An Introduction to Fourier Analysis
Part III

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Small print This is just a first draft for the course. The content of the course will be what I say, not what these notes say. Experience shows that skeleton notes (at least when I write them) are very error prone so use these notes with care. I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors. This document is written in LaTeX2e and should be accessible in tex, dvi, ps and pdf form from my home page `http://www.dpmms.cam.ac.uk/~twk/` together with a list of corrections.

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1 Some notes of explanation

Since the birth of the Lebesgue integral it has been clear that it is a much more powerful tool for studying Fourier analysis than the Riemann integral. However I shall try to make the course accessible to those who have not done measure theory (though they may have to take the statement of certain results on trust). If either those who know Lebesgue integration, or those who do not, feel that this leads to any problems they should raise them with me.

Because of the strong number theoretic bias of this course, I will not have the time to devote to the Fourier transform that, ideally, I would have wished. The reader must be aware that she is seeing only a limited number of aspects of Fourier analysis. Although I intend to reach the end of Section 13, I am not sure that I will I have time for the final two sections.

The exercises do not form part of the course. I hope that those who attempt them will find them reasonably easy, instructive and helpful both in understanding the course and helping the reader towards 'mathematical maturity' — but I may well be wrong.

2 Fourier series on the circle

We work on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (that is on the interval $[0, 2\pi]$ with the two ends 0 and 2π identified). If $f : \mathbb{T} \to \mathbb{C}$ is integrable[1] we write

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) \exp -int\, dt.$$

[1] That is to say Lebesgue integrable or Riemann integrable according to the reader's background.

We shall see (Lemma 10) that f is uniquely determined by its Fourier coefficients $\hat{f}(n)$. Indeed it is clear that there is a 'natural identification' (where natural is deliberately used in a vague sense)

$$f(t) \sim \sum_{r=-\infty}^{\infty} \hat{f}(r) \exp irt.$$

However, we shall also see that, even when f is continuous, $\sum_{r=-\infty}^{\infty} \hat{f}(r) \exp irt$ may fail to converge at some points t.

Fejér discovered that, although

$$S_n(f,t) = \sum_{r=-n}^{n} \hat{f}(r) \exp irt$$

may behave badly as $n \to \infty$, the average

$$\sigma_n(f,t) = (n+1)^{-1} \sum_{m=0}^{n} S_m(f,t) = \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \hat{f}(r) \exp irt$$

behaves much better. (We call $\sigma_n(f,t)$ the Fejér sum. We also write $S_n(f,t) = S_n(f)(t)$ and $\sigma_n(f,t) = \sigma_n(f)(t)$.)

Exercise 1 *Let a_1, a_2, ... be a sequence of complex numbers.*

(i) Show that, if $a_n \to a$, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \to a$$

as $n \to \infty$.

(ii) By taking an appropriate sequence of 0s and 1s or otherwise find a sequence a_n such that a_n does not tend to a limit as $n \to \infty$ but $(a_1 + a_2 + \cdots + a_n)/n$ does.

(iii) By taking an appropriate sequence of 0s and 1s or otherwise find a bounded sequence a_n such that $(a_1 + a_2 + \cdots + a_n)/n$ does not tend to a limit as $n \to \infty$.

In what follows we define

$$f * g(t) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t-s)g(s)\, ds$$

(for appropriate f and g).

Lemma 2 *If f is integrable we have*

$$S_n(f) = f * D_n$$
$$\sigma_n(f) = f * K_n.$$

where

$$D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)}$$
$$K_n(t) = \frac{1}{n+1}\left(\frac{\sin(\frac{n+1}{2}t)}{\sin(\frac{1}{2}t)}\right)^2$$

for $t \neq 0$.

The key differences between the Dirichlet kernel D_n and the Fejér kernel K_n are illustrated by the next two lemmas.

Lemma 3 *(i)* $\frac{1}{2\pi}\int_{\mathbb{T}} D_n(t)\,dt = 1.$
(ii) If $t \neq \pi$, then $D_n(t)$ does not tend to a limit as $n \to \infty$.
(iii) There is a constant $A > 0$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |D_n(t)|\,dt \geq A \log n$$

for $n \geq 1$.

Lemma 4 *(i)* $\frac{1}{2\pi}\int_{\mathbb{T}} K_n(t)\,dt = 1.$
(ii) If $\eta > 0$, then $K_n \to 0$ uniformly for $|t| \geq \eta$ as $n \to \infty$.
(iii) $K_n(t) \geq 0$ for all t.

The properties set out in Lemma 4 show why Fejér sums work so well.

Theorem 5 *(i) If $f : \mathbb{T} \to \mathbb{C}$ is integrable and f is continuous at t, then*

$$\sigma_n(f, t) \to f(t)$$

as $n \to \infty$.
(ii) If $f : \mathbb{T} \to \mathbb{C}$ is continuous, then

$$\sigma_n(f) \to f$$

uniformly as $n \to \infty$.

Exercise 6 *Suppose that $L_n : \mathbb{T} \to \mathbb{R}$ is continuous (if you know Lebesgue theory you merely need integrable) and*

(A) $\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} L_n(t)\,dt = 1$,

(B) If $\eta > 0$, then $L_n \to 0$ uniformly for $|t| \geq \eta$ as $n \to \infty$,

(C) $L_n(t) \geq 0$ for all t.

(i) Show that, if $f : \mathbb{T} \to \mathbb{C}$ is integrable and f is continuous at t, then

$$L_n * f(t) \to f(t)$$

as $n \to \infty$.

(ii) Show that, if $f : \mathbb{T} \to \mathbb{C}$ is continuous, then

$$L_n * f \to f$$

uniformly as $n \to \infty$.

(iii) Show that condition (C) can be replaced by

(C') There exists a constant $A > 0$ such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} |L_n(t)|\,dt \leq A$$

in parts (i) and (ii). [You need only give the proof in one case and say that the other is 'similar'.]

Exercise 7 *Suppose that $L_n : \mathbb{T} \to \mathbb{R}$ is continuous but that*

$$\sup_n \frac{1}{2\pi}\int_{\mathbb{T}} |L_n(t)|\,dt = \infty.$$

Show that we can find a sequence of continuous functions $g_n : \mathbb{T} \to \mathbb{R}$ with $|g_n(t)| \leq 1$ for all t, $L_n * g_n(0) \geq 0$ for all n and

$$\sup_n L_n * g_n(0) = \infty.$$

(i) If you know some functional analysis deduce the existence of a continuous function f such that

$$\sup_n L_n * f(0) = \infty.$$

(ii) Even if you can obtain the result of (i) by slick functional analysis there is some point in obtaining the result directly.

(a) Suppose that we have defined positive integers $n(1) < n(2) < \cdots < n(k)$, a continuous function g_k and a real number $\epsilon(k)$ with $2^{-k} > \epsilon(k) >$

0. *Show that there is an* $\epsilon(k+1)$ *with* $\epsilon(k)/2 > \epsilon(k+1) > 0$ *such that whenever* g *is a continuous function with* $\|g - g_k\|_\infty < 2\epsilon(k+1)$ *we have* $|L_{n(j)} * g(0) - L_{n(j)} * g_k(0)| \leq 1.$ *for* $1 \leq j \leq k$.

(b) Continuing with the notation of (a), show that there exists an $n(k+1) > n(k)$ *and a continuous function* g_{k+1} *with* $\|g_{k+1} - g_k\|_\infty \leq \epsilon(k+1)$ *such that* $|L_{n(k+1)} * g_{k+1}(0)| > 2^{k+1}$.

(c) By carrying out the appropriate induction and considering the uniform limit of g_k *obtain (i).*

(iii) Show that there exists a continuous function f *such that* $S_n(f, 0)$ *fails to converge as* $n \to \infty$. *(We shall obtain a stronger result later in Theorem 25.)*

Theorem 5 has several very useful consequences.

Theorem 8 (Density of trigonometric polynomials) *The trigonometric polynomials are uniformly dense in the continuous functions on* $\mathbb{T}$.

Lemma 9 (Riemann-Lebesgue lemma) *If* f *is an integrable function on* $\mathbb{T}$, *then* $\hat{f}(n) \to 0$ *as* $|n| \to \infty$.

Theorem 10 (Uniqueness) *If* f *and* g *are integrable functions on* $\mathbb{T}$ *with* $\hat{f}(n) = \hat{g}(n)$ *for all* n, *then* $f = g$.

Lemma 11 *If* f *is an integrable function on* $\mathbb{T}$ *and* $\sum_j |\hat{f}(j)|$ *converges, then* f *is continuous and* $f(t) = \sum_j \hat{f}(j) \exp ijt$.

As a preliminary to the next couple of results we need the following temporary lemma (which will be immediately superseded by Theorem 14).

Lemma 12 (Bessel's inequality) *If* f *is a continuous function on* $\mathbb{T}$, *then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^2 \, dt.$$

Theorem 13 (Mean square convergence) *If* f *is a continuous function on* $\mathbb{T}$, *then*

$$\frac{1}{2\pi} \int_{\mathbb{T}} |f(t) - S_n(f, t)|^2 \, dt \to 0$$

as $n \to \infty$.

Theorem 14 (Parseval's Theorem) *If f is a continuous function on $\mathbb{T}$, then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^2\, dt.$$

More generally, if f and g are continuous

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)^* = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(t)^*\, dt.$$

(The extension to all L^2 functions of Theorems 13 and 14 uses easy measure theory.)

Exercise 15 *If you use Lebesgue integration, state and prove Theorems 13 and 14 for $(L^2(\mathbb{T}), \|\ \|_2)$.*

If you use Riemann integration, extend and prove Theorems 13 and 14 for all Riemann integrable function.

Note the following complement to the Riemann-Lebesgue lemma.

Lemma 16 *If $\kappa(n) \to \infty$ as $n \to \infty$, then we can find a continuous function f such that $\limsup_{n\to\infty} \kappa(n)\hat{f}(n) = \infty$.*

The proof of the next result is perhaps more interesting than the result itself.

Lemma 17 *Suppose that f is an integrable function on $\mathbb{T}$ such that there exists an A with $|\hat{f}(n)| \leq A|n|^{-1}$ for all $n \neq 0$. If f is continuous at t, then $S_n(f,t) \to f(t)$ as $n \to \infty$.*

Exercise 18 *Suppose that $a_n \in \mathbb{C}$ and there exists an A with $|a_n| \leq A|n|^{-1}$ for all $n \geq 1$. Write*

$$s_n = \sum_{r=0}^{n} a_r.$$

Show that, if

$$\frac{s_0 + s_1 + \cdots + s_n}{n+1} \to s$$

as $n \to \infty$, then $s_n \to s$ as $n \to \infty$. (Results like this are called Tauberian theorems.)

Exercise 19 *(i) Suppose that $f : [-\pi, \pi) \to \mathbb{R}$ is increasing and bounded. Write $f(\pi) = \lim_{t\to 0} f(\pi - t)$. Show that*

$$\int_{-\pi}^{\pi} f(t) \exp it\, dt = \int_{0}^{\pi} (f(t) - f(t-\pi)) \exp it\, dt$$

and deduce that $|\hat{f}(1)| \leq (f(\pi) - f(-\pi))/2 \leq (f(\pi) - f(-\pi))$.

(ii) Under the assumptions of (i) show that

$$|\hat{f}(n)| \leq (f(\pi) - f(-\pi))/|n|$$

for all $n \neq 0$.

(iii) (Dirichlet's theorem) Suppose that $g = f_1 - f_2$ where $f_k : [-\pi, \pi) \to \mathbb{R}$ is increasing and bounded $[k = 1, 2]$. (It can be shown that functions g of this form are the, so called, functions of bounded variation.) Show that if g is continuous at t, then $S_n(g, t) \to f(t)$ as $n \to \infty$.

Most readers will already be aware of the next fact.

Lemma 20 *If $f : \mathbb{T} \to \mathbb{C}$ is continuously differentiable, then*

$$(f')\hat{\ }(n) = in\hat{f}(n).$$

This means that Lemma 17 applies, but we can do better.

Lemma 21 *If $f : \mathbb{T} \to \mathbb{C}$ is continuously differentiable, then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Here is a beautiful application due to Weyl of Theorem 8. If x is real, let us write $\langle x \rangle$ for the fractional part of x, that is, let us write

$$\langle x \rangle = x - [x].$$

Theorem 22 *If α is an irrational number and $0 \leq a \leq b \leq 1$, then*

$$\frac{\operatorname{card}\{1 \leq n \leq N \mid \langle n\alpha \rangle \in [a, b]\}}{N} \to b - a$$

as $N \to \infty$. The result is false if α is rational.

