How to Write a Part III Essay

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These *unofficial* notes replace an earlier set by Marj Batchelor which were becoming illegible through repeated photocopying. Many of the key pieces of advice are taken almost word for word from her notes though the elegant picture of a small crustacean has, I am afraid, vanished. I should be glad to have suggestions for additions, corrections and improvements (by e-mail to `twk@dpmms.cam.ac.uk` or otherwise) both from essay writers and essay markers. These notes were last revised in 2009.

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1 Introduction

It is as foolish to write an essay on essay writing as it is to lecture on lecturing or give a course on teaching. We do not learn to write mathematics by following a set of rules. We learn by imitating other mathematicians or avoiding their mistakes[1]. Eventually, with practice, we acquire our own voice.

On the other hand, there are a few tricks of the trade which will convert a ghastly expositor into a bad one, a bad expositor in to a moderate one and a moderate one into a good one. I shall try to give some of them. You may disagree with some or all of what I say. That does not matter. What matters is that you should think about the problems of mathematical writing.

1.1 What is the essay?

Every Part III student has the option of replacing a three hour examination paper by an essay. In the 'standard essay' you are asked to read two or three mathematics papers and then write a connected account of their contents. The essay is set and marked by an 'assessor' who also gives you advice on how to tackle it. Because Part III covers a wide range of subjects and is taught by a wide range of people this standard pattern may be modified in all sorts of ways

The essay option is not compulsory, partly to allow you to abandon your essay if it goes badly wrong and partly because few of the staff fancy supervising the essay of an unwilling student. However, there are many reasons why you should do the essay.

(1) It provides practice in reading and writing mathematics.

(2) With luck, it will help you choose a PhD topic. In any case it will help get you into a 'PhD state of mind'.

(3) It will bring you into contact with a member of the staff. This is a good thing in itself and may prove very useful when you need a reference. If you want to stay on to do a PhD here a good essay will be an excellent recommendation to possible research supervisors.

(4) It provides an alternative to the 'listen to the lecturer, study your notes, write out the exam' system of the rest of Part III.

Inspection of the Part III mark lists shows that *your essay will normally be among your best papers*[2]. The reasons for this are obvious. With an essay

[1]Clearly we should follow the example of good expositors and avoid the mistakes of bad ones. Who are your models of good exposition? Or do you think all mathematical exposition is of the same standard?

[2]This may not be true for the top 10 candidates who may well find it easier to get full marks on an exam than on an essay. However, if you are that good you ought to do the

you can judge your progress much more easily than with preparation for an exam. If you work hard at an essay you can expect to do well at it — even if you work hard preparing an exam you may still be unlucky on the day.

Sometimes students complain that they work harder on their essay than on their other papers. There are various answers to this.

(1) Perhaps they need to work harder on their other papers.

(2) Both students and assessors are explicitly advised by the Faculty Board that a good essay should not require more work than a 24 hour examinable course.

(3) (This is I think the real answer.) Work done studying for an examination is not directly comparable with work done on an essay. It is easier to work for five exams and one essay than for six exams because the essay requires different skills and provides a refreshing change.

1.2 Essay length

There is no set length for an essay but the standard advice is to aim for 5000 to 7000 words[3]. This does not mean that anybody will be worried if you aim for 7000 words and end up with 8000 but if you aim for 7000 words and end up with 10000 your assessor will certainly feel that your essay is too long. Would you be happy if you went to a lecture scheduled to last 50 minutes and slowly realised as the lecturer droned on that it was going to last for two hours? After a certain point writing more will only involve you in extra work without any reward. (Indeed, you may actually lose marks.)

In theory (see Littlewood's essay *Mathematics with a minimum of raw material* in [9]) two sentences could deserve a College Research Fellowship (and so, certainly a PhD). In the same way 2000 words, if they were the right words, could well constitute an $\alpha+$ essay but, in practice, much less than 5000 words would probably be considered a bit light-weight.

When I explain this point students invariably ask 'Do you count formulae?' This shows that they do not understand what I have just said. A figure like 6000 words is only a guideline. Nobody is going to actually count words so nobody cares what you do about formulae and diagrams. (Most students use too many formulae and not enough diagrams, but that is another matter.) I give a method for estimating the number of words in your essay in

essay on other, educational grounds and, in fact, the top candidates almost invariably do an essay.

[3]Unless your assessor suggests otherwise. This is one of the things you should discuss at your first meeting. It is unlikely that your assessor agrees with everything in these notes and since your task is to please him or her, your assessor's views are much more important than mine.

Section 2.1.

1.3 Shakespeare it is not

The essay is not an exercise in fine English but a preparation for writing mathematics papers. (The only way to learn to ride a bicycle is to ride a bicycle.) If English is not your first language you should rest assured that this will not handicap you[4]. The assessors are interested in clear thought and clear mathematics however expressed. For good or ill, English is likely to remain the lingua franca of mathematics for your lifetime. You may have to write your mathematics in English in the future. You should seize the opportunity to practice now.

