s)|\,ds\right)$$

as $\eta \to 0$.

Exercise 2.2.4

(i) We have, making the change of variable $s = t + a$

$$\begin{aligned}\hat{f}_a(n) &= \frac{1}{2\pi}\int_{\mathbb{T}} f(a+t)\exp(-int)\,dt\\ &= \exp(ina)\frac{1}{2\pi}\int_{\mathbb{T}} f(a+t)\exp(-in(a+t))\,dt\\ &= \exp(ina)\frac{1}{2\pi}\int_{\mathbb{T}} f(s)\exp(-ins))\,ds = \exp(ina)\hat{f}(n)\end{aligned}$$

Thus

$$S_n(f_a,t) = \sum_{r=-n}^{n}\hat{f}_a(r)\exp(irt) = \sum_{r=-n}^{n}\hat{f}(r)\exp(ir(t+a)) = S_n(f,a+t)$$

for all t and, in particular $S_n(f_a,0) = S_n(f,a)$.

(ii) This follows from Theorem 2.2.3 by translation using (i).

(iii) Since the countable union of sets of first category is of first category the set of continuous functions with bounded partial sums $S_n(f,e)$ at any point e of E is of first category.

(iv) Since the partial sums of a convergent sum are bounded quasi-all $f \in C(\mathbb{T})$ diverge on a dense subset of $\mathbb{T}$.

Exercise 2.2.5

(i) Since f_n is continuous, $x \in \mathbb{T} : f_n(x) \leq k$ is closed and the intersection of closed sets is closed,

$$E_k = \bigcap_{k=1}^{\infty}\{x \in \mathbb{T} : f_n(x) \leq k\}$$

is closed. By hypothesis $a_j \notin E_k$ so E_k has dense complement. Thus $E = \bigcup_{k=1}^{\infty} E_k$ is of first category. Since every point x where $f_n(x)$ converges lies in E, f_n diverges quasi-everywhere on $\mathbb{T}$.

(ii) Immediate from (i) and Exercise 2.2.4.

Exercise 2.3.2

(i) We have

$$\sigma_n(f,t) = \frac{1}{n}\sum_{j=0}^{n-1} S_j(f,t)) = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{r=-j}^{j}\hat{f}(r)\exp(irt)$$
$$= \sum_{r=-n}^{n}\sum_{j=|r|}^{n-1}\hat{f}(r)\exp(irt) = \sum_{r=-n+1}^{n-1}\frac{n-|r|}{n}\hat{f}(r)\exp(irt).$$

(ii) We have

$$\sigma_n(f,t) = \sum_{r=-n+1}^{n-1}\frac{n-|r|}{n}\hat{f}(r)\exp(irt)$$
$$= \sum_{r=-n+1}^{n-1}\frac{n-|r|}{n}\frac{1}{2\pi}\int_{\mathbb{T}} F(s)\exp(-irs)\,ds\exp(irt)$$
$$\frac{1}{2\pi}\int_{\mathbb{T}}\sum_{r=-n+1}^{n-1}\frac{n-|r|}{n}\exp(ir(t-s))f(s)\,ds$$
$$= f * K_n(t).$$

(iii) We have

$$\left(\sum_{r=0}^{n-1}\exp\Big(i((-((n-1)/2)+r)t)\Big)\right)^2 = \sum_{u=0}^{n-1}\sum_{v=0}^{n-1}\exp\Big(i((-((n-1)/2)+(v+u)t))\Big)$$
$$= \sum_{r=-n+1}^{n-1}(n-|r|)\exp(irt).$$

(iv) We have

$$\sum_{r=0}^{n-1}\exp\Big(i((-((n-1)/2)+r)0)\Big) = \sum_{r=0}^{n-1}1 = n$$

whilst, if $t \neq 0$, summing a geometric series gives

$$\sum_{r=0}^{n-1}\exp\Big(i((-((n-1)/2)+r)t)\Big) = \exp(-it(n-1)/2)\sum_{r=0}^{n-1}\exp(irt)$$
$$= \exp(-it(n-1)/2)\frac{\exp(int)-1}{\exp(it)-1}$$
$$= \frac{\exp(int/2)-\exp(-int/2)}{\exp(it/2)-\exp(-it/2)} = \frac{\sin(nt/2)}{\sin t/2}.$$

By (iii), this implies

$$K_n(t) = \begin{cases} \frac{1}{n}\left(\frac{\sin\left(nt/2\right)}{\sin t/2}\right)^2 & \text{for } t \neq 0 \\ n & \text{for } t = 0. \end{cases}$$

EXERCISE 2.3.6

If $|n| \geq N + 1$, then

$$|\hat{f}(n)| = |\hat{f}(n) - \hat{P}(n)| = |(\widehat{f - P})(n)| \leq \|f - P\|_\infty < \epsilon.$$

If $g \in C(\mathbb{T})$, then, by Fejér's theorem, given any $\epsilon > 0$, we can find a trigonometric polynomial P with $\|g - P\|_\infty < \epsilon$. If P has degree N,

$$|\hat{g}(n)| < \epsilon \text{ for } n \geq N + 1$$

Thus $\hat{g}(n) \to 0$ as $|n| \to \infty$.

Exercise 2.3.7

(i) Since f_n and g_n converge uniformly, it follows that $f, g \in C(\mathbb{T})$ and we can find an M such that $\|f_n\|_\infty, \|g_n\|_\infty \le M$ for all n. Thus

$$\begin{aligned}
|f_n * g_n(t) - f * g(t)| &= \frac{1}{2\pi}\left|\int_{\mathbb{T}} f_n(t-s)g_n(s) - f(t-s)g_n(s)\,ds\right| \\
&\le \frac{1}{2\pi}\int_{\mathbb{T}} |f_n(t-s)g_n(s) - f(t-s)g_n(s)|\,ds \\
&\le \frac{1}{2\pi}\int_{\mathbb{T}} (|f_n(t-s)g_n(s) - f(t-s)g_n(s)| \\
&\qquad + |f(t-s)g_n(s) - f(t-s)g_n(s)|\,ds \\
&\le M\|f_n - f\|_\infty + M\|g_n - g\|_\infty \to 0
\end{aligned}$$

uniformly as $n \to \infty$.

(ii) Let $P(t) = \sum_{j=-n}^{n} a_j \exp(ijt)$, $Q(t) = \sum_{j=-n}^{n} b_j \exp(ijt)$. Then

$$\begin{aligned}
P * Q(t) &= \frac{1}{2\pi}\int_{\mathbb{T}} \sum_{j=-n}^{n}\sum_{k=-n}^{n} a_j \exp(ij(t-s))b_k \exp(iks)\,ds \\
&= \sum_{j=-n}^{n}\sum_{k=-n}^{n} a_j b_k \exp(ijt)\frac{1}{2\pi}\int_{\mathbb{T}} \exp(i(j-k)s)\,ds \\
&= \sum_{j=-n}^{n} a_j b_j \exp(ijt)
\end{aligned}$$

as required.

(iii) Chose trigonometric polynomials f_n, g_n with $f_n \to f$, $g_n \to g$ uniformly. Since the uniform limit of continuous functions is continuous $f * g$ is continuous. We have $\widehat{f_n * g_n}(r) = \hat{f}_n(r)\hat{g}_n(r)$, $\widehat{f_n * g_n}(r) \to \widehat{f * g}(r)$, $\hat{f}_n(r) \to \hat{f}(r)$ and $\hat{g}_n(r) \to \hat{g}(r)$ so $\widehat{f * g}(r) = \hat{f}(r)\hat{g}(r)$.

Exercise 2.3.10

(i) $|\hat{f}(j)\exp ijt| \leq |\hat{f}(j)|$, so by the Weierstrass M-test, $\sum_{j=-n}^{n}\hat{f}(j)\exp ijt$ converges uniformly. The uniform limit of a sequence of continuous functions is continuous.

(ii) We recall that if a sequence of continuous function converges uniformly on $[a,b]$ to a limit then the integral of the limit is the limit of the integral. If $g_n, g \in C(\mathbb{T})$ and $g_n \to g$ then

$$\frac{1}{2\pi}\int_{\mathbb{T}} g_n(t)\,dt \to \frac{1}{2\pi}\int_{\mathbb{T}} g(t)\,dt.$$

Thus if $n \geq |m|$

$$\begin{aligned}\hat{f}(m) &= \frac{1}{2\pi}\int_{\mathbb{T}}\left(\sum_{j=-n}^{n}\hat{f}(j)\exp ijt\right)\exp(-imt)\,dt \\ &\to \frac{1}{2\pi}\int_{\mathbb{T}} h(t)\exp(-imt)\,dt = \hat{h}(m)\end{aligned}$$

as $n \to \infty$. Thus $\hat{f}(m) = \hat{h}(m)$ for all m and $h = f$ as required.

Exercise 2.4.2

(i) Observe that, since $\langle \mathbf{u}, \mathbf{v}\rangle$ and λ are real,

$$0 \leq \langle \lambda\mathbf{u} + \mathbf{v}, \lambda\mathbf{u} + \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle + 2\lambda\langle \mathbf{u}, \mathbf{v}\rangle\langle \mathbf{v}, \mathbf{v}\rangle.$$

If $\langle \mathbf{u}, \mathbf{u}\rangle = 0$, then $\mathbf{u} = \mathbf{0}$ and the required inequality is trivial. If not,

$$0 \leq \left(\lambda\langle \mathbf{u}, \mathbf{u}\rangle^{1/2} + \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle^{1/2}}\right)^2 - \frac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\langle \mathbf{u}, \mathbf{u}\rangle} + \langle \mathbf{v}, \mathbf{v}\rangle.$$

Setting

$$\lambda = -\langle \mathbf{u}, \mathbf{v}\rangle\langle \mathbf{u}, \mathbf{u}\rangle$$

we obtain

$$0 \leq -\frac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\langle \mathbf{u}, \mathbf{u}\rangle} + \langle \mathbf{v}, \mathbf{v}\rangle$$

which is the required inequality.

(ii) Choose θ so that $e^{i\theta}\langle \mathbf{a}, \mathbf{b}\rangle$ is real and set $\mathbf{u} = e^{i\theta}\mathbf{a}$, $\mathbf{v} = \mathbf{b}$ in (i).

(iii) We have $||\mathbf{a}|| \geq 0$ and

$$||\mathbf{a}|| = 0 \Rightarrow \langle \mathbf{a}, \mathbf{a}\rangle = 0 \Rightarrow \mathbf{a} = \mathbf{0}.$$

Since

$$\langle \lambda\mathbf{a}, \lambda\mathbf{a}\rangle = \lambda\lambda^*\langle \mathbf{a}, \mathbf{a}\rangle = |\lambda|^2||\mathbf{a}||^2$$

we have $||\lambda\mathbf{a}|| = |\lambda|||\mathbf{a}||$. Finally

$$\begin{aligned}||\mathbf{a} + \mathbf{b}||^2 &= \langle \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b}\rangle = ||\mathbf{a}||^2 + 2\Re\langle \mathbf{a}, \mathbf{b}\rangle + ||\mathbf{b}||^2\\ &\leq ||\mathbf{a}||^2 + 2||\mathbf{a}||||\mathbf{b}|| + ||\mathbf{b}||^2 = (||\mathbf{a}|| + ||\mathbf{b}||)^2,\end{aligned}$$

so $||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$.

Exercise 2.4.3

(i) We have

$$\langle f, g\rangle^* = \frac{1}{2\pi}\left(\int_{\mathbb{T}} f(t)g(t)^*\,dt\right)^* = \frac{1}{2\pi}\int_{\mathbb{T}}(f(t)g(t)^*)^*\,dt = \langle g, f\rangle.$$

(ii) We have

$$\langle \mathbf{f}, \mathbf{f}\rangle \geq 0 = \frac{1}{2\pi}\int_{\mathbb{T}}|f(t)|^2\,dt.$$

(iii) Recall that, if $h \in C(\mathbb{T})$ is real valued and positive, then $\frac{1}{2\pi}\int_{\mathbb{T}} h(t)\,dt = 0$ implies $h = 0$. Thus

$$\langle \mathbf{f}, \mathbf{f}\rangle = 0 \Rightarrow |f|^2 = 0 \Rightarrow f = 0.$$

(iv) and (v)
The linearity of the integral gives

$$\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$$

and

$$\langle \lambda f, g\rangle = \lambda\langle f, g\rangle.$$

Exercise 2.4.4

(i) We have

$$\langle e_r, e_s\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} \exp(i(r-s)t)\,dt = \begin{cases} 1 & \text{if } r = s,\\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle f, e_r\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\exp(-irt)\,dt = \hat{f}(r).$$

(ii)

$$\langle e_r, e_s\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} \exp(i(r-s)t)\,dt = \begin{cases} 1 & \text{if } r = s,\\ 0 & \text{otherwise.} \end{cases}$$

(iii) We have

$$\begin{aligned}
\left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_2^2 &= \left\langle f - \sum_{j=-n}^{n} a_j e_j, f - \sum_{j=-n}^{n} a_j e_j \right\rangle \\
&= \|f\|_2^2 + \sum_{j=-n}^{n} (a_j^*\langle f, e_j\rangle + a_j\langle e_j, f\rangle) + \sum_{j=-n}^{n} |a_j|^2 \\
&= \sum_{j=-n}^{n} |a_j - \hat{f}(j)|^2 + \|f\|_2^2 - \sum_{j=-n}^{n} |\hat{f}(j)|^2
\end{aligned}$$

(iv) Since $|a_j - \hat{f}(j)|^2 \geq 0$ we have

$$\left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_2 \geq \left\| f - \sum_{j=-n}^{n} \hat{f}(j) e_j \right\|_2$$

with equality if and only if $a_j = \hat{f}(j)$ for $-n \leq j \leq n$.

(v) Setting $a_j = \hat{P}(j)$ in (iv) yields

$$\begin{aligned}
\left\| f - \sum_{j=-n}^{n} \hat{f}(j) e_j \right\|_2 &\leq \left\| f - \sum_{j=-n}^{n} \hat{P}(j) e_j \right\|_2 = \left\| f - \sum_{j=-n}^{n} \hat{P}(j) e_j \right\|_2 \\
&\leq \frac{1}{2\pi}\int_{\mathbb{T}} f(t)|f(t) - P(t)|^2\,dt \leq \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\epsilon^2\,dt = \epsilon^2
\end{aligned}$$

so

$$\left\| f - \sum_{j=-n}^{n} \hat{f}(j) e_j \right\|_2 \leq \epsilon.$$

(vi) By Fejér's theorem, we may make ϵ arbitrarily small at the expense of choosing n sufficiently large. Thus

$$\left\| f - \sum_{j=-n}^{n} \hat{f}(j) e_j \right\|_2 \to 0$$

as $n \to \infty$.

(vii) If we set $a_j = \hat{f}(j)$ in part (iii), we obtain

$$\left\| f - \sum_{j=-n}^{n} \hat{f}_j e_j \right\|_2^2 = \|f\|_2^2 - \sum_{j=-n}^{n} |\hat{f}(j)|^2$$

so part (iv) yields

$$\sum_{j=-n}^{n} |\hat{f}(j)|^2 \to \|f\|_2^2$$

(viii) Exactly as in (iii),

$$\left\| \mathbf{f} - \sum_{j=1}^{n} a_j \mathbf{e}_j \right\|_2^2 = \|\mathbf{f}\|_2^2 + \sum_{j=1}^{n} |a_j - \langle \mathbf{f}, \mathbf{e}_j \rangle|^2 - \sum_{j=1}^{n} |\langle \mathbf{f}, \mathbf{e}_j \rangle|^2.$$

and taking $a_j = \langle \mathbf{f}, \mathbf{e}_j \rangle$, we have

$$\|\mathbf{f}\|_2^2 \geq \sum_{j=1}^{n} |\langle \mathbf{f}, \mathbf{e}_j \rangle|^2.$$

Thus, since an increasing sequence in $\mathbb{R}$ bounded above tends to a limit no larger than the bound, $\sum_{j=1}^{\infty} |\langle \mathbf{f}, \mathbf{e}_j \rangle|^2$ converges and

$$\|\mathbf{f}\|_2^2 \geq \sum_{j=1}^{\infty} |\langle \mathbf{f}, \mathbf{e}_j \rangle|^2.$$

Exercise 2.4.5

Let

$$f_n(t) = \begin{cases} -1 & \text{if } t \leq -n^{-1}, \\ nt & \text{if } -n^{-1} < t < n^{-1}, \\ 1 & \text{if } n^{-1} \leq t. \end{cases}$$

If $m \geq n$ we have $f_n(t) - f_m(t) = 0$ for $|t| \geq n^{-1}$ and $1 \geq |f_n(t) - f_m(t)|$ if $|t| \leq n^{-1}$. Thus

$$\|f_n - f_m\|_2^2 \leq \int_{-1/n}^{1/n} 1\, dt = 2/n.$$

Since $2/n \to 0$ as $n \to \infty$, the sequence is Cauchy.

Suppose, if possible, that $f_n \underset{\| \|_2}{\to} f$ for some $f \in C([-1, 1])$. If $1 > \delta > 0$, then, whenever $n \geq \delta^{-1}$,

$$0 \leq \int_\delta^1 |f(t) - 1|^2\, dt = \int_\delta^1 |f(t) - f_n(t)|^2\, dt \leq \|f - f_n\|_2^2 \to 0$$

as $n \to \infty$. Thus, by Exercise 1.2.9, $|f(t) - 1|^2 = 0$ for $t \geq \delta$, so $f(t) = 1$ for $t \geq \delta$. Since δ was arbitrary, $f(t) = 1$ for $t > 0$. Similarly $f(t) = -1$ for $t < 0$. Thus

$$\lim_{t \to 0+} f(t) = 1 \neq -1 = \lim_{t \to 0-} f(t)$$

and f is not continuous. The desired result follows by reductio ad absurdum.

Exercise 2.4.6

(i) By considering the standard norm on $\mathbb{C}^{N+M+1}$, we have

$$\left(\sum_{j=-M}^{N} |a_j + b_j|^2\right)^{1/2} \leq \left(\sum_{j=-M}^{N} |a_j|^2\right)^{1/2} + \left(\sum_{j=-M}^{N} |b_j|^2\right)^{1/2} \leq \|\mathbf{a}\|_2 + \|\mathbf{b}\|_2.$$

Allowing $M, N \to \infty$, we see that $\mathbf{a} + \mathbf{b} \in l^2$ with $\|\mathbf{a} + \mathbf{b}\|_2 \leq \|\mathbf{a}\|_2 + \|\mathbf{b}\|_2$.

(ii) If $\mathbf{a} \in l^2$ and $\lambda \in \mathbb{C}$ we have

$$\left(\sum_{j=-M}^{N} \|\lambda a_j|^2\right)^{1/2} = |\lambda|\left(\sum_{j=-M}^{N} |a_j|^2\right)^{1/2} \to |\lambda|\|\mathbf{a}\|$$

as M, $N \to \infty$, so $\lambda\mathbf{a} \in l^2$ and $\|\lambda\mathbf{a}\|_2 = |\lambda|\|\mathbf{a}\|_2$.

By definition, $\|\mathbf{a}\|_2 \geq 0$. Finally

$$\|\mathbf{a}\|_2 = 0 \Rightarrow \sum_{j=-\infty}^{\infty} |a_j|^2 = 0 \Rightarrow a_j = 0 \text{ for all } j \Rightarrow \mathbf{a} = \mathbf{0}.$$

Thus we have a norm.

(iii) By considering the standard inner product on $\mathbb{C}^{N+M+1}$, we have

$$\left(\sum_{j=-M}^{N} |a_j b_j^*|\right)^2 \leq \left(\sum_{j=-M}^{N} |a_j|^2\right)\left(\sum_{j=-M}^{N} |b_j|^2\right)^2 \leq \|\mathbf{a}\|_2\|\mathbf{b}\|_2.$$

Allowing $M, N \to \infty$ we see that that $\sum_{j=-\infty}^{\infty} a_j b_j^*$ is absolutely convergent and so convergent.

Since

$$\left(\sum_{j=-M}^{N} a_j b_j^*\right)^* = \sum_{j=-M}^{N} a_j^* b_j,$$

allowing M, $N \to \infty$ yields $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle^*$.

Automatically $\langle \mathbf{a}, \mathbf{a} \rangle = \|\mathbf{a}\|_2^2 \geq 0$ and

$$\langle \mathbf{a}, \mathbf{a} \rangle = 0 \Rightarrow \|\mathbf{a}\| = 0 \Rightarrow \mathbf{a} = \mathbf{0}.$$

Since

$$\sum_{j=-M}^{N} (\lambda a_j b_j^*) = \lambda \sum_{j=-M}^{N} a_j b_j^*$$

and

$$\sum_{j=-M}^{N} (a_j + c_j) b_j^* = \sum_{j=-M}^{N} a_j b_j^* + \sum_{j=-M}^{N} c_j b_j^*,$$

allowing M, $N \to \infty$ yields

$$\langle \lambda\mathbf{a}, \mathbf{b} \rangle = \lambda\langle \mathbf{a}, \mathbf{b} \rangle.$$

and

$$\langle \mathbf{a} + \mathbf{c}, \mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{c}, \mathbf{b} \rangle$$

(iv) Let E consist of $\mathbf{a}$ with $a_j \in \mathbb{Q}$ and only finitely many a_j non-zero. Since $\mathbb{Q}^N$ is countable and the countable union of countable sets is countable E is countable.

Suppose $\mathbf{a} \in l^2$. Let $e_j(n) \in \mathbb{Q}$ with $|e_j(n) - a_j| \leq (2n+1)^{-1}2^{-n}$ for $|j| \leq n$, and $e_j(n) = 0$ otherwise. Then $\mathbf{e}(n) \in E$ and

$$\|\mathbf{a} - \mathbf{e}(n)\|_2^2 = \sum_{|j| \leq n} |e_j(n) - a_j|^2 + \sum_{|j| \geq n+1} |a_j|^2 \leq 2^{-2n} + \sum_{|j| \geq n+1} |a_j|^2 \to 0$$

as $n \to \infty$. Thus E is dense in l^2.

EXERCISE 2.4.8

We first show that l^1 (which is a subset of the vector space of two way infinite sequences) is a normed vector space.

(i) We observe that

$$\sum_{j=-M}^{N} |a_j + b_j| \leq \sum_{j=-M}^{N} |a_j| + \sum_{j=-M}^{N} |b_j| \leq \|\mathbf{a}\|_1 + \|\mathbf{b}\|_1.$$

Allowing $M, N \to \infty$ we see that $\mathbf{a} + \mathbf{b} \in l^1$ with $\|\mathbf{a} + \mathbf{b}\|_1 \leq \|\mathbf{a}\|_1 + \|\mathbf{b}\|_1$.

If $\mathbf{a} \in l^1$ and $\lambda \in \mathbb{C}$ we have

$$\sum_{j=-M}^{N} |\lambda a_j| = |\lambda| \sum_{j=-M}^{N} |a_j| \to |\lambda|\|\mathbf{a}\|$$

as M, $N \to \infty$ so $\lambda\mathbf{a} \in l^1$ and $\|\lambda\mathbf{a}\|_1 = |\lambda|\|\mathbf{a}\|_1$.

By inspection, $\|\mathbf{a}\|_1 \geq 0$ and

$$\|\mathbf{a}\|_1 = 0 \Rightarrow \sum_{j=-\infty}^{\infty} |a_j| = 0 \Rightarrow a_j = 0 \text{ for all } j \Rightarrow \mathbf{a} = \mathbf{0}.$$

Thus we have a norm.

Next we prove separability. Let E consist of $\mathbf{a}$ with $a_j = x_j + iy_j$ where x_j, $y_j \in \mathbb{Q}$ and only finitely many a_j non-zero. Since $\mathbb{Q}^N$ is countable and the countable union of countable sets is countable, E is countable.

Suppose $\mathbf{a} \in l^1$. Let $e_j(n) \in \mathbb{Q}$ with $|e_j(n) - a_j| \leq (2n+1)^{-1}2^{-n}$ for $|j| \leq n$, and $e_j(n) = 0$ otherwise. Then $\mathbf{e}(n) \in E$ and

$$\|\mathbf{a} - \mathbf{e}(n)\|_1 = \sum_{|j|\leq n} |e_j(n) - a_j| + \sum_{|j|\geq n+1} |a_j| \leq 2^{-n} + \sum_{|j|\geq n+1} |a_j|^2 \to 0$$

as $n \to \infty$. Thus E is dense in l^1.

Finally we prove completeness. By Exercise 1.2.15 (i), it sufficient to show that if $\|\mathbf{a}(n) - \mathbf{a}(m)\|_1 \leq 2^{-n}$ for all $m \geq n$ then $\mathbf{a}(n)$ converges as $n \to \infty$. Following our usual method, we observe that $|a_j(n) - a_j(m)\| \leq 2^{-n}$ for all $m \geq n$ and so, since $\mathbb{C}$ is complete, $a_j(n) \to a_j$ for some $a_j \in \mathbb{C}$.

Next we note that since the sequence $\mathbf{a}(n)$ is Cauchy it is bounded with $\|\mathbf{a}(n)\|_1 \leq M$ say, so $\sum_{-P}^{Q} |a_j(n)| \leq M$ and, allowing $n \to \infty$, this gives $\sum_{-P}^{Q} |a_j| \leq M$ for all P, $Q > 0$. Allowing P, $Q \to \infty$ this shows that $\mathbf{a} \in l^1$.

Finally we have

$$\sum_{-N}^{N} |a_j - a_j(n)| \le \sum_{-N}^{N} |a_j - a_j(m)| + \sum_{-N}^{N} |a_j(m) - a_j(n)|$$

$$\le \sum_{-N}^{N} |a_j - a_j(m)| + ||\mathbf{a}_n - \mathbf{a}_m||$$

$$\le \sum_{-N}^{N} |a_j - a_j(m)| + 2^{-n}$$

Now, allowing $m \to \infty$, we obtain

$$\sum_{-N}^{N} |a_j - a_j(n)| \le 2^n$$

Allowing $N \to \infty$, we obtain $||\mathbf{a} - \mathbf{a}(n)||_1 \to 0$ and we are done.