(Of course this result may be deduced from the ergodic theorem and Theorem 8 itself can be deduced from the Stone-Weierstrass theorem but the techniques used can be extended in directions not covered by the more general theorems.)

Hurwitz used Parseval's theorem in a neat proof of the isoperimetric inequality.

Theorem 23 *Among all smooth closed non-self-intersecting curves of given length, the one which encloses greatest area is the circle.*

(Reasonably simple arguments show that the requirement of smoothness can be dropped.)

3 A Theorem of Kahane and Katznelson

We need to recall (or learn) the following definition.

Definition 24 *A subset E of $\mathbb{T}$ has (Lebesgue) measure zero if, given $\epsilon > 0$, we can find intervals I_j of length $|I_j|$ such that $\bigcup_{j=1}^{\infty} I_j \supseteq E$ but $\sum_{j=1}^{\infty} |I_j| < \epsilon$.*

There is a deep and difficult theorem of Carleson which tells us that if $f : \mathbb{T} \to \mathbb{C}$ is continuous (or even L^2), then the set

$$E = \{t \in \mathbb{T} \mid S_n(f, t) \nrightarrow f(t) \text{ as } n \to \infty\}$$

has measure 0. (We shall neither prove nor make use of this result which is included for information only.) Kahane and Katznelson proved a converse which though much easier to prove is still remarkable.

Theorem 25 (Kahane and Katznelson) *Given any subset E of $\mathbb{T}$ with measure zero, we can find a continuous function f such that*

$$\limsup_{n \to \infty} |S_n(f, t)| \to \infty$$

for all $t \in E$.

The theorem follows relatively simply from its 'finite version'.

Lemma 26 *Given any $K > 0$, we can find a $\epsilon(K) > 0$ such that if J_1, J_2, $\dots J_N$ is any finite collection of intervals with $\sum_{r=1}^{N} |J_r| < \epsilon(K)$ we can find a trigonometric polynomial P such that $\|P\|_\infty \le 1$ but*

$$\sup_n |S_n(P, t)| \ge K$$

for all $t \in \bigcup_{r=1}^{N} J_r$.

It is the proof of Lemma 26 which contains the key idea. This is given in the next lemma.

Lemma 27 *Let us define* $\log z$ *on* $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ *so that* $\log x$ *is real when* x *is real and positive. Suppose that* $1 > \delta > 0$ *and that* $\theta_1, \theta_2, \ldots, \theta_N \in \mathbb{R}$. *If we set*

$$\phi(z) = \log\left(N^{-1}\sum_{n=1}^{N}\frac{1+\delta}{1+\delta-ze^{-i\theta_n}}\right),$$

then ϕ *is a well defined analytic function on* $\{z \mid |z| < 1+\delta/2\}$ *such that*
(i) $|\Im\phi(z)| < \pi$ *for all* $|z| < 1+\delta/2$,
(ii) $\phi(0) = 0$,
(iii) $|\Re\phi(e^{i\theta})| \geq \log(\delta^{-1}/4N)$ *for all* $|\theta - \theta_n| \leq \delta/2$ *and* $1 \leq n \leq N$.

4 Many Dimensions

The extension of these ideas to higher dimensions can be either trivial or very hard. If $f : \mathbb{T}^m \to \mathbb{C}$ we define

$$\hat{f}(\mathbf{n}) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}}\ldots\int_{\mathbb{T}} f(\mathbf{t})\exp(-i\mathbf{n}.\mathbf{t})\,dt_1\ldots dt_m.$$

Very little is known about the convergence of

$$\sum_{u^2+v^2\leq N}\hat{f}(u,v)\exp(i(ux+vy))$$

as $N \to \infty$ even when f is continuous. (Of course, under stronger conditions, such as those in Exercise 34 below the matter becomes much easier.)

However the treatment of the sums of type

$$\sum_{|u|,|v|\leq N}\hat{f}(u,v)\exp(i(ux+vy))$$

is a straightforward. The following results are part of the course but will be left as exercises. (Of course, if you have trouble with them you can ask the lecturer to do them. If you are using Lebesgue integration work with L^∞ rather than L^1 functions.)

Lemma 28 *If we define* $\tilde{K}_n : \mathbb{T}^m \to \mathbb{R}$ *by*

$$\tilde{K}_n(t_1, t_2, \ldots, t_m) = \prod_{j=1}^{m} K_n(t_j),$$

then we have the following results.

(i) $\dfrac{1}{(2\pi)^m}\displaystyle\int_{\mathbb{T}^m} \tilde{K}_n(\mathbf{t})\,d\mathbf{t} = 1.$

(ii) If $\eta > 0$*, then*

$$\frac{1}{(2\pi)^m}\int_{|\mathbf{t}|\geq\eta} \tilde{K}_n(\mathbf{t})\,d\mathbf{t} \to 0$$

as $n \to \infty$*.*

(iii) $\tilde{K}_n(\mathbf{t}) \geq 0$ *for all* $\mathbf{t}$*.*

(iv) $\tilde{K}_n$ *is a (multidimensional) trigonometric polynomial.*

Lemma 29 *If* $f : \mathbb{T}^m \to \mathbb{C}$ *is integrable and* $P : \mathbb{T}^m \to \mathbb{C}$ *is a trigonometric polynomial, then*

$$P * f(\mathbf{x}) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} P(\mathbf{x}-\mathbf{t})f(\mathbf{t})\,d\mathbf{t}$$

is a trigonometric polynomial in $\mathbf{x}$*.*

Theorem 30 (Density of trigonometric polynomials) *The trigonometric polynomials are uniformly dense in the continuous functions on* $\mathbb{T}^m$*.*

Lemma 31 (Riemann-Lebesgue lemma) *If* f *is an integrable function on* $\mathbb{T}^m$*, then* $\hat{f}(\mathbf{n}) \to 0$ *as* $|\mathbf{n}| \to \infty$*.*

Theorem 32 (Uniqueness) *If* f *and* g *are integrable functions on* $\mathbb{T}^m$ *with* $\hat{f}(\mathbf{n}) = \hat{g}(\mathbf{n})$ *for all* $\mathbf{n}$*, then* $f = g$*.*

Lemma 33 *If* f *is an integrable function on* $\mathbb{T}$ *and* $\sum_{\mathbf{j}} |\hat{f}(\mathbf{j})|$ *converges, then* f *is continuous and* $f(\mathbf{t}) = \sum_{\mathbf{j}} \hat{f}(\mathbf{j}) \exp i\mathbf{j}.\mathbf{t}$*.*

Exercise 34 *Suppose that* $f : \mathbb{T}^m \to \mathbb{C}$ *is integrable and* $\sum_{(u,v)\in\mathbb{Z}^2} |\hat{f}(u,v)| < \infty$*. Show that*

$$\sum_{u^2+v^2\leq N} \hat{f}(u,v)\exp(i(ux+vy)) \to f(x,y)$$

uniformly as $N \to \infty$*.*

Theorem 35 (Parseval's Theorem) *If* f *is a continuous function on* $\mathbb{T}^m$*, then*

$$\sum_{n\in\mathbb{Z}^m} |\hat{f}(\mathbf{n})|^2 = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} |f(\mathbf{t})|^2\,d\mathbf{t}.$$

More generally, if f *and* g *are continuous,*

$$\sum_{n\in\mathbb{Z}^m} \hat{f}(\mathbf{n})\hat{g}(\mathbf{n})^* = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} f(\mathbf{t})g(\mathbf{t})^*\,d\mathbf{t}.$$

Exercise 36 *Prove the results from Lemma 28 to Theorem 35*

Exercise 37 *The extension of Lemma 17 to many dimensions is not required for the course but makes a nice exercise. The proof follows the one dimensional proof but is not quite word for word.*

(i) Suppose that f is a bounded integrable function on $\mathbb{T}^2$ such that there exists an A with $|\hat{f}(u,v)| \leq A(u^2+v^2)^{-1}$ for all $(u,v) \neq (0,0)$. Show that, if f is continuous at (s,t), then

$$\sum_{|u|,|v|\leq n} \hat{f}(u,v)\exp(i(us+vt)) \to f(s,t)$$

as $n \to \infty$

(ii) (This generalises Lemma 21.) Suppose that f is a twice differentiable function on $\mathbb{T}^2$ with $\dfrac{\partial^2 f(x,y)}{\partial x \partial y}$ continuous. Show that $\sum_{(u,v)\in\mathbb{Z}^2} |\hat{f}(u,v)| < \infty$.

(iii) State the correct generalisations of parts (i) and (ii) to higher dimensions.

We immediately obtain a striking generalisation of Weyl's theorem (Theorem 22).

Theorem 38 *Suppose that α_1, α_2, ..., α_M are real numbers. A necessary and sufficient condition that*

$$\frac{\operatorname{card}\{1 \leq n \leq N \mid (\langle n\alpha_1\rangle, \langle n\alpha_2\rangle, \dots, \langle n\alpha_M\rangle) \in \prod_{j=1}^M [a_j, b_j]\}}{N} \to \prod_{j=1}^M (b_j - a_j)$$

as $N \to \infty$ whenever $0 \leq a_j \leq b_j \leq 1$ is that

$$\sum_{j=1}^M n_j\alpha_j \notin \mathbb{Z} \text{ for integer } n_j \text{ not all zero.} \qquad \bigstar$$

If α_1, α_2, ..., α_M satisfy $\bigstar$ we say that they are independent. The multidimensional version of Weyl's theorem has an important corollary.

Theorem 39 (Kronecker's theorem) *Suppose that α_1, α_2, ..., α_M are independent real numbers. Then given real numbers β_1, β_2, ..., β_M and $\epsilon > 0$ we can find integers N, m_1, m_2, ..., m_M such that*

$$|N\alpha_j - \beta_j - m_j| < \epsilon$$

for each $1 \leq j \leq M$.

The result is false if α_1, α_2, ..., α_M are not independent.

We use this to obtain a theorem of Kolmogorov.

Theorem 40 *There exists a Lebesgue integrable (that is an L^1) function $f : \mathbb{T} \to \mathbb{C}$ such that*

$$\limsup_{n\to\infty} |S_n(f,t)| = \infty$$

for all $t \in \mathbb{T}$.

Although this result is genuinely one of Lebesgue integration it can be obtained by simple (Lebesgue measure) arguments from a result not involving Lebesgue integration.

Lemma 41 *Given any $K > 0$ we can find a trigonometric polynomial P such that*
(i) $\displaystyle\frac{1}{2\pi}\int_{\mathbb{T}} |P(t)|\,dt \leq 1$,
(ii) $\max_{n\geq 0} |S_n(P,t)| \geq K$ *for all $t \in \mathbb{T}$.*

In our discussion of Kronecker's theorem (Theorem 39) we worked modulo 1. In what follows it is easier to work modulo 2π. The readier will readily check that the definition and theorem that follow give the appropriate restatement of Kronecker's theorem.

Definition 42 *We work in $\mathbb{T}$. If $\alpha_1, \alpha_2, \ldots, \alpha_M$ satisfy*

$$\sum_{j=1}^{M} n_j\alpha_j \neq 0 \text{ for integer } n_j \text{ not all zero.} \qquad \bigstar$$

we say that they are independent.

Theorem 43 (Kronecker's theorem (alternative statement)) *Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_M$ are independent points in $\mathbb{T}$. Then, given complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_M$ with $|\lambda_j| = 1$ $[j = 1, 2, \ldots, M]$ and $\epsilon > 0$, we can find an integer N such that*

$$|\exp(iN\alpha_j) - \lambda_j| < \epsilon$$

for each $1 \leq j \leq M$.

Our construction requires some preliminary results.

Lemma 44 *If x_1, x_2, …, x_M are independent points in $\mathbb{T}$ and $t \in \mathbb{T}$ and*

$$\sum_{j=1}^{M} p_j(x_j - t) = 0 \text{ for integer } p_j \text{ not all zero}$$
$$\sum_{j=1}^{M} q_j(x_j - t) = 0 \text{ for integer } q_j \text{ not all zero}$$

then there exist $p, q \in \mathbb{Z} \setminus \{0\}$ such that $pq_j = qp_j$ for $1 \leq j \leq M$.