More generally

the less you want to do the essay, the more important it is that you do it.

The next time you write a substantial piece of mathematics it will probably be a paper for publication or a PhD thesis for examination. Take this one available opportunity for a dress rehearsal.

1.4 Nor is it research

Sometimes students ask why the essay could not be replaced by a research project. This question makes three basic assumptions:

(1) Reading mathematics is easy.
(2) Writing mathematics is easy.
(3) Mathematical research is easy.

Assumption (3) is clearly false. A PhD is a *three year* apprenticeship and the test of a good PhD is that it should contain one good and original idea. In my opinion (and in the opinion of most mathematicians I know) assumptions (1) and (2) are also false. The essay gives you the opportunity to practice reading and writing mathematics.

Of course, the assessor will be delighted if you find a new proof of some result or make a clever application of the method you describe but such things are not required for a good essay[5].

[4]If you do not believe me and you think that we are more interested in your English than your mathematics then you should still do the essay since you have even less chance of writing good English in a three hour exam.

[5]You should also remember that what seems new to you may not be new to your more knowledgeable assessor.

2 Technical points

2.1 How to count words

The standard way to count typed or handwritten words is statistical. Take a page of your essay, count the number of words on it and multiply by the number of pages. Of course this may be inaccurate by 10% or 20% but nobody cares about the exact length of your essay, your object is to provide a rough guide for your own use. In later life you may need to gauge the length of a paper or book. The publisher is then not interested in the number of words but in the number of published pages you will use. In accordance with the modern principle of giving customers what they want rather than what they need most word processors have a word counting facility.

2.2 Meetings with your assessor

You should normally have a talk with the assessor for your essay once before starting work on it, once when you have sketched out a full plan for it and once when you have completed your first draft. (Of course, individual assessors may wish to see you on fewer or more occasions or at different times.) In addition you should contact your assessor if a really serious problem crops up (for example if the main proof in the paper you are studying appears to be fatally flawed). Many assessors will confine their advice to cases when your project seems to them to be veering badly off course. If things are going well they will give you encouragement but nothing else. Other assessors discuss essays in detail with everybody. (If you go on to do a PhD you will find that PhD supervisors exhibit an even wider range of attitudes.)

2.3 How to contact your assessor

You know when your essay has reached one of the points when it needs to be discussed. Your assessor does not. It is therefore up to you to make contact with your assessor. Most assessors can be contacted by e-mail. Close study of the following two possible messages may help you draft your own.

Message A

Hi I'm Jean and I'm doing your essay. It would be real cool if we could meet sometime and discuss it.

Thought for the day. No dog is so short that its legs do not reach the ground.

Message B

Dear Dr Moreau

My name is Jean Brun and I am a student at St Judes. I have completed the first draft of my essay on 'Central Principles' for which you are the assessor. Could we meet to discuss it? I am free every afternoon except Thursday and from 11 to 1 on Tuesday, Thursday and Saturday. Would you like me to place a Xerox of my first draft in your pigeon hole in the department?

Thank you

If the assessor does not use e-mail send them a letter or contact them personally.

2.4 Unfair means

The nature of the Part III essay is such that cheating is unlikely to occur. However, here are some guidelines for you to observe.

(1) You **are** employing unfair means if you make substantial use of a source without making it clear that you are doing so, (I go into the mechanics of acknowledgement in the last section.)

(2) You **are** employing unfair means if you make substantial use of an unpublished source. Thus you may not use an essay that you wrote for some other purpose or adapt part of someone else's essay. (It is possible to imagine exceptions to this rule but you should not go against it without the **explicit** permission of your assessor.)

(3) You **are** employing unfair means if you ask someone other than the assessor to help you improve or correct the mathematics of your essay.

(4) You **are not** employing unfair means if you ask someone to read your essay with a view to removing grammatical errors and misprints of all kinds. (Use someone who is not an expert in the topic.)

(5) You **are not** employing unfair means if you talk about your essay with other students or give a seminar based on your essay with questions at the end.

If you are in any doubt whatsoever as to the propriety of anything that you wish to do you should **at once** consult your assessor or a senior member of staff.

2.5 Making a timetable

Every substantial piece of work involves crises. The printer may break down for five days, the book you want may be out of the library or you may suddenly realise that the proof you worked out with so much labour has a gaping hole in the middle. Your timetable must leave room for such crises.

Your timetable should also leave time for reflection. Every author's manual contains the same advice, often in the same words. 'When you have completed a draft let it sleep for a time before embarking on correction or rewriting.' If you attack the same problem over and over again without rest you will always attack it in the same way. If you step away from it for a while and then return you may discover a new method of approach.

Unfortunately you must hand in your essay by a certain date. (If you fail to do so, the examiners may simply refuse to consider it.) The obvious and correct advice is 'start as early as possible and aim to finish as soon as possible'. One plausible timetable would call for you to read up during the Christmas holidays, to write your first draft in the second term, revise in the Easter holidays and hand in at the beginning of the third term leaving you free to concentrate on examination revision in the five or six week run up to your exams.