Exercise 2.4.9

(i) We have

$$\begin{aligned}
&||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2 + i||\mathbf{x}+i\mathbf{y}||^2 - i||\mathbf{x}-i\mathbf{y}||^2\\
&\quad = (||\mathbf{x}||^2 + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{x},\mathbf{y}\rangle^* + ||\mathbf{y}||^2) - (||\mathbf{x}||^2 - \langle \mathbf{x},\mathbf{y}\rangle - \langle \mathbf{x},\mathbf{y}\rangle^* + ||\mathbf{y}||^2)\\
&\quad + i(||\mathbf{x}||^2 + i\langle \mathbf{x},\mathbf{y}\rangle - i\langle \mathbf{x},\mathbf{y}\rangle^* + ||\mathbf{y}||^2) - i(||\mathbf{x}||^2 - i\langle \mathbf{x},\mathbf{y}\rangle + i\langle \mathbf{x},\mathbf{y}\rangle^* + ||\mathbf{y}||^2)\\
&\quad = 2(\langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{x},\mathbf{y}\rangle^*) + 2(\langle \mathbf{x},\mathbf{y}\rangle - \langle \mathbf{x},\mathbf{y}\rangle^*)\\
&\quad = 4\langle \mathbf{x},\mathbf{y}\rangle
\end{aligned}$$

Similarly, for a real inner product space,

$$\begin{aligned}
&||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2\\
&\quad = (||\mathbf{x}||^2 + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{x},\mathbf{y}\rangle + ||\mathbf{y}||^2) - (||\mathbf{x}||^2 - \langle \mathbf{x},\mathbf{y}\rangle - \langle \mathbf{x},\mathbf{y}\rangle + ||\mathbf{y}||^2)\\
&\quad = 4\langle \mathbf{x},\mathbf{y}\rangle
\end{aligned}$$

(ii) We have

$$\begin{aligned}
4\langle \phi\mathbf{u},\phi\mathbf{v}\rangle_B &= ||\phi\mathbf{u}+\phi\mathbf{v}||_B^2 - ||\phi\mathbf{u}-\phi\mathbf{v}||_B^2 + i||\phi\mathbf{u}+i\phi\mathbf{v}||_B^2 - i||\phi\mathbf{u}-i\phi\mathbf{v}||_B^2\\
&= ||\phi(\mathbf{u}+\mathbf{v})||_B^2 - ||\phi(\mathbf{u}-\mathbf{v})||_B^2 + i||\phi(\mathbf{u}+i\mathbf{v})||_B^2 - i||\phi(\mathbf{u}-i\mathbf{v})||_B^2\\
&= ||\mathbf{u}+\mathbf{v}||_A^2 - ||\mathbf{u}-\mathbf{v}||_A^2 + i||\mathbf{u}+i\mathbf{v}||_A^2 - i||\mathbf{u}-i\mathbf{v}||_A^2\\
&= 4\langle \mathbf{u},\mathbf{v}\rangle_A
\end{aligned}$$

(iii) If $f \in C\mathbb{T})$, then we know, by Exercise 2.4.4, that $\theta(f) \in l^2$ and $||\theta(f)||_2 = ||f||_2$. We also have

$$\theta(\lambda f + \mu g)_n = (\lambda f + \mu g)^\wedge(n) = \lambda\hat{f}(n) + \mu\hat{g}(n) = \lambda\theta(f)_n + \mu\theta(g)_n$$

so θ is a linear isometry. By (ii), θ preserves inner products so

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)^* = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(t)^*\,dt$$

for all $f,\ g \in C(\mathbb{T})$.

Exercise 2.4.10

(i) If we set

$$\langle F, G\rangle_X = \|F+G\|_X^2 - \|F-G\|_X^2 + i\|F+iG\|_X^2 - i\|F-iG\|_X^2$$

for all $F,\ G \in X$ then

$$\langle f, g\rangle_X = \langle f, g\rangle$$

for all $f, g \in C(\mathbb{T})$. Further if $f_n,\ g_n \in C(\mathbb{T})$, $F,\ G \in X$ and we have $\|f_n - F\|, \|g_n - G\|_X \to 0$ then $\langle f, g\rangle \to \langle F, G\rangle_X$.

If $F, G \in X$ we can find $f_n,\ g_n \in C(\mathbb{T})$ such that $\|f_n - F\|, \|g_n - G\|_X \to 0$. Thus $\langle g_n, f_n\rangle \to \langle G, F\rangle_X$ and

$$\langle g_n, f_n\rangle = \langle f_n, g_n\rangle^* \to \langle F, G\rangle_X^*$$

whence $\langle G, F\rangle_X = \langle F, G\rangle_X^*$. A similar argument shows that

$$\langle(\lambda F + \mu H), G\rangle_X = \lambda\langle F, G\rangle_X + \mu\langle F, H\rangle_X$$

for all $\lambda,\ \mu \in \mathbb{C}$, $F,\ G,\ H \in X$.

We now observe that

$$\begin{aligned} 4\langle F, F\rangle_X &= \|F+F\|_X^2 - \|F-F\|_X^2 + i\|F+iF\|_X^2 - i\|F-iF\|_X^2 \\ &= 4\|F\|_X^2 - 0 + (i-i)2\|F\|_X^2 = 4\|F\|_X^2 \end{aligned}$$

so $\langle F, F\rangle_X \geq 0$ and

$$\langle F, F\rangle_X = 0 \Rightarrow \|F\|_X = 0 \Rightarrow F = 0.$$

Thus $\langle\ ,\ \rangle_X$ is an inner product extending our original inner product and giving rise to the norm $\|\ \|_X$.

(ii) Observe that

$$\|\theta(f_n) - \theta(f_m)\|_2 = \|f_n - f_m\|_X$$

so, since the f_n form a Cauchy sequence in X, the $\theta(f_n)$ form a Cauchy sequence in l^2. Since l^2 is complete, there exists a $\mathbf{c} \in l^2$ with $\theta(f_n) \to \mathbf{c}$ in l^2 as $n \to \infty$.

If $g_n \in C(\mathbb{T})$ and $g_n \to F$ then

$$\|\theta(g_n) - \theta(f_n)\|_2 = \|g_n - f_n\|_X \leq \|g_n - F\|_X + \|F - f_n\|_X \to 0$$

as $n \to 0$ so $\theta(g_n) \to \mathbf{c}$ in l^2 as $n \to \infty$.

Thus we may define $\tilde{\theta}(F) = \mathbf{c}$.

(iii) Observe that $f \to f$ in X.

(iv) If $F,\ G \in X$ we can find $f_n,\ g_n \in C(\mathbb{T})$, with $f_n \to F$, $g_n \to G$. in X,

We have $\theta(\lambda f_n + \mu g_n) = \lambda\theta f_n + \mu\theta g_n$ so, taking limits,

$$\tilde{\theta}(\lambda F + \mu G) = \lambda\tilde{\theta}F + \mu\tilde{\theta}(G)$$

Thus θ is linear.

Recall that θ preserves norm. Thus

$$\|\tilde{\theta} f_n\|_2 = \|\theta f_n\|_2 = \|f_n\|_X \to \|F\|_X$$

and so $\|\tilde{\theta} F\|_2 = \|F\|_X$. Thus $\tilde{\theta} : X \to l^2$ is linear and preserves norm.

(v) Observe that if $m \geq n$

$$\|P_n - P_m\|_X^2 = \sum_{n+1 \leq |j| \leq m} |a_j|^2 \leq \sum_{|j| \geq n+1} |a_j|^2 \to 0$$

as $n \to \infty$. Thus the P_n form a Cauchy sequence in X and so $P_n \to F$ for some $F \in X$. Writing

$$a_j(n) = \begin{cases} a_j & \text{for } |j| \leq n \\ 0 & \text{otherwise,} \end{cases}$$

we have $\theta P_n = \mathbf{a}(n)$ so $\theta P_n \to \mathbf{a}$ in l^2 and $\theta(F) = \mathbf{a}$.

(vi) Thus θ is a surjection and so a linear isometry.

Exercise 2.4.11

Since $\sum_{j=1}^{\infty} j^{-2}$ converges, $\mathbf{a} \in l^2$. If $t = k2^{-p}\pi$ then, whenever $N,\ M \geq 2^p$,

$$\left|\sum_{-M}^{N} a_r \exp(irt)\right| \geq 2\sum_{j=k}^{n} j^{-1} \to \infty$$

as $n \to \infty$. Thus

$$\sum_{j=-n}^{n} a_j \exp ijt \to \infty$$

for all t of the stated form.

Exercise 2.5.3

Observe that

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{T}} |t| \left(\frac{\sin(nt/2)}{t/2}\right)^4 dt &= 2n^4 \frac{1}{2\pi}\int_0^{\pi} t\left(\frac{\sin(nt/2)}{nt/2}\right)^4 dt \\
&= 2n^2 \frac{1}{2\pi}\int_0^{n\pi/2} u\left(\frac{\sin u}{u}\right)^4 du \\
&\leq 2n^2\left(\int_0^{\pi/2} u\left(\frac{\sin u}{u}\right)^4 du + \int_{\pi/2}^{\infty} u^{-3}\, du\right) \\
&\leq Cn^2
\end{aligned}$$

for some constant C and so, since $\lambda_n \geq A'n^3$, we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} |t| J_n(t)\, dt \leq Bn^{-1}$$

for some constant B.

EXERCISE 2.5.5

Choose m to be the the largest integer with $m \leq n/2 + 1$ and observe that $m \geq n/2$.

EXERCISE 2.5.7

(i) Observe that, if g exists then, by the fundamental theorem of the calculus,

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} g'(t)\,dt = \frac{g(2\pi) - g(0)}{2\pi} = 0.$$

(ii) Let

$$g(t) = \frac{1}{2\pi}\int_0^t f(s)\,ds - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds$$

for $|t| \leq \pi$,

(iii) Immediate.

(iv) Observe that, if f is once continuously differentiable, then

$$\widehat{f'}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)dt = \frac{1}{2\pi}\left[f(t)\right]_{-\pi}^{\pi} = 0$$

and use induction

Exercise 2.5.8

(i) Observe that, if we write
$$Q(t) = J_n * f(t) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)J_n(s)\,ds$$
as at the beginning of Theorem 2.5.4, then $\hat{Q}(0) = \hat{j}_n(0)\hat{f}(0) = 0.$ and the rest of the argument is unchanged.

(ii) Observe that, if we set
$$\tilde{Q}(t) = \sum_{1\leq|r|\leq n} \frac{\hat{Q}(r)}{ir}\exp(irt),$$
then $\tilde{Q}$ is a real valued trigonometric polynomial of degree at most n with $\tilde{Q}'(t) = Q(t)$ and $\tilde{Q}(Q) = 0$. Taking $f = g - \tilde{Q}$ in (i), we see that there exists a real trigonometric polynomial P of degree at most n with $\hat{P}(0) = 0$ such that
$$\|P - f\|_\infty \leq Cn^{-1}\|f'\|_\infty$$
that is to say
$$\|P - g + \tilde{Q}\|_\infty \leq Cn^{-1}\|g' - Q|_\infty.$$
Setting $R = P - \tilde{Q}$ gives the required result.

(iii) The result is true for $k = 1$ by part (i). Suppose it is true for $k = m$ and f is an $m + 1$ times continuously differentiable function with with $\hat{f}(0) = 0$. By the inductive hypothesis we can find a a real trigonometric polynomial Q of degree at most m with $\hat{Q}(0) = 0$ such that
$$\|Q - f'\|_\infty \leq Cn^{-m}\|f^{(m+1)}\|_\infty.$$
Since $\hat{f}(0) = \hat{Q}(0) = 0$, part (i) shows that there exists a real trigonometric polynomial P with
$$\|f - P\|_\infty \leq Cn^{-m-1}\|f^{(m+1)}\|_\infty$$
and completes the induction.

(iv) If $g = f - \hat{f}(0)$, then g is k times continuously differentiable with $\hat{g} = 0$ so we can find a real trigonometric polynomial R of degree at most n such that
$$\|g - R\|_\infty \leq C^k n^{-k}\|g^{(k)}\|.$$
Setting $P = R + \hat{f}(0)$ gives
$$\|f - P\|_\infty \leq C^k n^{-k}\|f^{(k)}\|.$$

Exercise 2.6.2

If $a < b$ and $c < d$ let $T : [a, b] \to [c, d]$ be given by

$$T(x) = c + \frac{d-c}{b-a}(x-a).$$

Suppose that Weierstrass's theorem is true for $[a, b]$. If $f \in C([c, d])$ and $\epsilon > 0$, then $f \circ T$ is continuous on $[a, b]$ so we may find a polynomial P with $|P(x) - f(T(x))| < \epsilon$ for all $x \in [a, b]$. Now $Q = P \circ T^{-1}$ is a polynomial and $|Q(t) - f(t)| = |P(T(t)) - f(t)| < \epsilon$ for all $t \in [c, d]$.

Exercise 2.6.3

(i) This is de Moivre's theorem.

$$\cos n\theta + i \sin n\theta = e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n.$$

(ii) We have

$$\begin{aligned}
\cos n\theta &= \Re(\cos n\theta + i\sin n\theta) = \Re(\cos\theta + i\sin\theta)^n \\
&= \Re\left(\sum_{r=0}^{n}\binom{n}{r} i^r (\cos\theta)^{n-r}(\sin\theta)^r\right) \\
&= \sum_{n\geq n-2u\geq 0}\binom{n}{2u}(-1)^u(\cos\theta)^{n-2u}(\sin\theta)^{2u} \\
&= \sum_{n\geq n-2u\geq 0}\binom{n}{2u}(-1)^u(\cos\theta)^{n-2u}(1-(\cos\theta)^2)^u = T_n(\cos\theta)
\end{aligned}$$

for some polynomial T_n of degree n.

(The uniqueness of T_n follows from the fact that any two polynomials of degree at most n which agree at $n+1$ points are equal.)

(iii) We have

$$\begin{aligned}
2^n = (1+1)^n + (1-1)^n &= \sum_{r=0}^{n}\binom{n}{r} + \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} \\
&= 2\sum_{n\geq n-2r\geq 0}\binom{n}{2r}
\end{aligned}$$

The coefficient of t^n in $T_n(t)$ is

$$\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} = 2^{n-1}$$

for $n \geq 1$. Note that $T_0 = 1$.

Exercise 2.6.4

(i) F is the composition of continuous functions, so continuous. We have

$$F(-\theta) = f(\cos(-\theta)) = f(\cos\theta) = F(\theta).$$

(ii) We have

$$|f(t) - Q(t)| = |\Re(f(t) - P(t))| \leq |f(t) - Q(t)|$$

so $\|P - f\|_\infty \leq \|Q - f\|_\infty$. If $P(t) = \sum_{j=-n}^{n} b_j \exp(ijt)$ then

$$Q(t) = \Re\left(\sum_{j=0}^{N} 2^{-1}(b_j + b_{-j}) \cos jt\right) = \sum_{j=0}^{N} a_j \cos jt$$

with $a_j = 2^{-1}\Re(b_j + b_{-j})$ real.

(iii) If N is sufficiently large we have $\|\sigma_N(F) - F\|_\infty < \epsilon$. so by applying part (ii) (or thinking about the properties of $\sigma_n F$) we can find real a_j such that

$$\left|f(\cos\theta) - \sum_{j=0}^{n} a_j T_j(\cos\theta)\right| < \epsilon$$

for all $\theta \in [0, \pi]$ whence

$$\left|f(t) - \sum_{j=0}^{n} a_j T_j(t)\right| < \epsilon$$

for all $t \in [-1, 1]$ and the result follows on taking $P(t) = \sum_{j=0}^{n} a_j T_j(t)$.

EXERCISE 2.6.5

Observe that, if $\theta \neq n\pi$, the 'function of a function rule' for differentiation shows us that F is differentiable with

$$\bigstar \qquad F'(\theta) = \sin\theta f'(\cos\theta).$$

If $\theta = \pi$ the left derivative

$$\lim_{h\to 0-} \frac{F(\pi + h) - F(\pi)}{h} = (\sin\pi) \times f'(1) = 0$$

and the right derivative

$$\lim_{h\to 0+} \frac{F(\pi + h) - F(\pi)}{h} = \lim_{h\to 0} \frac{F(\pi - h) - F(\pi)}{h} = (\sin\pi) \times (-f'(1)) = 0$$

so F is differentiable at π with derivative 0. A similar calculation shows that F is differentiable at 0 with derivative 0.

The formula $\bigstar$ shows that $|F'(t)| \leq |f(\cos t)| \leq \|f\|_\infty$ and so $\|F'\|_\infty \leq \|f'\|_\infty$.

Theorem 2.5.1 now tells us that there exists a constant K independent of n such that we can find a trigonometric polynomial Q of degree at most $n \geq 1$ with

$$\|Q(\theta) - F(\theta)\| \leq \frac{K}{n}\|F'\|_\infty \leq \frac{K}{n}\|f'\|_\infty$$

for all $\theta \in \mathbb{T}$. Part (ii) of Exercise 2.6.4 tells us that we can find real a_j such that such that

$$|f(\cos\theta) - \sum_{j=0}^{n} a_j T_j(\cos\theta)| < \frac{K}{n}\|f'\|_\infty$$

for all $\theta \in \mathbb{T}$ whence

$$|f(t) - \sum_{j=0}^{n} a_j T_j(t)| < \frac{K}{n}\|f'\|_\infty$$

for all $t \in [-1, 1]$ and the result follows on taking $P(t) = \sum_{j=0}^{n} a_j T_j(t)$.

Suppose that we do not know that $f(0) = 0$. Applying our result to $f - f(0)$ we see that we can find real polynomial Q of degree at most n with

$$\|(f - f(0)) - Q\|_\infty \leq Kn^{-1}\|(f - f(0))'\|_\infty.$$

Taking $P = q + f(0)$ we obtain

$$\|f - P\|_\infty \leq Kn^{-1}\|f'\|_\infty$$

as required.

EXERCISE 2.6.6

(Details may vary according to the definition of infinite integral used, but all respectable definitions will work.)

(i) The change of variable $u = K(t - s)$ gives

$$\int_{-\infty}^{\infty} E_K(t-s)\,dt = \int_{-\infty}^{\infty} K \times E(K(t-s))\,dt = \int_{-\infty}^{\infty} E(u)\,du = 1.$$

(ii) Let $|f(t)| \leq A$ for all t. Since $t \mapsto E_K(t-s)f(s)$ is continuous and $|E_K(t-s)f(s)| \leq AE_K(t-s)$, the required integral exists.

Given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(u) - f(v)| \leq \epsilon$ for all $|u - v| \leq \delta$. Thus

$$\begin{aligned}
|E_K * f(t) - f(t)| &= \left|\int_{-\infty}^{\infty} E_K(t-s)f(s)\,ds - \int_{-\infty}^{\infty} E_K(t-s)\,ds \times f(t)\right| \\
&= \left|\int_{-\infty}^{\infty} E_K(t-s)(f(s) - f(t)\,ds\right| \\
&\leq \int_{-\infty}^{\infty} E_K(t-s)|f(t) - f(s)|\,ds \\
&\leq \int_{|t-s|\leq\delta} E_K(t-s)|f(t) - f(s)|\,ds + \int_{|t-s|>\delta} E_K(t-s)|f(t) - f(s)|\,ds \\
&\leq \epsilon \int_{|t-s|\leq\delta} E_K(t-s)\,ds + 2A \int_{|t-s|>\delta} E_K(t-s)\,ds \\
&= \epsilon \int_{|t-s|\leq\delta} E_K(t-s)\,ds + 2A \int_{u\geq\delta} E_K(u)\,du \\
&\leq \epsilon + 2AK \int_{u\geq\delta} E(Ku)\,du \\
&= \epsilon + 2A \int_{|s|\geq K\delta} E(s) \to \epsilon
\end{aligned}$$

uniformly as $K \to \infty$. Since ϵ was arbitrary,

$$E_K * f(t) \to f(t)$$

uniformly as $K \to \infty$.

Exercise 2.6.7

(i) Observe that, if $|s| \geq 1$, then $u(t-s)f(s) = 0$ for all t.

(ii) Observe that, if $|x| \leq 2$ and $n \geq 8(|K|+1)^2$,

$$\begin{aligned}\left|E_K(x) - \sum_{j=0}^{n} \frac{(-1)^j K^{2j}}{j!} x^{2j}\right| &= \left|\sum_{j=n+1}^{n} \frac{(-1)^j K^{2j}}{j!} x^{2j}\right| \leq \sum_{j=n+1}^{\infty} \frac{|K|^{2j}}{j!} |x|^{2j} \\ &\leq \sum_{j=n+1}^{\infty} \frac{|2K|^{2j}}{j!} \leq \frac{|2K|^{n+1}}{(n+1)!} \sum_{j=0}^{\infty} 2^{-j} \\ &\leq 2\frac{|2K|^{n+1}}{(n+1)!} \to 0\end{aligned}$$

as $n \to \infty$.

Thus, provided only that n is sufficiently large, we may take

$$P(x) = \sum_{j=0}^{n} \frac{(-1)^j K^{2j}}{j!} x^{2j}$$

We now have

$$\begin{aligned}\|P * g(t) - E_K * g(t)| &= \left|\int_{-\infty}^{\infty} (P(t-s) - E_K(t-s)) g(s)\, ds\right| \\ &= \left|\int_{-1}^{1} (P(t-s) - E_K(t-s)) g(s)\, ds\right| \\ &\leq \int_{-1}^{1} |P(t-s) - E_K(t-s)||g(s)|\, ds \\ &\leq 2\epsilon \sup_{x \in \mathbb{R}} |g(x)|\end{aligned}$$

for all $t \in [-1, 1]$.

(iii) If $Q_m(t) = t^m$ then

$$Q_m * g(t) = \int_{-1}^{-1} (t-s)^m g(s)\, ds = \sum_{j=0}^{m} \binom{m}{j} (-s)^j \int_{-1}^{-1} t^{m-j} g(s)\, ds$$

is a polynomial of degree at most m in t. By linearity

$$\left(\sum_{m=0}^{n} a_m Q_m\right) * g = \sum_{m=0}^{n} a_m Q_m * g$$

is a polynomial in t.

(iv) If $f \in C([-1, 1])$ and $\epsilon > 0$, define $g \in C(\mathbb{R})$ by $g(t) = 0$ for $|t| \geq 2$, g linear on $[1, 2]$ and on $[-2, -1]$, $g(t) = f(t)$ on $[-1, 1]$. By Exercise 2.6.6, we can find $K > 0$ such that $|g(t) - E_K * g(t)| < \epsilon/2$ for all $t \in [-1, 1]$. By part (ii) of this question we can find a polynomial Q such that $|Q * g(t) - E_K * g(t)| < \epsilon/2$ for all $t \in [-1, 1]$ and so

$$|Q * g(t) - f(t)| = |Q * g(t) - g(t)| < \epsilon$$

for all $t \in [-1, 1]$. We know that $P = Q*g$ is a polynomial, so Theorem 2.6.1 follows.

Exercise 2.6.8

(i) Since f is continuous on a closed bounded interval, it is uniformly continuous. Thus, given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$|f(s) - f(t)| < \epsilon \text{ for all } s, t \in [0, 1] \text{ with } |s - t| < \delta.$$

Let $n > \delta^{-1}$. Then, if $t \in [0, 1]$, we know that $t \in [r/n, (r + 1)/n]$ for some integer r with $0 \leq r \leq n - 1$. We thus have $t = \lambda(r/n) + (1 - \lambda)(r + 1)/n$ for some λ with $0 \leq \lambda \leq 1$. Thus

$$\begin{aligned}|f_n(t) - f(t)| &= |\lambda f(r/n) + (1 - \lambda)f((r + 1)/n) - \lambda f(t) - (1 - \lambda)f(t)| \\ &\leq \lambda|f(r/n) - f(t)| + (1 - \lambda)|f((r + 1)/n) - f(t)| \\ &< \lambda\epsilon + (1 - \lambda)\epsilon = \epsilon.\end{aligned}$$

Thus $f_n \to f$ uniformly.

(ii) The two functions g and $\sum_{r=0}^{n} g(r/n)\Delta_{r,n}$ are linear on each interval $[r/n, (r + 1)/n]$ and take the same value at each point r/n. They are thus equal.

(iii) Direct computation, considering each of the cases $t \leq (r - 1)/n$, $(r - 1)/n < t \leq r/n$, $r/n < t \leq (r + 1)/n$ and $(r + 1)/n < t$ in turn.

(vi) Suppose that we can find appropriate Q_m

Let $f \in C([-1, 1])$ and $\epsilon > 0$. By (i) and (ii), we can find an n and real a_r such that

$$\left|f(t) - \sum_{r=0}^{n} a_r\Delta_{r,n}(t)\right| < \epsilon/2$$

for all $t \in [-1,1]$. By (iv),

$$\left|\sum_{r=0}^{n} a_r\Delta_{r,n}(t) - \frac{n}{2}\sum_{r=0}^{n} a_r(Q_m(t-(r+1)/n) + Q_m(t-(r-1)/n) - 2Q_m(t-r/n))\right|$$

$$\leq \frac{n}{2}\sum_{r=0}^{n} |a_r|\Big(\big|Q_m(t-(r+1)/n) - |t-(r+1)/n|$$

$$+ \big|Q_m(t-(r-1)/n) - |(t-(r-1)/n|\big| + 2\big|Q_m(t-r/n)) - 2|r-t/n|\big|\Big)$$

$$\leq 2n\sum_{r=0}^{n} |a_r| \sup_{s\in[-2,2]} \big|Q_m(s) - |s|\big| < \epsilon/2$$

for all $t \in [-1,1]$, provided only m is sufficiently large.

For such an m,

$$P(t) = \frac{n}{2}\sum_{r=0}^{n} a_r(Q_m(t-(r+1)/n) + Q_m(t-(r-1)/n) - 2Q_m(t-r/n))$$

defines a polynomial P with $|P(t) - f(t)| < \epsilon$ for all $t \in [-1]$.

(v) Suppose that such a P_n exists. Then $Q_m(t) = 2P_m(t/2)$ defines a sequence with the properties required for (iv).

Exercise 2.6.10

(i) Observe that
$$|t|^2 - (1 - (1 - \delta)(1 - t^2)) = -\delta(1 - t^2) \to 0$$
uniformly for $t \in [-1, 1]$ as $\delta \to 0+$. Thus, since $x \to \sqrt{x}$ is continuous and so uniformly continuous on $[0, 1]$,
$$(1 - (1 - \delta)(1 - t^2))^{1/2}\Big| \to |t|$$
uniformly for $t \in [-1, 1]$ as $\delta \to 0+$, so we are done.

(ii) By part (i), there is an $1 > \delta > 0$ such that
$$\Big||t| - (1 - (1 - \delta)(1 - t^2))^{1/2}\Big| < \epsilon/2$$
for all $|t| \leq 1$. By the paragraph preceding the exercise, we can find an n and $a_j \in \mathbb{R}$ such that
$$\left|\sum_{r=0}^{n} a_r x^r - (1 - x)^{1/2}\right| < \epsilon/2$$
for $|x| \leq 1 - \delta$. Thus
$$\left|\sum_{r=0}^{n} a_r(1 - \delta)(1 - t^2)^r - (1 - (1 - \delta)(1 - t^2)^{1/2})\right| < \epsilon/2$$
for $|t| \leq 1$.

Writing $P(t) = \sum_{r=0}^{n} a_r(1 - \delta)(1 - t^2)^r$, we see that P is a polynomial with
$$\Big||t| - P(t)\Big| \leq \Big||t| - (1 - (1 - \delta)(1 - t^2))^{1/2}\Big| + \Big|P(t) - (1 - (1 - \delta)(1 - t^2))^{1/2}\Big| < \epsilon$$
for $t \in [-1, 1]$. We thus have P of the form promised by Lemma 2.6.9.

Exercise 2.6.11

Observe that

$$\int_a^b \sum_{j=0}^n a_j t^j f(t)\,dt = \sum_{j=0}^n a_j \int_a^b t^j f(t)\,dt = 0.$$

Thus $\int_a^b P(t)f(t)\,dt = 0$ for any polynomial P.

Using Weierstrass's theorem we can find polynomials P_n with $P_n \to f$ uniformly on $[a, b]$. Thus

$$0 = \int_a^b P_n(t)f(t)\,dt \to \int_a^b f(t)^2\,dt$$

and $\int_a^b f(t)^2\,dt = 0$. Since $t \mapsto f(t)^2$ is continuous and positive this implies that $f(t)^2 = 0$ for all $t \in [a, b]$ and so $f = 0$.

Exercise 2.6.12

$\|p\|_I = \sup_{t\in I} |p(t)|$ is well defined since a continuous function on a closed bounded interval is bounded. Three of the four rules for norms follow the standard pattern.

(a) $|p(t)| \geq 0$ for all $t \in I$, so $\|p\|_I \geq 0$.

(b) $|\lambda p(t)| = |\lambda||p(t)|$ for all $t \in I$, so $\|\lambda p\|_I = |\lambda|\|p\|_I$.

(c) $|p(t) + q(t)| \leq |p(t)| + |q(t)|$ for all $t \in I$ so $\|p + q\|_I \leq \|p\|_I + \|q\|_I$ for all $p,\ q \in \mathcal{P}$.

The remaining rule depends on the observation that if $\|p\|_I = 0$ then p vanishes on the infinite set I. Since a non-zero polynomial has only finitely many zeros, it then follows that $p = 0$.