Lemma 45 *If x_1, x_2, …, x_M are independent points in $\mathbb{T}$ and $t \in \mathbb{T}$, then one of the following must hold:-*

(a) There exists a $i \neq 1$ such that the points $x_j - t$ with $j \neq i$ are independent.

(b) $x_1 - t$ is a rational multiple of 2π and the points $x_j - t$ with $j \neq 1$ are independent.

Lemma 46 *(i) If I is an open interval in $\mathbb{T}$ and x_1, x_2, …, x_m are independent we can find $x_{m+1} \in I$ such that x_1, x_2, …, x_{m+1} are independent.*

(ii) Given an integer $M \geq 1$ we can find independent points x_1, x_2, …, x_M in $\mathbb{T}$ such that

$$|x_j - 2\pi j/M| \leq 10^{-3}M^{-1}$$

Lemma 47 *If M and x_1, x_2, …, x_M in $\mathbb{T}$ are as in Lemma 46 (ii), then setting*

$$\mu = M^{-1}\sum_{j=1}^{M} \delta_{x_j}$$

we have the following two results.

(i) $\max_{n\geq 0} |S_n(\mu, t)| \geq 100^{-1}\log M$ for each $t \in \mathbb{T}$,

(ii) There exists an N such that $\max_{N\geq n\geq 0} |S_n(\mu, t)| \geq 200^{-1}\log M$ for each $t \in \mathbb{T}$.

Remark 1 If you wish you may treat $S_n(\mu, t)$ as a purely formal object. However, it is better for later work to think what it actually is.
Remark 2 Factors like 10^{-3} and 100^{-1} in Lemmas 46 (ii) and 47 are more or less chosen at random and are not ‘best possible’.

A simple argument using Lemma 47 now gives Lemma 41 and we are done.

Exercise 48 *Show, by considering Fejér sums or otherwise, that we cannot find a continuous function f such that $S_n(f, t) \to \infty$ uniformly as $n \to \infty$.*

5 Some simple geometry of numbers

We need the following extension of Theorem 35.

Lemma 49 *If A is a well behaved set and f is the characteristic function of A (that is $f(x) = 1$ if $x \in A$, $f(x) = 0$ otherwise), then*

$$\sum_{n \in \mathbb{Z}^m} |\hat{f}(\mathbf{n})|^2 = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} |f(\mathbf{t})|^2 \, d\mathbf{t}.$$

If you know Lebesgue measure then this is obvious (for bounded measurable sets, say) since a simple density argument shows that Parseval's Theorem (Theorem 35) holds for every $f \in L^1 \cap L^2$. If we restrict ourselves to Riemann integration it is obvious what sort of approximation argument we should use but the technical details are typically painful.

Exercise 50 *EITHER (i) Give the detailed proof of Lemma 49 in terms of Lebesgue measure.*

OR (ii) Give the detailed proof of Lemma 49 in terms of Riemann integration in the special case when A is a sphere.

We use Parseval's Theorem (in the form of Lemma 49) to give Siegel's proof of Minkowski's theorem.

Theorem 51 (Minkowski) *Let Γ be an open symmetric convex set in $\mathbb{R}^m$ with volume V. If $V > 2^m$, then $\Gamma \cap \mathbb{Z}^m$ contains at least two points.*

The reader will recall that Γ is convex if

$$\mathbf{x}, \mathbf{y} \in \Gamma \text{ and } 0 \leq \lambda \leq 1 \Rightarrow \lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in \Gamma$$

and that Γ is symmetric if

$$\mathbf{x} \in \Gamma \Rightarrow -\mathbf{x} \in \Gamma.$$

It is not entirely obvious (though it is true) that every open convex set has a (possibly infinite) volume in the sense of Riemann. Readers who wish to use Riemann integration may add the words 'well behaved' to the statement of Minkowski's theorem.

Lemma 52 *If $V \leq 2^m$ there exists an open symmetric convex set Γ in $\mathbb{R}^m$ with volume V such that $\Gamma \cap \mathbb{Z}^m = \{\mathbf{0}\}$.*

To prove Minkowski's theorem (Theorem 51) it suffices to prove an essentially equivalent result.

Theorem 53 *Let Γ be a bounded open symmetric convex set in $\mathbb{R}^m$ with volume V. If $V > 2^m(2\pi)^m$, then $\Gamma \cap (2\pi\mathbb{Z})^m$ contains at least two points.*

We need the following simple but crucial result.

Lemma 54 *If Γ is symmetric convex set and $\mathbf{x}, \mathbf{x} - 2\mathbf{y} \in \Gamma$, then $\mathbf{y} \in \Gamma$.*

By applying Parseval's theorem to $f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^m} \mathbb{I}_{\Gamma/2}(\mathbf{x} - 2\pi\mathbf{k})$ we obtain the following results.

Lemma 55 *Let Γ be a bounded open symmetric convex set in $\mathbb{R}^m$ with volume V such that $\Gamma \cap (2\pi\mathbb{Z})^m$ only contains $\mathbf{0}$. Then*

$$2^{-m} \sum_{\mathbf{k}\in\mathbb{Z}^m} \left| \int_{\mathbb{R}^m} \mathbb{I}_\Gamma(\mathbf{x}) e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{x} \right|^2 = (2\pi)^m V. \qquad \bigstar$$

where $\mathbb{I}_\Gamma$ is the characteristic function of Γ.

Minkowski's theorem follows at once by considering the term with $\mathbf{k} = \mathbf{0}$ in equation $\bigstar$.

Here is a simple application of Minkowski's theorem.

Lemma 56 *Suppose that a, b, c, d are real numbers with $ad - bc = 1$. Given $l > 0$ and $\epsilon > 0$ we can find integers m and n such that*

$$|an + bm| \leq (1+\epsilon)l, \ \ |cn + dm| \leq (1+\epsilon)l^{-1}.$$

Taking $c = x$, $a = 1$, $b = 0$, $d = -1$ and thinking carefully we obtain in quick succession.

Lemma 57 *If x is real there exist n and m integers with $n \neq 0$ such that*

$$\left| x - \frac{m}{n} \right| \leq \frac{1}{n^2}.$$

Lemma 58 *If x is real there exist infinitely many pairs of integers n and m with $n \neq 0$ such that*

$$\left| x - \frac{m}{n} \right| \leq \frac{1}{n^2}.$$

Here is another simple consequence.

Lemma 59 (Quantitative version of Dirichlet's theorem) *If $\mathbf{x} \in \mathbb{R}^m$, then, given $l > 0$ and $\epsilon > 0$, we can find $n,\ n_1,\ n_2,\ \ldots,\ n_m \in \mathbb{Z}$ such that*

$$|nx_j - n_j| \leq l^{-1}$$

for $1 \leq j \leq m$ and $|n| \leq l^m$.

We conclude our collection of consequences with Legendre's four squares theorem.

Theorem 60 (Legendre) *Every positive integer is the sum of at most 4 squares.*

Lemma 61 *We cannot reduce 4 in the statement of Legendre's theorem (Theorem 60).*

We need an observation of Euler.

Lemma 62 *If x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3 are real, then*

$$\begin{aligned}(x_0^2+&x_1^2+x_2^2+x_3^2)(y_0^2+y_1^2+y_2^2+y_3^2)\\&=(x_0y_0-x_1y_1-x_2y_2-x_3y_3)^2+(x_0y_1+x_1y_0+x_2y_3-x_3y_2)^2+\\&\quad(x_0y_2-x_1y_3+x_2y_0+x_3y_1)^2+(x_0y_3+x_1y_2-x_2y_1+x_3y_1)^2.\end{aligned}$$

Exercise 63 *In the lectures we will use quaternions to prove Lemma 62. Prove the equality by direct verification.*

Lemma 64 *Legendre's four square theorem will follow if we can show that every odd prime is the sum of at most four squares.*

We shall also need the volume of a 4 dimensional sphere. A simple argument gives the volume of a unit sphere in any dimension.

Lemma 65 *Let V_n be the (n-dimensional) volume of an n dimensional unit sphere.*

(i) If $f : [0,\infty) \to \mathbb{R}$ is a continuous function with $t^{n+2}f(t) \to 0$ as $t \to \infty$, then

$$\int_{\mathbb{R}^n} f(\|\mathbf{x}\|)\,dV(\mathbf{x}) = V_n \int_0^\infty f(t)nt^{n-1}\,dt.$$

(ii) $V_{2k} = \dfrac{\pi^k}{k!}$, $V_{2k+1} = \dfrac{k!2^{2k+1}\pi^k}{(2k+1)!}$.

Finally we need the apparently more general version of Minkowski's theorem obtained by applying a linear map.

Theorem 66 (Minkowski for general lattices) *We work in $\mathbb{R}^m$. Let Λ be a lattice with fundamental region of volume L and let Γ be an open symmetric convex set with volume V. If $V > 2^m L$, then $\Gamma \cap \Lambda$ contains at least two points.*

We now turn to the proof of the fundamental lemma.

Lemma 67 *Every odd prime is the sum of at most four squares.*

We begin with a simple lemma.

Lemma 68 *Let p be an odd prime.*

(i) If we work in $\mathbb{Z}_p$, then the set $\{u^2 : u \in \mathbb{Z}_p\}$ has at least $(p+1)/2$ elements.

(ii) We can find integers u and v such that $u^2 + v^2 \equiv -1 \bmod p$.

We now introduce a lattice.

Lemma 69 *Let p, u and v be as in Lemma 68. If*

$$\Lambda = \{(n, m, a, b) \in \mathbb{Z}^4 \,:\, nu + mv \equiv a,\ mu - nv \equiv b \bmod p\}$$

then Λ is a lattice with fundamental region of volume p^2.

If $(n, m, a, b) \in \Lambda$, then $n^2 + m^2 + a^2 + b^2 \equiv 0 \bmod p$.

We can now prove Lemma 67 and with it Theorem 60.

Exercise 70 *(i) Recall that if p is a prime, then the multiplicative group $(\mathbb{Z}_p \setminus \{0\}, \times)$ is cyclic. (This is the subject of Exercise 110.) Deduce that if $p = 4k+1$, then there is an element u in $(\mathbb{Z}_p \setminus \{0\}, \times)$ of order 4. Show that $u^2 = -1$.*

(ii) If p and u are as in (i) show that

$$\Lambda = \{(n, m) \in \mathbb{Z}^2 \,:\, m \equiv n \bmod p\}$$

is a lattice and deduce that there exist n, m with $n^2 + m^2 = p$. (This is a result of Fermat. Every prime congruent to 1 modulo 4 is the sum of two squares.)

6 A brief look at Fourier transforms

If time permits we will look at Fourier transforms in sections 14 and 15. If $f : \mathbb{R} \to \mathbb{C}$ is integrable on each finite interval $[a, b]$ (in the Riemann or Lebesgue sense) and $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ we[2] shall say that f is *appropriate*. (This is non-standard notation.) If f is appropriate define

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t) \exp(-i\lambda t)\, dt.$$

[2] The majority of my auditors who know Lebesgue integration will prefer the formulations '$f \in L^1(\mathbb{R})$' or 'f measurable and $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.

Lemma 71 *If f is appropriate, $\hat{f} : \mathbb{R} \to \mathbb{C}$ is continuous and bounded.*

Our first problem is that even when f is appropriate $\hat{f}$ need not be.

Example 72 *If f is the indicator function of $[a, b]$ (that is, $f(x) = 1$ if $x \in [a, b]$, $f(x) = 0$ otherwise), then*

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)| \, d\lambda = \infty.$$

This turns out not to matter very much but should be borne in mind.

When we try to imitate our treatment of Fourier series we find that we need to interchange the order of integration of two infinite integrals. If we use Lebesgue integration we can use a very powerful theorem.

Theorem 73 (Fubini's theorem) *If $f : \mathbb{R}^2 \to \mathbb{C}$ is measurable and either of the two integrals*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dx \, dy \ \text{ and } \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dy \, dx$$

exists and is finite, then they both do and the integrals

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy \ \text{ and } \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx.$$

exist and are finite and equal.

If we use Riemann integration, then we have a slogan.

Pretheorem 74 *If $f : \mathbb{R}^2 \to \mathbb{C}$ is well behaved and either of the two integrals*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dx \, dy \ \text{ and } \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dy \, dx$$

is finite, then they both are and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy \ \text{ and } \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx.$$

In every case that we need the pretheorem can be turned into a theorem but the proofs become more and more tedious as we weaken the conditions on f.