3 How to read a paper

Why are bookshops filled with 'self-help' books? Evidently because people buy them. Why do people buy 'self-help' books? One reason must be to reassure themselves that they are not alone and that whatever their problem, from excessive shyness through to poor punctuation, it is one shared with many other people. I can give very little help with your problems in reading mathematics, but I can assure you that you share them with most other mathematicians.

3.1 Why are mathematics papers hard to read?

You are hardly likely to be doing Part III if you have not been able to understand most of what your lecturers told you in the past. It therefore comes as a shock when you try to read mathematics papers and find them hard to follow. Of course, the main reason why mathematics papers are hard to read is that mathematics is hard but there are good reasons why they are harder to understand than lectures.

(1) A mathematics paper stakes out a claim. Thus the writer will prove the strongest version of the theorem that he or she can. Frequently a slightly weaker theorem is much easier to prove and contains the basic idea.

(2) A mathematics paper emphasises novelty. A theorem or its proof is best understood in context but a paper will concentrate on what is new and not waste time discussing the known context.

(3) A mathematics paper is a private enterprise which contributes to a communal good. In a lecture course, the lecturer is like a conductor blending

the contribution of many individuals into a harmonious whole. A single paper represents the contribution of the double bass or the triangle.

(4) A mathematics paper contains new mathematics. The writer may not fully understand what is important and what is unimportant in what he or she has done. Similarly he or she may not understand what is truly difficult and what merely seems so.

3.2 How do we read a proof?

It seems to me that most mathematicians approach the job of reading a long proof as follows.

(1) What does the theorem mean? One way of trying to find out is to try it out on a few simple examples.

(2) Let us try and show it is false. By trying to construct counter-examples we get some idea of how the theorem works.

(3) Once we have convinced ourselves that there are no counter-examples let us try and prove it for ourselves.

(4) If we cannot prove it then there must be a counter-example. Return to (2).

(5) After repeated cycles through (2) and (3) we admit we cannot do it ourselves and look at what the author says.

(6) The initial steps of the author's proof should, if we have tried (3) sufficiently often, be familiar but at some point something new will turn up.

(7) Perhaps this new point is the key? Return to (3).

(8) After repeated cycles through (2) to (7) we reach the end of the proof.

3.3 Understanding

After a recent TV programme on Andrew Wiles, one of the Arts Fellows at my college told me that he now felt that he almost understood the proof of Fermat's last theorem. If you feel that you have understood a proof except for a few calculations and some technicalities you are closer in spirit to him than is altogether proper. You have not understood a theorem until you can prove it yourself.

3.4 Some papers are just long proofs

If a paper is just a long proof then we should treat it as such. First we must identify the central theorem. Having done so we try to prove it as before. Now the point at which we say 'Ah, I did not think of that' may be a reference to a previous lemma. We now know (or think we know) the point of the lemma and we try to prove it. By repeated use of this technique we can identify the structure of lemmas and definitions which support the theorem and, ultimately, obtain the full proof of the main theorem.

Much of mathematics is automatic writing, only by trying to do as much of the proof yourself as you can will you identify the key steps which are not automatic.

3.5 But all papers have context

Mathematicians ask two questions about theorems — how and why. How do you prove it and why should you prove it. Thus, given a theorem, we may ask:

(1) What simpler results does it generalise?

(2) How can you use it prove other things? Can you give examples?

(3) Does it generalise? If not, what is the obstacle? What are the counter-examples which demonstrate the obstacle?

(4) What is the next step in the development of the subject? What are the open questions?

Even if the paper does not consider these questions your essay should do so.

3.6 A possible moral

If you are a great mathematician like Kolmogorov you may have so many ideas that you have no time to spend in presenting them. In any case, if you are a great mathematician, people will not grudge the work required to read your papers. If you are a mathematician with nothing to say then no matter how clearly you write and how inviting your presentation, people will not read your papers.

If you are a middle ranking mathematician (particularly if you are starting out in your profession) the number of people who actually read your papers will depend on how clearly written they are as well as on what they say. Most middle ranking mathematicians (and quite a lot of high ranking mathematicians) have papers which they feel have been unfairly neglected. Sometimes their opinion is at fault (we are not the best judges of our own children), often it is a question of fashion or bad luck, but sometimes the paper just fails to communicate its point.

Erdős says 'Everyone writes. Nobody reads.' Creative mathematicians are more interested in their own ideas than in other people's. Reading mathematics is hard. You have to write so as to catch and hold the attention of an unwilling audience. You have read other people's papers. Can you do better?

4 How to write your essay

Much of what follows is a more or less lightly modified version of Marj Batchelor's advice.