Let

$$f(t) = \begin{cases} 0 & \text{if } t \in I \\ 2(t - 1/4) & \text{if } t \notin I \cup J \\ 1 & \text{if } t \in J \end{cases}.$$

By the Weierstrass approximation theorem we can find $p_n \in \mathcal{P}$ with $p_n \to f$ uniformly on $[0, 1]$ and so

$$\|p_n - 0\|_I,\ \|p_n - 1\|_J \to 0$$

as $n \to \infty$.

Exercise 2.7.2

Suppose that $\|T_n(\mathbf{u})\|$ is bounded for each $\mathbf{u} \in U$. Then the hypotheses of Theorem 2.7.1 are satisfied for

$$\mathcal{T} = \{T_n : n \geq 1\}$$

and so we can find a K such that $\|T_n\| \leq K$ for all n contrary to our initial hypothesis.

Exercise 2.7.3

If $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{F}$, then

$$\begin{aligned} T(T^{-1}(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2)) &= \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 \\ &= \lambda_1 T(T^{-1}\mathbf{v}_1) + \lambda_2 T(T^{-1}\mathbf{v}_2) \\ &= T(\lambda_1 T^{-1}\mathbf{v}_1 + \lambda_2 T^{-1}\mathbf{v}_2), \end{aligned}$$

so, since T is injective,

$$T^{-1}(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) = \lambda_1 T^{-1}\mathbf{v}_1 + \lambda_2 T^{-1}\mathbf{v}_2.$$

Exercise 2.7.4

If $\mathbf{a}, \mathbf{b} \in l^\infty$ and $\lambda, \mu \in \mathbb{F}$, then there exist N and M such that $a_n = 0$ for $n \geq N$ and $b_n = 0$ for $n \geq M$. Thus $\lambda a_n + \mu b_n = 0$ for $n \geq \max\{M, N\}$ and $\lambda\mathbf{a} + \mu\mathbf{b} \in c_{00}$. Thus c_{00} is indeed a subspace of l^∞.

$$T(\lambda\mathbf{a} + \mu\mathbf{b}) = (\lambda a_1 + \mu b_1, \ldots, n^{-1}(\lambda a_n + \mu b_n), \ldots) = \lambda T\mathbf{a} + \mu T\mathbf{b}$$

so $T : c_{00} \to c_{00}$ is linear. Similarly

$$S(a_1, a_2, \ldots a_n, \ldots) = (a_1, 2a_2, \ldots na_n, \ldots)$$

defines a linear map $c_{00} \to c_{00}$. Since $ST = TS = I$, T is invertible with inverse S.

We note that

$$\|T\mathbf{a}\|_\infty = \sup_{n\geq 1} n^{-1}|a_n| \leq \sup_{n\geq 1} |a_n| = \|\mathbf{a}\|_\infty$$

so T is continuous. However if we write $\mathbf{e}_n$ for the element of c_{00} whose nth coordinate is 1 and all others 0, we have $\|n^{-1/2}\mathbf{e}_n\|_\infty = n^{-1/2} \to 0$ but $\|S(n^{-1/2}\mathbf{e}_n)\|_\infty = n^{1/2} \to \infty$ so $T^{-1} = S$ is not continuous.

Exercise 2.7.8

Theorem Suppose that $(U, \| \|_U$ is a complete normed space and $(V, \| \|_V)$ is normed space. Suppose that we can find $C > 0$ and λ with $1 > \lambda \geq 0$ such that, whenever $\mathbf{v} \in V$, there exists a $\mathbf{u} \in U$ with $\|\mathbf{u}\|_U \leq C\|\mathbf{v}\|_V$ and $\|T\mathbf{u} - \mathbf{v}\|_V \leq \lambda\|\mathbf{v}\|$. Then T is surjective.

The proof just copies the last paragraph of the proof of Theorem 2.7.6.

If $\mathbf{v} \in V$, we set $\mathbf{x}_0 = \mathbf{0}$ and find inductively $\mathbf{x}_n$, $\mathbf{u}_n \in U$ such that

$$\|\mathbf{x}_{n+1}\|_U \leq C \left\| \mathbf{v} - T\left(\sum_{j=0}^{n} \mathbf{x}_j\right) \right\|_V$$

$$\left\| \mathbf{v} - T\left(\sum_{j=0}^{n+1} \mathbf{x}_j\right) \right\|_V \leq \lambda \left\| \mathbf{v} - T\left(\sum_{j=0}^{n} \mathbf{x}_j\right) \right\|_V .$$

By induction

$$\|\mathbf{x}_n\|_U \leq \lambda^n \|\mathbf{v}\|$$

and so

$$\left\| \mathbf{v} - T\left(\sum_{j=0}^{n} \mathbf{x}_j\right) \right\|_V \leq C\lambda^n \|\mathbf{v}\|_V.$$

Since $\| \|_U$ is complete, $\sum_{j=0}^{n} \mathbf{x}_j$ converges to some $\mathbf{u} \in U$ with

$$\|\mathbf{u}\| \leq \sum_{j=0}^{n} \|\mathbf{x}_j\|_U \leq 2(1-\lambda)^{-1}\|\mathbf{v}\|_V.$$

Thus

$$\begin{aligned}
\|\mathbf{v} - T\mathbf{u}\|_V &\leq \left\| \mathbf{v} - T\left(\sum_{j=0}^{n} \mathbf{x}_j\right) \right\|_V + \left\| T\left(\sum_{j=0}^{n} \mathbf{x}_j\right) - T\mathbf{u} \right\|_V \\
&\leq \left\| \mathbf{v} - T\left(\sum_{j=0}^{n} \mathbf{x}_j\right) \right\|_V + \|T\| \left\| \left(\sum_{j=0}^{n} \mathbf{x}_j\right) - \mathbf{u} \right\|_V \to 0
\end{aligned}$$

so $T\mathbf{u} = \mathbf{v}$ and we are done.

Exercise 2.7.9★

Exercise 2.7.11

(i) If $(x, y) \in \Gamma(f)$, then $x \in X$ and $y = f(x)$, so $\tilde{f}(x) = (x, y)$. Thus $\tilde{f} : X \to \Gamma(f)$ is surjective. Again

$$\tilde{f}(x) = \tilde{f}(x') \Rightarrow (x, f(x)) = (x', f(x')) \Rightarrow x = x'$$

so $\tilde{f}$ is injective.

(ii) Suppose that $(x_n, f(x_n)) \underset{d}{\to} (x, y)$. Then $d_X(x_n, x) \to 0$, so, by the continuity of f, $d_Y(f(x_n), f(x)) \to 0$. Since $d_Y(f(x_n), f(x)) \to 0$, the uniqueness of limits gives $y = f(x)$, so $(x, y) = (x, f(x)) \in \Gamma(f)$. Thus $\Gamma(f)$ is closed.

(iii) The argument of (ii) shows that

$$E_+ = \{(x, 1/x) : x > 0\} \text{ and } E_- = \{(x, 1/x) : x < 0\}$$

are closed. Since $\{(0, 0)\}$ is closed, $\Gamma f = E_+ \cup E_- \cup \{0\}$ is closed. However $1/n \to 0$, but $f(1/n) = n \nrightarrow 0 = f(0)$ so f is not continuous.

Exercise 2.7.12

Let $\mathbf{u}$, $\mathbf{u}' \in U$, $\mathbf{v}$, $\mathbf{v}' \in U$. and $\lambda \in \mathbb{F}$.

(i) $\|(\mathbf{u}, \mathbf{v})\|_{U\times V} = \|\mathbf{u}\|_U + \|\mathbf{v}\|_V \geq 0$.

(ii) $\|(\mathbf{u}, \mathbf{v})\|_{U\times V} = 0 \Rightarrow \|\mathbf{u}\|_U = \|\mathbf{v}\|_V = 0 \Rightarrow (\mathbf{u}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$.

(iii)

$$\begin{aligned}\|\lambda(\mathbf{u}, \mathbf{v})\|_{U\times V} &= \|(\lambda\mathbf{u}, \lambda\mathbf{v})\|_{U\times V} = \|\lambda\mathbf{u}\|_U + \|\lambda\mathbf{v}\|_V \\ &= |\lambda|\|\mathbf{u}\|_U + |\lambda|\|\mathbf{v}\|_V = |\lambda|\|(\mathbf{u}, \mathbf{v})\|_{U\times V}.\end{aligned}$$

(iv) We have

$$\begin{aligned}\|(\mathbf{u}, \mathbf{v}) + (\mathbf{u}', \mathbf{v}')\|_{U\times V} &= \|(\mathbf{u} + \mathbf{u}', \mathbf{v} + \mathbf{v}')\|_{U\times V} = \|\mathbf{u} + \mathbf{u}'\|_U + \|\mathbf{v} + \mathbf{v}'\|_V \\ &\leq \|\mathbf{u}\|_U + \|\mathbf{u}'\|_U + \|\mathbf{v}\|_V + \|\mathbf{v}'\|_V \\ &= \|(\mathbf{u}, \mathbf{v})\|_{U\times V} + \|(\mathbf{u}', \mathbf{v}')\|_{U\times V}\end{aligned}$$

If the original norms are complete the completeness of the product norm follows from the completeness of the associated product metric. Alternatively observe that if the sequence $(\mathbf{u}_n, \mathbf{v}_n)$ is Cauchy for $\| \ \|_{U\times V}$ then the sequences $(\mathbf{u}_n)$ is Cauchy for $\| \ \|_U$ and so converges to some $\mathbf{u} \in U$. Similarly $(\mathbf{v}_n)$ converges to some $\mathbf{v} \in V$ for the norm $\| \ \|_V$. Thus $(\mathbf{u}_n, \mathbf{v}_n)$ converges to $(\mathbf{u}, \mathbf{v})$ in the product norm.

Exercise 2.7.14

If $(u_n, Tu_n) \to (u, v)$ in the product norm, then $u_n' \to v$ and $u_n \to u$ uniformly on $[0,1]$ so, by a standard result, u is differentiable with derivative $u' = v$. Thus $(u, v) = (u, Tu)$. However, if we take $s_n(t) = \sin 2\pi nt$, $s_n(t) = \cos 2\pi nt$, we have $s_n \in U$, $\|s_n\|_U = 1$, yet

$$\|Ts_n\|_{U\times V} = \|(s_n, nc_n)\|_{U\times V} \geq \|nc_n\|_V = n \to \infty$$

as $n \to \infty$.

This does not contradict the closed graph theorem since $(U, \|\ \|_U)$ is not complete. (By Exercise 1.5.10 (ii), if U was complete then U would be closed. But we know that the polynomials lie in U and form a dense subset of V, yet $V \neq U$.)

Exercise 3.1.5★

Exercise 3.2.3

Observe first that f_n is unambiguously defined on the edges of the closed square $S_n(u, v)$ with vertices

$$(u2^{-n}, v2^{-n}),\ ((u+1)2^{-n}, v2^{-n}),\ ((u+1)2^{-n}, (v+1)2^{-n})\ ((u+1)2^{-n}, v2^{-n})$$

and is continuous on $S_n(u, v)$ for each (u, v), so f_n is defined and continuous everywhere.

A continuous function on $[0,1]^2$ is uniformly continuous. Thus, given $\epsilon > 0$, we can find an N such that $|f(s,t) - f(s',t')| < \epsilon$ whenever $|s - s'|,\ |t - t'| \leq 2^N$ If $n > N$,

$$\begin{aligned}
&\Big|f_n((u+s)2^{-n}, (v+t)2^{-n}) - f((u+s)2^{-n}, (v+t)2^{-n})\Big| \\
&\quad \leq (1-s)(1-t)\Big|f(u2^{-n}, v2^{-n}) - f((u+s)2^{-n}, (v+t)2^{-n})\Big| \\
&\quad + s(1-t)\Big|f((u+1)2^{-n}, v2^{-n}) - f((u+s)2^{-n}, (v+t)2^{-n})| \\
&\quad + (1-s)t\Big|f(u2^{-n}, (v+1)2^{-n}) - f((u+s)2^{-n}, (v+t)2^{-n})| \\
&\quad + st\Big|f((u+1)2^{-n}, (v+1)2^{-n}) - f((u+s)2^{-n}, (v+t)2^{-n})| \\
&\quad \leq (1-s)(1-t)\epsilon + s(1-t)\epsilon + (1-s)t + \epsilon st = \epsilon.
\end{aligned}$$

Thus $f_n \to f$ uniformly.

(ii) We can find $g_n \in \mathcal{A}_n$ such that

$$|g_n(u, v) - f_n(u, v)| \leq 2^{-n}$$

for all $u,\ v \in \mathbb{Z}$ with $0 \le u, v \le n-1$ and so, by a calculation similar to the one just done,

$$\|g_n - f_n\|_\infty \le 2^{-n}.$$

Thus $\bigcup_{n=1}^\infty \mathcal{A}_n$ is uniformly dense in $C([0,1]^2)$. Since

$$\mathcal{A}_{n,m} = \{g \in \mathcal{A}_n \ : \ \|g\|_\infty \le m\} \text{ is finite}\}.$$

it follows that

$$\bigcup_{n=1}^\infty \mathcal{A}_n = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty \mathcal{A}_{n,m}$$

is countable.

Exercise 3.3.1

(i) Routine verification that d is a metric (in fact we have a norm on Y, but a metric on X). For example

$$\begin{aligned} d(\boldsymbol{\phi}, \boldsymbol{\psi}) + d(\boldsymbol{\psi}, \boldsymbol{\tau}) &\ge \|\phi_j - \psi_j\|_\infty + \|\psi_j - \tau_j\|_\infty \\ &\ge \|\phi_j - \tau_j\|_\infty \end{aligned}$$

so, taking a maximum over j,

$$d(\boldsymbol{\phi}, \boldsymbol{\psi}) + d(\boldsymbol{\psi}, \boldsymbol{\tau}) \ge d(\boldsymbol{\phi}, \boldsymbol{\tau})$$

and we have the triangle inequality.

Let us write

$$\boldsymbol{\phi}_n = (\phi_{0,n}, \phi_{1,n}, \dots \phi_{4,n})$$

and so on. If $\boldsymbol{\phi}_n$ is Cauchy for d, then $\phi_{j,n}$ is Cauchy for $\|\ \|_\infty$ so there exist continuous functions $\phi_j : [0,1] \to \mathbb{R}$ with $\phi_{j,n} \to \phi_j$ uniformly $[4 \ge j \ge 0]$ and so with $d(\boldsymbol{\phi}_n, \boldsymbol{\phi}) \to 0$. Thus (Y, d) is complete.

(ii) Suppose $\boldsymbol{\phi}_n \in X$ for each n and $d(\boldsymbol{\phi}_n, \boldsymbol{\phi}) \to 0$. Then if $[4 \ge j \ge 0]$ and $0 \le s \le t \le 1$. $\phi_{j,n}(s) \le \phi_{j,n}(t)$ and

$$\phi_{j,n}(s) \to \phi_j(s),\ \phi_{j,n}(t) \to \phi_j(t)$$

so $\phi_j(s) \le \phi_j(t)$. Thus $\boldsymbol{\phi} \in X$.

We have shown that X is a closed subset of Y. Since a closed subset of a complete metric space is complete under the restriction metric (X, d) is complete.

Exercise 3.4.3

Since the rationals are dense, we can find distinct rational $u_{r,j}$ with $\left|u_{r,j} - \phi_j(5r+j)N^{-1}\right| \leq \epsilon/20$. Since ★ tells us that

$$\left|\phi_j(5r+j)N^{-1}) - \phi_j(t)\right| \leq \epsilon/20.$$

when $t \in [0,1]$, $|t - (5r+j)N^{-1}| \leq 10N^{-1}$, $0 \leq j \leq 4$ we are done.

Exercise 3.4.4

(Not that the increasing bits are liable to be steep because N is liable to be very large compared with ϵ^1,)

Observe that if $0 \leq t \leq 1$ and $|t - (5r+j)N^{-1}| \leq 10N^{-1}$ and $t \in [0,1]$, we have

$$|u_{r,j} - \psi_j(t)| \leq \epsilon/5$$

so, if $|t - (5r+j)N^{-1}| \leq 5N^{-1}$, we have

$$|\phi((5r+j)N^{-1}) - \psi_j(t)| = |u_{r,j} - \psi_j(t)| \leq \epsilon/5.$$

But $|\psi_j(x) - \psi_j(x')| \leq \epsilon/20$ whenever $|x - x'| \leq 10N^{-1}$ so, taking $(5r+j)N^{-1} = x'$ and $x = t$ we have

$$|\phi(t) - \psi_j(t)| < \epsilon/2$$

for all $|t - (5r+j)N^{-1}| \leq 5N^{-1}$. Thus $|\phi(t) - \psi_j(t)| < \epsilon/2$ for all $t \in [0,1]$ and we are done.

Exercise 3.4.5

If $v \neq v'$ then, since $v - v' \in \mathbb{Q}$ and μ is irrational, we have $\mu(v - v')$ irrational, but

$$\mu(v - v') = u' - u \in \mathbb{Q}.$$

The result follows by reductio ad absurdum.

3. Exercise 4.1.3

(i) We have $Ah + B \geq 0$ for $h > 0$, so, allowing $h \to 0+$, we have $B \geq 0$. We have $Ah + B \leq 0$ for $h < 0$ so, allowing $h \to 0-$, we have $B \leq 0$. Thus $B = 0$.

(ii) We have $Ah^2 + Bh + C \geq 0$ for $h > 0$ so, allowing $h \to 0+$, we have $C \geq 0$. We have $Ah^2 + Bh + C \leq 0$ for $h < 0$ so, allowing $h \to 0-$, we have $C \leq 0$. Thus $C = 0$ and $Ah^3 + Bh^2 \geq 0$ for all h.

We now have $Ah + B \geq 0$ for all h, so, allowing $h \to 0$, we have $B \geq 0$.

We now have $A + Bh^{-1} \geq 0$ if $h > 0$. Allowing $h \to \infty$, we get $A \geq 0$. We also have $A + Bh^{-1} \leq 0$ if $h < 0$. Allowing $h \to -\infty$, we get $A \leq 0$. Thus $A = 0$.

(iii) The argument of (ii) still shows that $C = 0$, $B \geq 0$. However, if $\delta = 1/10$, $h^3 + h^2 \geq 0$ and $-h^3 + h^2 \geq 0$ for all $|h| \leq \delta$, so, general we cannot say anything about the sign of A.

Exercise 4.1.4

(i) Taking $g = f$ and using Exercise 1.2.9, we see that $f(t)^2 = 0$ so $f(t) = 0$ for all $t \in [a, b]$.

(ii) Since f is continuous on the closed bounded interval $[a, b]$, f is bounded and we can find an M with $M \geq |f(t)|$ for all $t \in [a, b]$.

If $n \geq 1 + 2(b - a)^{-1}$, let g_n be the function with $g_n(a) = g_n(b) = 0$, $g_n(t) = f(t)$ for $t \in [a+1/n, b-1/n]$ and g_n linear on $[a, a+1/n]$, $[b-1/n, b]$. Then

$$\begin{aligned}\left|\int_a^b f(t)^2\,dt\right| &= \left|\int_a^b (f(t)^2 - g_n(t)f(t)\,dt\right|\\ &\leq \int_a^{a+1/n} |f(t)^2 - g_n(t)f(t)|\,dt| + \int_{b-1/n}^b |f(t)^2 - g_n(t)f(t)|\,dt\\ &\leq \frac{2M^2}{n} + \frac{2M^2}{n} = \frac{4M^2}{n} \to 0\end{aligned}$$

as $n \to \infty$ so $\int_a^b f(t)^2\,dt = 0$ and the argument of (i) applies.

(iii) Suppose that $g : [a, b] \to \mathbb{R}$ is continuous with $g(a) = g(b) = 0$. By Weierstrass's theorem, we can find a polynomial Q_n such that $|Q_n(t) - g(t)| < 1/n$ for $t \in [a, b]$. If we set

$$P_n(t) = (Q_n(t) - Q(a)) + \frac{t - b}{b - a} Q_n(b),$$

then P_n is a polynomial with $P_n(a) = P_n(b) = 0$ and

$$|P_n(t) - Q_n(t)| \leq |Q(a)| + |Q(b)| \leq 2/n$$

for $t \in [a, b]$. Thus

$$0 = \int_a^b f(t)P_n(t)\,dt \to \int_a^b f(t)g(t)\,dt$$

and the required result follows from (ii).

Exercise 4.1.5

Write $\tau = \psi - \phi$.

$$\iint_D |\nabla\psi|^2\,dx\,dy = \iint_D |\nabla(\phi + \tau)|^2\,dx\,dy = \iint_D \nabla(\phi + \tau) \cdot \nabla(\phi + \tau)\,dx\,dy$$
$$= \iint_D |\nabla\phi|^2\,dx\,dy + \iint_D |\nabla\tau|^2\,dx\,dy + 2\iint_D \nabla\phi \cdot \nabla\tau\,dx\,dy.$$

But, using the divergence theorem (assumed applicable),

$$\begin{aligned}\iint_D \nabla\phi \cdot \nabla\tau\,dx\,dy &= \iint_D \nabla\tau \cdot \nabla\phi - (\nabla^2\phi)\tau\,dx\,dy \\ &= \iint_D \nabla(\tau\nabla\phi)\,dx\,dy \\ &= \int_{\partial D} \tau\nabla\phi \cdot \mathbf{n} ds = 0\end{aligned}$$

since $\tau = 0$ on ∂D.

Thus

$$\iint_D |\nabla\psi|^2\,dx\,dy = \iint_D |\nabla\phi|^2\,dx\,dy + \iint_D |\nabla\tau|^2\,dx\,dy \geq \iint_D |\nabla\phi|^2\,dx\,dy.$$

Exercise 4.2.4

Set $\phi = \phi_1 - \phi_2$ and apply Theorem 4.2.3 (ii).

EXERCISE 4.2.6

We have

$$\frac{\partial\phi}{\partial x}(x,y) = \frac{x}{(x^2+y^2)^{1/2}} F'((x^2+y^2)^{1/2})$$

and

$$\frac{\partial^2\phi}{\partial x^2}(x,y) = \left(\frac{1}{(x^2+y^2)^{1/2}} - \frac{x^2}{(x^2+y^2)^{3/2}}\right) F'((x^2+y^2)^{1/2}) + \frac{x^2}{x^2+y^2} F''((x^2+y^2)^{1/2}).$$

The corresponding partial derivatives with respect y are obtained by interchanging x and y. Thus

$$\begin{aligned}
\nabla^2\phi(\mathbf{x}) &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \\
&= \left(\frac{2}{(x^2+y^2)^{1/2}} - \frac{x^2+y^2}{(x^2+y^2)^{3/2}}\right) F'((x^2+y^2)^{1/2}) + \frac{x^2+y^2}{x^2+y^2} F''((x^2+y^2)^{1/2}) \\
&= \left(\frac{2}{r} - \frac{1}{r}\right) F'(r) + F''(r) = \frac{1}{r} F'(r) + F''(r) = \frac{1}{r}\frac{d}{dr}(rF'(r)).
\end{aligned}$$

Also

$$\begin{aligned}
|\nabla\phi(\mathbf{x})|^2 &= \left(\frac{\partial\phi}{\partial x}(x,y)\right)^2 + \left(\frac{\partial\phi}{\partial y}(x,y)\right)^2 \\
&= \frac{x^2+y^2}{x^2+y^2}(F'((x^2+y^2)^{1/2}))^2 = |F'(r)|^2.
\end{aligned}$$

Exercise 4.2.8

If $f \in \mathcal{F}$, then $t \mapsto t^2 f'(t)^2$ is continuous and positive so

$$\int_0^1 t^2 f'(t)^2\,dt \geq 0$$

with equality if and only if $t^2 f'(t)^2 = 0$ for all $t \in [0,1]$. However, if $t^2 f'(t)^2 = 0$ for all $t \in [0,1]$, then $f'(t) = 0$ for all $t \in (0,1]$ so, by the mean value theorem, $f(0) = f(1)$, contrary to the definition of $\mathcal{F}$.

On the other hand, if $n \geq 2$ and we set

$$h_n(t) = \begin{cases} n^2 t & \text{if } 0 \leq t \leq 1/n \\ 2n - n^2 t & \text{if } 1/n < t \leq 2/n \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_n(t) = \int_0^t g_n(s)\,ds,$$

then, using the fundamental theorem of the calculus, $f_n' = g_n$ and $f_n \in \mathcal{F}$. However

$$\int_0^1 t^2 f_n(t)^2\,dt \leq \int_0^{2/n} 4n^{-2}n^2\,dt = 8n^{-1} \to 0$$

as $n \to \infty$ so

$$\inf_{f \in \mathcal{F}} \int_0^1 t^2 f'(t)^2\,dt = 0.$$

EXERCISE 4.2.9

(i) Done at the start of complex variable courses as the Cauchy–Riemann equations.

$$\begin{aligned}\nabla u.\nabla u &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} - i(\frac{\partial v}{\partial y}\right) = |g'(z)|^2\end{aligned}$$

(ii) Let $g(z) = \sum_{j=0}^{\infty} a_n z^n$. Then, since $\sum_{n=0}^{\infty} |a_n| < \infty$, g is well defined and continuous on

$$\bar{D} = \{z : |z| \leq 1\}$$

and analytic on D. Thus, by (i), $\phi(x, y) = \Re g(x + iy)$ solves the stated Dirichlet problem.

Since

$$\begin{aligned}\sum_{n=1}^{N}\sum_{m=1}^{M} |na_n z^n||ma_n z^n| &= \sum_{n=1}^{N} |na_n z^{n-1}| \sum_{m=1}^{M} |na_m z^n| \\ &\leq \left(\sum_{n=1}^{\infty} |na_n R^n|\right)^2 < \infty,\end{aligned}$$

standard theorems justify the following calculation

$$\begin{aligned}\iint_{\bar{D}(R)} \nabla\psi\cdot\nabla\psi\,dx\,dy &= \int\int_{\bar{D}(R)} |g'(x+iy)|^2\,dx\,dy \\ &= \int_0^R \int_0 2\pi 2\pi r \sum_{n=1}^{\infty} na_n r^{n-1}\exp(i(n-1)\theta) \sum_{m=1}^{\infty} ma_m r^{m-1}\exp(-i(m-1)\theta)\,d\theta\,dr \\ &= 2\pi \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \int_0^R nma_n a_m r^{n+m-1}\,dr \int_0^{2\pi} \exp(i(n-m)\theta)\,d\theta \\ &= 2\pi \sum_{n=1}^{\infty} na_n^2 R^{2n}.\end{aligned}$$

Finally observe that, if we choose a_n as suggested ,then if $n = 2^m$

$$na_n^2 R^{2n} = 2^{2m}2^{-2m}R^{2^{2^m+1}} = R^{2^{2^m+1}}$$

so

$$\iint_{\bar{D}(R)} \nabla\psi\cdot\nabla\psi\,dx\,dy = 2\pi \sum_{m=1}^{\infty} R^{2^{2^m+1}} \to \infty$$

as $R \to 1-$.

Exercise 4.3.2

(i) $\rho(x, y) = \min\{\eta, d(x, y)\} \geq 0$

(ii) $\rho(x, y) = 0 \Rightarrow \min\{\eta, d(x, y)\} = 0 \Rightarrow d(x, y) = 0 \Rightarrow x = y$.

(iii) $\rho(x, y) = \min\{\eta, d(x, y)\} = \min\{\eta, d(y, x)\} = \rho(y, x)$.

(iv)If $d(x, y) \geq \eta$ or $d(y, z) \geq \eta$, then

$$\rho(x, y) + \rho(y, z) \geq \eta \geq \rho(x, z).$$

If $d(x, y) < \eta$ and $d(y, z) < \eta$, then

$$\rho(x, y) + \rho(y, z) = d(x, y) + d(y, z) \geq d(x, z) \geq \rho(x, z)$$

Thus the triangle inequality holds.