Exercise 75 *In this exercise we use Riemann integration and derive a simple Fubini type theorem.*

(i) If I and J are intervals on $\mathbb{R}$ (so I could have the form $[a,b]$, $[a,b)$, (a,b) or (a,b)) and we write $\mathbb{I}_{I\times J}(x,y)=1$ if $(x,y)\in I\times J$, $\mathbb{I}_{I\times J}(x,y)=0$, otherwise show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{I}_{I\times J}(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{I}_{I\times J}(x,y)\,dy\,dx.$$

(ii) Suppose that I_r and J_r are intervals on $\mathbb{R}$ and that $\lambda_r\in\mathbb{C}$ $[1\leq r\leq n]$. If $f=\sum_{r=1}^{n}\mathbb{I}_{I_r\times J_r}$ show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx.$$

(iii) Suppose that $f:\mathbb{R}^2\rightarrow\mathbb{C}$ is continuous and that I and J are intervals on $\mathbb{R}$. If $g(x,y)=\mathbb{I}_{I\times J}(x,y)f(x,y)$ show using (ii) that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)\,dy\,dx.$$

(iv) Suppose that $f:\mathbb{R}^2\rightarrow\mathbb{C}$ is continuous and that there exists a real constant A such that

$$|f(x,y)|\leq A(1+x^2)^{-1}(1+y^2)^{-1}. \qquad \bigstar$$

Show, using (ii), that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx.$$

Conditions like $\bigstar$ imposing some rate of decrease at infinity play an important role in Fourier analysis.

In section 14 (if we reach it) we shall adopt a slightly more sophisticated approach to the Fourier transform than that given in the next exercise but the results are sufficient for many purposes and the exercise gives an interesting review of earlier work. We shall need the following definition.

Definition 76 *We say that f is piecewise continuous if, for each $R>0$, f is continuous at all but finitely many points of $(-R,R)$ and $f(t)=\lim_{h\rightarrow 0+}f(t-h)$ for all t.*

(Different authors use different definitions. They are all the same 'in spirit' but not 'in logic'.)

Exercise 77 *If f is appropriate and $R > 0$ we define*

$$\sigma_R(f,t) = \frac{1}{2\pi}\int_{-R}^{R}\left(1-\frac{|\lambda|}{R}\right)\hat{f}(\lambda)\exp(i\lambda t)\,d\lambda.$$

(It will become clear that this is the analogue of the Fejér sum.)

(i) (For users of the Lebesgue integral) By adapting the proof of Theorem 5 show that if $f \in L^1$ and f is continuous at t then

$$\sigma_R(f,t) \to f(t)$$

as $R \to \infty$. Is the result necessarily true if f is not continuous at t? Give reasons.

(i') (For users of the Riemann integral) By adapting the proof of Theorem 5 show that if f is continuous and there there exists a real constant A such that

$$|f(x)| \leq A(1+x^2)^{-1}$$

for all x, show that

$$\sigma_R(f,t) \to f(t)$$

for all t.

Without going into detail, convince yourself that the result continues to hold if we replace the condition 'f continuous' by the condition 'f piecewise continuous' and the conclusion by

$$\sigma_R(f,t) \to f(t)$$

at all t where f is continuous. (All we need is a slight extension of Exercise 75 (iv).)

(ii) (For users of the Lebesgue integral) Suppose that f and g are piecewise continuous L^1 functions. Show that, if $\hat{f}(\lambda) = \hat{g}(\lambda)$ for all λ, then $f(t) = g(t)$ for all t.

(ii') (For users of the Riemann integral) Suppose f and g are piecewise continuous and there there exists a real constant A such that

$$|f(x)|, |g(x)| \leq A(1+x^2)^{-1}$$

for all x. Show that, if $\hat{f}(\lambda) = \hat{g}(\lambda)$ for all λ, then $f(t) = g(t)$ for all t.

7 Infinite products

Our object in the next few lectures will be to prove the following remarkable theorem of Dirichlet on primes in arithmetic progression.

Theorem 78 (Dirichlet) *If a and d are strictly positive coprime integers, then there are infinitely many primes of the form $a + nd$ with n a positive integer.*

(Obviously the result must fail if a and d are not coprime.)

There exist a variety of proofs of special cases when d has particular values but, so far as I know, Dirichlet's proof of his theorem remains, essentially, the only approachable one. In particular there is no known reasonable[3] elementary (in the technical sense of not using analysis) proof.

Dirichlet's method starts from an observation of Euler.

Lemma 79 *If s is real with $s > 1$, then*

$$\prod_{\substack{p \text{ prime} \\ p \leq N}} \left(1 - \frac{1}{p^s}\right)^{-1} \to \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Using this result, we get a new proof of the existence of an infinity of primes.

Theorem 80 (Euclid) *There exist an infinity of primes.*

This suggests that it may be worth investigating infinite products a bit more.

Definition 81 *Let $a_j \in \mathbb{C}$. If $\prod_{n=1}^{N}(1 + a_n)$ tends to a limit L as $N \to \infty$, we say that the* infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges *to a value L and write*

$$\prod_{n=1}^{\infty}(1 + a_n) = L.$$

If the infinite product $\prod_{n=1}^{\infty}(1+|a_n|)$ converges, then we say that $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent.

The next result was removed from the first year of the Tripos a couple of years before I took it.

Lemma 82 *Let $a_j \in \mathbb{C}$.*

(i) $\prod_{n=1}^{\infty}(1 + a_n)$ is absolutely convergent if and only if $\sum_{n=1}^{\infty} a_n$ is.

(ii) If $\prod_{n=1}^{\infty}(1 + a_n)$ is absolutely convergent and $1 + a_n \neq 0$ for each n, then the infinite product converges and

$$\prod_{n=1}^{\infty}(1 + a_n) \neq 0.$$

[3] In the sense that most reasonable people would call reasonable. Selberg produced a (technically) elementary proof which may be found in his collected works.

Exercise 83 *Find $a_j \in \mathbb{C}$ such that $\prod_{n=1}^{\infty}(1+a_n)$ is not absolutely convergent but is convergent to a non-zero value.*

We shall only make use of absolute convergent infinite products.

Exercise 84 *If $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent and $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection (that is, σ is a permutation of $\mathbb{N}$) show that $\prod_{n=1}^{\infty}(1+a_{\sigma(n)})$ is absolutely convergent and*

$$\prod_{n=1}^{\infty}(1+a_{\sigma(n)}) = \prod_{n=1}^{\infty}(1+a_n)$$

Whilst this is a useful result to know, we shall make no essential use of it. When we write $\sum_{p \text{ prime}}$ or $\prod_{p \text{ prime}}$ we mean the primes p to be taken in order of increasing size.

Using Lemma 82 we obtain the following strengthening of Euclid's theorem.

Theorem 85 (Euler) $\displaystyle\sum_{p\ prime}\frac{1}{p} = \infty.$

Since we wish to consider infinite products of functions it is obvious that we shall need an analogue of the Weierstrass M-test for products, obvious what that analogue should be and obvious how to prove it.

Lemma 86 *Suppose U is an open subset of $\mathbb{C}$ and that we have a sequence of functions $g_n : U \to \mathbb{C}$ and a sequence of positive real numbers M_n such that $M_n \geq |g_n(z)|$ for all $z \in U$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\prod_{n=1}^{N}(1+g_n(z))$ converges uniformly on U.*

Later we shall need to consider $\sum n^{-s}$ with s complex. To avoid ambiguity, we shall take $n^{-s} = \exp(-s\log n)$ where $\log n$ is the real logarithm of n.

Lemma 87 *If $\Re s > 1$ we have*

$$\prod_{p\ prime}(1-p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$$

both sides being absolutely convergent for each s and uniformly convergent for $\Re s > 1+\epsilon$ for each fixed $\epsilon > 0$.

We now detour briefly from the main argument to show how infinite products can be used to answer a very natural question. 'Can we always find an analytic function with specified zeros?' (We count multiple zeros multiply in the usual way.) Naturally we need to take account of the following fact.

Lemma 88 *If z_1, z_2, ... are distinct zeros of an analytic function which is not identically zero, then $z_n \to \infty$ as $n \to \infty$.*

A little thought suggests the path we ought to take, though we may not see how to reach it. A way to reach the path is provided by the Weierstrass primary function $E(z, m)$.

Definition 89 *If m is a strictly positive integer*

$$E(z, m) = (1 - z)e^{z + z^2/2 + z^3/3 + \cdots + z^m/m}.$$

Lemma 90 *The function $E(\ ,m) : \mathbb{C} \to \mathbb{C}$ is analytic with a unique zero at 1. Given $\epsilon > 0$ we can find an M such that*

$$|1 - E(z, m)| \leq \epsilon$$

for all $m \geq M$ and $|z| \leq 1/2$.

Theorem 91 (Weierstrass) *If k is a positive integer and $z_1, z_2, \ldots$ is a sequence of non-zero complex numbers with $z_n \to \infty$, then we can choose $n(j) \to \infty$ so that*

$$F(z) = z^k \prod_{j=1}^{\infty} E\big(z/z_j, n(j)\big)$$

is a well defined analytic function with a zero of order k at 0, and zeros at the z_j (multiple zeros counted multiply) and no others.

Lemma 92 *If f_1 and f_2 are analytic functions on $\mathbb{C}$ with the same zeros (multiple zeros counted multiply), then there exists an analytic function g such that*

$$f_1(z) = e^{g(z)} f_2(z).$$

Lemma 93 *If $z_1, z_2, \ldots$ and $w_1, w_2, \ldots$ are sequences of complex numbers with $z_j, w_j \to \infty$ as $j \to \infty$ and $z_j \neq w_k$ for all j, k, then there exists a meromorphic function with zeros at the z_j and poles at the w_k (observing the usual multiplicity conventions).*

Extra exercise *(i) Show that we can find an A such that*

$$|1 - E(z, m)| \leq A|z|^{m+1}$$

for $|z| \leq 1/2$.

(ii) If k is a positive integer and $z_1, z_2, \ldots$ is a sequence of non-zero complex numbers with $z_n \to \infty$, then

$$F(z) = z^k \prod_{j=1}^{\infty} E(z/z_j, j)$$

is a well defined analytic function with a zero of order k at 0, and zeros at the z_j (multiple zeros counted multiply) and no others.

Exercise 94 *(It may be helpful to attack parts of this question non-rigorously first and then tighten up the argument second.)*

(i) If C_N is the contour consisting of the square with vertices

$$\pm(N+1/2) \pm (N+1/2)i$$

described anti-clockwise, show that there is a constant K such that

$$|\cot \pi z| \leq K$$

for all $z \in C_N$ and all integers $N \geq 1$.

(ii) By integrating an appropriate function round the contour C_N, or otherwise, show that, if $w \notin \mathbb{Z}$,

$$\sum_{n=-N}^{n=N} \frac{1}{w-n} \to \pi \cot \pi w.$$

(iii) Is it true that, if $w \notin \mathbb{Z}$,

$$\sum_{n=-M}^{n=N} \frac{1}{w-n} \to \pi \cot \pi w,$$

as $M, N \to \infty$? Give reasons.

(iv) Show that

$$P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

is a well defined analytic function and that there exists an analytic function g such that

$$\sin \pi z = e^{g(z)} P(z).$$

(v) Find a simple expression for $P'(z)/P(z)$.
[Hint: If $p(z) = \prod_{j=1}^{N}(z - \alpha_j)$, what is $p'(z)/p(z)$?]

Find a related expression for $\frac{d}{dz} \sin \pi z / \sin \pi z$.

(vi) Show that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(vii) Find a similar expression for $\cos \pi z$. *(These results are due to Euler.)*

Exercise 95 *(This makes use of some of the techniques of the previous exercise.) (i) Show that the infinite product*

$$g(z) = \prod_{n=1}^{\infty} e^{z/n} \left(1 - \frac{z}{n}\right)$$

exists and is analytic on the whole complex plane.

(ii) Show that

$$g'(z) = g(z) \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Explain why $\sum_{n=1}^{\infty}(\frac{1}{z-n} + \frac{1}{n})$ *is indeed a well defined analytic function on* $\mathbb{C} \setminus \mathbb{Z}$.

(iii) By using (ii), or otherwise, show that

$$g(z+1) = -Azg(z) \qquad (*)$$

for some constant A.