4.1 Your purpose

Your only purpose should be to teach your readers a little bit about your subject. You may think that your sole purpose is to obtain an alpha for your Part III exam and you may argue that the only person who is likely to read this essay will be your assessor. However, this one of the only exercises (perhaps the only exercise) you will have in writing mathematics before you have to write for a real audience in a PhD thesis or journal article and you need to make the most of it. Moreover, if you pretend to write your essay for your colleagues with the sole intent of making it easy for them to understand the material then you will certainly please your assessor too.

4.2 Your topic

Unless you have a clear idea of what you want to explain you will certainly fail to explain it. Is the centre of your essay a theorem to be proved or a method to be illustrated? If it is a method, which examples will show it in its best light, which examples does it fail to cope with, which other methods should it be compared with? Every talk, article, lecture course or book must have a focus. In a short talk the focus is usually achieved by concentrating on a single result. In an essay you may wish to present a small collection of *related* results. However there is *absolutely no point* in writing down everything you know or even the most difficult thing you know about your topic. A statue is a block of marble from which material has been removed. It is better to write up a simple result well, illustrating it with examples which you have invented and calculations you have done, than to do a shoddy job of copying somebody else's highly technical paper.

4.3 The crustacean style

Remember that your purpose is to explain. Consider also that your reader is short of time, tired, and possibly stupid as well. Your aim is to make it easy for your reader to find out what she needs in a minimum of time, The secret of achieving this is to adopt a crustacean rather than a vertebrate philosophy of presentation. Vertebrates have their skeleton hidden within. This philosophy is appropriate not only to detective novels ('So far we have assumed that Sir Horace could not have reached Hangdog Hall in time. But the vicar mentioned (page 33) that the train to Maplethorpe was running late and we know from our own experience (page 77) that the train slows down as it reaches Blackberry Cutting so it would be perfectly possible for an experienced mountaineer (pages 56 and 103) like Sir Horace ...') but for any novel in which you do not want the reader to be aware of how the plot is planned. In this style it is acceptable, even laudable to bury key pieces of information in the middle of interior paragraphs. It is a fault if the reader is consciously aware of the construction details.

In the crustacean style, however, the structure is on the outside, and the organisation is evident at a glance. First paragraphs of sections describe what follows in the section, first sentences of paragraphs indicate what will follow within the paragraph and important information is made to stand out visibly on the page. This style is used in front page reporting and is to be used by you.

Here are some phrases typical of the crustacean style:

'The key point of the proof is to show that f is a well defined isomorphism.'

'We first show that f is well defined.'

'Next we show that f is a morphism.'

'Since f is clearly surjective we need only check that f is injective which we do by looking at the kernel.'

'This completes the proof that f is a well defined isomorphism.'

'The next three lemmas are entirely routine and show that our results on continuous functions can be extended to distributions.'

'Lemma 3 can be improved to show that the growth is no faster than polynomial but we only need some bound depending on n alone.'

'This is the only point in the argument where we use Axiom A.'

Often it is better to say 'By Theorem 7.3 which says that all snarks are boojums' than 'By Theorem 7.3' or 'Since all snarks are boojums'. In the same way 'We show that G is Abelian' may be less helpful to the reader than 'We show that the group G of translations is Abelian' or 'We show that the group G defined at the start of Section 2 is Abelian'. Not all readers have perfect memories.

4.4 Your outline

Once you have decided what your subject is you must decide how to present it. Which points should you make in the introduction? Which definitions will you need and where should they be put? What notation are you going to use? (If you use i for the identity map and go on to talk about complex numbers you, or at least your readers, may have problems.) Which lemmas will you need to prove the central theorem and in which order should they come? Should the counter-examples be presented early to show how strong the main theorem is or late to show which avenues for generalisation are blocked?

The standard advice with which I have no reason to disagree says that you should start by writing down a paragraph in the style of an undergraduate syllabus.

> Inversion theorems of classical Fourier Analysis for $\mathbb{R}$ and $\mathbb{T}$. Definition of a Locally Compact Abelian Group. Statement (without proof) of existence of Haar measure. Definition of character. Inversion theorem corresponds to existence of 'sufficiently many' characters. Proof (follow Rudin) of inversion theorem for LCA group (giving parallels with classical case). Statement structure theorem and brief sketch proof.

Next write out the statements of your main definitions, examples, lemmas and theorems in the order that you intend to give them. You have now decided your strategy leaving your tactics (the proofs and the connecting discussions) for the first draft. At this point you should consult the assessor to check that your plans are reasonable.

Do remember that the logical order is not necessarily the pedagogic order. Your object is to keep the reader interested and to ensure that he or she can understand the general sweep of the argument without bothering with all the details. For example if your topic is Theorem A and Corollary B but the proof of Theorem A depends on Lemmas 1, 2 and 3 which have long and complicated proofs, it may be best to order your outline in the following way.

Section 1. Introduction
Section 2. Statements of Lemmas 1, 2 and 3.
Section 3. Statement and proof of Theorem A.
Section 4. Statement and proof of Corollary B.
Section 5. Discussion of Corollary B.
Sections 6, 7, 8, ... Proofs of Lemmas 1, 2 and 3.