Since $d(x, y) \geq \rho(x, y)$,

$$x_n \underset{d}{\rightarrow} x \Rightarrow x_n \underset{\rho}{\rightarrow} x$$

and every Cauchy sequence for d is a Cauchy sequence for ρ.

If $x_n \underset{\rho}{\rightarrow} x$, then we can find a N such that $\rho(x_n, x) < \eta$ and so $d(x_n, x) = \rho(x_n, x)$ for $n \geq N$, whence $x_n \underset{d}{\rightarrow} x$.

If (x_n) is Cauchy for ρ, then we can find a N such that $\rho(x_n, x_m) < \eta$ and so $d(x_n, x_m) = \rho(x_n, x_n)$ for n, $m \geq N$, whence (x_n) is Cauchy for d.

Exercise 4.3.4

If $x_1,\ x_2,\ \ldots, x_n \in \mathbb{R}$ then

$$2 + \sum_{j=1}^{n} |x_j| \notin \bigcup_{j=1}^{n} B(x_j, 1)$$

so $\mathbb{R}$ with the usual metric, which we know to be complete, is not bounded.

$(0, 1)$ is not complete with the usual metric, since it is not a closed subset of $\mathbb{R}$ with the usual metric. However, if $\epsilon > 0$, we know, by the axiom of Archimedes, that we can find an integer n with $n \geq 2\epsilon^{-1}$. The balls $B(r/n, \epsilon) = (0, 1) \cap (r/n - \epsilon, r/n + \epsilon)$ with $1 \leq r \leq n - 1$ cover $(0, 1)$.

Exercise 4.3.5

Given $\delta > 0$ we can find $x_1,\ x_2,\ \dots,\ x_n$ with $\bigcup_{j=1}^{n} B(x_j, \delta/2)$. Let

$$\Gamma = \{j \ : \ B(x_j, \delta/2) \cap E \neq \varnothing$$

For each $j \in \Gamma$, choose a $y_j \in B(x_j, \delta/2) \cap E$ and observe that $B(y_j, \delta) \supseteq B(y_j, \delta/2)$. Thus $\bigcup_{j\in\Gamma} B(y_j, \delta) \supseteq E$ and the required result follows.

Exercise 4.3.8

If the finite intersection property holds and $\mathcal{U}$ is a collection of open sets with $\bigcup_{U\in\mathcal{U}} U = X$, then $\mathcal{F} = \{X \setminus U \ : \ U \in \mathcal{U}\}$ is a collection of closed sets with $\bigcap_{F\in\mathcal{F}} F = \varnothing$. By the finite intersection property, $\mathcal{F}$ must contain a finite collection F_1, F_2,..., F_n with $\bigcap_{j=1}^{n} F_j = \varnothing$. We have $U_j = X \setminus F_j \in \mathcal{U}$ and $\bigcup_{j=1}^{n} U_j = X$.

Exercise 4.3.10

(i) Suppose that U is open in (X, d). Then, if $e \in U \cap E$, we have $e \in U$, so there exists a $\delta > 0$ such that, if $y \in X$ and $d(e, y) < \delta$, we have $y \in U$. Thus, if $y \in E$ and $d_E(e, y) < \delta$, we have $y \in E \cap U$.

Suppose now that V is open in (E, d_E). Set

$$U = \bigcup \{B_d(v, \delta) \ : \ v \in E,\ \delta > 0,\ B_d(v, \delta) \subseteq V\}.$$

Since U is the union of open sets in (X, d), U is open in (X, d). We need to show that $V = U \cap E$. Observe first that, if $v \in V$, then, since V is open in (E, d_e), there exists a $\delta > 0$ with $B_d(v, \delta) \subseteq V$ and so $v \in U$. Thus $U \supseteq V$ and so $U \cap E \supseteq V$. On the other hand, if $u \in U \cap E$, then we can find a $v \in E$ and a $\delta > 0$ such that $u \in B(v, \delta) \subseteq V$ whence $V \supseteqq U \cap E$ and we are done.

(ii) Suppose that E is a compact subset of some metric space (X, d). If $\mathcal{U}$ is a collection of open sets in X with $\bigcup_{U\in\mathcal{U}} U \supseteq E$, then

$$\mathcal{V} = \{U \cap E \ : \ U \in \mathcal{U}\}$$

is a collection of open sets in E with $\bigcup_{V\in\mathcal{V}} V = E$. By compactness, we can find $U_i \in \mathcal{U}$ such that, writing $V_i = U_i \cap E$, we have $\bigcup_{i=1}^{n} U_i = E$ and so $\bigcup_{i=1}^{n} U_i \supseteq E$.

Suppose conversely that E has the property stated in the question. Then if

$$\mathcal{V} = \{U \cap E \ : \ U \in \mathcal{U}\}$$

is a collection of open sets in E we can associate with each $V \in \mathcal{V}$ an open set U_V in (X, d) such that $V = U_V \cap E$. We have $\bigcup_{V \in \mathcal{V}} U_V \supseteq E$ so we can find $V(1),\ V(2),\ \ldots,\ V(n) \in \mathcal{V}$ such that $\bigcup_{j=1}^{n} U_{V(j)} \supseteq E$ and so $\bigcup_{j=1}^{n} V(j) = E$.

Exercise 4.3.10

(i) Suppose that E is compact. Consider $x_n \in E$ such that $d(x_n, x) \to 0$ as $n \to \infty$. If E is compact, we can find $n(j) \to \infty$ and $e \in E$ such that $d_E(x_{n(j)}, e) \to 0$ as $j \to \infty$. Automatically, $d(x_{n(j)}, e) \to 0$, so, by the uniqueness of limits, $x = e \in E$. Thus E is closed.

With the usual metric $\mathbb{R}$ is a closed subset of itself which is not compact (since not bounded).

(ii) (Using the the Heine–Borel property.) Let E be a closed subset of compact metric space (X, d). If $\mathcal{U}$ is an open cover of E (that is to say $\bigcup_{U \in \mathcal{U}} U \supseteq E$ then, since E is closed, $\mathcal{U} \cup (X \setminus E)$ is an open cover of X, so has a finite subcover U_1, U_2, ..., U_n, $X \setminus E$ with $U_j \in \mathcal{U}$. We have $\bigcup_{j=1}^{n} U_j \supseteq E$ so we are done.

(Using the Bolzano–Weierstrass property.) Let E be a closed subset of compact metric space (X, d). If we have a sequence $x_n \in E$ then $x_n \in X$, so we may find $n(j) \to \infty$ and $x \in E$ such that $x_{n(j)} \to x$. But E is closed so $x \in E$.

(Using completeness and totally boundedness.) Let E be a closed subset of compact metric space (X, d). We know that (X, d) is complete and totally bounded. Since a subset of a totally bounded set is totally bounded (see Exercise 4.3.5), E is totally bounded. Since a closed subset of a complete space is complete, E is complete.

(Using the finite intersection property.) Suppose that $e_n \in E$, $e_n \to e$. By extracting a subsequence, if necessary, we may suppose that $d(e_n, x) < 1/n$. Let $\tilde{B}_n = \{e \in E \,:\, d(e, x) \leq 1/n\}$. We observe that $\tilde{B}_n$ is closed in (E, d_E), and

$$e_n \in \tilde{B}_n = \bigcap_{j=1}^{n} \tilde{B}_j$$

so the collection of B_n is has non-empty finite intersections so $\bigcap_{j=1}^{\infty} \tilde{B}_j \neq$. Since

$$\bigcap_{j=1}^{\infty} \tilde{B}_j \subseteq \bigcap_{j=1}^{\infty} \bar{B}(e, 1/j) = \{e\}$$

it follows that

$$\{e\} = \bigcap_{j=1}^{n} \tilde{B}_j \subseteq E$$

and we are done.

Exercise 4.3.12

In any metric space a totally bounded set is bounded. If E is a bounded subset of $\mathbb{R}^n$, then $E \subseteq B(0, R)$ for some $R > 0$. But, if $\delta > 0$, we can find an integer $N < 2^{-n}\delta$ and we observe that

$$B((x_1, x_2, \dots x_n), \delta) \supseteq \prod_{j=1}^{n} [x_j - 1/N, x_j + 1/N]$$

so $\bigcup_{\mathbf{u}\in\Gamma} B(\mathbf{u}, \delta) \supseteq B(0, R)$ where Γ is the collection of $\mathbf{u} \in \mathbb{R}^n$ with each coordinate u_j satisfying $Nu_j \in \mathbb{Z}$ and with $|u_j| \leq R + 1$. Thus bounded and totally bounded are equivalent for $\mathbb{R}^n$ with the usual metric.

We already know that a subset of a complete metric space is complete if and only if it is closed.

Theorem 4.3.1 now follows from the equivalence of conditions (i) and (iii) in Theorem 4.3.6.

Exercise 4.3.13

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) &\Leftrightarrow f(x) \in \bigcup_{B\in\mathcal{B}} B \\ &\Leftrightarrow f(x) = b \text{ for some } b \in B \text{ for some } B \in \mathcal{B} \\ &\Leftrightarrow x \in F^{-1}(B) \text{ for some } B \in \mathcal{B} \\ &\Leftrightarrow x \in \bigcup_{B\in\mathcal{B}} f^{-1}(B) \end{aligned}$$

so $f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) = \bigcup_{B\in\mathcal{B}} f^{-1}(B)$.

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{B\in\mathcal{B}} B\right) &\Leftrightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B \\ &\Leftrightarrow f(x) = b \text{ for some } b \in B \text{ for all } B \in \mathcal{B} \\ &\Leftrightarrow x \in F^{-1}(B) \text{ for all } B \in \mathcal{B} \\ &\Leftrightarrow x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B) \end{aligned}$$

so $f^{-1}\left(\bigcap_{B\in\mathcal{B}} B\right) = \bigcap_{B\in\mathcal{B}} f^{-1}(B)$.

$$\begin{aligned} y \in f\left(\bigcup_{A\in\mathcal{A}} A\right) &\Leftrightarrow y = f(a) \text{ for some } a \in A \text{ for some } A \in \mathcal{A} \\ &\Leftrightarrow y = f(a) \text{ for some } a \in \bigcup_{A\in\mathcal{A}} A \\ &\Leftrightarrow y = f\left(\bigcup_{A\in\mathcal{A}} A\right) \end{aligned}$$

so $f\left(\bigcup_{A\in\mathcal{A}}\right) = \bigcup_{A\in\mathcal{A}} f(A)$.

$$\begin{aligned} y \in f\left(\bigcup_{A\in\mathcal{A}} A\right) &\Rightarrow y = f(a) \text{ for some } a \in A \text{ for all } A \in \mathcal{A} \\ &\Rightarrow y \in f(A) \text{ for all } A \in \mathcal{A} \Rightarrow y \in \bigcap_{A\in\mathcal{A}} f(A) \end{aligned}$$

so $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$.

Let $X = \{1, 2\}$, $Y = \{1\}$, $\mathcal{A} = \{\{1\}, \{2\}\}$ with $A_1 = \{1\}$, $A_2 = \{2\}$, $f(1) = f(2) = 1$. Then

$$f(A_1) \cap f(A_2) = \{1\} \neq \varnothing = f(\varnothing) = f(A_1 \cap A_2).$$

Exercise 4.3.16

We have $d(x,y)+d(y,z)\geq d(x,z)$, so $d(y,z)\geq d(x,z)-d(x,y)$. Further
$$d(y,z)+d(x,z)=d(y,z)+d(z,x)\geq d(y,x)=d(x,y)$$
so $d(y,z)\geq d(x,y)-d(x,z)$. Thus $|d(x,y)-d(x,z)|\leq d(y,z)$.

It follows that, if $\epsilon>0$, then $\epsilon>d(e,e')\Rightarrow\epsilon>|d(e,a)-d(e',a)|$, so f is continuous.

If E is compact then, since f is continuous, f has an infimum attained at some point $e_0\in E$.

The point e_0 need not be unique. If $X=\mathbb{R}$ with the usual metric, $E=\{-1,1\}$ and $a=0$, then E is closed and bounded, so compact, but 1 and -1 are both closest points to 0.

We know that $d(x,y)$ is bounded, so we may consider $M=\sup_{x,y\in X}d(x,y)$. Chose a_n, $b_n\in X$ so that $d(a_n,b_n)\geq M-1/n$. By the Bolzano–Weierstrass property we can find $n(j)\to\infty$ with $a_{n(j)}\underset{d}{\to}a$ for some $a\in X$ and then $j(k)\to\infty$ with $b_{n(j(k))}\underset{d}{\to}b$ for some $b\in X$. Since
$$\begin{aligned}M\geq d(a,b)&\geq d(a_{n(j(k))},b_{n(j(k)})-d(a_{n(j(k))},a)-d(b_{n(j(k))})\\&\geq M-1/n(j(k))-d(a_{n(j(k))},a)-d(b_{n(j(k))})\to M\end{aligned}$$
a and b have the required property.

Exercise 4.3.17

Let $\epsilon>0$. Since $f:X\to Y$ is continuous, it follows that, given $x\in X$, we can find a $\delta_x>0$ such that
$$D(x,y)<2\delta_x\Rightarrow\rho(f(x),f(y))<\epsilon/2.$$
The open balls $B(x,\delta_x)$ cover X (since $x\in B(x,\delta_x)$) and so by compactness we can find n and x_1, x_2, … x_n such that $\bigcup_{j=1}^{n}B(x_j,\delta_j)=X$. Set $\delta=\min_{1\leq j\leq n}\delta_{x_j}$ so $\delta>0$.

If $d(x,y)<\delta$, we know that $x\in B(x_J,\delta_J)$ for some J with $\leq J\leq n$. The triangle inequality now tells us that $y\in B(x_J,\delta_J+\delta)\subseteq B(x_J,2\delta_J)$. Thus $\rho(f(x_J,x),\ f(x_J,y))<\epsilon/2$ and another use of the triangle inequality gives $\rho(f(x),f(y))<\epsilon$.

Exercise 4.4.1

We know that the collection of functions V of all functions $f : X \to \mathbb{F}$ is a vector space for the appropriate pointwise operations, so we only need to check that $C_{\mathbb{F}}(X)$ is a subspace of V.

Automatically, $0 \in C_{\mathbb{F}}(X)$. If $f,\ g \in_{\mathbb{F}} (X)$, $x \in X$ and $\epsilon > 0$, we can find a $\delta > 0$ such that

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)|, |g(x) - g(y)| < \epsilon/2,$$

so

$$d(x,y) < \delta \Rightarrow |(f+g)(x) - (f+g)(y)| < \epsilon.$$

Thus $f + g \in C_{\mathbb{F}}(X)$. If $\lambda \in \mathbb{F}$, $f \in C_{\mathbb{F}}(X)$, $x \in X$ and $\epsilon > 0$, we can find a $\delta > 0$ such that

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon/(|\lambda| + 1)$$

so

$$d(x,y) < \delta \Rightarrow |\lambda f(x) - \lambda f(y)| < \epsilon.$$

Thus $\lambda f \in C_{\mathbb{F}}(X)$.

If $f \in C_{\mathbb{F}}(X)$ then $|f|$ is a continuous function (observe that $||f(x)| - |f(y)|| \le |f(x) - f(y)|$) so, by Theorem 4.3.15 attains a maximum $\|f\|_\infty$. Since

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\ &\le \|f\|_\infty |g(x) - g(y)| + \|g\|_\infty |f(x) - f(y)|, \end{aligned}$$

$fg \in C_{\mathbb{F}}(X)$ whenever $f, \in C_{\mathbb{F}}(X)$.

We have $|f(x)| + |g(x)| \ge |f(x) + g(x)|$ for all $x \in X$, so taking a supremum we have $\|f\|_\infty + \|g\|_\infty \ge \|f + g\|_\infty$. The remaining properties of a norm are immediate.

Since

$$|f(x)g(x)| \le |f(x)| \|g\|_\infty \le |f\|_\infty \|g\|_\infty$$

we have $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$.

If $X = [0,1]$ with the usual metric and $f(x) = x$, $g(x) = 1 - x$ then $\|f\|_\infty = \|g\|_\infty = 1$ but $\|fg\|_\infty = 1/4$.

Exercise 4.4.4

Immediate. If $f_n, gn \in \mathcal{A}$, $\lambda \in \mathbb{F}$, and $f_n \to f$, $g_n \to g$ uniformly then

$$\|\lambda f_n - \lambda f\|_\infty = |\lambda|\|f - f_n\|_\infty \to 0,$$
$$\|(f_n + g_n) - (f + g)\|_\infty \leq \|f_n - f\|_\infty + \|g_n - g\|_\infty \to 0,$$

$$\begin{aligned}\|f_n g_n) - fg\|_\infty &\leq \|f_n g_n - f_n g\|_\infty + \|f_n g - fg\|_\infty \\ &\leq \|f_n\|_\infty \|g - g_n\|_\infty + \|g\|_\infty \|f - f_n\|_\infty \\ &\leq (\|f\|_\infty + \|f - f_n\|_{infty})\|g - g_n\|_\infty + \|g\|_\infty \|f - f_n\|_\infty \to 0\end{aligned}$$

as $n \to \infty$.

Exercise 4.4.5

(i) By symmetry we may suppose $a \geq b$. Now

$$\max\{a, b\} + \min\{a, b\} = a + b$$

automatically and

$$\max\{a, b\} - \min\{a, b\} = a - b = |a - b|,$$

so we have proved the first two formulae

Using these results,

$$\begin{aligned}\max\{a, b\} &= \frac{1}{2}((\max\{a, b\} + \min\{a, b\}) + (\max\{a, b\} - \min\{a, b\}) \\ &= \frac{1}{2}((a + b) + |a - b|)\end{aligned}$$

and

$$\begin{aligned}\min\{a, b\} &= \frac{1}{2}((\max\{a, b\} + \min\{a, b\}) - (\max\{a, b\} - \min\{a, b\}) \\ &= \frac{1}{2}((a + b) - |a - b|).\end{aligned}$$

(ii) If $a \neq b$, then

$$\det\begin{pmatrix}1 & a \\ 1 & b\end{pmatrix} = b - a \neq 0.$$

(Or we could just solve the given system.)

Exercise 4.4.7

Set
$$f_n(x) = \begin{cases} 1 - nx & \text{for } 0 \leq 1/n, \\ 0 & \text{for } 1/n \leq x \leq 1. \end{cases}$$
and $f(0) = 1$, $f(x) = 0$ otherwise.

Exercise 4.4.8

(i) We have
$$\sum_{j=0}^{n} a_j e^{ijt} \sum_{k=0}^{m} b_k e^{ikt} = \sum_{u=0}^{n+m} \left(\sum_{j+k=u} a_j b_k \right) e^{iut},$$
so $f, g \in \mathcal{P} \Rightarrow fg \in \mathcal{P}$. Even simpler arguments show
$$f, g \in \mathcal{P} \Rightarrow f + g \in \mathcal{P}, \text{ and } \lambda \in \mathbb{C},\ f \in \mathcal{P} \Rightarrow \lambda f \in \mathcal{P},$$
$1 \in \mathcal{P}$ and, if $s \neq t$ (working in $\mathbb{T}$), we have $e^{it} \neq e^{is}$.

However, if $p(t) = \sum_{j=0}^{n} a_j e^{ijt}$
$$\begin{aligned}
&\int_0^1 \left| \sum_{j=0}^{n} a_j e^{ijt} - e^{-it} \right|^2 dt \\
&\quad = \int_0^1 \left(\sum_{j=0}^{n} a_j e^{ijt} - e^{-it} \right) \left(\sum_{j=0}^{n} a_j e^{ijt} - e^{-it} \right)^* dt \\
&\quad = \int_0^1 \left(\sum_{j=0}^{n} a_j e^{ijt} - e^{-it} \right) \left(\sum_{j=0}^{n} a_j^* e^{-ijt} - e^{i=t} \right)^* dt \\
&\quad = \int_0^1 \left(\sum_{0<|j|\leq n}^{n} c_j e^{ijt} + \sum_{j=0}^{n} |a_j|^2 \right) + 1\, dt \\
&\quad = \sum_{j=0}^{n} |a_j|^2 + 1 \geq 1.
\end{aligned}$$

Thus $\|p - e_{-1}\|_\infty^2 \geq 1$ and so $\|p - e_{-1}\|_\infty \geq 1$ for all $p \in \mathcal{P}$. It follows that $\operatorname{Cl} \mathcal{P} \neq C_{\mathbb{C}}(\mathbb{T})$.

(ii) The Cauchy-Riemann equations tell us that $z \mapsto z^*$ is not analytic, so, if $g(z) = z^*$, then $g \notin \operatorname{Cl} \mathcal{Q}$ and so we have $\operatorname{Cl} \mathcal{Q} \neq C_{\mathbb{C}}(\bar{D})$.

Exercise 4.4.10

(i) Observe that $\Re f = (f + f^*)/2$ so, using condition (iii), $\Re\mathcal{A} \subseteq \mathcal{A}$. If $f \in A$, then $\Im f = \Re(-if) \in \Re\mathcal{A}$.

(ii) Certainly $1 \in \Re\mathcal{A}$. If $f, g \in \Re\mathcal{A}$and $\lambda \in \mathbb{R}$, then $f, g \in A$ so $f \times g, f + g, \lambda f \in A$ and $f \times g, f + g, \lambda f \in \Re\mathcal{A}$. If $x \neq y$ then we can find $F \in \mathcal{A}$ such that $F(x) \neq F(y)$. At least one of the statements $\Re F(x) \neq \Re F(y)$ or $\Im F(x) \neq \Im F(y)$ must be true, so there exists an $f \in \Re\mathcal{A}$ with $f(x) \neq f(y)$. Thus the conditions for the real Stone–Weierstrass theorem hold and $\Re\mathcal{A}$ is uniformly dense in $C_{\mathbb{R}}(X)$.

(iii) If $G \in C_{\mathbb{C}}(X)$, part (ii) tells that we can find $u_n, v_n \in \Re\mathcal{A}$ with $u_n \to \Re G$, $v_n \to \Im G$ uniformly. Since $\Re\mathcal{A} \subseteq \mathcal{A}$, $g_n = u_n + iv_n \in \mathcal{A}$ and we have $g_n \to G$ uniformly.

(iv) We used the fact that $f \in \mathcal{A} \Rightarrow f^* \in \mathcal{A}$ to show that $\Re\mathcal{A}$ was an algebra, that $\Re\mathcal{A}$ separated points and $\Re\mathcal{A} \subseteq \mathcal{A}$.

EXERCISE 4.4.11

(i) The limit of the product is the product of the limits, so

$$\begin{aligned}(x_n, y_n) \to (x, y) \Rightarrow x_n \to x,\ y_n \to y \Rightarrow f(x_n) \to f(x),\ g(y_n) \to g(y)\\ \Rightarrow f(x_n)g(y_n) \to f(y)g(y)\end{aligned}$$

as $n \to \infty$.

(ii) Let $\mathcal{A}$ be the collection of functions of the form $\sum_{j=1}^{n} f_j \otimes g_j$. It is an algebra. If $(x_1, y_1) \neq (x_2, y_2)$ then at least one of the statements $x_1 \neq x_2$, $y_1 \neq y_2$ must be true. Suppose, without loss of generality $x_1 \neq x_2$. If $f(x) = x$, $g(y) = 1$ then $f \otimes g \in \mathcal{A}$ and $f \otimes g(x_1, y_1) = x_1 \neq x_2 = f \otimes g(x_2, y_2)$. (If we work over $\mathbb{C}$ we also need to note that since

$$\left(\sum_{j=1}^{n} f_j \otimes g_j\right)^* = \sum_{j=1}^{n} f_j^* \otimes g_j^*,$$

$\mathcal{A}$ is algebraically closed under conjugation.)

The Stone–Weierstrass theorem now tells us that $C_{\mathbb{F}}(X, Y)$ is the uniform closure of $\mathcal{A}$, which is the result we are asked to prove.

(iii) Since $F : [a, b]\times[c, d] \to \mathbb{C}$ is continuous, it is uniformly continuous, so, given $\epsilon > 0$, we can find a $\delta > 0$ such that $|x - x'|$, $|y - y'| < \delta$ implies

$$|F(x, y) - F(x', y')| < \epsilon.$$

Thus, if $|y - y'| < \delta$,

$$\begin{aligned}|u(y) - u(y')| &= \left|\int_a^b F(x, y)\, dx - \int_a^b F(x, y')\, dx\right| = \left|\int_a^b F(x, y) - F(x, y')\, dx\right|\\ &\leq \int_a^b |F(x, y) - F(x, y')|\, dx \leq (b - a)\epsilon.\end{aligned}$$

Since ϵ was arbitrary, u is continuous.

(iv) Observe that

$$\begin{aligned}&\int_c^d \left(\int_a^b f(x)g(y)\, dx\right) dy = \int_c^d g(y)\left(\int_a^b f(x)\, dx\right) dy = \left(\int_a^b f(x)\, dx\right) \int_c^d g(y)\, dy\\ &= \left(\int_c^d g(y),\, dy\right)\int_a^b f(x)\, dx = \int_a^b \left(\int_c^d g(y)f(x)\, dy\right) dx = \int_a^b \left(\int_c^d f(x)g(y)\, dy\right) dx\end{aligned}$$

Thus, by the linearity of the integral,

$$\int_c^d \left(\int_a^b \sum_{j=1}^{n} f_j(x)g_j(y)\, dx\right) dy = \int_c^d \left(\int_a^b \sum_{j=1}^{n} f_j(x)g_j(y)\, dx\right) dy.$$

If $F \in C([a,b] \times [c,d])$, the given any $\epsilon > 0$ we can find $p \in \mathcal{A}$ such that $\|F - p\|_\infty < \epsilon$. Now

$$\left| \int_c^d \left(\int_a^b F(x,y)\,dx \right) dy - \int_c^d \left(\int_a^b p(x,y)\,dx \right) dy \right| = \left| \int_c^d \left(\int_a^b F(x,y) - p(x,y)\,dx \right) dy \right|$$

$$\leq \int_c^d \left| \int_a^b F(x,y) - p(x,y)\,dx \right| dy \leq \int_c^d \int_a^b |F(x,y) - p(x,y)|\,dx\,dy$$

$$\leq \int_c^d \int_a^b \epsilon\,dx\,dy = (d-c)(b-a)\epsilon$$

and, similarly,

$$\left| \int_a^b \left(\int_c^d F(x,y)\,dy \right) dx - \int_a^b \left(\int_c^d p(x,y)\,dy \right) dx \right| \leq (d-c)(b-a)\epsilon.$$

Thus

$$\left| \int_c^d \left(\int_a^b F(x,y)\,dx \right) dy - \int_a^b \left(\int_c^d F(x,y)\,dy \right) dx \right| \leq 2(d-c)(b-a)\epsilon$$

and, since ϵ was arbitrary,

$$\int_c^d \left(\int_a^b F(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d F(x,y)\,dy \right) dx.$$

Exercise 4.4.12

(i) The open balls $B(x, 1/n)$ cover X, so we can find a finite subset E_n such that the $B(x, 1/n)$ with $x \in E_n$. The set $\bigcup_{n=1}^{\infty} E_n$ is a countable dense subset of X.

(ii) We observe that

$$d(x, y) < \epsilon \Rightarrow |d(x, a) - d(y, a)| < \epsilon \Rightarrow |f_a(x) - f_a(y)| < \epsilon.$$

$\mathcal{A}$ is algebraically closed under addition and multiplication and contains the constant functions. If $u \neq v$ we can find $a \in A$ such that $d(a, u) < \min\{2d(u, v), 1\}$ so $d(a, u) < d(a, v)$ and $f_a(u) \neq f_a(v)$. Thus A separates points and the Stone–Weierstrass theorem tells us that $\mathcal{A}$ is uniformly dense in $C_{\mathbb{R}}(X)$.