(iv) By considering a particular value of z, *or otherwise, show that* A *is real and positive and*

$$\sum_{n=1}^{N} \frac{1}{n} - \log N \to \log A$$

as $N \to \infty$. *Deduce the existence of Euler's constant*

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} n^{-1} - \log N\right)$$

and rewrite $(*)$ *as*

$$g(z+1) = -e^{\gamma} zg(z).$$

(v) Find a simple expression for $zg(z)g(-z)$. *Use* $(*)$ *to show that* $\sin \pi z$ *is periodic.*

8 Fourier analysis on finite Abelian groups

One of Dirichlet's main ideas is a clever extension of Fourier analysis from its classical frame. Recall that classical Fourier analysis deals with formulae like

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n(t)$$

where $e_n(t) = \exp(int)$. The clue to further extension lies in the following observation.

Lemma 96 *Consider the Abelian group* $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ *and the subgroup* $S = \{z : |z| = 1\}$ *of* $(\mathbb{C} \setminus \{0\}, \times)$*. The continuous homomorphisms* $\theta : \mathbb{T} \to S$ *are precisely the functions* $e_n : \mathbb{T} \to S$ *given by* $e_n(t) = \exp(int)$ *with* $n \in \mathbb{Z}$*.*

Exercise 97 *(i) Find (with proof) all the continuous homomorphisms*

$$\theta : (\mathbb{R}, +) \to (S, \times).$$

What is the connection with Fourier transforms?

(ii) (Only for those who know Zorn's lemma[4].) Assuming Zorn's lemma show that any linearly independent set in a vector space can be extended to a basis. If we consider $\mathbb{R}$ *as a vector space over* $\mathbb{Q}$*, show that there exists a linear map* $T : \mathbb{R} \to \mathbb{R}$ *such that* $T(1) = 1$, $T(\sqrt{2}) = 0$*. Deduce the existence of a function* $T : \mathbb{R} \to \mathbb{R}$ *such that* $T(x + y) = T(x) + T(y)$ *for all* $x, y \in \mathbb{R}$ *which is not continuous (with respect to the usual metric). Show that, if we accept Zorn's lemma, there exist discontinuous homomorphisms* $\theta : (\mathbb{R}, +) \to (S, \times)$*.*

This suggests the following definition.

Definition 98 *If* G *is a finite Abelian group, we say that a homomorphism* $\chi : G \to S$ *is a* character*. We write* $\hat{G}$ *for the collection of such characters.*

In this section we shall accumulate a substantial amount of information about $\hat{G}$ by a succession of small steps.

Lemma 99 *Let* G *be a finite Abelian group.*

(i) If $x \in G$ *has order* m *and* $\chi \in \hat{G}$*, then* $\chi(x)$ *is an* m*th root of unity.*

(ii) $\hat{G}$ *is a finite Abelian group under pointwise multiplication.*

[4] And, particularly, those who only know Zorn's lemma.

To go further we consider, for each finite Abelian group G, the collection $C(G)$ of functions $f : G \to \mathbb{C}$. If G has order $|G|$, then $C(G)$ is a vector space of dimension $|G|$ which can be made into a complex inner product space by means of the inner product

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)g(x)^*.$$

Exercise 100 *Verify the statements just made.*

Lemma 101 *Let G be a finite Abelian group. The elements of $\hat{G}$ form an orthonormal system in $C(G)$.*

Does $\hat{G}$ form an orthonormal basis of $C(G)$? The next lemma tells us how we may hope to resolve this question.

Lemma 102 *Let G be a finite Abelian group. The elements of $\hat{G}$ form an orthonormal basis if and only if, given an element $x \in G$ which is not the identity, we can find a character χ with $\chi(x) \neq 1$.*

The way forward is now clear.

Lemma 103 *Suppose that H is a subgroup of a finite Abelian group G and that $\chi \in \hat{H}$. If K is a subgroup of G generated by H and an element $a \in G$, then we can find a $\tilde{\chi} \in \hat{K}$ such that $\tilde{\chi}|H = \chi$.*

Lemma 104 *Let G be a finite Abelian group and x an element of G of order m. Then we can find a $\chi \in \hat{G}$ with $\chi(x) = \exp 2\pi i/m$.*

Theorem 105 *If G is a finite Abelian group, then $\hat{G}$ has the same number of elements as G and they form an orthonormal basis for $C(G)$.*

Lemma 106 *If G is a finite Abelian group and $f \in C(G)$, then*

$$f = \sum_{\chi\in\hat{G}} \hat{f}(\chi)\chi$$

where $\hat{f}(\chi) = \langle f, \chi\rangle$.

Exercise 107 *Suppose that G is a finite Abelian group. Show that if we define $\theta_x : \hat{G} \to \mathbb{C}$ by $\theta_x(\chi) = \chi(x)$ for $\chi \in \hat{G}$, $x \in G$, then the map $\Theta : G \to \hat{\hat{G}}$ given by $\Theta(x) = \theta_x$ is an isomorphism.*

If we now identify x with θ_x (and, so, G with $\hat{\hat{G}}$) show that

$$\hat{\hat{f}}(x) = |G|^{-1}f(x^{-1})$$

for all $f \in C(G)$ and $x \in G$.

We have now done all that that is required to understand Dirichlet's motivation. However, it seems worthwhile to make a slight detour to put 'computational' bones on this section by exhibiting the structure of G and $\hat{G}$.

Lemma 108 *Let $(G, \times)$ be an Abelian group.*

(i) Suppose that $x, y \in G$ have order r and s with r and s coprime. Then xy has order rs.

(ii) If G contains elements of order n and m, then G contains an element of order the least common multiple of n and m.

Lemma 109 *Let $(G, \times)$ be a finite Abelian group. Then there exists an integer N and an element k such that k has order N and, whenever $x \in G$, we have $x^N = e$.*

Exercise 110 *Let p be a prime. Use Lemma 109 together with the fact that a polynomial of degree k can have at most k roots to show that the multiplicative group $(\mathbb{Z}_p \setminus \{0\}, \times)$ is cyclic.*

Lemma 111 *With the hypotheses and notation of Lemma 109, we can write $G = K \times H$ where K is the cyclic group generated by x and H is another subgroup of K.*

As usual we write C_n for the cyclic group of order n.

Theorem 112 *If G is a finite Abelian group, we can find $n(1)$, $n(2)$, $\ldots n(m)$ with $n(j+1)$ dividing $n(j)$ such that G is isomorphic to*

$$C_{n(1)} \times C_{n(2)} \times \ldots C_{n(m)}.$$

Lemma 113 *If we have two sequences $n(1)$, $n(2)$, $\ldots n(m)$ with $n(j+1)$ dividing $n(j)$ and $n'(1)$, $n'(2)$, $\ldots n'(m')$ with $n'(j+1)$ dividing $n'(j)$, then*

$$C_{n(1)} \times C_{n(2)} \times \ldots C_{n(m)} \text{ is isomorphic to } C_{n'(1)} \times C_{n'(2)} \times \ldots C_{n'(m')}$$

if and only if $m = m'$ and $n(j) = n'(j)$ for each $1 \leq j \leq m$.

It is easy to identify $\hat{G}$.

Lemma 114 *Suppose that*

$$G = C_{n(1)} \times C_{n(2)} \times \ldots C_{n(m)}$$

with $C_{n(j)}$ a cyclic group of order $n(j)$ generated by x_j. Then the elements of $\hat{G}$ have the form $\chi_{\omega_{n(1)}^{r(1)}, \omega_{n(2)}^{r(2)}, \ldots \omega_{n(m)}^{r(m)}}$ with $\omega_{n(j)} = \exp\big(2\pi i/n(j)\big)$ and

$$\chi_{\omega_{n(1)}^{r(1)}, \omega_{n(2)}^{r(2)}, \ldots \omega_{n(m)}^{r(m)}}(x_1^{s(1)} x_2^{s(2)} \ldots x_m^{s(m)}) = \omega_{n(1)}^{r(1)s(1)} \omega_{n(2)}^{r(2)s(2)} \ldots \omega_{n(m)}^{r(m)s(m)}.$$

My readers will see that $\hat{G}$ is isomorphic to G, but the more sophisticated algebraists will also see that this is *not a natural isomorphism* (whereas G and $\hat{\hat{G}}$ are *naturally isomorphic*). Fortunately such matters are of no importance for the present course.

9 The Euler-Dirichlet formula

Dirichlet was interested in a particular group. If d is a positive integer consider $\mathbb{Z}/(d)$ the set of equivalence classes

$$[m] = \{r : r \equiv m \mod d\}$$

under the usual multiplication modulo d. We set

$$G_d = \{[m] : m \text{ and } d \text{ coprime}\}$$

and write $\phi(d)$ for the order of G_d (ϕ is called Euler's totient function).

Lemma 115 *The set G_d forms a finite Abelian group under standard multiplication.*

The results of the previous section show that, if $[a] \in G_d$ and we define $\delta_a : G_d \to \mathbb{C}$ by

$$\begin{aligned} \delta_a([a]) &= 1 \\ \delta_a([m]) &= 0 \qquad \text{if } [m] \neq [a], \end{aligned}$$

then

$$\delta_a = \phi(d)^{-1} \sum_{\chi \in G_d} \chi([a])^* \chi.$$

We now take up the proof of Dirichlet's theorem in earnest. We shall operate under the standing assumption that a and d are positive coprime integers and our object is to show that the sequence

$$a,\ a+d,\ a+2d,\ \ldots, a+nd,\ \ldots$$

contains infinitely many primes. Following Euler's proof that there exist infinitely many primes we shall seek to prove this by showing that

$$\sum_{\substack{p \text{ prime} \\ p=a+nd \text{ for some } n}} \frac{1}{p} = \infty.$$

Henceforward, at least in the number theory part of the course p will be a prime, $\sum_p$ will mean the sum over all primes and so on.

In order to simplify our notation it will also be convenient to modify the definition of a character. From now on, we say that χ is a character if χ is a map from $\mathbb{N}$ to $\mathbb{C}$ such that there exists a character (in the old sense) $\tilde{\chi} \in \hat{G}_d$ with

$$\begin{aligned}\chi(m) &= \tilde{\chi}([m]) \qquad \text{if } m \text{ and } d \text{ are coprime}\\ \chi(m) &= 0 \qquad \text{otherwise.}\end{aligned}$$

We write $\sum_\chi$ to mean the sum over all characters and take χ_0 to be the character with $\chi_0([m]) = 1$ whenever m and d are coprime.

Lemma 116 *(i) If χ is a character, then $\chi(m_1 m_2) = \chi(m_1)\chi(m_2)$ for all $m_1, m_2 \geq 0$.*

(ii) If $\chi \neq \chi_0$, then $\sum_{m=k+1}^{k+d} \chi(m) = 0$.

(iii) If $\delta_a(m) = \phi(d)^{-1} \sum_\chi \chi(a)^* \chi(m)$ then $\delta_a(m) = 1$ when $m = a + nd$ and $\delta_a(m) = 0$ otherwise.

(iv) $\displaystyle\sum_{p=a+nd} p^{-s} = \phi(d)^{-1} \sum_\chi \chi(a)^* \sum_p \chi(p) p^{-s}.$

Lemma 117 *The sum $\sum_{p=a+nd} p^{-1}$ diverges if $\sum_p \chi(p)p^{-s}$ remains bounded as s tends to 1 through real values of $s > 1$ for all $\chi \neq \chi_0$.*

We now prove a new version of Euler's formula.

Theorem 118 (Euler-Dirichlet formula) *With the notation of this section,*

$$\prod_p (1 - \chi(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

both sides being absolutely convergent for $\Re s > 1$.

To link $\prod_p (1-\chi(p)p^{-s})^{-1}$ with $\sum_p \chi(p)p^{-s}$ we use logarithms. (If you go back to our discussion of infinite products, you will see that this is not unexpected.) However, we must, as usual, be careful when choosing our logarithm function. For the rest of the argument, log will be the function on

$$\mathbb{C} \setminus \{x : x \text{ real and } x \leq 0\}$$

defined by $\log(re^{i\theta}) = \log r + i\theta$ $[r > 0,\ -\pi < \theta < \pi]$.

Lemma 119 *(i) If* $|z| \leq 1/2$ *, then* $|\log(1-z)+z| \leq |z|^2$.