In this way the reader learns as much as is needed to get to the point in as brief a way as possible. Details of proofs are sentenced to the end of the paper to be consulted if needed. This technique is particularly important when giving talks. *Always put your important points at the beginning of your talk.* Observation shows that halfway through the average seminar most of the audience are asleep, catching up on correspondence, thinking about their own mathematics or trying to prove your result by a slicker method. In addition all talks take 50% longer to give than expected (even after allowing for this rule) so if you leave important results until last you will have to engage in an undignified scramble to reach them.

4.5 Navigation

Remember that few people read mathematics papers straight through. You must make it easy for your reader to skip bits (with the intention, of course, of coming back to them later) or to refer back to some previous point without rereading the whole essay. Good layout will help but the necessary signposts should be incorporated into your prose. If you make sure that new information is never buried in the middle of a paragraph your readers can hop about your essay with the confidence that all they need will be evident.

4.6 The introduction

For many mathematicians the introduction is the place where they dump a survey of the literature and all the definitions and trivial remarks they can find. If you are sure that everybody will have to read your paper or you expect that nobody will, this is the easiest way to construct a paper. If you think that you have something worthwhile to say but are modest enough to doubt whether your potential readers know this, you will use your introduction as an advertisement and a map.

You would like to address your reader as follows:-

> Since you have read my title and my abstract I can assume that you are interested to hear what I have to offer. My main theorem is the following. You may need the following two definitions to understand it. It is important because it does so and so. You may also be interested in the following corollary and in the lemma I use to establish the theorem. If you are not interested in these then I am afraid you will not be interested in the rest and we part company with no hard feelings.
>
> If you are interested, a proper respect for my predecessors means that I must explain briefly how my work depends on theirs. Now let me sketch the plan of my paper. Section 2 contains definitions and preliminary computations. Section 3 is devoted to the proof of the key lemma in the case of the real line. Section 4 extends the lemma to general locally compact Abelian groups. The details are very technical and not needed if we only wish to prove the main theorem for standard groups like the line and

> the circle. Section 5 contains the proof of the main theorem. Section 6 contains examples showing that our main theorem is, in some sense, best possible.

Academic convention prevents you from being quite as direct as this but the crustacean style requires you to get as close as possible. As a rule of thumb the introduction should occupy about 1/7 of a mathematical paper but in a short paper it may well occupy rather more and in a long paper rather less.

If you only wish to write your introduction once you will have to make it the last thing you write. It is very rare for mathematical paper to turn out as planned and the introduction describes what you have written and not what you wished to write. My personal preference is to write the introduction first and then rewrite it repeatedly as the exposition progresses. This requires more work but makes sure that you keep the general plan constantly in mind.

4.7 The conclusion

The introduction had to be an advertisement to tempt the customer to buy and a map to help the explorer navigate. At the end of your essay the customer has already bought the goods and the explorer has completed the journey. There is no logical reason why we should trouble with an ending just as there is no logical reason why we should say thank you at the end of a transaction or goodbye at the end of a meeting but there are human reasons why we should.

Your reader has worked hard to understand your work. Can you not reward him or her with some final insight or a tantalising open question so that your essay ends on a high note? Or, if this is not possible, should you not review the path you have taken together? Remember the preacher's advice 'First I tell them what I'm going to say. Then I tell them. Then I tell them what I've said.'

What applies to the essay as a whole applies to its component sections. I am tempted to reverse my advice on the crustacean style. Last paragraphs of sections summarise the section, last sentences of paragraphs summarise the paragraph. This advice should not be taken literally but I am sure that most mathematics lecture courses would be improved if mathematicians took more trouble with the beginning and ending of each lecture.

4.8 A note on omissions

In a talk it may well be desirable to omit long calculations and long lists of conditions which are not required to understand the central idea of the proof. In a paper you have to present the complete argument however repulsive the details. Similarly, in your essay you must give proofs in full although you may well precede such a proof by an example or a proof of a simple case to help the reader. In talking about the background to your paper (for example if the purpose of your essay is to use a technique to do some calculations) or in describing developments of your results you may well choose to summarise and give a reference to the literature. Think of your topic as as a tree, you can *assume* the roots and *prune* the branches but you cannot remove a section of the trunk.

Of course, there are exceptions to this rule. There are branches of mathematics where full proofs of major results do not exist or are just too complicated and some where, to the outsider at least, it appears that practitioners leap from assertion to assertion like chamois on a mountainside. However, if you do decide to omit a proof on the grounds that it is too hard for the reader examine your conscience and ask whether you do not mean that it is too hard for you. Such omissions are precisely the kind of thing you should discuss with your assessor.

4.9 Layout is important

The layout of a paper is how it looks on the page — margins, indentation, how many lines are skipped between paragraphs, which headings are underlined and so on. Here the general principle is

$$\boxed{\text{do what is necessary to make the important bits stand out.}}$$

If you write your paper by hand most problems of layout will settle themselves. If you use LaTeX then the system will settle most problems for you. If you are a TeX fan then you do not need (or, what comes to almost the same thing, you think you do not need) any advice. If you use a typewriter or a non-TeX based word processing system then you need all the help you can get.