Let $\mathcal{P}_0$ be the collection of functions of the form

$$\lambda_0 + \sum_{j=1}^{m} \lambda_j f_{a(j)}$$

with $\lambda_j \in \mathbb{Q}$, $a(j) \in A$ and let $\mathcal{A}_0$ be the collection of finite products $g_1 g_2 \ldots g_u$ with $g_v \in \mathcal{P}_0$. Then $\mathcal{P}_0$ and $\mathcal{A}_0$ are countable. Given any $f \in \mathcal{A}$ and any $\epsilon > 0$ we can find $g \in \mathcal{A}_0$ with $\|f - g\|_\infty < \epsilon$. Thus $\mathcal{A}_0$ is a countable dense subset of $C_{\mathbb{R}}(X)$ and we are done.

(iii) If $\mathcal{B}$ is a countable dense subset of $C_{\mathbb{R}}(X)$ then

$$\{f + ig, \ \ f, \ g \in \mathcal{B}\}$$

is a countable dense subset of $C_{\mathbb{C}}(X)$.

(iv) No. Lots of different ways of seeing this. For example, set

$$g_n(x) = \max\{1, 10n^2|x - 1/n|\}$$

Since $n^{-1} - (n + 1)^{-1} \leq 2(n + 1)^{-1}$, $\|g_n - g_m\|_\infty = 1$ for all $n \neq m$. (Also direct consequence of Theorem 4.5.10).

Exercise 4.5.2

Let $\|\ \|_A$, $\|\ \|_B$, $\|\ \|_C$ be norms on a vector space V over $\mathbb{F}$. If we have $K_1, K_2, K_3, K_4 > 0$ such that

$$K_1\|\mathbf{u}\|_A \geq \|\mathbf{u}\|_B \geq K_2\|\mathbf{u}\|_A,\ K_3\|\mathbf{u}\|_A \geq \|\mathbf{u}\|_C \geq K_4\|\mathbf{u}\|_A$$

for all $\mathbf{u} \in V$, then

$$K_2^{-1}K_3\|\mathbf{u}\|_B \geq K_3\|\mathbf{u}\|_A \geq \|\mathbf{u}\|_C \geq K_4\|\mathbf{u}\|_A \geq K_1^{-1}K_4\|\mathbf{u}\|_B$$

for all $\mathbf{u} \in V$.

Exercise 4.5.2

Let $\|\ \|$ be a norm on a finite dimensional space V over $\mathbb{F}$. Then there exist $A \geq B > 0$ such that

$$A\|\mathbf{v}\|_2 \geq \|\mathbf{v}\| \geq B\|\mathbf{v}\|_2.$$

for all $\mathbf{v} \in V$.

If $(\mathbf{v}_n)$ is Cauchy for $\|\ \|$, then

$$\|v_n - v_m\|_2 \leq B^{-1}\|v_n - v_m\| \to 0$$

as $n, m \to \infty$ so $(\mathbf{v}_n)$ is Cauchy for $\|\ \|_2$ and so converges to some $\mathbf{v}$ for that norm. It follows that

$$\|\mathbf{v}_n - \mathbf{v}\| \leq A\|\mathbf{v}_n - \mathbf{v}\|_2 \to 0$$

so we are done.

Exercise 4.5.4

(i) Writing $\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{j=1}^{\infty} u_j v_j$, we have

$$\begin{aligned}\|\mathbf{v}_1 + \mathbf{v}_2\|_2^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_2^2 &= \langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2\rangle + \langle \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\rangle \\ &= \langle \mathbf{v}_1, \mathbf{v}_1\rangle + 2\langle \mathbf{v}_1, \mathbf{v}_2\rangle + \langle \mathbf{v}_2, \mathbf{v}_2\rangle + \langle \mathbf{v}_1, \mathbf{v}_1\rangle - 2\langle \mathbf{v}_1, \mathbf{v}_2\rangle + \langle \mathbf{v}_2, \mathbf{v}_2\rangle \\ &= 2\|\mathbf{v}_1\|_2^2 + 2\|\mathbf{v}_2\|_2^2.\end{aligned}$$

(ii) In particular,

$$\begin{aligned}\min\{\|\mathbf{v}_1 + \mathbf{v}_2\|_2^2, \|\mathbf{v}_1 - \mathbf{v}_2\|_2^2\} &\leq \tfrac{1}{2}(\|\mathbf{v}_1 + \mathbf{v}_2\|_2^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_2^2) \\ &= \|\mathbf{v}_1\|_2^2 + \|\mathbf{v}_2\|^2\end{aligned}$$

(iii) Let $P(n)$ be the statement that, if $\mathbf{v}_j \in l^2$ $[1 \leq j \leq 2^n]$ we can find $\zeta_j \in \{-1, 1\}$ $[1 \leq j \leq 2^n]$ such that

$$\left\|\sum_{j=1}^{2^n} \zeta_j \mathbf{v}_j\right\|_2^2 \leq \sum_{j=1}^{2^n} \|\mathbf{v}_j\|_2^2.$$

Taking $\zeta_1 = 1$ we see that $P(0)$ is true.

Suppose that $P(n)$ is true and $\mathbf{v}_j \in l^2$ $[1 \leq j \leq 2^{n+1}]$. Then we can find $\alpha_j \in \{-1, 1\}$ $[1 \leq j \leq 2^n]$, $\beta_j \in \{-1, 1\}$ $[2^n + 1 \leq j \leq 2^{n+1}]$, such that writing

$$\mathbf{w}_1 = \sum_{j=1}^{2^n} \alpha_j \mathbf{v}_j,\ \mathbf{w}_2 = \sum_{j=2^n+1}^{2^{n+1}} \beta_j \mathbf{v}_j,$$

we have

$$\|\mathbf{w}_1\| \leq \sum_{j=1}^{2^n} \|\mathbf{v}_j\|_2^2,\ \|\mathbf{w}_2\| \leq \sum_{j=2^n+1}^{2^{n+1}} \|\mathbf{v}_j\|_2^2.$$

By (iii), we can find $\gamma \in \{-1, 1\}$ such that

$$\|\mathbf{v}_1 + \gamma \mathbf{v}_2\|_2^2 \leq \|\mathbf{w}_1\|_2^2 + \|\mathbf{w}_2\|^2.$$

Setting $\zeta_j = \alpha_j$ for $1 \leq j \leq 2^n$ and $\zeta_j = \gamma\beta_j$ for $2^n + 1 \leq j \leq 2^{n+1}$, we have $\zeta_j \in \{-1, 1\}$ $[1 \leq j \leq 2^{n+1}]$ and

$$\left\|\sum_{j=1}^{2^{n+1}} \zeta_j \mathbf{v}_j\right\|_2^2 \leq \sum_{j=1}^{2^{n+1}} \|\mathbf{v}_j\|_2^2.$$

Thus $P(n + 1)$ is true and the induction is complete.

(iv) Automatically

$$\left\|\sum_{j=1}^{N} \zeta_j \mathbf{u}_j\right\|_1 = \sum_{j=1}^{N} |\zeta_j| = \sum_{j=1}^{N} 1 = N.$$

(v) Suppose that $T : l^1 \to l^2$ and there exists a constant $K > 0$ such at

$$K\|\mathbf{u}\|_1 \geq \|T\mathbf{u}\|_2 \geq K^{-1}\|\mathbf{u}\|_1$$

for all $\mathbf{u} \in l^1$. Let $\mathbf{u}_j$ be as in part (iv) and take $\mathbf{v}_j = \mathbf{u}_j$. If we choose ζ_j as in (iii), then

$$\begin{aligned} 2^{2n} &= \left\| \sum_{j=1}^{2^n} \zeta_j \mathbf{u}_j \right\|_1^2 \leq K^2 \left\| T\left(\sum_{j=1}^{2^n} \zeta_j \mathbf{u}_j \right) \right\|_2^2 = K^2 \left\| \sum_{j=1}^{2^n} \zeta_j \mathbf{v}_j \right\|_2^2 \\ &\leq K^2 \sum_{j=1}^{2^n} \|\mathbf{v}_j\|_2^2 \leq K^4 \sum_{j=1}^{2^n} \|\mathbf{u}_j\|_1^2 = K^4 2^n, \end{aligned}$$

so $K^4 \geq 2^n$ for all n, which is impossible.

Exercise 4.5.5

Observe that, if $\mathbf{a} \in l^1$, then

$$\|\mathbf{a}\|_1^2 \geq \left(\sum_{j=1}^{n} |a_j| \right)^2 \geq \sum_{j=1}^{n} |a_j|^2$$

for all n, so $\mathbf{a} \in l^2$ and $\|\mathbf{a}\|_1^2 \geq \|\mathbf{a}\|_2^2$, whence $\mathbf{a} \in l^2$.

Since l^1 is a vector space and l^2 is algebraically closed under addition and scalar multiplication, l^2 is a subspace of l^1. However it is not a closed subspace under the norm $\| \ \|_2$. If we set

$$\mathbf{u}_n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots),$$

then $\mathbf{u}_n \underset{\| \ \|_2}{\to} \mathbf{u}$ where

$$\mathbf{u} = (1, 1/2, 1/3, \ldots) \notin l^1.$$

Exercise 4.5.7

E is algebraically closed under addition and scalar multiplication and so a subspace. If we set

$$\mathbf{u}_n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots),$$

then $\mathbf{u}_n \underset{\| \ \|_1}{\to} \mathbf{u}$ where

$$\mathbf{u} = (1, 1/2, 1/3, \ldots) \notin E.$$

EXERCISE 4.5.8

(i) If we choose a basis $\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_n$ then

$$\left\| \sum_{j=1}^{n} x_j \mathbf{e}_j \right\|_2^2 = \sum_{j=1}^{n} |x_j|$$

defines a norm on V, so, by Theorem 4.5.1, there exists an $A > 0$ with $A\|\mathbf{v}\|_V \geq \|\mathbf{v}\|_1$.

Thus, if $\mathbf{v} = \sum_{j=1}^{n} v_j \mathbf{e}$,

$$\begin{aligned} \|Tv\|_W &= \left\| T\left(\sum_{j=1}^{n} v_j \mathbf{e} \right) \right\|_W = \left\| \left(\sum_{j=1}^{n} v_j T\mathbf{e} \right) \right\|_W \leq \sum_{j=1}^{n} |v_j| \|T\mathbf{e}_j\|_W \\ &\leq \|\mathbf{v}\|_1 \max_{1 \leq j \leq n} \|T\mathbf{e}_j\|_W \leq A\|\mathbf{v}\|_V \max_{1 \leq j \leq n} \|T\mathbf{e}_j\|_W \end{aligned}$$

so T is continuous.

(ii) Let $\| \ \|$ and $\| \ \|_*$ be two norms on a finite dimensional space V over $\mathbb{F}$. The identity map $I : V \to V$ given by $I\mathbf{v} = \mathbf{v}$ linear map so by the statement at the beginning of the exercise I is continuous and we can find a $K > 0$ such that

$$K\|\mathbf{v}\| \geq \|I\mathbf{v}\|_* = \|\mathbf{v}\|_*$$

for all $\mathbf{v} \in V$. Exactly the same argument shows that we can find a $K' > 0$ such that $K'\|\mathbf{v}\|_* \geq \|\mathbf{v}\|$ for all $\mathbf{v} \in V$.

(iii) Just take $T\mathbf{a} = \sum_{j=1}^{\infty} a_j$.

EXERCISE 4.5.9

We observe that

(i) $\|(x, y)\|_{[n]} = n|x| + n^{-1}|y| \geq 0$ and $\|(x, y)\|_{[n]} = 0 \Rightarrow (x, y) = (0, 0)$,

(ii) $\|\lambda(x, y)\|_{[n]} = \|(\lambda x, \lambda y)\| = n|\lambda||x| + n^{-1}|\lambda||y| = |\lambda|\|(x, y)\|_{[n]}$.

(iii) $\|(x, y) + (x', y')\|_{[n]} = \|(x + x', y + y')\| = n|x + x'| + n^{-1}|y + y'| \geq n(|x| + |x'|) + n^{-1}(|y| + |y'|) = \|(x, y)\|_{[n]} + \|(x', y')\|_{[n]}$

so we have a norm. We note that

$$\|(1, 0)\|_{[n]} = n\|(1, 0)\|_{[1]} \text{ and } \|(0, 1)\|_{[1]} = n\|(0, 1)\|_{[n]}$$

Exercise 4.5.13

(i)$\Rightarrow$(ii) If V is finite dimensional over $\mathbb{F}$, then V may be identified with $\mathbb{F}^n$ and so may be equipped with the Euclidean norm $\| \|_2$. By Theorem 4.5.1 any sequence $\mathbf{x}_n$ in the unit ball $\tilde{B}$ for our norm $\| \|$ is bounded in the norm $\| \|_2$ and so contains a subsequence $\mathbf{x}_{n(j)}$ convergent to some $\mathbf{x}$ in the $\| \|_2$ norm and so in our norm. The closed unit ball is closed, so $\mathbf{x} \in \bar{B}$. Thus $\bar{B}$ is compact.

(ii)$\Rightarrow$(iii) Since $\bar{B}$ is compact, re-scaling shows that

$$\bar{B}(r) = \{\mathbf{x} \in V : \|\mathbf{x}\| \leq r\} = \{r\mathbf{x} \in V : \mathbf{x} \in \bar{B}\}$$

is compact. Any bounded set lies in some $\bar{B}(r)$ and any closed subset of a bounded set is compact.

(iii)$\Rightarrow$(iv) The closed unit ball $\bar{B}$ is bounded and closed. $\mathbf{0} \in \operatorname{Int} \bar{B}$.

(iv)$\Rightarrow$(i) If V contains a compact set with non-empty interior, then, by translation, there is a compact set E with $\mathbf{0} \in \operatorname{Int} E$. Thus we can find $\delta > 0$ such that

$$\{\mathbf{x} \in V : \|\mathbf{x}\| < 2\delta\} \subseteq E.$$

It follows that

$$\bar{B}(\delta) = \{\mathbf{x} \in V : \|\mathbf{x}\| \leq \delta\} \subseteq E.$$

Since a closed subset of a compact set is compact $\bar{B}(\delta)$ is compact and so, by re-scaling, the closed unit ball is compact and, by Theorem 4.5.10, V must be finite dimensional.

Exercise 4.6.5

Let $f_n(t) = \cos nt$. Then the set E of such f_n is uniformly bounded since $\|f_n\|_\infty = 1$ but not uniformly equicontinuous since

$$|f_n(0) - f_n(\pi/n)| = 2 \nrightarrow 0, \text{ but } |0 - \pi/n)| \to 0.$$

The set of constant functions is uniformly equicontinuous but not uniformly bounded.

Exercise 4.6.6

Let Γ be the collection of continuous piecewise linear functions g with

$$g(r2^{-m}) \in \{s2^{-n-2} \ : \ s \in \mathbb{Z}, |s| \leq 2^{n+2}\}.$$

for $0 \leq r \leq 2^m$

If $f \in \mathcal{G}$ let $g \in \mathcal{G}$ be chosen so that $|g(r2^{-m}) - f(r2^{-m})| \leq 2^{-n-3}$ for each r. We observe that

$$\begin{aligned} |g(t) - g(r2^{-m})| &\leq |g(r2^{-m}) - g((r+1)2^{-m})| \\ &\leq 2^{-n-2} + |f(r2^{-m}) - f((r+1)2^{-m})| \leq 2^{-n} + 2^{-n-2} \end{aligned}$$

so

$$\begin{aligned} |g(t) - f(t)| &\leq |g(t) - g(r2^{-m})| + |f(t) - f(r2^{-m})| + |g(r2^{-m}) - f(r2^{-m})| \\ &\leq 2^{-n+1} + 2^{-n-1} < 2^{-n+2} \end{aligned}$$

for $t \in [r2^{-m}, (r+1)2^{-m}]$ $[0 \leq r \leq 2^n - 1]$.

Thus $\|g - f\|_\infty < 2^{2-n}$ and, since Γ is finite, we are done.

Exercise 4.6.7★

Exercise 4.6.8

(i) We claim that the f_m form a Cauchy sequence in $(C(X), \|\ \|_\infty)$. For, if $\epsilon > 0$, then we can find a $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

for all $f \in \mathcal{F}$. Since E is dense, we can find an J such that

$$\bigcup_{j=1}^{J} B(x_j, \delta) = X$$

Since each $f_m(x_j)$ converges, we can now find an N such that

$$|f_u(x_j) - f_v(x_j)| < \epsilon$$

for all u, $v \geq N$ and all $1 \leq j \leq J$.

If $t \in X$, we can find a j with $1 \leq j \leq J$ such that $t \in B(x_j, \delta)$ and so

$$|f_u(t) - f_v(t)| \leq |f_u(t) - f_u(x_j)| + |f_v(t) - f_v(x_j)| + |f_u(x_j) - f_v(x_j)| < 3\epsilon$$

for all u, $v \geq N$. Since ϵ was arbitrary we have shown that the f_j form a Cauchy sequence and so converge in the uniform norm to a continuous function.

(ii) Take $N_0(m) = m$. Once $N_{j-1}(m)$ has been defined, we know that the $f_{N_{j-1}(m)}(x_j)$ form a bounded subset of $\mathbb{F}$ and so by the theorem of Bolzano–Weierstrass we can find a sequence $N_j(m)$ such that the $N_j(m)$ form a strictly increasing subsequence of the $N_{j-1}(m)$ and $f_{N_j(m)}(x_j)$ tends to a limit as $m \to \infty$.

We now observe that $N_m(m)$ is a subsequence of $N_j(m)$ for $m \geq j$, so $f_{N_m(m)}(x_j)$ tends to a limit as $m \to \infty$.

(iii) By (ii), any sequence in $\mathcal{F}$ has a subsequence satisfying the conditions of (i) and thus uniformly convergent. Since $\mathcal{F}$ is closed the limit lies in $\mathcal{F}$ and we are done.

Exercise 4.6.10

If $\operatorname{Cl}\mathcal{F}$ is compact, it is uniformly bounded and uniformly equicontinuous so the subset $\mathcal{F}$ is uniformly bounded and uniformly equicontinuous.

If $\mathcal{F}$ is uniformly bounded, then there exists a K such that $\|f\|_\infty \leq K$ for all $f \in \mathcal{F}$. If $g \in \operatorname{Cl}\mathcal{F}$ then we can find a sequence $f_n \in \mathcal{F}$ with $\|f_n - g\|_\infty \to 0$ and so $\|g\|_\infty \leq K$. Thus $\operatorname{Cl}\mathcal{F}$ is uniformly bounded.

If $\mathcal{F}$ is uniformly equicontinuous then, given ϵ we can find a $\delta > 0$ such that $|f(s) - f(t)| \leq \epsilon$ whenever $d(s,t) \leq \delta$. If $g \in \operatorname{Cl}\mathcal{F}$ then we can find a sequence $f_n \in \mathcal{F}$ with $\|f_n - g\|_\infty \to 0$. Automatically

$$\epsilon \geq |f_n(s) - f_n(t)| \to |g(s) - g(t)|$$

whenever $d(s,t) \leq \delta$. Thus $\operatorname{Cl}\mathcal{F}$ is uniformly equicontinuous.

Thus, if $\mathcal{F}$ uniformly bounded and uniformly equicontinuous, we have $\operatorname{Cl}\mathcal{F}$ uniformly bounded and uniformly equicontinuous. Since $\operatorname{Cl}\mathcal{F}$ is closed the theorem of Arzelá–Ascoli tells us that it is compact.

Exercise 4.6.11

(i) Observe that, if $f \in \mathcal{F}_M$, then $\|f\|_\infty \leq M$, so $\mathcal{F}_M$ is uniformly bounded. The mean value theorem tells us that, if $f \in \mathcal{F}_M$, then

$$|t - s| \leq M^{-1}\epsilon \Rightarrow |f(t) - f(s)| \leq \epsilon$$

so $\mathcal{F}_M$ is uniformly equicontinuous. By Lemma 4.6.9, $\operatorname{Cl}\mathcal{F}_M$ is compact.

(ii) Let $f_n(t) = \cos nt$. We have $\|f_n\|_\infty = 1$ but $|f_n(0) - f_n(\pi/n)| = 2 \nrightarrow 0$ and $|0 - \pi/n)| \to 0$. Thus uniform equicontinuity fails.

The set of constant functions f has $\|f'\|_\infty = 0$, but is not uniformly bounded.

(iii) If $n \geq 2$, let

$$g_n(x) = \begin{cases} nx & \text{for } 0 \leq x \leq n^{-1} \\ 1 & \text{for } 1/n < x < \pi - 1/n \\ n(\pi - x) & \text{for } \pi - 1/n \leq x \leq \pi \end{cases}$$

and $g_n(-x) = -g_n(x)$. If we set

$$f_n(t) = \int_0^t g_n(x)\,dx,$$

then $f_n(t)$ is differentiable with derivative $f_n'(t) = g_n(t)$. Thus $f_n \in \mathcal{M}$ for $M \geq 1$. We have $f_n \to g$ uniformly, so $g \in \operatorname{Cl}\mathcal{F}_M$ when $M \geq 1$.

Exercise 4.7.1

(i) By the Weierstrass M-test, $d(\mathbf{x}, \mathbf{y})$ is well defined. We observe that

(a) $d(\mathbf{x}, \mathbf{y}) \geq 0$

(b) $d(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow d_j(x_j, y_j) = 0 \ \forall j \Rightarrow x_j = y_j \ \forall j \Rightarrow \mathbf{x} = \mathbf{y}$

(c) Since $d_j(x_j, y_j) = d_j(x_j, y_j)$, we have $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.

(d) We have

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) &= \sum_{j=1}^{\infty} a_j(d_j(x_j, y_j) + d_j(y_j, z_j)) \\ &= \sum_{j=1}^{\infty} a_j d_j(x_j, z_j) = d(\mathbf{x}, \mathbf{z}) \end{aligned}$$

Thus d is a metric.

If we have no condition on the d_j, the sum may fail to converge. As an example, $(X_j, d_j) = (\mathbb{R}, e)$ with e the standard metric and $x_j = 0$, $y_j = 1$ for all j.

(ii) By (i), it suffices to check that $\min\{1, d_j(x_j, y_j)\}$ is a metric on X_j with $\min\{1, d_j(x_j, y_j)\} \leq 1$. This is immediate. (The triangle inequality was verified in Exercise 4.3.2.)

(iii) Immediate from (ii).

(iv) By (i) it suffices to check that

$$\tau_j(x_j, y_j) = \frac{d_j(x_j, y_j)}{1 + d_j(x, y)}$$

defines a metric on X_j with $\tau_j(x_j, y_j) \leq 1$. This is immediate except for the triangle inequality. We check the the triangle inequality as follows.

Observe that if $u,\ v \geq 0$,

$$\begin{aligned} &u(1 + v)(1 + u + v) + v(1 + u)(1 + u + v) - (u + v)(1 + u)(1 + v) \\ &\qquad = u(1 + u + 2v + uv + v^2) + v(1 + v + 2u + uv + u^2) \\ &\qquad\qquad - u(1 + u + v + uv) - v(1 + u + v + uv) = u(v + v^2) + v(u + u^2) \geq 0 \end{aligned}$$

so

$$\frac{u}{1 + u} + \frac{v}{1 + v} \geq \frac{u + v}{1 + u + v}$$

and

$$\frac{d_j(x_j, y_j)}{1 + d_j(x, y)} + \frac{d_j(y_j, z_j)}{1 + d_j(y_j, z_j)} \geq \frac{d_j(x_j, y_j) + d_j(y_j, z_j)}{1 + d_j(x_j, y_j) + d(y_j, z_j)}.$$

Further, if we write

$$g(t) = \frac{t}{1+t}$$

we have

$$g'(t) = \frac{1}{(1+t)^2}$$

so g is increasing and the triangle inequality for d_j gives

$$\frac{d_j(x_j, y_j) + d_j(y_j, z_j)}{1 + d_j(x_j, y_j) + d_j(y_j, z_j)} \geq \frac{d_j(x_j, z_j)}{1 + (d_j(x_j, z_j)}$$

so

$$\frac{d_j(x_j, y_j)}{1 + d_j(x, y)} + \frac{d_j(y_j, z_j)}{1 + d_j(y_j, z_j)} \geq \frac{d_j(x_j, z_j)}{1 + d_j(x_j, z_j)}$$

as required.

Exercise 4.7.2

(i) Since $2^j d(\mathbf{x}(n), \mathbf{x}) \geq d_j(x_j(n), x_j)$ we have

$$\mathbf{x}(n) \underset{d}{\rightarrow} \mathbf{x} \Rightarrow x_j(n) \underset{d_j}{\rightarrow} x_j.$$

Suppose conversely that $x_j(n) \underset{d_j}{\rightarrow} x_j$ for each j. If $\epsilon > 0$, we can find an M such that $\epsilon > 2^{-M+1}$. We can now find an N such that

$$d_j(x_j(n), x_j) < \epsilon/2$$

for all $n \geq N$ and $j \leq M$ so

$$\begin{aligned} d(\mathbf{x}(n), \mathbf{x}) &= \sum_{j=1}^{M} a_j d_j(x_j(n), x_j) + \sum_{j=M+1}^{\infty} a_j d_j(x_j(n), x_j) \\ &\leq \epsilon/2 + \sum_{j=M+1}^{\infty} a_j < \epsilon, \end{aligned}$$

Thus $\mathbf{x}(n) \underset{d}{\rightarrow} \mathbf{x}$.

(ii) Just observe that

$$d_j(x(n), x) \rightarrow 0 \Leftrightarrow \min\{1, d_j(x_j(n), x_j) \rightarrow 0$$

and

$$\frac{d_j(x_j(n), x)}{1 + d_j(x_j(n), x)} \rightarrow 0 \Leftrightarrow d_j(x_j(n), x) \rightarrow 0$$

as $n \rightarrow \infty$.

(iii) Since $2^j d(\mathbf{x}(n), \mathbf{x})(m) \geq d_j(x_j(n), x_j(m))$, if the sequence $\mathbf{x}(n)$ is Cauchy for d, then the sequence $x_j(n)$ is Cauchy for d_j.

Suppose conversely that the sequence $x_j(n)$ is Cauchy for each j. If $\epsilon > 0$ we can find an M such that $\epsilon > 2\sum_{j=M}^{\infty} a_j$. We can now find an N such that

$$d_j(x_j(n), x_j(m)) < \frac{\epsilon}{2\sum_{j=1}^{M} a_j}$$

for all n, $m \geq N$ and $j \leq M$, so

$$d(\mathbf{x}(n), \mathbf{x}(m)) = \sum_{j=1}^{M} a_j d_j(x_j(n), x_j(m)) + \sum_{j=M+1}^{\infty} a_j d_j(x_j(n), x_j(m))$$

$$\leq \epsilon/2 + \sum_{j=M+1}^{\infty} a_j < \epsilon$$

so the $\mathbf{x}(n)$ form a Cauchy sequence.

(iv) Just observe that

$$d_j(x(n), x(m)) \rightarrow 0 \Leftrightarrow \min\{1, d_j(x_j(n), x_j(m)) \rightarrow 0$$

and

$$\frac{d_j(x_j(n), x_j(m))}{1 + d_j(x_j(n), x_m)} \to 0 \Leftrightarrow d_j(x_j(n), x_j(m)) \to 0$$

as n, $m \to \infty$.

Exercise 4.7.3

(i) Observe that

$$\begin{aligned} \mathbf{x}(n) \underset{\rho_1}{\to} \mathbf{x} &\Rightarrow x_j(n) \underset{d_j}{\to} x_j \ \forall\ j \\ &\Rightarrow \mathbf{x}(n) \underset{\rho_2}{\to} \mathbf{x} \end{aligned}$$

so ι is continuous. Exactly the same argument shows that ι^{-1} is continuous.