(ii) If $\epsilon > 0$*, then* $\sum_p \log(1-\chi(p)p^{-s})$ *and* $\sum_p \chi(p)p^{-s}$ *converge uniformly in* $\Re s \geq 1+\epsilon$*, whilst*

$$\left|\sum_p \log(1-\chi(p)p^{-s}) + \sum_p \chi(p)p^{-s}\right| \leq \sum_{n=1}^{\infty} n^{-2}.$$

We have thus shown that if $\sum_p \log(1-\chi(p)p^{-s})$ remains bounded as $s \to 1+$, then $\sum_p \chi(p)p^{-s}$ does. Unfortunately it is not possible to equate $\sum_p \log(1-\chi(p)p^{-s})$ with $\log(\prod_p(1-\chi(p)p^{-s})^{-1})$.

However, we can refresh our spirits by proving Dirichlet's theorem in some special cases.

Example 120 *There are an infinity of primes of the form* $3n+1$ *and* $3n+2$*.equate* $\sum_p \log(1-\chi(p)p^{-s})$ *with* $\log(\prod_p(1-\chi(p)p^{-s})^{-1})$*.*

Exercise 121 *Use the same techniques to show that there are an infinity of primes of the form* $4n+1$ *and* $4n+3$*.*

10 Analytic continuation of the Dirichlet functions

Dirichlet completed his argument withouequate $\sum_p \log(1-\chi(p)p^{-s})$ with $\log(\prod_p(1-\chi(p)p^{-s})^{-1})$.t having to consider $\sum_{n=1}^{\infty}\chi(n)n^{-s}$ for anything other than real s with $s > 1$. However, as we have already seen, $\sum_{n=1}^{\infty}\chi(n)n^{-s} = L(s,\chi)$ is defined and well behaved in $\Re s > 1$. Riemann showed that it is advantageous to extend the definition of analytic functions like $L(s,\chi)$ to larger domains.

There are many ways of obtaining such *analytic continuations*. Here is one.

Lemma 122 *If* $f : \mathbb{R} \to \mathbb{C}$ *is bounded on* $\mathbb{R}$ *and locally integrable*[5]*, then*

$$F(s) = \int_1^{\infty} f(x)x^{-s}\,dx$$

is a well defined analytic function on the set of s *with* $\Re s > 1$*.equate* $\sum_p \log(1-\chi(p)p^{-s})$ *with* $\log(\prod_p(1-\chi(p)p^{-s})^{-1})$*.*

[5]Riemann or Lebesgue at the reader's choice

Lemma 123 *(i) If $\chi \neq \chi_0$ and $S(x) = \sum_{1 \leq m \leq x} \chi(m)$, then $S : \mathbb{R} \to \mathbb{C}$ is bounded and locally integrable. We have*

$$\sum_{n=1}^{N} \chi(n) n^{-s} \to s \int_1^\infty S(x) x^{-s-1}\, dx$$

as $N \to \infty$ for all s with $\Re s > 1$.

(ii) If $S_0(x) = 0$ for $x \leq 0$ and $S_0(x) = \sum_{1 \leq m \leq x} \chi_0(m)$, then, writing

$$T_0(x) = S_0(x) - d^{-1}\phi(d)x,$$

we see that $T_0 : \mathbb{R} \to \mathbb{R}$ is bounded and locally integrable. We have

$$\sum_{n=1}^{N} \chi(n) n^{-s} \to s \int_1^\infty T_0(x) x^{-s-1}\, dx + \frac{\phi(d)s}{d(s-1)}$$

as $N \to \infty$ for all s with $\Re s > 1$.

Lemma 124 *(i) If $\chi \neq \chi_0$, there exists an function $L(s, \chi)$ analytic on $\{s \in \mathbb{C} : \Re s > 0\}$ such that $\sum_{n=1}^\infty \chi(n) n^{-s}$ converges to $L(s, \chi)$ on $\{s \in \mathbb{C} : \Re s > 1\}$.*

(ii) There exists a meromorphic function $L(s, \chi_0)$ analytic on $\{s \in \mathbb{C} : \Re s > 0\}$ except for a simple pole, residue $\phi(d)/d$ at 1 such that $\sum_{n=1}^\infty \chi_0(n) n^{-s}$ converges to $L(s, \chi)$ for $\Re s > 1$.

Exercise 125 *(i) Explain carefully why $L(\ , \chi_0)$ is defined uniquely by the conditions given.*

(ii) Show that $\sum_{n=1}^\infty \chi_0(n) n^{-s}$ diverges for s real and $1 \geq s > 0$.

We now take up from where we left off at the end of the previous section.

Lemma 126 *(i) If $\Re s > 1$, then $\exp(-\sum_p \log(1 - \chi(p)p^{-s}) = L(s, \chi)$.*

(ii) If $\Re s > 1$, then $L(s, \chi) \neq 0$.

(iii) There exists a function $\operatorname{Log} L(s, \chi)$ analytic on $\{s : \Re s > 1\}$ such that $\exp(\operatorname{Log} L(s, \chi)) = L(s, \chi)$ for all s with $\Re s > 1$.

(iv) If $\chi \neq \chi_0$ and $L(1, \chi) \neq 0$, then $\operatorname{Log} L(s, \chi))$ tends to a finite limit as $s \to 1$ through real values with $s > 1$.

(v) There is a fixed integer M_χ such that

$$\operatorname{Log} L(s, \chi) + \sum_p \log(1 - \chi(p)p^{-s}) = 2\pi M_\chi$$

for all $\Re s > 1$.

(vi) If $\chi \neq \chi_0$ and $L(1, \chi) \neq 0$, then $\sum_p \chi(p) p^{-s}$ remains bounded as $s \to 1$ through real values with $s > 1$.

We mark our progress with a theorem.

Theorem 127 *If $L(1,\chi) \neq 0$ for all $\chi \neq \chi_0$ then there are an infinity of primes of the form $a+nd$.*

Since it is easy to find the characters χ in any given case and since it is then easy to compute $\sum_{n=1}^{N} \chi(n)n^{-1}$ and to estimate the error $\sum_{n=N+1}^{\infty} \chi(n)n^{-1}$ to sufficient accuracy to prove that $L(1,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-1} \neq 0$, it now becomes possible to prove Dirichlet's theorem for any particular coprime a and d.

Exercise 128 *Choose a and d and carry out the program just suggested.*

However, we still need to show that the theorem holds in all cases.

11 $L(1,\chi)$ is not zero

Our first steps are easy.

Lemma 129 *(i) If s is real and $s > 1$, then*

$$\prod_{\chi} L(s,\chi) = \exp(-\sum_{p}\sum_{\chi} \log(1-\chi(p)p^{-s}).$$

(ii) If s is real and $s > 1$, then $\prod_\chi L(s,\chi)$ is real and $\prod_\chi L(s,\chi) \geq 1$.
(iii) $\prod_\chi L(s,\chi) \nrightarrow 0$ as $s \to 1$.

Lemma 130 *(i) There can be at most one character χ with $L(1,\chi) = 0$.*
(ii) If a character χ takes non-real values then $L(1,\chi) \neq 0$.

We have thus reduced the proof of Dirichlet's theorem to showing that if χ is a character with $\chi \neq \chi_0$ which only takes the values 1, -1 and 0, then $L(1,\chi) \neq 0$. There are several approaches to this problem, but none are short and transparent. We use a proof of de la Vallée Poussin which is quite short, but not, I think, transparent.

Lemma 131 *Suppose that the character $\chi \neq \chi_0$ and only takes the values 1, -1 and 0. Set*

$$\psi(s) = \frac{L(s,\chi)L(s,\chi_0)}{L(2s,\chi_0)}.$$

(i) The function ψ is well defined and meromorphic for $\Re s > \frac{1}{2}$. It is analytic except, possibly, for a simple pole at 1.
(ii) If $L(1,\chi) = 0$, then 1 is a removable singularity and ψ is analytic everywhere on $\{s : \Re s > \frac{1}{2}\}$.
(iii) We have $\psi(s) \to 0$ as $s \to \frac{1}{2}$ through real values of s with $s \geq \frac{1}{2}$.

Lemma 132 *We adopt the hypotheses and notation of Lemma 131. If $\Re s > 1$, then the following is true.*

(i) $\psi(s) = \prod_{\chi(p)=1} \frac{1+p^{-s}}{1-p^{-s}}$.

(ii) We can find subsets Q_1 and Q_2 of $\mathbb{Z}$ such that

$$\prod_{\chi(p)=1} (1+p^{-s}) = \sum_{n\in Q_1} n^{-s}$$
$$\prod_{\chi(p)=1} (1-p^{-s})^{-1} = \sum_{n\in Q_2} n^{-s}.$$

(iii) There is a sequence of real positive numbers a_n with $a_1 = 1$ such that

$$\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Lemma 133 *We adopt the hypotheses and notation of Lemmas 131 and 132.*

(i) If $\Re s > 1$, then

$$\psi^{(m)}(s) = \sum_{n=1}^{\infty} a_n(-\log n)^m n^{-s}.$$

(ii) If $\Re s > 1$, then $(-1)^m \psi^{(m)}(s) > 0$.

(iii) If ψ has no pole at 1, then, if $\Re s_0 > 1$ and $|s - s_0| < \Re s_0 - 1/2$, we have

$$\psi(s) = \sum_{m=0}^{\infty} \frac{\psi^{(m)}(s_0)}{m!}(s-s_0)^m.$$

(iv) If ψ has no pole at 1, then $\psi(s) \nrightarrow 0$ as $s \rightarrow \frac{1}{2}$ through real values of s with $s \geq \frac{1}{2}$.

We have proved the result we set out to obtain.

Lemma 134 *If a character $\chi \neq \chi_0$ only takes real values then $L(1,\chi) \neq 0$.*

Theorem 135 *If $\chi \neq \chi_0$, then $L(1,\chi) \neq 0$.*

We have thus proved Theorem 78. If a and d are strictly positive coprime integers, then there are infinitely many primes of the form $a + nd$ with n a positive integer.

12 Chebychev and the distribution of primes

On the strength of numerical evidence, Gauss was led to conjecture that the number $\pi(n)$ of primes less than n was approximately $n/\log n$. The theorem which confirms this conjecture is known as the prime number theorem. The first real progress in this direction was due to Chebychev[6]. We give his results, not out of historical piety, but because we shall make use of them in our proof of the prime number theorem. (Note the obvious conventions that n is an integer with $n \geq 1$, $\prod_{n<p\leq 2n}$ means the product over all primes p with $n < p \leq 2n$ and so on. It is sometimes useful to exclude small values of n.)

Lemma 136 *(i)* $2^n < \binom{2n}{n} < 2^{2n}$.

(ii) $\binom{2n}{n}$ *divides* $\prod_{p<2n} p^{[(\log 2n)/(\log p)]}$ *and* $\prod_{n<p\leq 2n} p$ *divides* $\binom{2n}{n}$.

(iii) We have $\pi(2n) > (\log 2)n/(\log 2n)$.

(iv) There exists a constant $A > 0$ *such that* $\pi(n) \geq An(\log n)^{-1}$.

(v) There exists a constant B' *such that* $\sum_{p\leq n} \log p \leq B'n$.

(vi) There exists a constant B *such that* $\pi(n) \leq Bn(\log n)^{-1}$.

We restate the main conclusions of Lemma 136.

Theorem 137 (Chebychev) *There exist constants* A *and* B *with* $0 < A \leq B$ *such that*

$$An(\log n)^{-1} \leq \pi(n) \leq Bn(\log n)^{-1}.$$

Riemann's approach to the prime number theorem involves considering $\theta(n) = \sum_{p\leq n} \log p$ rather than $\pi(n)$.

Lemma 138 *Let* Q *be a set of positive integers and write* $\alpha(n) = \sum_{q\in Q, q\leq n} 1$ *and* $\beta(n) = \sum_{q\in Q, q\leq n} \log q$.

(i) There exist constants A *and* B *with* $0 < A \leq B$ *such that*

$$An(\log n)^{-1} \leq \alpha(n) \leq Bn(\log n)^{-1}.$$

if and only if there exist constants A' *and* B' *with* $0 < A \leq B$ *such that*

$$A'n \leq \beta(n) \leq B'n.$$

(ii) We have $n^{-1}(\log n)\alpha(n) \to 1$ *as* $n \to \infty$ *if and only if* $n^{-1}\beta(n) \to 1$ *as* $n \to \infty$.

Lemma 139 *If* $n^{-1}\theta(n) \to 1$ *as* $n \to \infty$, *then* $n^{-1}(\log n)\pi(n) \to 1$ *as* $n \to \infty$.

[6] His preferred transliteration seems to have been Tchebycheff, but he has been overruled.

13 The prime number theorem

We start by recalling various facts about the Laplace transform.