Here are a couple of general principles.

(1) All definitions and notational conventions should be clearly set out. Leaving spaces before and after definitions greatly increases their visibility. The same goes for lemmas, examples and theorems.

(2) When in doubt, display. Most mathematicians include some very short formulae in their text. 'If $x = 2$ then $\int_3^x f(t)\,dt$ is negative.' However,

even moderately short formulae like $A = \bigcup_{i=1}^{\infty}(A_i \cap B)$ may become ugly and difficult to read. It is much better to use a displayed equation

$$A = \bigcup_{i=1}^{\infty}(A_i \cap B).$$

If you are writing for a journal the printer may be anxious to save space but, so far as your essay is concerned, paper is cheap. (Nowadays, even publishers may approve of displayed equations since they make the job of electronic typesetting much easier.) Next time you have to attend a seminar with overheads covered with minute detail (or even better when you are preparing an overhead) remember

SPACE aids LEGIBILITY. LEGIBILITY aids COMPREHENSION.

4.10 But layout is no substitute for clear explanation

Few blessings are unalloyed and TEX is no exception. Mathematicians now spend hours discussing the difference between

$$\int_0^1 g(x)dx \text{ and } \int_0^1 g(x)\,dx$$

or trying to move a subscript 2.2 ems to the right. This is a harmless way of wasting time like train spotting or building a model of Canterbury Cathedral out of matchsticks but has no positive benefits. One reason is that most mathematicians have the visual sense of a dead codfish. (However, many mathematicians smell rather less and a few have better conversation.) The more fundamental reason is that, provided a certain standard of legibility is attained, content is more important than presentation.

Let me reiterate my advice on a crustacean style. Clarity depends on strategic choices. For the essay as a whole you must find an order of lemmas, examples and explanations which carries the reader along the path of understanding. Within each proof you must find an order of steps which, if possible, will show your readers why the theorem is true, or if this is not possible, at least convince them that it is true. These are the strategic decisions a mathematician must make — not fiddling tactical decisions about the spacing in formulae. I do not say that if you make the right strategic decisions then the essay will write itself (though this is, I think, almost the case) but I do say that if you make the wrong strategic decisions no amount of tactical brilliance whether in fine writing or fine printing will make your essay readable.

4.11 Stylistic points

Although your chief concern should be strategy, here are some 'tactical' points of style. They are adapted from the *Notes for Contributors* of the London Mathematical Society. Not only are they good sense but they are the kind of points referees tend to insist on.

(i) A paper should be written in clear, unambiguous and grammatical language. Thus a 'sentence' like 'Let $x = 3$, then $x^3 = 27$.' is unacceptable because it has two main verbs. A necessary, though not a sufficient condition of grammatical correctness is that your paper should sound OK when read aloud.

(ii) Words such as 'assume', 'suppose', 'show', 'imply', ... should usually be followed by 'that'.

(iii) Where 'if' is used to introduce a conditional clause, it should usually be followed by 'then' at the appropriate point, as in 'If $x = 3$, then $x^3 = 27$'.

(iv) Sentences should begin with words, not mathematical symbols. Formulae should never be separated merely by punctuation. Either place at least one word between the formulae or display as a vertical list. Thus you should replace
'f is defined by $f(x) = 0$ $[x < 0]$, $f(0) = \frac{1}{2}$, $f(x) = 1$ $[x > 0]$'
by
'We set $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 0$. We take $f(0) = \frac{1}{2}$.'
or better,
'We set

$$\begin{aligned} f(x) &= 0 \qquad \text{if } x < 0, \\ f(0) &= \tfrac{1}{2}, \\ f(x) &= 1 \qquad \text{if } x > 0.\text{'} \end{aligned}$$

The last suggestion takes up more space but can be read at a glance.

(v) You already know from your lectures and texts that most abbreviations are for the benefit of the writer and not the reader. The use of LC saves the writer's wrist but leaves the reader wondering 'locally compact' or perhaps 'locally convex' or perhaps 'Löwner-Carleson'. Of course, if the reader has been following with attention he or she will remember that 20 pages back (or was it 25?) you gave a list of abbreviations but you should do the work not the reader. Mathematicians have caused themselves much misery by ill chosen notation and most abbreviations are, almost by definition, ill chosen.

(vi) Do not write things which look like nonsense even if close textual study shows that they are not. Your reader has enough trouble without adding artificial difficulties. The London Mathematical Society *Notes* give the following examples of 'unnecessarily disturbing usages':

The number of prime divisors of 30=3.

$\exists 0 \leq i \leq n$ with $f(i) > 0$.

Let $f(g)$ be the left (right) quotient.

Let $A \ni a$.

Consider the open interval $]a, b[$.