(ii) Take $X_j = [0, 1]$ $a_j = 2^{-j}$ and d_j the usual Euclidean metric. Take $\sigma(2r) = 4r$,

$$\sigma(2r - 1) = \min\{s : s \neq \sigma(t) \text{ for } t \leq 2r - 2\}$$

Then if $y_{4n} = 1$ and $y_j = 0$ for $j \neq 4n$ for some particular n,

$$\rho(\mathbf{y}, \mathbf{0}) = 2^{-4n} = 2^{-2n}\rho_\sigma(\mathbf{y}, \mathbf{0}).$$

Thus ρ and ρ_σ are not Lipschitz equivalent.

Exercise 4.7.4

We have

(a) $d(\mathbf{x}, \mathbf{y}) \geq 0$.

(b) $d(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow d_j(x_j, y_j) = 0\ \forall j \Rightarrow x_j = y_j\ \forall j \Rightarrow \mathbf{x} = \mathbf{y}$

(c) Since $d_j(x_j, y_j) = d_j(y_j, x_j)$ we have $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.

(d) $d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geq d_j(x_j, y_j) + d_j(y_j, z_j) \geq d_j(x_j, z_j)$ for all j so

$$d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{z})$$

Thus d is a metric.

However, if we take $x_j(j) = 1$, $x_j(k) = 0$ for all $k \neq j$ we have

$$x_j(n) \to 0 \text{ as } n \to \infty \text{ for all } j$$

but

$$\mathbf{x}_n \underset{d}{\nrightarrow} \mathbf{0} \text{ as } n \to \infty.$$

Exercise 4.7.6

(i) If x_n forms a Cauchy sequence in X then, since X is complete, $x_n \underset{d}{\to} x$ as $n \to \infty$. Since f is continuous $f(x_n) \underset{\rho}{\to} f(x)$ in Y, so, since any convergent sequence is Cauchy, $f(x_n)$ forms a Cauchy sequence.

(ii) If $(n+1)^{-2} > d(x, 1/n)$, then $x = 1/n$ so $\rho(x, 1/n) = 0$. Thus ι is continuous. If $1 > \rho(x, 1/n)$, then $x = 1/n$ so $d(x, 1/n) = 0$. Thus ι^{-1} is continuous. We have shown that ι is a homeomorphism.

We have $d(1/n, 1/m) \leq 1/n$ for all $1 \leq n \leq m$ so the sequence $1/n$ is Cauchy in (X, d). However $\rho(1/n, 1/m) = 1$ for $m \neq n$ so the sequence $\iota(1/n)$ is not Cauchy in (X, ρ).

Exercise 4.8.1

(i) d_* is a discrete metric (see Exercise 1.2.11). Any sequence in $\mathbb{D}_2$ must contain an infinite number entries with value 0 and/or an infinite number of entries with value 1. Suppose $x_j = x$ infinitely often Then we can find a strictly increasing sequence $n(j)$ with $x_{n(j)} = x$ and so with $x_{n(j)} \underset{d_*}{\to} x$. Thus $(\mathbb{D}_2, d_*)$ is a compact metric space.

(ii) Either look case by case or remark that the integers modulo 2 form a group under addition.

EXERCISE 4.8.2

(i) Quote Theorem 4.7.5 to show compactness.

If $\boldsymbol{\omega} \in \mathbb{D}^\infty$ and $\delta > 0$ choose n such that $\delta > 3^{1-n}$ now take $\omega'_j = \omega_j$ for $j \neq n$, but $\omega'_n \neq \omega_n$. We have $\boldsymbol{\omega}' \neq \boldsymbol{\omega}$, but $d(\boldsymbol{\omega}, \boldsymbol{\omega}') = 2 \times 3^{-n} < \delta$. Thus $\boldsymbol{\omega}$ is not an isolated point.

(ii) Standard

(a) Since $(\omega_j + \omega'_j) + \omega'_j = \omega_j + (\omega'_j + \omega'_j)$ we have $(\boldsymbol{\omega} + \boldsymbol{\omega}') + \boldsymbol{\omega}'' = \boldsymbol{\omega} + (\boldsymbol{\omega}' + \boldsymbol{\omega}'')$.

Similarly

(b) $\boldsymbol{\omega} + \boldsymbol{\omega}' = \boldsymbol{\omega}' + \boldsymbol{\omega}$.

(c) $\boldsymbol{\omega} + \mathbf{0} = \boldsymbol{\omega}$.

(d) $\boldsymbol{\omega} + \boldsymbol{\omega} = \mathbf{0}$.

EXERCISE 4.8.3

(i) Set

$$\omega'_j(n) = \omega''_j(n) = \omega_j \text{ for } j \leq n$$

and

$$\omega'_j(n) = 0,\ \omega''_j(n) = 1 \text{ for } j \geq n + 1.$$

Then $\boldsymbol{\omega}'(n), \boldsymbol{\omega}''(n) \to \boldsymbol{\omega}$, but

$$g(\boldsymbol{\omega}'(n)) = 1 \to 1,\ g(\boldsymbol{\omega}''(n)) = 0 \to 0$$

as $n \to \infty$, so g is not continuous at any $\boldsymbol{\omega} \in \mathbb{D}^\infty$.

(ii) If η_j does not take the value 1 infinitely often then we can find an N with $\eta_j = 0$ for all $j \geq N$ so $g(\boldsymbol{\eta}) \geq 1/N$ and thus $g(\boldsymbol{\eta}) \neq 0$. Set

$$\omega_j(n) = \begin{cases} \eta_j(n) & \text{for } j \leq n \\ 1 & \text{for } j \geq n + 1 \end{cases}.$$

Then $\boldsymbol{\omega}(n) \to \boldsymbol{\eta}$, but $h(\boldsymbol{\omega}(n)) = 0 \nrightarrow h(\boldsymbol{\eta}$. Thus h is not continuous at $\boldsymbol{\eta}$.

If η_j does take the value 1 infinitely often, then given $\epsilon > 0$ we can find an N with $\epsilon > N^{-1}$ and $\eta_N = 1$. If $d(\boldsymbol{\omega}, \boldsymbol{\eta}) < 2 \times 3^{-N}$ then $\omega_N = \eta_N = 1$ so $h(\boldsymbol{\omega}) \leq N^{-1}$ and

$$|h(\boldsymbol{\omega}) - h(\boldsymbol{\eta}| = h(\boldsymbol{\omega}) < \epsilon.$$

Thus h is continuous at $\boldsymbol{\eta}$.

EXERCISE 4.8.4

(i) The sum $\sum_{\omega_j \neq 0} 2 \times 3^{-j}$ converges by comparison with a geometric series.

If $\boldsymbol{\omega} = \boldsymbol{\omega}'$ there is nothing to prove, so, without loss of generality we may suppose $\omega_j = \omega'_j$ for $1 \leq j \leq n - 1$, and $\omega_n = 1$, $\omega'_n = 0$. We now have $f(\boldsymbol{\omega}) > f(\boldsymbol{\omega}')$,

$$3^{-n+1} \geq 2 \times 3^{-n} + \sum_{j=n+1} 2 \times 3^{-j} \geq f(\boldsymbol{\omega}) - f(\boldsymbol{\omega}') \geq 2 \times 3^{-n} - \sum_{j=n+1} 2 \times 3^{-j} \geq 3^{-n}$$

and

$$3^{-n+1} \geq 2 \times 3^{-n} + \sum_{j=n+1} 2 \times 3^{-j} = d(\boldsymbol{\omega}, \boldsymbol{\omega}') \geq 2 \times 3^{-n}$$

so

$$\frac{3}{2} d(\boldsymbol{\omega}, \boldsymbol{\omega}) \geq |f(\boldsymbol{\omega}) - f(\boldsymbol{\omega}')| \geq \frac{1}{3} d(\boldsymbol{\omega}, \boldsymbol{\omega})$$

so f is automatically continuous and injective

(ii) The continuous image of a compact set is compact. Since f is a homeomorphism, the fact that $\mathbb{D}^\infty$ has no isolated points implies that E has no isolated points.

(iii) Just observe that

$$\begin{aligned} 2m3^{-n} &= \sum_{j=1}^{n} 2\epsilon_j 3^{-j} \leq \sum_{j=1}^{\infty} 2\epsilon_j 3^{-j} \\ &\leq \sum_{j=1}^{n} 2\epsilon_j 3^{-j} + \sum_{j=n+1}^{\infty} 2 \times 3^{-j} = (2m+1)3^{-n} \end{aligned}$$

(iv) By (iii), if $n \geq 1$, $f(\omega) \in [2m3^{-n}, (2m+1)3^{-n}]$ for some integer m.

If $e \in E$, then, choosing n such that $3^{-n+2} < \delta$, we know that $e \in [2m3^{-n}, (2m+1)3^{-n}]]$ for some m. If $m = 0$, let $x = 3^{-n} + \frac{1/2}{3}^{-n}$. Otherwise, take $x = (2m - \frac{1}{2})3^{-n}$. We have $x \notin E$ but $|x - e| < \delta$.

Exercise 4.8.5

(i) Suppose that $1/8 > \epsilon > 0$. Then

$$|x - x'|, |y - y'| < \epsilon/2 \Rightarrow |A(x,y) - A(x',y')| < \epsilon,\ |B(x) - B(x')| = |x - x'| < \epsilon.$$

(ii) Suppose $\epsilon > 0$, then we can find an N such that $3^{-N-1} < \epsilon$. We now have

$$\begin{aligned} d(\omega,\omega'), d(\zeta,\zeta') < 3^{-N-1} &\Rightarrow \omega_j = \omega'_1,\ \zeta_j = \zeta_1 \text{ for } j \leq N \\ &\Rightarrow \omega_j + \zeta_j = \omega'_1 + \zeta'_1 \text{ for } j \leq N \\ &\Rightarrow d(\omega + \zeta, \omega' + \zeta') \leq 3^{-N-1} < \epsilon \end{aligned}$$

(iii) Observe that $(\omega_j + \zeta_j) - \zeta_j = \omega_j$.

Exercise 4.8.6

(i) We have

$$\boldsymbol{\omega}, \boldsymbol{\zeta} \in H_n \Rightarrow \omega_j = \boldsymbol{\zeta}_j = 0 \ \forall j \le n$$
$$\Rightarrow \omega_j + \boldsymbol{\zeta}_j = 0 \ \forall j \le n \Rightarrow \boldsymbol{\omega} + \boldsymbol{\zeta} \in H_n$$

and $\mathbf{0} \in H_n$.

(ii) We have

$$\boldsymbol{\omega} \in H_n| \Rightarrow d(\mathbf{0}), \boldsymbol{\omega}) = \sum_{j=n+1}^{\infty} 2 \times 3^{-j} d_*(0, \omega_j) \le \sum_{j=n+1}^{\infty} 2 \times 3^{-j} = 3^{-n}$$

and

$$\boldsymbol{\omega} \notin H_n \Rightarrow d(\mathbf{0}), \boldsymbol{\omega}) \ge 2 \times 3^{-n}$$

Thus H_n is the open ball $B(\mathbf{0}, 2 \times 3^{-n})$ and the closed ball $\bar{B}(\mathbf{0}, 3^{-n})$

(iii) By definition, the

$$\{\boldsymbol{\omega} \in \mathbb{D}_2^{\infty} \ : \ \omega_r = \eta_r \text{ for all } r \le n\}$$

are disjoint for the 2^n possible choices $\eta_r \in \mathbb{D}_2$ $[1 \le r \le n]$ and if $\boldsymbol{\zeta} \in \mathbb{D}$

$$\boldsymbol{\zeta} \in \{\boldsymbol{\omega} \in \mathbb{D}_2^{\infty} \ : \ \omega_r = \zeta_r \text{ for all } r \le n\}.$$

Part (ii) and the translational invariance of d show that each coset is open and closed. Further

$$d(\boldsymbol{\omega}, \boldsymbol{\omega}') \le 3^{-n} \Rightarrow d(\boldsymbol{\omega}' - \boldsymbol{\omega}, \mathbf{0}) \le 3^{-n} \Rightarrow \boldsymbol{\omega}' - \boldsymbol{\omega} \in H_n \Rightarrow \boldsymbol{\omega}' \in \boldsymbol{\omega} + H_n.$$

(iv) $(\mathbb{D}_2^{\infty}, d)$ is compact so f is uniformly continuous. Thus we can find a $\delta > 0$ such that

$$|f(\boldsymbol{\omega}) - f(\boldsymbol{\omega}')| < \epsilon$$

whenever $d((\boldsymbol{\omega}), (\boldsymbol{\omega}')) < \delta$. By (ii) we can find an n such that $3^{-n-2} < \delta$ and so by (ii), (iii) and the translational invariance of d

$$|f(\boldsymbol{\omega}) - f(\boldsymbol{\omega}')| < \epsilon$$

whenever $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ belong to the same coset of H_n.

Exercise 4.8.7

(i) Suppose $|f(\omega)| \leq M$ for all ω. Then

$$\sum_{J \in \mathcal{H}_n} a_J \mathbb{I}_J(\omega) \leq f(\omega)\ \forall \omega \Rightarrow a_J \leq M\ \forall J \in \mathbb{H}_n \Rightarrow 2^{-n} \sum_{J \in \mathcal{H}_n} a_J \leq M$$

and, similarly,

$$\sum_{J \in \mathcal{H}_n} (-M) \mathbb{I}_J(\omega) \leq f(\omega)\ \forall \omega \Rightarrow 2^{-n} \sum_{J \in \mathcal{H}_n} a_J \geq -M$$

Since the supremum of a non-empty set bounded above always exists, $\mathcal{I}_n^L(f)$ exists. Similarly $\mathcal{I}_n^U(f)$ exists.

(ii) If $g = \sum_{J \in \mathcal{H}_n} a_J \mathbb{I}_J$ then $g = \sum_{K \in \mathcal{H}_{n+1}} b_K \mathbb{I}_K$ with

$$2^{-n} \sum_{J \in \mathcal{H}_n} a_J = 2^{-(n+1)} \sum_{K \in \mathcal{H}_{n+1}} b_K.$$

Thus

$$\mathcal{I}_n^L(f) \leq \mathcal{I}_{n+1}^L(f) \leq \mathcal{I}_{n+1}^U(f) \leq \mathcal{I}_n^U(f).$$

If

$$\sum_{J \in \mathcal{H}_n} b_J \mathbb{I}_J \leq f \leq \sum_{J \in \mathcal{H}_n} a_J \mathbb{I}_J(\omega) \leq f$$

then $b_J \leq a_J$ for all J and

$$2^{-n} \sum_{J \in \mathcal{H}_n} b_J \leq 2^{-n} \sum_{J \in \mathcal{H}_n} a_J.$$

Thus

$$\mathcal{I}_n^L(f) \leq \mathcal{I}_n^U(f)$$

for all f and n.

(iii) An increasing sequence bounded above converges. A decreasing sequence bounded below converges. If $x_n \geq y_n$ and $x_n \to x$, $y_n \to y$ then $x \geq y$.

(vi) Let $\epsilon > 0$. By Exercise 4.8.6 (iv), we can find an n such that

$$|f(\omega) - f(\omega')| < \epsilon$$

whenever ω and ω' belong to the same coset of H_n. We have

$$\mathcal{I}_n^U(f) - \mathcal{I}_n^L(f) \leq \epsilon$$

so $\epsilon \geq \mathcal{I}^U(f) - \mathcal{I}^L(f) \geq 0$. Since ϵ was arbitrary, the required result follows.

Exercise 4.8.8

(i) If $\boldsymbol{\eta} \in K$ where $K \in \mathcal{H}_n$ then

$$\mathbb{I}_K \geq \mathbb{I}_{\{\eta\}} \geq 0$$

so

$$2^{-n} \geq \mathcal{I}^U \mathbb{I}_{\{\eta\}} \geq \mathcal{I}^L \mathbb{I}_{\{\eta\}} \geq 0$$

and $\mathbb{I}_{\{\eta\}} \in \mathcal{R}$.

However, if V is open with $\boldsymbol{\eta} \in V$, then we can find $\boldsymbol{\omega} \in V$ with $\boldsymbol{\omega} \neq \boldsymbol{\eta}$ and so with

$$\mathbb{I}_{\{\eta\}}(\boldsymbol{\eta}) - \mathbb{I}_{\{\eta\}}(\boldsymbol{\omega}) = 1$$

(ii) If $J \in \mathcal{H}_n$, then, since J is open and E and E^c are both dense,

$$a_J \mathbb{I}_J \geq \mathbb{I}_{J \cap E},\ \mathbb{I}_{J \cap E^c} \geq b_J \mathbb{I}_J \Rightarrow aJ \geq 1 \geq 0 \geq b_J.$$

Thus $\mathcal{I}_n^U(\mathbb{I}_E) \geq 1$ and $\mathcal{I}_n^L(\mathbb{I}_E) \leq 0$, so $\mathbb{I}_E \notin \mathcal{R}$.

EXERCISE 4.8.9

(i) Observe first that $\mathcal{I}^L(-f) = -\mathcal{I}^U(-f)$, so it is sufficient consider the case $\lambda \geq 0$. If $\lambda \geq 0$, then

$$\mathcal{I}^L(\lambda f) = \lambda \mathcal{I}^L(f) \text{ and } \mathcal{I}^U(\lambda f) = \lambda \mathcal{I}^U(f).$$

Thus, if $f \in \mathcal{R}$ and $\lambda \in \mathbb{R}$, then $\lambda f \in \mathcal{R}$ and

$$\int_{\mathbb{D}_2^\infty} \lambda f(\omega)\, d\omega = \lambda \int_{\mathbb{D}_2^\infty} f(\omega)\, d\omega.$$

Next observe that if

$$\sum_{J \in \mathcal{H}_n} a_J \mathbb{I}_J \leq f \text{ and } \sum_{J \in \mathcal{H}_n} b_J \mathbb{I}_J \leq g$$

then

$$\sum_{J \in \mathcal{H}_n} (a_J + b_J) \mathbb{I}_J \leq f + g$$

so

$$\mathcal{I}_n^L(f + g) \leq \mathcal{I}_n^L f + \mathcal{I}_n^L g$$

and

$$\mathcal{I}^L(f + g) \geq \mathcal{I}^L(f) + \mathcal{I}^L(g).$$

Similarly,

$$\mathcal{I}^R(f + g) \geq \mathcal{I}^R(f) + \mathcal{I}^R(g),$$

so, if $f,\ g \in \mathcal{R}$, then $f + g \in \mathcal{R}$ and

$$\int_{\mathbb{D}_2^\infty} f(\omega) + g(\omega) +\ d\omega = \int_{\mathbb{D}_2^\infty} g(\omega)\, d\omega + \int_{\mathbb{D}_2^\infty} f(\omega)\, d\omega.$$

(ii) Observe that $\sum_{J \in \mathcal{H}_n} 0 \times \mathbb{I}_J(\omega) \leq f$, so $\mathcal{I}_n^L(f) \geq 0$, so $\mathcal{I}^L(f) \geq 0$.

(iii) Observe that $\sum_{J \in \mathcal{H}_n} 1 \times \mathbb{I}_J(\omega) = 1$.

(iv) Observe that

$$\sum_{J \in \mathcal{H}_n} a_J \mathbb{I}_J(\omega) \leq f(\omega) \Leftrightarrow \sum_{J \in \mathcal{H}_n} a_{J+\eta} \mathbb{I}_J(\omega) \leq f(\omega + \eta)$$

and

$$\sum_{J \in \mathcal{H}_n} a_{J+\eta} = \sum_{J \in \mathcal{H}_n} a_J$$

so $\mathcal{I}_n^L(f_\eta) = \mathcal{I}_n^L(f)$ and $\mathcal{I}^L(f_\eta) = \mathcal{I}^L(f)$. A similar argument shows that $\mathcal{I}^R(f_\eta) = \mathcal{I}^R(f)$, so the required result follows.

(v) If $f,\ g \in \mathcal{R}$ we set

$$\int_{\mathbb{D}_2^\infty} f + ig = \int_{\mathbb{D}_2^\infty} f + i \int_{\mathbb{D}_2^\infty} g$$

and argue as in the traditional case.

Exercise 4.8.11

(i) It is immediate that $\chi(t) = \exp(i\lambda t)$ gives a homomorphism. The argument of Lemma 4.8.10 (omitting the last paragraph) shows that only such χ are homomorphsms.

(ii) If $\chi : \mathbb{T}^2 \to S^1$ is a continuous homorphism, then

$$\phi(s) = \chi(s, 0)$$

defines a continuous homomorphism $\phi : \mathbb{T} \to S^1$, so $\chi(s, 0) = \exp(ins)$ for some $n \in \mathbb{Z}$. Similarly $\chi(0, t) = \exp(ims)$ for some $m \in \mathbb{Z}$, so

$$\chi(s, t) = \chi((s, 0) + (0, t)) = \chi(s, 0)\chi(0, t) = \exp\left(i(nt + ms)\right).$$

Direct verification shows that every χ of this form is indeed a continuous homomorphism.

Exercise 4.8.13

Simple verification.

(i)

$$\chi_j(\omega + \omega') = \begin{cases} 1 & \text{if } \omega_j = \omega'_j, \\ -1 & \text{if } \omega_j \neq \omega'_j \end{cases}$$

and

$$\chi_j(\omega)\chi_j(\omega') = \begin{cases} 1 & \text{if } \omega_j = \omega'_j, \\ -1 & \text{if } \omega_j \neq \omega'_j \end{cases}$$

so χ_j is a homomorphism.

If $E \subseteq S^1$, then $\chi_j^{-1}(E)$ is the union of cosets in $\mathcal{H}_j$, so open. Thus χ_j is continuous.

(ii) $\chi_j(\omega) = \zeta_j^{\omega_j}$ so $\chi(\omega) = \prod_{j=1}^{N} \chi_j(\omega_j)$. The product of continuous functions is continuous and the composition of homomorphisms is a homomorphism.

Exercise 4.8.14

Direct verification.

(i) Observe that e_0 is the unit and that e_{-n} is the inverse of e_n.

(ii) Observe that $\chi_0(\omega) = 1$ for all ω defines the unit and that

$$\phi(\omega)^2 = \chi_0(\omega).$$

Since $\omega = -\omega$ and $\chi(\omega) \in \{-1, 1\}$ we have $\chi(\omega) = \chi(-\omega) = \chi(\omega)^{-1}$.

Exercise 4.8.15

Direct verification.

(i) If $\chi \neq \chi_0$ then there there exists an N such that

$$\chi(\omega) = \left(\prod_{j=1}^{N-1} \zeta_j^{\omega_j}\right)(-1)^{\omega_N}$$

so there exists a collection $\mathcal{E}$ of exactly 2^{N-1} members of $\mathcal{H}_N$ such that

$$\chi(\omega) = \begin{cases} 1 & \text{if } \omega \in K \text{ for some } K \in \mathcal{E} \\ -1 & \text{if } \omega \in K \text{ for some } K \notin \mathcal{E} \end{cases}$$

so

$$\int_{\mathbb{D}_2^\infty} \chi(\omega)\, d\omega = 2^{-N}(2^{N-1} - 2^{N-1})$$

We have

$$\int_{\mathbb{D}_2^\infty} \chi_0(\omega)\, d\omega = \int_{\mathbb{D}_2^\infty} 1\, d\omega = 1$$

(ii) Just check the rules given in the definition of an inner product.

(iii) If $\chi_1 \neq \chi_2$, then $\chi_1 \cdot \chi_2$ is a character, but not the unit character. If $\chi_1 = \chi_2$, then $\chi_1 \cdot \chi_2 = \chi_0$. Thus

$$\int_{\mathbb{D}_2^\infty} \chi_1(\omega)\chi_2(\omega)^*\, d\omega = \int_{\mathbb{D}_2^\infty} \chi_1 \cdot \chi_2(\omega)$$
$$= \begin{cases} 0 & \text{if } \chi_1 \neq \chi_2 \\ 1 & \text{if } \chi_1 = \chi_2. \end{cases}$$

Exercise 4.8.16

Simple verification.

(i) We know that $\chi_0(\omega) = 1$ for all ω, so $\chi_0 \in G_n$. We know that $\chi^{-1} = \chi$. Finally

$\chi_1,\chi_2 \in G_n \Rightarrow \chi_1 \cdot \chi_2(\omega) = \chi_1(\omega)\chi_2(\omega) = 1^2 = 1\ \forall \omega \in H_n \Rightarrow \chi_1 \cdot \chi_2 \in G_n$

(ii) Either $\chi = \chi_0$

$$\chi(\omega)] = \left(\prod_{j=1}^{N-1} \zeta_j^{\omega_j}\right) \times (-1)^{\omega_N}$$

with not all the $zeta_j$ taking the value 1 so χ is not constant on H_N. Thus $N \leq n$ if $\chi \in G_n$.

The previous paragraph shows that if $\chi \in G_n$, then $\chi(\omega) = \prod_{j=1}^{n} \zeta_j^{\omega_j}$ for some $\zeta_j \in \{-1, 1\}$. The converse is immediate.

Since each choice of ζ_j produces a different χ, G_n has exactly 2^n elements.

(iii) Immediate.

(iv) We saw in Lemma 4.8.12 that $\widehat{\mathbb{D}_2^\infty} = \bigcup_{n=0}^{\infty} G_n$ and we know that the countable union of countable sets is countable.

EXERCISE 4.8.17

(i) We have (remembering that $\chi(-\boldsymbol{\eta}) = \chi(\boldsymbol{\eta})^* = \chi(\boldsymbol{\eta})$)

$$\begin{aligned} S_A(f, \omega) &= \sum_{\chi\in A} \hat{f}(\chi)\chi(\omega) = \sum_{\chi\in A} \int_{\mathbb{D}_2^\infty} f(\boldsymbol{\eta})\chi(-\boldsymbol{\eta})\, d\boldsymbol{\eta}\chi(\omega) \\ &= \int_{\mathbb{D}_2^\infty} \sum_{\chi\in A} f(\boldsymbol{\eta})\chi(\omega - \boldsymbol{\eta})\, d\boldsymbol{\eta} = \int_{\mathbb{D}_2^\infty} f(\boldsymbol{\eta}) D_A(\omega - \boldsymbol{\eta})\, d\boldsymbol{\eta}, \end{aligned}$$

(ii) If $\boldsymbol{\eta} \in H_n$, then $\chi(\boldsymbol{\eta}) = 1$ so that

$$D_{H_n}(\boldsymbol{\eta}) = \sum_{\chi\in H_n} 1 = 2^n$$

If $\boldsymbol{\eta} \notin H_n$, then we can find an $1 \leq r \leq n$ such that $\eta_r = 1$. Now

$$\chi(\omega) = \left(\prod_{j\neq r, 1\leq j\leq n} \zeta_j^{\omega_j}\right)\zeta_r^{\omega_r}$$

where $\zeta_j \in \{1, -1\}$. We observe that 2^{n-1} members χ of H_n have $\zeta_r = 1$ and 2^{n-1} members χ of H_n have $\zeta_r = -1$. Thus

$$D_{H_n}(\boldsymbol{\eta}) = 2^{n-1} - 2^{n-1} = 0.$$

(iii) By (i) and (ii),

$$\begin{aligned} S_{H_n}(f, \omega) &= \int_{\mathbb{D}_2^\infty} f(\boldsymbol{\eta}) D_{H_n}(\omega - \boldsymbol{\eta})\, d\boldsymbol{\eta} = \int_{\mathbb{D}_2^\infty} f(\omega)\mathbb{I}_{\boldsymbol{\eta}+H_n})\, d\omega \\ &= C_{\boldsymbol{\eta}+H_n}\mathbb{I}_{\boldsymbol{\eta}+H_n} \end{aligned}$$

(iv) Let $\epsilon > 0$. Since f is continuous and so uniformly continuous, we can find an N such that if $n \geq N$ $|f(\omega) - f(\omega')| < \epsilon$ for all $\omega \in J$ and all $J \in G_n$. Thus fixing n with $n \geq N$ for the time being, we can find a_J with

$$a_J - \epsilon \leq f(\omega) \leq a_J + \epsilon$$

for all $J \in G_n$. We now have

$$a_J - \epsilon \leq C_J \leq a_J + \epsilon$$

so, if $\boldsymbol{\eta} \in J$,

$$|S_{G_n}(f, \boldsymbol{\eta}) - f(\boldsymbol{\eta})| \leq |f(\boldsymbol{\eta}) - a_J| + |a_J - C_J| \leq 2\epsilon.$$

Since J was arbitrary,

$$|S_{G_n}(f, \boldsymbol{\eta}) - f(\boldsymbol{\eta})| \leq 2\epsilon$$

for all $\boldsymbol{\eta}$. Since ϵ was arbitrary,

$$S_{G_n}(f, \boldsymbol{\eta}) \to f(\boldsymbol{\eta})$$

uniformly on $\mathbb{D}_2^\infty$ as $n \to \infty$.