Definition 140 *If a is a real number, let us write $\mathcal{E}_a$ for the collection of piecewise continuous functions $F : \mathbb{R} \to \mathbb{C}$ such that $F(t) = 0$ for all $t < 0$ and $F(t)e^{-at} \to 0$ as $t \to \infty$. If $F \in \mathcal{E}_a$, we define the* Laplace transform *of F by*

$$(\mathcal{L}F)(z) = \int_{-\infty}^{\infty} F(t)\exp(-zt)\,dt$$

for $\Re z > a$.

Lemma 141 *If $F \in \mathcal{E}_a$, then $(\mathcal{L}F)(z)$ is well defined.*

Lemma 142 *(i) If $F \in \mathcal{E}_a$, then $(\mathcal{L}F)(z)$ analytic on $\{z \in \mathbb{C} : \Re z > a\}$.*

(ii) We define the Heaviside function H by writing $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geq 0$. If $a \in \mathbb{R}$ and $b \geq 0$ set $H_{a,b}(t) = H(t-b)e^{at}$. Then $H_{a,b} \in \mathcal{E}_a$ and $\mathcal{L}H_{a,b}(z)$ can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at a.

Exercise 143 *(Uses Exercise 77.) Let $F, G \in \mathcal{E}_a$ for some $a \in \mathbb{R}$.*

(i) Suppose that there exists a $b > a$ such that $(\mathcal{L}F)(z) = (\mathcal{L}G)(z)$ for all z with $\Re z = b$. Show that $F = G$.

(ii) Suppose that there exist distinct $z_n \in \mathbb{C}$ with $\Re z_n > a$ $[n \geq 0]$ such that $z_n \to z_0$ and $(\mathcal{L}F)(z_n) = (\mathcal{L}G)(z_n)$ $[n \geq 0]$. Show that $F = G$.

Engineers are convinced that the converse to Lemma 142 (i) holds in the sense that if $F \in \mathcal{E}_a$ has a Laplace transform f which can be extended to a function $\tilde{f}$ analytic on $\{z \in \mathbb{C} : \Re z > b\}$ [a, b real, $a \geq b$], then $F \in \mathcal{E}_b$. Unfortunately, this is not true, but it represents a good heuristic principle to bear in mind in what follows. Number theorists use the Mellin transform

$$\mathcal{M}F(z) = \int_0^{\infty} F(t)t^{z-1}\,dt$$

in preference to the Laplace transform but the two transforms are simply related.

Exercise 144 *Give the relation explicitly.*

Riemann considered the two functions

$$\Phi(s) = \sum_p p^{-s}\log p$$

and the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Both of these functions are defined for $\Re s > 1$ but Riemann saw that they could be extended to analytic functions over a larger domain.

The next lemma is essentially a repeat of Lemmas 123 (ii) and 124 (ii).

Lemma 145 *(i) Let $S_0(x) = 0$ for $x \leq 0$ and $S_0(x) = \sum_{1 \leq m \leq x} 1$. If*

$$T_0(x) = S_0(x) - x,$$

then T_0 is bounded and locally integrable. We have

$$\sum_{n=1}^{N} n^{-s} \to s \int_1^{\infty} T_0(x) x^{-s-1}\, dx + \frac{s}{s-1}$$

as $N \to \infty$ for all s with $\Re s > 1$.

(ii) There exists a meromorphic function ζ analytic on $\{s \in \mathbb{C} : \Re s > 0\}$ except for a simple pole, residue 1 at 1 such that $\sum_{n=1}^{\infty} n^{-s}$ converges to $\zeta(s)$ for $\Re s > 1$.

(iii) If $\Re s > 1$, then

$$\sum_{p \leq N} \frac{\log p}{p^s} \to s \int_1^{\infty} \theta(x) x^{-s-1}\, dx$$

as $N \to \infty$.

The use of s rather than z goes back to Riemann. Riemann showed that ζ can be extended to a meromorphic function over $\mathbb{C}$, but we shall not need this.

How does this help us study Φ?

Lemma 146 *(i) We have $\prod_{p<N}(1-p^{-s})^{-1} \to \zeta(s)$ uniformly for $\Re s > 1+\delta$ whenever $\delta > 0$.*

(ii) We have

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log p}{p^s - 1}$$

for all $\Re s > 1$.

(iii) We have

$$\Phi(s) = -\frac{\zeta'(s)}{\zeta(s)} - \sum_p \frac{\log p}{(p^s-1)p^s}$$

for all $\Re s > 1$.

(iv) The function Φ *can be analytically extended to a meromorphic function on* $\{s : \Re s > \frac{1}{2}\}$. *It has a simple pole at 1 with residue* 1 *and simple poles at the zeros of* ζ *but nowhere else.*

The next exercise is long and will not be used later but is, I think, instructive.

Exercise 147 *(i) Show by grouping in pairs that* $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ *converges to an analytic function* $g(s)$ *in the region* $\{s : \Re s > 0\}$.

(ii) Find A and B such that $g(s) = A\zeta(s)+B2^{-s}\zeta(s)$ *for all* $\Re s > 1$. *Why does this give another proof that* ζ *can be extended to an analytic function on* $\{s : \Re s > 0\}$*?*

(iii) Show that $g(1/2) \neq 0$ *and deduce that* $\zeta(1/2) \neq 0$.

(iv) By imitating the arguments of Lemma 146, show that we we can find an analytic function G *defined on* $\{s : \Re s > 1/3\}$ *such that*

$$\Phi(s) = -\frac{\zeta'(s)}{\zeta(s)} - \Phi(2s) - G(s).$$

Deduce that Φ *can be extended to a meromorphic function on* $\{s : \Re s > 1/3\}$.

(v) Show, using (iii), that Φ *has a pole at* $1/2$.

(vi) Show that the assumption that $|\sum_{p<N}\log p - N| \leq AN^{1/2-\epsilon}$ *for some* $\epsilon > 0$ *and* $A > 0$ *and all large enough* N *leads to the conclusion that* Φ *can be analytically extended from* $\{s : \Re s > 1\}$ *to an everywhere analytic function on* $\{s : \Re s > 1/2 - \epsilon\}$.

(vii) Deduce that if $\epsilon > 0$ *and* $A > 0$

$$|\sum_{p<N}\log p - N| \geq AN^{1/2-\epsilon} \text{ for infinitely many values of } N.$$

It is well known that Riemann conjectured that ζ has no zeros in $\{s : \Re s > 1/2\}$ and that his conjecture is the most famous open problem in mathematics. The best we can do is to follow Hadamard and de la Vallée Poussin and show that ζ has no zero on $\{s : \Re s = 1\}$. Our proof makes use of the slightly unconventional convention that if h and g are analytic in a neighbourhood of w, $g(w) \neq 0$ and $h(z) = (z-w)^k g(z)$, then h has a zero of order k at w. (The mild unconventionality arises when $k = 0$.)

Lemma 148 *Suppose that* ζ *has a zero of order* μ *at* $1 + i\alpha$ *and a zero of order* ν *at* $1 + 2i\alpha$ *with* α *real and* $\alpha > 0$. *Then the following results hold.*

(i) ζ *has a zero of order* μ *at* $1 - i\alpha$ *and a zero of order* ν *at* $1 - 2i\alpha$.

(ii) As $\epsilon \to 0$ through real positive values of ϵ

$$\begin{aligned} \epsilon\Phi(1+\epsilon \pm i\alpha) &\to -\mu \\ \epsilon\Phi(1+\epsilon \pm 2i\alpha) &\to -\nu \\ \epsilon\Phi(1+\epsilon) &\to 1. \end{aligned}$$

(iii) If $s = 1 + \epsilon$ with ϵ real and positive, then

$$\begin{aligned} 0 &\leq \sum_p p^{-s} \log p (e^{(i\alpha \log p)/2} + e^{-(i\alpha \log p)/2})^4 \\ &= \Phi(s+2i\alpha) + \Phi(s-2i\alpha) + 4(\Phi(s+i\alpha) + \Phi(s-i\alpha)) + 6\Phi(s). \end{aligned}$$

(iv) We have $0 \leq -2\nu - 8\mu + 6$.

Theorem 149 *If $\Re s = 1$, then $\zeta(s) \neq 0$.*

We note the following trivial consequence.

Lemma 150 *If we write*

$$T(s) = \frac{\zeta'(s)}{\zeta(s)} - (s-1)^{-1},$$

then given any $R > 0$ we can find a $\delta(R)$ such that T has no poles in

$$\{z : \Re z \geq 1 - \delta(R),\ |\Im z| \leq R\}.$$

We shall show that the results we have obtained on the behaviour of ζ suffice to show that

$$\int_1^X \frac{\theta(x) - x}{x^2}\, dx$$

tends to a finite limit as $X \to \infty$. The next lemma shows that this is sufficient to give the prime number theorem.

Lemma 151 *Suppose that $\beta : [1, \infty) \to \mathbb{R}$ is an increasing (so integrable) function.*

(i) If $\lambda > 1$, $y > 1$ and $y^{-1}\beta(y) > \lambda$, then

$$\int_y^{\lambda y} \frac{\beta(x) - x}{x^2}\, dx \geq A(\lambda)$$

where $A(\lambda)$ is a strictly positive number depending only on λ.

(ii) If

$$\int_1^X \frac{\beta(x) - x}{x^2}\, dx$$

tends to limit as $X \to \infty$, then $x^{-1}\beta(x) \to 1$ as $x \to \infty$.

We need a couple of further preliminaries. First we note a simple consequence of the Chebychev estimates (Theorem 137).

Lemma 152 *There exists a constant* $1 > K > 0$ *such that*

$$\frac{|\theta(x) - x|}{x} \leq K$$

for all x *sufficiently large.*

Our second step is to translate our results into the language of Laplace transforms. (It is just a matter of taste whether to work with Laplace transforms or Mellin transforms.)

Lemma 153 *Let* $f(t) = \theta(e^t)e^{-t} - 1$ *for* $t \geq 0$ *and* $f(t) = 0$ *otherwise. Then*

$$\mathcal{L}f(z) = \int_{-\infty}^{\infty} f(t)e^{-tz}\, dt$$

is well defined and

$$\mathcal{L}f(z) = \frac{\Phi(z+1)}{z+1} - \frac{1}{z}$$

for all $\Re z > 0$.

The statement $\int_1^\infty (\theta(x) - x)/x^2\, dx$ *convergent is equivalent to the statement that* $\int_{-\infty}^\infty f(t)\, dt$ *converges.*

We have reduced the proof of the prime number theorem to the proof of the following lemma.

Lemma 154 *Suppose* Ω *is an open set with* $\Omega \supseteq \{z : \Re z \geq 0\}$, $F : \Omega \to \mathbb{C}$ *is an analytic function and* $f : [0, \infty] \to \mathbb{R}$ *is bounded locally integrable function such that*

$$F(z) = \mathcal{L}f(z) = \int_0^\infty f(t)e^{-tz}\, dt$$

for $\Re z > 0$. *Then* $\int_0^\infty f(t)\, dt$ *converges.*

This lemma and its use to prove the prime number theorem are due to D. Newman. (A version will be found in [1].)

14 The Fourier transform and Heisenberg's inequality

In this section we return to the Fourier transform on $\mathbb{R}$. We follow a slightly different path to that mapped out in Section 6. I shall state results using

Lebesgue measure but students using Riemann integration will find appropriate modifications as exercises.

We pay particular attention to the Gaussian (or heat, or error) kernel $E(x) = (2\pi)^{-1/2}\exp(-x^2/2)$.

Lemma 155 $\hat{E}(\lambda) = (2\pi)^{1/2}E(\lambda)$.

Exercise 156 *If I prove Lemma 155 I shall do so by setting up a differential equation. Obtain Lemma 155 by complex variable techniques.*

We use the following neat formula.

Lemma 157 *If $f, g \in L^1(\mathbb{R})$ then the products $\hat{f} \times g,\ f \times \hat{g} \in L^1(\mathbb{R})$ and*

$$\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{infty}^{\infty} f(\lambda)\hat{g}(\lambda)\,d\lambda.$$

Exercise 158 *(For those using Riemann integration. You will need to refer back to the exercises in Section 6.) Suppose that f and g are continuous and there exists a real constant A such that*

$$|f(x)|, |g(x)| \leq A(1+x^2)^{-1}$$

for all x and

$$|\hat{f}(\lambda)|, |\hat{g}(\lambda)| \leq A(1+\lambda^2)^{-1}$$

for all λ. Show that

$$\int_{-\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{-\infty}^{\infty} f(\lambda)\hat{g}(\lambda)\,d\lambda.$$

Without going into detail, convince yourself that the hypothesis 'f and g are continuous' can be replaced by 'f and g are piecewise continuous'.