Not all of these usages may disturb you but all of them disturb some people. In an undergraduate lecture course the lecturer has a captive audience who have to fall in with his or her notational conventions. Writers of mathematical papers have no such power over their audience. The reader who does not like your style or is bored with your content can just stop reading. Since readers will not labour to understand your meaning, you must labour to make your meaning clear to them.

5 Starting, keeping going and stopping

5.1 Word processing

You do not need to use a word processor. Your assessor will be just as happy with a reasonable handwritten manuscript.

However, many of you will choose to use a mathematical word processing system. The standard advice, with which I firmly concur, is *do not use word processing for your first draft*. There are several reasons for this advice.

(1) The mechanics of mathematical word processing will distract you from the much more important task of mathematical essay writing.

(2) Word processing encourages you to forget the global shape of your work and concentrate on the local. This may lead to logical mistakes like the omission of lemmas and, in long proofs, to circular or incomplete arguments. It may also encourage large disparities in the amount of space allotted to the various parts of the essay.

(3) More generally, word processing encourages prolixity, repetition and slack construction.

With experience you may learn how to avoid these problems and compose directly at the keyboard, but you do not yet have experience.

5.2 The first draft

You may find it easiest to get started with an ordinary block of of lined paper (wide spaced for preference), several soft pencils (easy to erase), a good pencil sharpener and a very large, very good eraser[6]. Skip every other line. This is partly a psychological gimmick to get you past page 1 in a hurry but its main purpose is to make changes easier at a later stage.

5.3 The daily task

Decide how much you expect to write in a day. Now halve it. Now halve it again. This is your daily task, say 250 words or 1 page or 1000 keystrokes. These quantities may seem ridiculously small but if you write 250 words *each day* for three weeks you will have a 5000 word essay, if you write one handwritten page *each day* for 4 months you will have something the length of a PhD thesis and if you type 1000 keystrokes *each day* for 2 years you will have a substantial book.

Now comes the difficult bit. You must do your daily task *each day.* If it takes you an hour, congratulations, you have the rest of the day free for other things. If it takes all afternoon, you have the evening free. If it takes longer, you must cancel your candle lit dinner, make excuses to your drinking club and leave your opera seat unoccupied; completing your daily task takes priority even if it takes until midnight.

Never do more than your daily task. Experience shows that if you do three times as much one day you will take four days off as a reward. By working on your essay *each day* you keep it constantly in front of your subconscious so that it can work while you do not.

You may be worried by the thought of leaving your essay in the middle of some argument. Let me quote from Littlewood's *The Mathematician's Art of Work* [10].

> Most people need half an hour or so before being able to concentrate fully. I once came across some wise advice on this, and have taken it. The natural impulse towards the end of a day's work is to finish the immediate job: this is of course right if stopping would mean doing work all over again. But try to end in the middle of something; in a job of writing out, stop in the middle of a sentence. The usual recipe for warming-up is to run over the latter part of the previous day's work; this dodge is a further improvement.

[6] Instead of a wastepaper basket you should have a very large file in which you place every sheet of paper that you discard. Sometimes first thoughts turn out to be better than second.

5.4 What to do when you are stuck

Here are some suggestions.

(1) Make a note of what ought to follow and skip to the next section.

(2) Go back to some section which you have previously skipped and work on that bit.

(3) Go to a blackboard and explain the point to an invisible but patient audience.

(4) Is your organisation wrong? Perhaps you should have proved some lemma earlier? Go back and insert it. (Rather than rewriting use scissors and paste to rearrange your manuscript.) Perhaps you can break down one horrible, complicated section into three simpler ones.

(5) Perhaps you are pursuing a side issue. Is the result or argument essential for your essay? If it is you have no choice but to press on. If not, leave it out. What the writer does not enjoy writing the reader will not enjoy reading.

(6) If nothing seems to work you may have to face an unpleasant truth. It turns out to be very difficult indeed to give a clear account of woolly or incorrect mathematics. It may be that you have not reached the required stage of understanding which would permit you to write up. The only remedy is to stop writing and start thinking.

As you might expect this happens quite frequently when mathematicians write up their research. In these cases the correct remedy may be to let the paper rest for a week and then reconsider matters. You are writing to a deadline and though this remedy remains the best it will only be possible if you have left plenty of slack in your timetable.

5.5 Transferring the first draft to your word processor

If you are going to use a word processor you will need to transfer your hand-written document to the computer. How long this will take you depends on your temperament, your typing skills and your familiarity with mathematical word processing. However, I recommend that you plan on the assumption that you can transfer your work only twice as fast as you can write it. (Thus if it takes you 20 days to write your first draft you should allow 10 days for the transfer.) If you are already familiar with mathematical word processing this may be a very pessimistic estimate (but you will already have the experience to make a better one). If you are not, or if you revise as you type, I do not think it will be far wrong.

5.6 Standard advice on word processing

Make a backup copy at the end of each session so that when (I do not say if) you delete your file by mistake you only lose 24 hours' work. Never eat at a place named Mom's, never play cards with a man named Doc and never, never use an automatic replacement facility[7] or spelling 'corrector'[8]. Modern programs allow you to view your document on screen without printing it out — help save the forests.