Exercise 4.8.18

We have

$$\|f - \sum_{\chi \in A} \hat{f}(\chi)\|_2^2 = \|f\|_2^2 - \sum_{\chi \in A} |\hat{f}(\chi)|^2$$

for any finite $A \subseteq \widehat{\mathbb{D}_2^\infty}$. Given $\epsilon > 0$ we can find an M such that

$$\|f - \sum_{\chi \in G_M} \hat{f}(\chi)\|_\infty \leq \epsilon^{1/2}$$

so

$$\|f\|_2^2 - \sum_{\chi \in A} |\hat{f}(\chi)|^2 = \|f - \sum_{\chi \in G_M} \hat{f}(\chi)\|_2^2 \leq \epsilon$$

The conditions on $A(j)$ imply that we can find an N such that $A_n \supseteq G_M$ for all $n \geq N$.

Now

$$\sum_{\chi \in A_n} |\hat{f}(\chi)|^2 \geq \sum_{\chi \in G_M} |\hat{f}(\chi)|^2$$

so

$$0 \leq \|f - \sum_{\chi \in A(n)} \hat{f}(\chi)\|_2^2 = \|f\|_2^2 - \sum_{\chi \in A(n)} \hat{f}(\chi)|^2 \leq \|f\|_2^2 - \sum_{\chi \in G_M} |\hat{f}(\chi)|^2 \leq \epsilon$$

for all $n \geq N$, so we are done.

EXERCISE 4.9.4

(i) $\{1/4, 3/4\}$

(ii) If $E = \{(x, y) : x \in [1/4, 3/4],\ y = 1/2\}$ then, if $(x, y) \in E$, we have $(x, y) = (x, 1/2)$ and

$$B((x, y), \delta) \cap E = \{(1/2, t), :, \text{with } t \in [0, 1] \cap (x - \delta, x + \delta)\}$$

contains more than one point.

(iii) Suppose that the statement in the first sentence is false. Then $[s, t] \subseteq E$ and $B((s + t)/2, (t - s)/2) \subseteq E$ so E is not perfect. Thus the statement must be true.

If $x \in E$ and $x \neq \{0, 1\}$, then, given $\delta > 0$, we can find $b < x < a$ with $a, , b \notin E$ and $b - x,\ x - a < \delta$ Setting $U = (a, b)$, $V = [0, 1] \setminus V$ we see that that the conditions of the definition are satisfied. The cases $x = 0$ and $x = 1$ are dealt with similarly.

EXERCISE 4.9.6★

EXERCISE 4.9.7

(i) $1 > \lambda_1 \geq \lambda_1\lambda_2 \ldots \lambda_n \geq \lambda \geq 0$ so $1 > \lambda \geq 0$.

(ii) $\lambda_1\lambda_2 \ldots \lambda_n = (2/3)^n \to 0$.

(iii) Let $\beta = (1 - \alpha)/2$, $\lambda_1 = \alpha + \beta$ and

$$\lambda_j = \frac{\alpha + \beta^j}{\alpha + \beta^{j-1}} \text{ for } j \geq 2$$

Then $1 > \lambda_j > 0$ and

$$\lambda_1\lambda_2 \ldots \lambda_n \alpha + \beta^n \to \alpha.$$

EXERCISE 4.9.9

$x = 0$, $u = 1$, $v = -1$, $E = \{x\}$.

Exercise 4.9.10

(i) If $x,\ y \in X$ and $e \in E$, then

$$d(x,e) + d(x,y) \geq d(y,e)$$

so taking the supremum over $e \in E$, $d(x,E) + d(x,y) \geq d(y,E)$. Similarly $d(y,E) + d(x,y) \geq d(x,E)$ so

$$|d(x,E) - d(y,E)| \leq d(x,y)$$

and so the map $x \mapsto d(x,E)$ is continuous.

(ii) Choose $e \in E$ such that $d(e,F) = \inf_{y\in E} d(y,F)$. The function $\alpha : F \to \mathbb{R}$ given by $\alpha(x) = d(x,F)$ is continuous and E is compact so α attains its minimum at $f \in F$ say. Thus $d(e,f) = d(e,F) = \inf_{y\in E} d(y,F)$.

(iii) No. Work in $[0,1]$ with the usual metric. Let $E = [3/8, 5/8]$ $F = \{1/4, 3/4\}$. Then, taking $e_1 = 3/8$, $e_2 = 5/8$ $f_1 = 1/4$ and $f_2 = 3/4$,

$$d(e_1,f_1) = d(e_2,f_2) = 1/8 = \inf_{y\in E} d(y,F).$$

Exercise 4.9.11

(i) If $x \in E$, $y \in F$, then $d(x,y) \geq 0$, so $d(x,F) \geq 0$ for all $x \in E$ and thus $\tau(E,F) \geq 0$.

(ii) Suppose $x \in E \cap F$. Since $d(x,x) = 0$, we have $d(x,F) = 0$ so $\tau(E,F) = 0$.

(iii) We have

$$\tau(E,F) = \inf_{x\in E}\inf_{y\in F} d(x,y) = \inf_{y\in F}\inf_{x\in E} d(x,y) = \inf_{x\in E}\inf_{y\in F} d(y,x) = \tau(F,E).$$

(iv) Work in $\mathbb{R}$ with the usual metric. Take $E = \{0,1\}$, $F = \{1,2\}$, $G = \{2,3\}$.

EXERCISE 4.9.12

Observe that, if $c \in E$ then

$$2\max\{d(a,c), d(b,c)\} \geq d(a,c) + d(b,c) \geq d(a,b)$$

so $\tau_2(E,E) \geq \tau_1(E,E) \geq d(a,b)/2$.

EXERCISE 4.9.13

Much as in Exercise 4.9.10 observe that $x \mapsto d(x,E)$ is continuous so (since E is compact) the map is bounded and attains its bounds.

EXERCISE 4.9.14

(i) $d(x,y) \geq 0$ for all x, y so

$$\sigma(E,F) = \sup_{x\in E}\inf_{y\in F} d(x,y) \geq 0.$$

(ii) If $e \in E \setminus F$, then $\sigma(E,F) \geq d(e,F) > 0$. Thus

$$\sigma(E,F) = 0 \Rightarrow E \subseteq F$$

On the other hand,

$$E \subseteq F \Rightarrow \inf_{y\in F} d(x,y) = d(x,x) = 0 \forall x \in E \Rightarrow \sigma(E,F) = 0,$$

so we have the required result.

(iii) Work in $[0,1]$ with the usual metric. If $E = \{1/2\}$, $F = \{1/4, 1/2\}$, then $\sigma(E,F) = 0$, $\sigma(F,E) = 1/4$.

EXERCISE 4.9.17

(i) $\rho(E,F) = \sigma(E,F) + \sigma(F,E) \geq 0 + 0 = 0$.

(ii) $\rho(E,F) = 0 \Leftrightarrow \sigma(E,F) = \sigma(F,E) = 0 \Leftrightarrow E \subseteq F,\ F \subseteq E \Leftrightarrow E = F$.

(iii) $\rho(E,F) = \sigma(E,F) + \sigma(F,E) = \sigma(F,E) + \sigma(E,F) = \rho(F,E)$.

(iv) We have

$$\begin{aligned}\rho(E,G) = \sigma(E,G) + \sigma(G,E) &\leq \sigma(E,F) + \sigma(F,G) + \sigma(G,F) + \sigma(F,E)\\ &= \sigma(E,F) + \sigma(F,E) + \sigma(F,G) + \sigma(G,F) = \rho(E,F) + \rho(F,G)\end{aligned}$$

Exercise 4.9.27

If two spaces are homeomorphic to the same space, then they are homeomorphic to each other.

If E is compact totally disconnected set in a metric space (X, d), then (E, d_E) with d_E the restriction of d to E^2 is a compact totally disconnected space.

Exercise 4.9.29

U is open so $V = U^c$ is closed and vice versa.

Exercise 4.9.31

(i) If $x \in W_j$, then, since W_j is open, we can find an $\eta > 0$ such that $B(x, \eta) \subseteq W_j$. Since X is perfect we can find a $y \in X$ such that $y \in B(x, \eta)$ and $y \neq x$. Automatically $y \in W_j$. Thus W_j is perfect.

If x, $y \in W_j$ and $x \neq y$. Then, since X is totally disconnected, we can find U, V open with $U \cap V = \emptyset$, $U \cup V = X$, $x \in U$ and $Y \in V$. Setting $U' = U \cap W_j$, $V' = V \cap W_j$, we have U', V' open subsets of W_j with $U' \cap V' = \emptyset$, $U' \cup V' = X$, $x \in U'$ and $Y \in V'$. Thus W_j is totally disconnected.

(ii) If W is clopen perfect and totally disconnected then we can find disjoint clopen sets U and V with $U \cap V = W$. Since U and V are clopen perfect and totally disconnected we can find disjoint clopen sets V' and V' with $U' \cap V' = V$. By repeating this process we can find k disjoint clopen sets U_r such that $\bigcup_{r=1}^{k} U_r = W$.

Now take the W_j constructed in Lemma 4.9.30 and find $m - N - 1$ disjoint clopen sets U_j $[N \leq j \leq M]$ with $\bigcup_{j=N}^{M} U_j = W_N$. If we set $U_j = W_j$ $[1 \leq j \leq N - 1]$ we have the required objects.

(iii) Take $M = 2^m$ in (ii). Let $I_{\mathbf{w}} = U_{\theta(\mathbf{w})}$ where $\theta(\mathbf{w}) = 1 + \sum_{r=1}^{m} w_r 2^r$ and set $I_{\mathbf{u}}(r) = I_{(\mathbf{u},0)}(r + 1) \cup I_{(\mathbf{u},1)}(r + 1)$ for all $\mathbf{u} \in \mathbb{D}_2^r$ and all $1 \leq r \leq m - 1$.

Exercise 5.1.1

In the notation of Theorem 1.4.3

$$x'(t) = f(x(t), t) \text{ with } x(0) = 0,$$

where $f(x, t) = x^{2/3}$. But, then,

$$\frac{|f(u, 0) - f(0, 0)|}{|u|} = |u|^{1/3} \to \infty$$

so the Lipschitz condition breaks down in any open interval containing 0.

Exercise 5.1.2

$$|f(x, y)| \leq (1 + R^2) \text{ for } |X| \leq R.$$

We have, by the function of a function rule,

$$G'(t) = g'(\tan t)\tan'(t) = \frac{1 + \tan^2 t}{1 + \tan^2 t} = 1$$

so

$$\frac{d}{dt}(G(t) - t) = 0$$

and, by the mean value theorem,

$$G(t) - t = G(0) - 0 = 0$$

for all $t|t| < \pi/2$. Thus $g(\tan t) = t$ for all $|t| < \pi/2$, and $g(t) = \tan^{-1}(t)$ for $|t| < \pi/2$. Thus $g(t) \to \infty$ as $t \to \frac{\pi}{2}$ from below and our equation has no solution on $\mathbb{R}$.

Exercise 5.1.4

(i) Since $f_{,1}(x,t)$ is continuous on the compact set $[x_0-1,x_0+1]\times[t_0-1,t_0+1]$, it is bounded by M_0, say, so, by the mean value theorem,

$$|f(u,t)-f(v,t)|\leq M_0|u-v|$$

for all $t\in[t_0-\delta_0,t_0+\delta_0]$ and all $u,\ v\in[x_0-\eta_0,x_0+\eta_0]$.

(ii) The if part is automatic. To prove the only if part observe that, if f is locally Lipschitz, then, given (y,s), we can find $\eta_{y,s},\delta_{y,s}>0$ and $K_{y,s}>0$ such that, writing

$$R_{y,s}=[y-4\eta_{y,s},+4\eta_{y,s}]\times[s-\delta_{y,s}/4,s+\delta_0],$$

we have

$$(u,t),(v,t)\in R_{y,s}\Rightarrow|f(u,t)-f(v,t)|\leq K_{y,s}|u-v|.$$

The open sets $U_{y,s}=(y-\eta_{y,s},y+\eta_{y,s})\times(s-\delta_{y,s},s+\delta_{y,s})$ cover the compact set $E=[-a,a]\times[-a,a]$, so we can find a finite set of points $(y_j,s_j)\in E$ with $1\leq j\leq n$ such that $\bigcup_{1\leq j\leq n}U_{y_j,s_j}\supseteq E$. Take $K_a=\max_{1\leq j\leq n}K_{y_j,s_j}$ and $\delta_a=\min_{1\leq j\leq n}\delta_{y_j,sj}$. Then, if $t\in[-a,a]$, $|u|$, $|v|\leq a$ and $|u-v|\leq\delta_a$, we know that $(t,u),(t,v)\in R_{yr,s_r}$ for some $1\leq r\leq n$ so

$$|f(u,t)-f(v,t)|\leq K_{y_r,s_r}|u-v|\leq K_a|u-v|.$$

Finally, we observe that, if $t\in[-a,a]$, $|u|$, $|v|\leq a$, then, without loss of generality, we may suppose that $u\leq v$. We can find

$$u=u_1\leq u_2\leq\ldots\leq u_m=v\text{ with }u_{p+1}-u_p<\delta_a$$

and so

$$|f(u,t)-f(v,t)|\leq\sum_{p=1}^{m-1}|f(u_{p+1},t)-f(u_p,t)|\leq\sum_{p=1}^{m-1}K_a|u_{p+1}-u_p|=K_a|u-v|.$$

Exercise 5.1.5

(i) Since g is is continuous and $[-b,b]\times[-b,b]$ is compact, g is uniformly continuous and bounded by M, say. Automatically $|g(x,y)|\leq M$ for $(x,y)\in[-b,b]\times\mathbb{R}$ and so $|f(x,y)|\leq M$ for all $(x,y)\in\mathbb{R}^2$.

Now suppose $b>\epsilon>0$. Then by uniform continuity we can find a $\delta>0$ such that

$$|g(x,y)-g(x',y')|<\epsilon/4$$

whenever $\|(x,y)-(x',y')\|<\delta$, and $(x,y),(x',y')\in[-b,b]\times[-b,b]$.

If $(x,y),(x',y')\in[-b,b]\times\mathbb{R}$ and $y\geq -b\leq y'$ and $\|(x,y)-(x',y')\|<\delta$, then $\|(x,y)-(x,b)\|,\|(x,b)-(x',y')\|<\delta$ and so

$$|h(x,y)-h(x',y')|\leq|h(x,y)-h(x,b))+|h(x,b)-f(x',y')|\leq\epsilon/2$$

A similar or easier argument applies to the other possible cases so

$$|h(x,y)-h(x',y')|\leq\epsilon/2$$

whenever $\|(x,y)-(x',y')\|<\delta$, and $(x,y),(x',y')\in[-b,b]\times[-b,b]$. Essentially the same arguments now give the required result for f.

(ii) First observe that that $h(u,t)-h(v,t)|\leq K|u-v|$, for all $|u|$, $|v|\leq b$ and all t since, for example, if $t\geq b$ we have

$$|h(u,t)-h(v,t)|=|h(u,b)-h(v,b)|\leq K|u-v|.$$

We now prove that $|f(u,t)-b_b(v,t)|\leq K|u-v|$, by checking cases. For example, if $v\leq -b\leq u\leq b$ we have

$$|f(u,t)-f(v,t)|=|h(u,t)-h(-b,t)|\leq K|u-(-b)|\leq K|u-v|.$$

Exercise 5.1.6

Observe that, if $g(t) \geq 0$, we have

$$\frac{d}{dt}e^{-Kt}(g(t)+1) = e^{-Kt}\big(g'(t)-K(g(t)+1)\big) \leq e^{-Kt}\big(f(g(t),t)-K(g(t)+1)\big) \leq 0$$

and if $g(t) \leq 0$

$$\frac{d}{dt}e^{-Kt}(-g(t)+1) = e^{-Kt}\big(-g'(t)-K(-g(t)+1)\big) \geq e^{-Kt}\big(-f(g(t),t)-K(-g(t)+1)\big) \geq 0.$$

Thus $e^{-Kt}(|g(t)|+1)$ is a decreasing function, and

$$|x_0|+1 = e^0(|g(0)|+1) \geq e^{-Kt}(|g(t)|+1)$$

whence

$$(|x_0|+1)e^{Kt} \geq |g(t)|+1 \geq |g(t)|$$

for all $t \geq 0$.

A similar argument shows that

$$(|x_0|+1)e^{Kt} \geq |g(t)|+1 \geq |g(t)|$$

for all $t \leq 0$ and the final sentence of the exercise follows by translation.

Exercise 5.1.8★

Exercise 5.1.9

We have $f(u,t) = (1+u^2)$, but

$$\frac{1+u^2}{1+|u|} \to \infty \text{ as } u \to \infty$$

Exercise 5.1.10

Let
$$f(u,t) = \begin{cases} u^\beta & \text{for } u \geq 0 \\ -(-u)^\beta & \text{for } u < 0 \end{cases}$$
Since
$$\frac{f(u,t)}{u} \to 0 \text{ as } u \to 0$$
and using standard rules for $u \neq 0$
$$\frac{\partial f}{\partial u}(u,t) = \begin{cases} \beta u^{\beta-1} & \text{for } u \geq 0 \\ -\beta(-u)^{\beta-1} & \text{for } u < 0 \end{cases}$$
so f is continuously differentiable everywhere and so locally Lipschitz. Automatically $|f(u,t)| \leq (1+|u|)^\beta$.

Suppose $g'(t) = f(g(t),t)$ in some open interval containing 0 and that $g(0) > 0$. We have (in that interval)
$$-1 = \frac{g'(t)}{g(t)^\beta} = \frac{d}{dt}\frac{1}{\gamma g(t)^\gamma}$$
with $\gamma = \beta - 1$, so
$$-t + C = \frac{1}{\gamma g(t)^\gamma}$$
with $C = (\gamma g(0)^\gamma)^{-1}$. Thus we have solution in $(-\infty, C)$, but $g(t) \to \infty$ as $t \to C$. If $g(0) < 0$, a symmetry argument shows that again no solution exists for all t.

Exercise 5.1.12

(i) The condition restricts growth of x so it does not escape through the upper or lower sides of the rectangle.

(ii) Follow the proof of Lemma 5.1.7 with $x_0 = 0$, $S = \delta$, $T = \eta$ using the mean value theorem to give $|g(t)| < \eta$ for $|t| \leq \eta$.

Exercise 5.1.14

Observe that $f_n(u,t) = h_n(u)$ where h_n is continuous and piecewise differentiable with left and right derivatives at the finite number of points where it is not differentiable. Further $h'_n(u) \to 0$ as $|u| \to \infty$. It follows that the functions f_n are continuous and we can find a $K_n > 0$ such that

$$|f_n(u,t) - f_n(v,t)| \leq K_n |u - v|$$

for all $u,\ v,\ t \in \mathbb{R}$.

We have, by easy calculation or guesswork followed by verification,

$$g_{2n-1}(t) = 3n^{1/2}|t| \text{ for } |t| \leq n^{-1}$$

so $g_{2n-1}(n^{-1}) = g_{2n-1}(-n^{-1}) = 3n^{-1/2}$ and

$$g_{2n-1}(t) = |t|^{3/2} \text{ for } t \geq n^{-1}.$$

By inspection

$$g_{2n}(t) = |t| \text{ for all } t.$$

If we take $G_1(t) = |t|^{3/2}$ and $G_2(t) = 0$ the remaining statements are true by inspection.

Exercise 5.2.3

Write $\Omega_1 \sim \Omega_2$ if Ω_1 and Ω_2 are conformally equivalent.

If the identity map $\iota : \Omega \to \Omega$ is a conformal mapping, so $\Omega \sim \Omega$.

If $f : \Omega_1 \to \Omega_2$ is conformal the f has inverse f^{-1} and $f^{-1}$is conformal. Thus $\Omega_1 \sim \Omega_2 \Rightarrow \Omega_2 \sim \Omega_1$.

If $f : \Omega_1 \to \Omega_2$ is conformal and $g : \Omega_2 \to \Omega_3$ the $g \circ f$ is a bijective analytic map $\Omega_1 \to \Omega_3$ whose inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is also analytic. Thus

$$\Omega_1 \sim \Omega_2,\ \Omega_2 \sim \Omega_3 \Rightarrow \Omega_1 \sim \Omega_3.$$

Thus conformal equivalence is indeed an equivalence relation.

Exercise 5.2.6

Suppose $f : \Omega_1 \to \Omega_2$ is a conformal map and Ω_1 is simply connected.

If $z, w \in \Omega_2$, then $f^{-1}(z), f^{-1}(w) \in \Omega_1$ so we can find a continuous $\gamma : [0, 1] \to \Omega_1$ with $\gamma(0) = f^{-1}(z), \gamma(1) = f^{-1}(w)$. Now $f \circ \gamma$ is a continuous function from $[0, 1]$ into Ω_2 with $f \circ \gamma(0) = z$ and $f \circ \gamma(1) = w$. Thus Ω_2 is path connected.

Let us write $F = f^{-1}$. If Γ is closed contour in Ω_2 let C be the closed contour $F \circ \Gamma$. If $g : \Omega_2 \to \mathbb{C}$ is analytic, then

$$\int_\Gamma g(z)\,dz = \int_C g(F(w))F'(w)\,dw = 0$$

since the map $w \mapsto g(F(w))F'(w)$ is analytic and Ω_1 is simply connected. Thus Ω_2 is simply connected.

Exercise 5.2.7

(i) Let C be the closed contour formed by C_1 followed by C_2 reversed. Since $z \mapsto 1/z$ is analytic in Ω and Ω is simply connected,

$$0 = \int_C \frac{dw}{w} = \int_{C_1} \frac{dw}{w} - \int_{C_2} \frac{dw}{w}$$

and

$$\int_{C_1} \frac{dw}{w} = \int_{C_2} \frac{dw}{w}.$$

Thus the definition given is unambiguous once α_0 is chosen.

(ii) Let Γ be a path within Ω from z_0 to z and Γ' the path γ followed by $\Gamma(h)$.

$$L(z+h) - L(z) = \int_{\Gamma'} \frac{dw}{w} - \int_{\Gamma} \frac{dw}{w} = \int_{\Gamma(h)} \frac{dw}{w}$$

so

$$L(z+h) - L(z) - \frac{h}{z} = \int_{\Gamma(h)} \left(\frac{1}{w} - \frac{1}{z}\right) dw$$

whence

$$\begin{aligned}\left|[L(z+h) - L(z) - \frac{h}{z}\right| &\leq \text{length}(\Gamma(h)) \times \sup_{w \in \Gamma(h)} \left|\frac{1}{w} - \frac{1}{z}\right| \\ &\leq |h| \left|\frac{1}{z+h} - \frac{1}{z}\right| \to 0\end{aligned}$$

as $h \to 0$. Thus L is analytic on Ω with

$$L'(z) = \frac{1}{z}.$$

(iii) We have

$$\frac{d}{dz}\frac{\exp L(z)}{z} = \frac{L'(z)\exp L(z)}{z} - \frac{\exp L(z)}{z^2} = \frac{\exp L(z)}{z^2} - \frac{\exp L(z)}{z^2} = 0,$$

so

$$\frac{\exp L(z)}{z} = \frac{\exp L(z_0)}{z_0} = \exp \alpha_0 z_0 = 1$$

and $\exp L(z) = z$ for all $z \in \Omega$.

(iv) The composition of analytic functions is analytic so S is analytic on Ω. We have

$$S(z)^2 = \exp(2L(z)/2) = z.$$

(v) Observe that S_1 and S_2 are continuous and never zero. Thus S_1/S_2 is continuous. Further

$$\left(\frac{S_1(z)}{S_2(z)}\right)^2 = 1$$

so $S_1(z)/S_2(z) \in \{-1, 1\}$. If $z, , w \in \Omega$, then we can find $\gamma : [0, 1] \to \Gamma$ continuous with $\gamma(0) = z$, $\gamma(1) = w$.

$$\frac{S_1 \circ \gamma(t)}{S_2 \circ \gamma(t)} \mapsto \{-1, 1\}$$

is a continuous function taking only two values so is constant. Thus $S_1/S_2 = 1$ or $S_1/S_2 = -1$.

(vi) Similar to (v). We have

$$\exp(L_1(z) - L_2(z)) = 1$$

so $L_1(z) - L_2(z) \in 2\pi i\mathbb{Z}$. Taking γ as in (iv) we have

$$\frac{1}{2\pi i}(L_1 \circ \gamma(t) - L_2 \circ \gamma(t)$$

a continuous map of $[0, 1]$ into the integers. It must therefore be constant and the stated result follows.

Exercise 5.2.10★

Exercise 5.2.13

$S = T_2 T_1^{-1}$.

Take $U = T_2^{-1} P T_1^{-1}$ so $P = T_2 U T_1$.

Exercise 5.2.15

If θ is real,

$$\left|\frac{a-e^{i\theta}}{a^*e^{i\theta}-1}\right| = \left|\frac{a-e^{i\theta}}{a^*-e^{-i\theta}}\right| = \frac{|a-e^{i\theta}|}{|(a-e^{i\theta})^*|} = 1$$

Thus, since T_a maps circles to circles or straight lines, T_a maps the unit circle to itself. Since $T_a(0) = a$ it follows that T_a maps the interior of the unit disc to itself, so T_a is conformal map of D itself. We have $T_a(0) = a$ and $T_a(a) = 0$ by direct calculation.

We have

$$T_a'(z) = \frac{1}{a^*z-1} - a^*\frac{z-a}{(a^*z-1)^2}$$

so

$$T_a'(0) = -1 + a^*a = |a^2| - 1.$$

If $|a| > 1$ it remains true (with the same proof) that T_a maps the unit circle to the unit circle, but now the interior point 0 is mapped to the exterior point a. Thus T_a maps $D \setminus \{1/a^*\}$ conformally to the 'exterior', $\bar{D}^c = \{z \in \mathbb{C} : |z| > 1\}$ of D. (We sometimes say that $1/a^*$ goes to ∞.)

If $|a| = 1$ then $a = e^{i\phi}$ for some real ϕ and $T_a(z) = e^{i\phi} = a$ for all $z \neq a$ (and $T_a(a)$ is undefined).

Exercise 5.2.17

Observe that

$$T_{a,\theta}(z) = 0 \Leftrightarrow z = a$$

so

$$T_{a,\theta} = T_{b,\phi} \Rightarrow T_{b,\phi}(a) = 0 \Rightarrow b = a.$$

Further if $a \neq 0$

$$T_{a,\theta} = T_{a,\phi} \Rightarrow T_{a,\theta}(0) = T_{a,\phi}(0) \Rightarrow ae^{i\theta} = ae^{i\phi} \Rightarrow \theta - \phi \equiv 0 \pmod{2\pi}$$

This proves the only if statement. The converse is immediate.