By taking $f = E_h$ where $E_h(x) = h^{-1}E(h^{-1}(x))$ $[h > 0]$ in Lemma 157 we obtain a nice pointwise inversion result.

Theorem 159 *If $f, \hat{f} \in L^1$ and f is continuous at t, then $\hat{\hat{f}}(t) = 2\pi f(-t)$.*

Exercise 160 *(For those using Riemann integration.) Suppose that f is piecewise continuous and there exists a real constant A such that*

$$|f(x)| \leq A(1+x^2)^{-1}$$

for all x and

$$|\hat{f}(\lambda)| \leq A(1+\lambda^2)^{-1}$$

for all λ. Show that if f is continuous at t, then $\hat{\hat{f}}(t) = 2\pi f(-t)$.

Exercise 161 *Suppose that f satisfies the conditions of Theorem 159 (if you use Lebesgue integration) or Exercise 160 (if you use Riemann integration). If t is a point where $f(t+) = \lim_{h\to 0+} f(t+h)$ and $f(t-) = \lim_{h\to 0+} f(t-h)$ both exist, show that*

$$\hat{\hat{f}}(-t) = \pi(f(t+) + f(t-)).$$

Lemma 162 (Parseval's equality) *If f and $\hat{f}$ are continuous and integrable and $\hat{\hat{f}}(t) = 2\pi f(-t)$ for all t then*

$$\int_{\mathbb{R}} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\lambda)|^2\,d\lambda.$$

(A les ad hoc version of Parseval's equality is given in Exercise 163 (v).)

Exercise 163 *(This requires Lebesgue measure.) The present course is rather old fashioned, not least in the way it thinks of Fourier transforms $\hat{f}$ in terms of its values $\hat{f}(\lambda)$ at points λ, rather than an object in its own right. Here is one of several ways in which a more general view gives a more elegant theory.*

(i) Let S be the set of infinitely differentiable functions f with

$$x^n f^{(m)}(x) \to 0$$

as $|x| \to \infty$ for all integers $n, m \geq 0$. Show that, if $f \in S$, then $\hat{f} \in S$.

(ii) Let $\mathbb{I}_{[a,b]}(x) = 1$ for $x \in [a,b]$, $\mathbb{I}_{[a,b]}(x) = 0$ otherwise. Show that, if E_h is defined as above, then

$$\|\mathbb{I}_{[a,b]} - E_h * \mathbb{I}_{[a,b]}\|_2 \to 0$$

as $h \to 0+$. Deduce, or prove otherwise, that S is L^2 norm dense in L^2.

(iii) By taking $g = \hat{f}$ in Lemma 157 show that

$$\|\hat{f}\|_2^2 = 2\pi\|f\|_2^2$$

for all $f \in S$.

(iii) Deduce that there is a unique continuous mapping $\mathcal{F} : L^2 \to L^2$ with $\mathcal{F}(f) = \hat{f}$ for all $f \in S$. (Uniqueness is easy but you should take care proving existence.)

(iv) Show that $\mathcal{F} : L^2 \to L^2$ is linear and that

$$\|\mathcal{F}(f)\|_2^2 = 2\pi\|f\|_2^2$$

for all $f \in L^2$.

If we define $\mathcal{J} : L^2 \to L^2$ by $(\mathcal{J}f)(t) = f(-t)$ show that $\mathcal{F}^2 = 2\pi\mathcal{J}$.

(v) If we wish to work in L^2, it makes sense to use a different normalising factor and call $\mathcal{G} = (2\pi)^{-1/2}\mathcal{F}$ the Fourier transform. Show that $\mathcal{G}^4 = I$ and that $\mathcal{G} : L^2 \to L^2$ is a bijective linear isometry.

(vi) (Parseval's equality) Show that, if we work in L^2

$$\int_{\mathbb{R}} \mathcal{G}f(\lambda)(\mathcal{G}g)^*(\lambda)\,d\lambda = \int_{\mathbb{R}} f(t)g(t)^*\,dt.$$

We now come to one of the key facts about the Fourier transform (some would say one of the key facts about the world we live in).

Theorem 164 (Heisenberg's inequality) *If f is reasonably well behaved, then*

$$\frac{\int_{-\infty}^{\infty} \lambda^2|\hat{f}(\lambda)|^2\,d\lambda}{\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2\,d\lambda} \times \frac{\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx}{\int_{-\infty}^{\infty} |f(x)|^2\,dx} \geq \frac{1}{4}.$$

If equality holds, then $f(x) = A\exp(-bx^2)$ for some $b > 0$.

Exercise 165 *Write down explicit conditions for Theorem 164.*

The extension of Heisenberg's inequality to all $f \in L^2$ is given in Section 2.8 of the beautiful book [3] of Dym and McKean.

15 The Poisson formula

The following remarkable observation is called Poisson's formula.

Theorem 166 *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous function such that $\sum_{m=-\infty}^{\infty} |\hat{f}(m)|$ converges and $\sum_{n=-\infty}^{\infty} |f(2\pi n + x)|$ converges uniformly on $[-\pi, \pi]$. Then*

$$\sum_{m=-\infty}^{\infty} \hat{f}(m) = 2\pi \sum_{n=-\infty}^{\infty} f(2\pi n).$$

It is possible to adjust the hypotheses on f in Poisson's formula in various ways, though some hypotheses there must be. We shall simply think of f as 'well behaved'. The following rather simple lemma will suffice for our needs.

Lemma 167 *If $f : \mathbb{R} \to \mathbb{C}$ is a twice continuously differentiable function such that $\int_{-\infty}^{\infty} |f(x)|\,dx$, $\int_{-\infty}^{\infty} |f'(x)|\,dx$ and $\int_{-\infty}^{\infty} |f''(x)|\,dx$ converge whilst $f'(x) \to 0$ and $x^2 f(x) \to 0$ as $|x| \to \infty$, then f satisfies the conditions of Theorem 166.*

Exercise 168 *(i) By applying Poisson's formula to the function f defined by $f(x) = \exp(-t|x|/2\pi)$, show that*

$$2(1-e^{-t})^{-1} = \sum_{n=-\infty}^{\infty} 2t(t^2 + 4\pi^2 n^2)^{-1}.$$

(ii) By expanding $(t^2+4\pi n^2)^{-1}$ and (carefully) interchanging sums, deduce that

$$2(1-e^{-t})^{-1} = 1 + 2t^{-1} + \sum_{m=0}^{\infty} c_m t^m$$

where $c_{2m} = 0$ and

$$c_{2m+1} = a_{2m+1} \sum_{n=1}^{\infty} n^{-2m}$$

for some value of a_{2m+1} to be given explicitly.

(iii) Hence obtain Euler's formula

$$\sum_{n=1}^{\infty} n^{-2m} = (-1)^{m-1} 2^{2m-1} b_{2m-1} \pi^{2m}/(2m-1)!$$

for $m \geq 1$, where the b_m are defined by the formula

$$(e^y - 1)^{-1} = y^{-1} - 2^{-1} + \sum_{n=1}^{\infty} b_n y^n/n!$$

(The b_n are called Bernoulli numbers.)

Exercise 169 *Suppose f satisfies the conditions of Lemma 167. Show that*

$$K \sum_{m=-\infty}^{\infty} \hat{f}(Km) = 2\pi \sum_{n=-\infty}^{\infty} f(2\pi K^{-1} n)$$

for all $K > 0$. What is the corresponding result when $K < 0$?

By letting $K \to 0+$, deduce that

$$\hat{\hat{f}}(0) = 2\pi f(0).$$

(There is some interest in just seeing that this is so but it is more profitable to give a rigorous proof.) Deduce in the usual way that

$$\hat{\hat{f}}(t) = 2\pi f(-t)$$

for all t.

Poisson's formula has a particularly interesting consequence.

Lemma 170 *If $g : \mathbb{R} \to \mathbb{C}$ is twice continuously differentiable and $g(t) = 0$ for $|t| \geq \pi$, then g is completely determined by the values of $\hat{g}(m)$ for integer m.*

Taking $g = \hat{f}$ and remembering the inversion formula we obtain the following result.

Pretheorem 171 *If $f : \mathbb{R} \to \mathbb{C}$ is a well behaved function with $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \pi$, then f is determined by its values at integer points.*

We call this a pretheorem because we have not specified what 'well behaved' should mean.

The simplest approach is via the *sinc function*

$$\operatorname{sinc}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(ix\lambda)\, d\lambda.$$

We state the most immediately useful properties of sinc.

Lemma 172 *(i)* $\operatorname{sinc}(0) = 1$,
(ii) $\operatorname{sinc}(n) = 0$ *if* $n \in \mathbb{Z}$ *but* $n \neq 0$.

(We note also that although, strictly speaking, $\widehat{\operatorname{sinc}}(\lambda)$ is not defined for us, since $\int |\operatorname{sinc}(x)|\, dx = \infty$, we are strongly tempted to say that $\widehat{\operatorname{sinc}}(\lambda) = 1$ if $|\lambda| < \pi$ and $\widehat{\operatorname{sinc}}(\lambda) = 0$ if $|\lambda| > \pi$.)

We can, at once, prove that Pretheorem 171 is best possible.

Lemma 173 *If $\epsilon > 0$, then we can find an infinitely differentiable non-zero f such that $\hat{f}(\lambda) = 0$ for $|\lambda| > \pi + \epsilon$, but $f(n) = 0$ for all $n \in \mathbb{Z}$.*

Exercise 174 *In Lemma 173 show that we can take $f \in S$ where S is the class discussed in Exercise 163.*

We can also show how to recover the function of Pretheorem 171 from its values at integer points.

Theorem 175 *Suppose $f : \mathbb{R} \to \mathbb{C}$ is a continuous function with $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$. If $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \pi$, then*

$$\sum_{n=-N}^{N} f(n) \operatorname{sinc}(t - n) \to f(t)$$

as uniformly as $N \to \infty$.

Thus Pretheorem 171 holds under very general conditions. We state it in a lightly generalised form.

Theorem 176 (Shannon's Theorem) *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous function with $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$ and that $K > 0$. If $\hat{f}(\lambda) = 0$ for $|\lambda| \geq K$, then f is determined by its values at points of the form $n\pi K^{-1}$ with $n \in \mathbb{Z}$.*

Theorem 176 belongs to the same circle of ideas as Heisenberg's inequality. It is the key to such devices as the CD.

16 References and further reading

If the elegance and variety of a subject is to be judged by the elegance and variety of the (best) texts on that subject, Fourier Analysis must surely stand high. On the pure side the books of Helson [4] and Katznelson [6] would be my first choice for introductions and this course draws on both. If you wish to think about applications, the obvious text is that of Dym and McKean [3]. The next two recommendations are irrelevant to Part III but, if you go on to work in any field involving classical analysis, Zygmund's treatise [10] is a must and, if you would like a first glimpse at wavelets, (unmentioned in this course) Babarah Hubbard's popularisation *The World According to Wavelets* [5] is splendid light reading.

There is an excellent treatment of Dirichlet's theorem and much more in Davenport's *Multiplicative Number Theory* [2]. [The changes between the first and second editions are substantial but do not affect that part which deals with material in this course.] If you wish to know mThore about the Riemann zeta-function you can start with [9].

In preparing this course I have also used [7] and [8] since I find the author sympathetic.

References

[1] J. Bak and D. J. Newman *Complex Analysis* Springer, New York, 1982.

[2] H. Davenport *Multiplicative Number Theory* (2nd Edition), Springer, New York, 1980.

[3] H. Dym and H. P. McKean *Fourier Series and Integrals* Academic Press, 1972.

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[5] B. Hubbard *The World According to Wavelets* A. K. Peters, 1996.

[6] Y. Katznelson *An Introduction to Harmonic Analysis* Wiley, 1963. [There is a CUP reprint]

[7] T. W. Körner *Fourier Analysis* CUP, 1988.

[8] T. W. Körner *Exercises for Fourier Analysis* CUP, 1993.

[9] S. J. Patterson *An Introduction to the Theory of the Riemann zeta-function* CUP, 1988.

[10] A. Zygmund *Trigonometric Series* (2 Volumes) CUP, 1959.