Exercise 5.2.18

(i) Let $S(a) = b$. We have shown that there is a Möbius map T_b which maps D conformally to D and b to 0. Choose $\theta \in \mathbb{R}$ with $\exp(i\theta)(T_b S)'(0)$ real and positive and set $RT(z) = \exp(i\theta)T_b S$, $V = R(U)$.

(ii) Observe that $Q = T_1 T_2^{-1}$ gives a conformal map of D to itself with $Q(0) = 0$ and

$$Q'(0) = \frac{1}{T_2'(a)} T_1'(a) = 1.$$

Thus, by Schwarz's lemma, $Q(z) = z$ and $T_1 = T_2$.

(iii) Let $P = RT^{-1}$. Then $P : D \to D$ is an analytic function with $P(0) = 0$

$$P'(0) = \frac{R'(a)}{T'(a)}.$$

By Schwarz's lemma $|P'(0)| \leq 1$ with equality only if P is a rotation. The stated result follows.

Exercise 5.2.20

Take $T_1(z) = z^{1/2}$ where we define $(r\exp(i\theta))^{1/2} = r^{1/2}\exp(i\theta/2)$ for $r > 0$ (choosing $r^{1/2} > 0$) and $2\pi > \theta > 0$. Then T_1 maps Ω conformally to

$$\Omega_1 = \{z : |z| < 1,\ \Im(z) > 0\}.$$

Take $T_2(z) = 1/z$. Then T_2 maps Ω_1 conformally to

$$\Omega_2 = \{z : \Re(z) > 1/2,\ \Im(z) < 0\}.$$

Take $T_3(z) = (z - 1/2)^2$. Then T_3 maps Ω_2 conformally to

$$\Omega_3 = \{z : \Im z < 0\}.$$

Take $T_4(z) = (z - i)^{-1}$. Then T_4 maps Ω_3 conformally to

$$\Omega_4 = \{z : |z - i/2| \leq 1/2\}.$$

Take $T_5 = 2(z + i/2)$. Then T_5 maps Ω_5 conformally to D.

Setting $T = T_5T_4T_3T_2T_1$ gives the desired conformal map.

By inspection there is open sets U and V with $U \supseteq \{-1/2, 1/2\}$ such that $T_5T_4T_3T_2$ is a continuous bijection from U to V with continuous inverse. Thus, if S has the properties stated, then there will be a continuous function

$$T : \Omega \cup \{1/4\} \to \mathbb{C}$$

with $T(z) = z^{1/2}$ for all $z \in \Omega$. Now

$$T(1/4 + \eta) = (1/4 + i\eta)^{1/2} \to 1/2 \text{ and } S(1/4 - i\eta) \to -1/2$$

as $\eta \to 0$ though positive real values so T is not continuous.

The first stated result follows by reductio ad absurdum.

To obtain the final result recall that there is Möbius map

$$Q(z) = \exp(i\theta)\frac{z - a}{a^*z - 1}$$

with $|a| < 1$ such that $S = Q\tilde{T}$. Since Q is a well defined continuous function then if $\tilde{T}$ had a continuous extension to $\tilde{t}$ on $\mathrm{Cl}(\Omega)$, T would have the continuous extension $T = Q\tilde{T}$

Exercise 5.3.1

(i) The rationals are countable and countable unions of countable sets are countable so $\mathcal{B}$ is countable.

If $z \in \Omega$, then, since Ω is open, we can find a $\delta > 0$ such that $D(z, 3\delta) \subseteq \Omega$. Since the rationals are dense in $\mathbb{R}$ we can find $u,\ v \in \mathbb{C}$ such that

$$|(u + iv) - z| < \delta.$$

Thus $z \in D(u+iv, \delta)$ and $D(u+iv, 2\delta) \subseteq \Omega$. We have shown that $\bigcup_{B\in\mathcal{B}} B = \Omega$.

Let the members of $\mathcal{B}$ be $B_1, B_2, B_3, \ldots$. If we set $K_n = \bigcup_{j=1}^{n} \operatorname{Cl} B_j$, then $K_n \subseteq K_{n+1} \subseteq \Omega$ and K_n is closed and bounded so compact. We have

$$\bigcup_{j=1}^{n} B_j = \int K_n \subseteq \Omega \text{ and } \bigcup_{j=1}^{\infty} B_j = \Omega$$

so $\bigcup_{n=1}^{\infty} \operatorname{Int} K_n = \Omega$.

(ii) $L \subseteq \Omega = \bigcup_{n=1}^{\infty} \operatorname{Int} K_n$ so, by the compactness of L, we can find an N such that

$$L \subseteq \bigcup_{n=1}^{N} \operatorname{Int} K_n \subseteq K_N.$$

Exercise 5.3.2

The first part is routine. We have $\left|2^{-m}\max\{1, \sup_{z\in K_m}|f(z)-g(z)|\right| \leq 2^{-m}$ so $\sum_{m=1}^{\infty} 2^{-m}\max\{1, \sup_{z\in K_m}|f(z)-g(z)|\}$ converges by comparison. Thus $d_{\mathcal{K}}$ is well defined.

$$d_{\mathcal{K}}(f,g) = 0 \Rightarrow \sup_{z\in K_n}|f(z)-g(z)| \text{ for all } n$$
$$\Rightarrow f(z) = g(z) \text{ for } z \in K_n, \text{ for all } n \Rightarrow f(z) = g(z) \text{ for } z \in \Omega.$$

The verification that $d_{\mathcal{K}}(f,f) = 0$, $d_{\mathcal{K}}(f,g) = d_{\mathcal{K}}(g,f)$ is immediate. Since

$$|f(z)-g(z)| + |g(z)-h(z)| \geq |f(z)-h(z)|$$

we have

$$\max\{1, \sup_{z\in K_m}|f(z)-g(z)|\} + \max\{1, \sup_{z\in K_m}|g(z)-h(z)|\} \geq \max\{1, \sup_{z\in K_m}|f(z)-u(z)|\}$$

and $d_{\mathcal{K}}(f,g) + d_{\mathcal{K}}(g,h) \geq d_{\mathcal{K}}(f,h)$.

Suppose f_r is a Cauchy sequence in $A(\Omega), d_{\mathcal{K}})$. Then

$$|f_p(z) - f_q(z)| \leq 2^n d_{\mathcal{K}}(f_p, f_q) \to 0$$

as p, $q \to \infty$. Thus, by the general principle of uniform convergence, f_p converges uniformly on each $K_n$to a continuous function $f : \Omega = \bigcup_{n=1}^{\infty} K_n \to \mathbb{C}$. By Morrera's theorem, f is analytic on each $\operatorname{Int} K_n$ and so on $\bigcup_{n=1}^{\infty} K_n = \Omega$. Since that f_p converges uniformly on each K_n we have $d_{\mathcal{K}}(f, f_p \to 0$ as $p \to \infty$.

Exercise 5.3.3

We showed that $(i) \Leftrightarrow (ii)$ in the course of doing the previous exercise.

Now suppose L is a compact subset of Ω. We have $\bigcup_{n=1}^{\infty} \operatorname{Int} K_n = \Omega \supseteq L$ so, since L is compact, there must be an N such that $K_N \supseteq L$. Thus condition (i) implies condition (iii). Condition (iii) automatically implies condition (ii).

Exercise 5.3.4

We just apply exercise 5.3.3 repeatedly. Since condition (i) implies condition (iii) and condition (ii) implies condition (i),

$$\begin{aligned} d_{\mathcal{K}}(f_n, f) \to 0 &\Rightarrow f_n \to f \text{ uniformly on } L_j \text{ for each } j \\ &\Rightarrow d_{\mathcal{L}}(f_n, f) \to 0. \end{aligned}$$

Thus the identity map $\iota : (A(\Omega), d_{\mathcal{K}}) \to (A(\Omega), d_{\mathcal{L}})$ is continuous. Similarly ι^{-1} is continuous and so the identity map is a homeomorphism.

In much the same way,

$$d_{\mathcal{K}}(g_n, g_m) \to 0 \Rightarrow g_n - g_m \text{ uniformly on } L_j \text{ for each } j \quad \Rightarrow d_{\mathcal{L}}(g_n, g_m) \to 0.$$

Exercise 5.3.5

Suppose that $L \subseteq \Omega$ is compact. If $z \in L$ we can find a $\delta(z) > 0$ such that $D(z, 2\delta(z)) \subseteq \Omega$. Since $\bigcup_{z \in L} D(z, 4\delta(z)) \supseteq L$ compactness show that there are $z_1, z_2, \dots, z_M \in L$ such that $\bigcup_{m=1}^{M} D(z, \delta(z_m)) \supseteq L$. Setting $delta = \min_{1 \le m \le M} \delta_m$ we see that $D(z, 3\delta) \subset \Omega$ for all $z \in L$. Now $K = \operatorname{Cl}\left(\bigcup_{z \in L} D(z, 2\delta)\right)$ closed and bounded so compact and $K \subseteq \Omega$ so $f_n \to f$ uniformly on K.

Write $C(a)$ for the circular contour described by $z = a + \delta e^{i\theta}$ as θ runs from 0 to 2π By Cauchy's integral formula

$$\begin{aligned} |f_n'(a) - f'(a)| &= \left| \frac{1}{2\pi} \int_{C(a)} \frac{f_n(z) - f(z)}{(z-a)^2} \, dz \right| \\ &\le \frac{1}{2\pi} \operatorname{length} C(a) \times \sup_{z \in C(a)} \left| \frac{f_n(z) - f(z)}{(z-a)^2} \right| \\ &= \delta^{-1} \sup_{z \in C(a)} |f_n(z) - f(z)| \le \delta^{-1} \sup_{z \in K} \to 0. \end{aligned}$$

Thus f_n' converges uniformly to f' on L and, since L was arbitrary, the stated result follows.

Exercise 5.3.6

We use a standard compactness argument.

Suppose (ii) holds and we have a compact set $L \subseteq \Omega$. Since

$$\bigcup_{w\in L} D(w,\delta_w) \supseteq L$$

and L is compact, we can find $w_j \in L$ $[1 \leq j \leq N]$ with

$$\bigcup_{j=1}^{N} D(w,\delta_w) \supseteq L.$$

Taking $M(L) = \max_{1\leq j\leq N}^{N} \delta_{w_j}$, we see that (i) holds.

Suppose (i) holds. If $w \in \Omega$ then we can find a $\delta_w > 0$ such that $D(w, 2\delta_w) \subseteq \Omega$. Taking $L = \operatorname{Cl} D(w,\delta_w)$ and $M_w = M(L)$, we see that (i) holds.

Exercise 5.3.9

By the general principle of uniform convergence (or by repeating its proof more or less word by word), if (Y,ρ) is a metric space, the functions $f_n : Y \to \mathbb{C}$ are continuous $\sup_{y\in Y} \rho(f_n(y), f_m(y))$ exists (that is to say, is finite) for each n and m, then, if $\sup_{m\geq n\geq N} \sup_{y\in Y} \rho((f_n(y), f_m(y)) \to 0$, there exists a continuous function $f : Y \to \mathbb{C}$ with $\sup_{y\in Y} \rho(f_n(y), f(y)) \to 0$.

If a sequence $f_n \in X$ is Cauchy in $(X, \|\ \|_X)$, then the first paragraph tells us that the functions f_n satisfy the form of the general principle of uniform convergence given above and so $f_n \to f$ uniformly for some continuous function. But the uniform limit of analytic functions is analytic so f is analytic. It is immediate that $\sup_{|z|<1} |f(z)|$ is finite so $f \in X$ and $\|f_n - f\|_X \to 0$. Thus X is complete.

Let $e_n(z) = z^n$. If $n > m$

$$\|e_m - e_n\|_X = \sup_{|z|<1} |z^n - z^m| = \sup_{|z|<1} |z^m||z^{n-m} - 1| \geq \sup_{0\geq r<1} r^m(1 + r^{n-m}) \geq 2$$

so the sequence e_n does not converge, Thus X is not compact.

If $f_n \in Y$ and $f_n \underset{\|\ \|_X}{\to} f$, then

$$|f(z)| \leq \sup_{n\geq 1} |f_n(z)| \leq 1 \text{ for all } |z| < 1,$$

so Y is closed. The counter-example in the previous paragraph shows that Y is not compact.

Exercise 5.3.13

Note that P and Q are bijections of D to itself. We have

$$P(D),\ W(D),\ Q^1(D) \subseteq D$$

so $PWQ^{-1} \subseteq D$. However

$$PWQ^{-1}(Q(1/2)) = P(1/4) = PWQ^{-1}(Q(-1/2)),$$

and $Q(1/2) \neq Q(-1/2)$ so PWQ^1 is not injective. Schwartz's lemma tells us that if $PWQ^{-1}(0) = 0$, then $|(PWQ^{-1})'(0)| < 1$.

If $a \in Q(\Omega)$

$$a = QSP^{-1}(b) \Leftrightarrow Q^{-1}(a) = SP^{-1}(b) \Leftrightarrow WQ^{-1}(a) = P^{-1}(b) \Leftrightarrow PWQ^{-1}9b)$$

so $(QSP^{-1})^{-1} = PWQ^{-1}$. Now if $0 \in P(\Omega) \cap QS(\Omega)$, then

$$1 = (QSP^{-1})'(0) \times ((QSP^{-1})^{-1})'(0) = (QSP^{-1})'(0) \times (PWQ^{-1})'(0)$$

so $|(QSP^{-1})'(0)| > 1$. Taking $Q = T_{S(a)}^{-1}$, $P = T_a^{-1}$ gives the stated result.

Exercise 5.4.4

If E cannot be covered by a finite set of closed balls of radius ϵ then, since $\operatorname{Cl} E \supseteq E$, it follows ClE cannot. If E can be covered by a finite set of closed balls of radius ϵ, then the same set covers $\operatorname{Cl}(E)$

Exercise 5.4.5

(i) We observe that

$$\|f\|_* = 0 \Rightarrow \|f\|_\infty = 0 \Rightarrow f = 0.$$

The remaining conditions for a norm are immediate.

Now suppose that we have a Cauchy sequence f_n for $\| \ \|$. The f_n is Cauchy in the uniform norm ans converges in the uniform norm to a continuous function f, say. The continuous function $\frac{\partial f_n}{\partial x_j}$ is Cauchy in the uniform norm and so converges in the uniform norm to a continuous function g_j. We know that if a continuous function (in the variable t_j, say) converges uniformly and its continuous derivative converges uniformly then the function is continuously differentiable with derivative the uniform limit of the derivatives. Thus f has continuous partial derivative and

$$\frac{\partial f_n}{\partial x_j} \to \frac{\partial f}{\partial x_j}$$

uniformly as $n \to \infty$. A function with continuous partial derivatives is continuously differentiable and we have shown that $\|f_n - f\|_1 \to 0$. Thus we have a complete norm.

(ii) The mean value theorem tells us that

$$|f(\mathbf{t}) - f(\mathbf{s})| \leq \|Df\|_\infty d(\mathbf{s}, \mathbf{t}) \leq \|f\|_* d(\mathbf{s}, \mathbf{t})$$

EXERCISE 5.4.6

Direct calculation shows that $\| \|_A$ is a norm. For example

$$\|D(f+g)(\mathbf{t})\|_A = \|D(f)(\mathbf{t}) + D(g)(\mathbf{t})\|_A \leq \|D(f)(\mathbf{t})\|_A + \|D(g)(\mathbf{t})\|_A$$

so

$$\sup_{\mathbf{t}\in\mathbb{T}^n} \|D(f+g)(\mathbf{t})\|_A \geq \sup_{\mathbf{t}\in\mathbb{T}^n} \|D(f)(\mathbf{t})\|_A + \sup_{\mathbf{t}\in\mathbb{T}^n} \|D(g)(\mathbf{t})\|_A$$

so

$$\begin{aligned}\|f+g\|_{**} &= \|f+g\|_\infty + \sup_{\mathbf{t}\in\mathbb{T}^n} \|Df(\mathbf{t}) + Dg(\mathbf{t})\|_A \\ &\leq \|f\|_\infty + \|g\|_\infty + \sup_{\mathbf{t}\in\mathbb{T}^n} \|D(f)(\mathbf{t})\|_A + \sup_{\mathbf{t}\in\mathbb{T}^n} \|D(g)(\mathbf{t})\|_A = \|f\|_{**} + \|g\|_{**}\end{aligned}$$

Theorem 4.5.1 tells us that there exist constants $C_1, C_2 \geq 1$ such that

$$C_1\|D(f)f(\mathbf{t})\|_\infty \geq \|D(f)(\mathbf{t})\|_A \geq C_2\|D(f)f(\mathbf{t})\|_\infty$$

Taking suprema over all $\mathbf{t} \in \mathbb{T}^n$ and noting that $C_1\|f\|_\infty \geq \|f\|_\infty \geq C_2\|f\|_\infty$ gives the final result.

EXERCISE 5.4.8

We first show that $H(\epsilon, \bar{B})$ is invariant under translation

We use the assumptions and notation of Definition 5.4.3 to show that $H(\epsilon, E + h) = H(\epsilon, E)$ (where $h \in C(\mathbb{T}^n)$ and $E + h = \{g + h \,:\, g \in E\}$). This follows from the observation that

$$\bigcup_{m=1}^{M} \tilde{B}(f_j, \epsilon) \supseteq E \Leftrightarrow \bigcup_{n=1}^{M} \tilde{B}(f_j + h, \epsilon) \supseteq E + h.$$

We now turn to scaling and show that, with the assumptions and notation of Definition 5.4.3, $H(\epsilon, \rho E) = H(\rho\epsilon, E)$ (where $\rho > 0$ and $\rho E = \{\rho g \,:\, g \in E\}$. This follows from the observation that

$$\bigcup_{m=1}^{M} \tilde{B}(f_j, \epsilon) \supseteq E \Leftrightarrow \bigcup_{n=1}^{M} \tilde{B}(\rho f_j, \rho\epsilon) \supseteq \rho E.$$

This shows that (using Theorem 5.4.7) we have

$$H(\epsilon, \rho B_n) = H(\rho^{-1}\epsilon, B_n) \geq C_n \rho^n \epsilon^{-n}$$

and so by Exercise 5.4.4

$$H(\epsilon, \rho \bar{B}_n) \geq C_n \rho^n \epsilon^{-n}$$

for all $0 < \rho\epsilon < 1/2$. The full result now follows from the translational invariance proved in the first paragraph.

Exercise 5.5.1

R is well defined because continuous functions which are zero outside bounded interval are bounded.

(i) We have

$$\|F_q\|_* = \delta \sup_{\mathbf{t}\in\mathbb{R}^n} |F(t)| + 2\pi q \sum_{j=1}^{n} \sup_{\mathbf{t}\in\mathbb{R}^n} \left|\frac{\partial F}{\partial x_j}(\mathbf{t})\right| = q\|F\|_*,$$

so $F_q \subseteq \epsilon q\delta RB$.

(ii) We have

$$F_q(\mathbf{t} + 2\pi q^{-1}\mathbf{k}) = 0$$

for all $\mathbf{t} \notin I_{\mathbf{k}} = \prod_{j=1}^{q}(2\pi q^{-1}k_j, 2\pi q^{-1}(k_j+1))$ and

$$\|F_{q,\theta}\|_* \leq \delta qRB.$$

Thus $F_{q,\theta} \in \delta qRB$.

(iii) Further, if $\theta \neq \theta'$ there is a k such that $\theta_k \neq \theta'_k$. We may suppose $\theta_k = 1$ and $\theta_{k'} = -1$ so

$$F_{q,\theta}(\approx) - F_{q,\theta'}(\approx = 2Fq(2\pi q^{-1}k_j, 2\pi q^{-1}(k_j+1)$$

and

$$\|F_{q,\theta} - F_{q,\theta'}\|_\infty = 4\delta.$$

Thus the balls in $C(\mathbb{T}^n)$ with radius δ centres $F_{q,\theta}$ and $F_{q,\theta'}$ do not intersect. Since Θ_q has 2^q members we conclude that δqRB cannot be covered by 2^q uniform balls of radius δ.

(iv) By rescaling as in Exercise 5.4.8 B cannot be covered by 2^q uniform balls of radius $q^{-n}R^{-n}\delta^{-n}$.

(v) Thus if B can be covered by N uniform balls of radius $(qR\delta)^{-n}$, we have $N \geq 2^q$ so

$$\log N \geq q \log 2.$$

If η is sufficiently small we can always choose q and δ such that $q \geq \eta^{-1/n}\epsilon^{-1}$ and $qR\delta < \epsilon$. With these choices, $\log N \geq \eta\epsilon^{-n}$ and we have the result of Theorem 5.4.7.

Exercise 5.5.3

We have

$$\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} H_n(\mathbf{t})\,d\mathbf{t} = \frac{1}{(2\pi)^m}\int_{\mathbb{T}}\int_{\mathbb{T}}\dots\int_{\mathbb{T}}\prod_{k=1}^{m} J_N(t_k)\,dt_1dt_2\dots dt_m$$
$$= \prod_{j=1}^{m}\frac{1}{(2\pi)}\int_{\mathbb{T}} J_n(t_j)\,dt_j = 1$$

Similarly, if r is fixed with $1 \leq r \leq m$, then

$$\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} |t_r|H_n(\mathbf{t})\,d\mathbf{t}$$
$$= \prod_{j\neq r}\frac{1}{(2\pi)}\int_{\mathbb{T}} J_n(t_j)\,dt_j \times \frac{1}{(2\pi)}\int_{\mathbb{T}} |t_r|J_n(t_r)\,dt_r \leq Bn^{-1}$$

We know by Pythagoras (or squaring both sides) that $\|\mathbf{t}\| \leq \sum_{k=1}^{m}|t_k|$ so

$$\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} \|\mathbf{t}\|J_n(\mathbf{t})\,d\mathbf{t} \leq \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m}\sum_{j=1}^{m}|t_j|J_n(\mathbf{t})\,d\mathbf{t} \leq B^m n^{-1}.$$

(ii) Since H_n is a trigonometric polynomial in $C(\mathbb{T}^m)$ of degree $4(n-1)$,

$$Q(t) = H_n * f(t) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} f(\mathbf{t}-\mathbf{s})H_n(\mathbf{s})\,d\mathbf{s}$$

defines a real trigonometric polynomial in $C(\mathbb{T}^m)$ (that is to say a multinomial) of degree at most $4(n-1)$.

By the mean value theorem,

$$|f(\mathbf{t}) - f(\mathbf{s})| \leq \|Df\|_\infty|\mathbf{t}-\mathbf{s}|$$

so

$$|Q(\mathbf{f}) - f(\mathbf{t})| = \left|\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m}(f(\mathbf{t}-\mathbf{s}) - f(\mathbf{t}))H_n(\mathbf{s}\,d\mathbf{s}\right|$$
$$\leq \|Df\|_\infty\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m}|\mathbf{s}|J_n(\mathbf{s})\,d\mathbf{s} \leq Bn^{-1}.$$

(iii) If $n \leq 4$ and we take $Q = 0$ we have

$$\|Q - f\|_\infty \leq 4n^{-1}\|f'\|_\infty.$$

If $n > 4$ we take u to be the greatest integer with $4(u-1) \leq n$ and choose Q of degree at most $4(u-1)$ with

$$\|Q - f\|_\infty \leq Bu^{-1}\|f'\|_\infty.$$

We have $n \geq 8u$ so

$$\|Q - f\|_\infty \leq 8Bn^{-1}\|f'\|_\infty.$$

Exercise 5.5.4

(i) We have

$$\left\|\mathbf{x} - \frac{1}{1+\epsilon}\mathbf{y}\right\| = (1+\epsilon)^{-1}\|(1+\epsilon)\mathbf{x} - \mathbf{y}\|$$

$$\leq (1+\epsilon)^{-1}(\|\mathbf{x}-\mathbf{y}\| + \epsilon\|\mathbf{x}\|) \leq 2\epsilon$$

and $\|\mathbf{y}\| \leq 1+\epsilon$.

(ii) By scaling, we may suppose $|f\|_\infty = 1$. By a trigonometric polynomial Q of degree at most n with $\|f - Q\|_\infty \leq A(m)n^{-1}$ and so, setting $P = (1 + A(m)n^{-1}P$, we have trigonometric polynomial of degree at most n with $\|P\|_\infty \leq 1$ and

$$\|f - P\|_\infty \leq 2A(m)n^{-1}.$$

Exercise 5.5.5

(i) Immediate,

(ii) We have

$$\|P - Q\|_\infty \le \sum |a_{\mathbf{j}} - b_{\mathbf{j}}| \le \text{maximum number of non zero terms} \times 2M^{-n-1}$$
$$= (2M-1)^n \times 2M^{-n-1} \le 2^{n+1}M^{-1}$$

so

$$\|f - Q\|_\infty \le AM^{-1} + 2^{n+1}M^{-1} = A'M^{-1}$$

with $A' = A + 2^{n+1}$.

(iii) The number of possible values that $a_{\mathbf{j}}$ can take is no more than $(2^{M+1} + 1)^2$ and the number of non-zero $a_{\mathbf{j}}$ is no more than $(2M-1)^n$. Thus

$$\operatorname{card} \Gamma'_M \le \big((2M^{2n+1} + 1)^2\big)^{(2M-1)^n}$$

(iv) We have (for example)

$$\log \operatorname{card} \Gamma'_M \le 2\big(\log(2M^{2n+1} + 1)\big)(2M-1)^n \le 2\big(\log(2M^{2n+2})\big)(2M)^n$$
$$= 2^{n+1}(log 2 + (2n+2)\log M)M^n \le A''M^n \log M$$

for an appropriate A''. But we know that the $\operatorname{card}\Gamma'$ uniform balls radius $A'M^{-1}$ centred on the points of Γ' cover B, so we are done.

(v) We may take $A' \ge 1$.. Now we can always find an integer M with $2A'\epsilon^{-1} \ge M \ge A'\epsilon^{-1}$. Applying the final formula of (iii) we know that B can be covered by N balls of radius at most ϵ (and so by N balls of radius exactly ϵ) with

$$\log N \le A'' \frac{(2A'\epsilon^{-1})^n}{\log}(2A\epsilon^{-1}) \le A'''\epsilon^{-n}\log \epsilon^{-1}$$

for an appropriate constant A'''.

'You are old, father William,' the young man said,
  'And your hair has become very white;
And yet you incessantly stand on your head —
  Do you think, at your age, it is right?

'In my youth,' father William replied to his son,
  'I feared it would injure the brain;
But now that I'm perfectly sure I have none,
  Why, I do it again and again.'

'You are old,' said the youth, 'as I mentioned before,
  And have grown most uncommonly fat;
Yet you turned a back-somersault in at the door —
  Pray, what is the reason of that?'

'In my youth,' said the sage, as he shook his grey locks,
  'I kept all my limbs very supple
By the use of this ointment — one shilling the box —
  Allow me to sell you a couple.'

'You are old,' said the youth, 'and your jaws are too weak
  For anything tougher than suet;
Yet you finished the goose, with the bones and the beak —
  Pray, how did you manage to do it?'

'In my youth,' said his father, 'I took to the law,
  And argued each case with my wife;
And the muscular strength, which it gave to my jaw,
  Has lasted the rest of my life.'

'You are old,' said the youth; one would hardly suppose
  That your eye was as steady as ever;
Yet you balanced an eel on the end of your nose —
  What made you so awfully clever?'

'I have answered three questions, and that is enough,'
  Said his father; 'don't give yourself airs!
Do you think I can listen all day to such stuff?
  Be off, or I'll kick you down stairs!'

You are old father William Lewis Carroll