5.7 Revision

Once you have your first draft on the word processor print it out onto *real* paper. The go back to your room or the library with *real* scissors and paste, plenty of extra paper and four or five fine pens of various vivid colours. Decide what revisions are necessary (be ruthless) and go back to the computer and make them. Let your essay sleep for a week and then repeat the process. Continue until you are happy with your work or you run out of time. 'A work of art is never finished, it is merely abandoned.' (But see the note 5.11 on stopping.)

If you do not use a word processor your tools will be the xerox machine, a pen and black ink, snowpak and sellotape but the principle remains the same.

5.8 Revise, rewrite or reject

Much revision consists of adding or removing punctuation, changing a word here and there, inserting an extra step in an argument and so on. If you find yourself doing more than this you should rewrite the offending passage from scratch[9]. It will not take as long as you think it will and the new version will flow better than a hacked about original. Piecemeal revision is also more likely to produce inconsistencies.

Remember that there is an alternative to revision and rewriting. It is called omission. If something does not fit in with the flow of the essay or reads badly however often you rewrite it then, perhaps, it should not form part of the essay. The hardest thing for an author to do is to leave out a passage over which he or she has sweated blood — but often it is the right thing to do. How many lectures have you heard which were too short and

[7]Since the effect of replacing sin by cos is to produce 'cosce the effect of replacing cos by cos'.

[8]On the other hand a spell checker is invaluable when used with a good dictionary.

[9]But keep a copy of the original in case you change your mind.

how many too long? How many books on your shelves are too short and how many too long?

5.9 Know yourself

Halmos is one of the great mathematical expositors. Here is some advice from his essay *How To Write Mathematics* [4].

> In the first draft ... I recommend that you spill your heart, write quickly, violate all rules, write with hate or with pride, be snide, be confused, be 'funny' if you must, be unclear, be ungrammatical — just keep on writing. When you come to rewrite, however, and however often that may be necessary, do not edit but rewrite. It is tempting to use a red pencil to indicate insertions, deletions, and permutations, but in my experience it leads to catastrophic blunders. Against human impatience, against the all too human partiality everyone feels towards his own words, a red pencil is much too feeble a weapon. You are faced with a first draft that any reader except yourself would find all but unbearable; you must be merciless about changes of all kinds and, especially, about wholesale omissions. Rewrite means write again — every word.

This is attractive advice until you realise that Halmos rewrote (that is wrote again — every word) each of his books three times and followed this by a massive revision. (This is an underestimate, parts of his books were rewritten six or seven times.)

Every book that Halmos wrote was a success and some were outstanding. The problem with his advice is that most mathematicians dislike writing, hate revising and consider rewriting a confession of failure. The word processor encourages endless minor changes but discourages root and branch revision. Under these circumstances my advice is as follows.

> Revision, however thorough you believe it to be, leaves most things unaltered. You may haul large chunks of prose from one place to another but the patterns embedded in those chunks remain unchanged and the overall structure is altered only in the crudest way. However hard you wield the red pencil most of your sentences will retain the the form you originally gave them.
>
> Thus the last draft that you write out in full (and for most mathematicians this will also be the first) must be as perfect as you can make it. If you are dissatisfied with anything, rewrite

> it before starting revision. Once you start revision you will find that what you thought was perfect requires a multitude of large and small corrections but at least you can concentrate on the job in hand.

Be a Halmos if you can, but, if you cannot, recognise your limitations and act accordingly.

5.10 Abandoning

From time to time you will be filled with disgust and despair at what you have written. You may be right but it is much more probable that you are wrong since almost everybody goes through such a phase from time to time. (It gets worse when you are doing a PhD.)

Even if you are right, it probably makes sense to continue. Provided that your essay shows evidence of hard work and thought it should receive a decent mark. The alternative is to try and work up a course that you did not much like (otherwise you would already have decided to take it for examination) to that fairly high level at which you can be confident of a decent mark in your exam. In any case you should go to your assessor for a second opinion before deciding to abandon.

5.11 Stopping

You are writing an essay, not a rocket guidance system. Your essay does not have to be perfect. The maximum mark you can get for an essay is 100 and a decent essay will get 65. Once you have got the mathematics and the exposition settled, repeated minor changes will have little effect on your mark[10]. On the other hand working on your weaker exam subjects can boost your total marks very substantially. Once your final version is complete leave it and get on with your exam revision.

6 Sources

By comparison with practitioners in other disciplines, mathematicians are very lax in acknowledging sources. Undergraduate texts customarily contain no references and even graduate texts announce that ‘No attempt has been made to trace the sources of the various theorems proved here’. Mathematics papers normally open with a cluster of references intended to establish

[10] And TEXtual changes will have no effect whatever.

the importance of the problem treated and the originality of the approach adopted but otherwise only contain references to results required but not proved in the paper itself (and these references will be further limited to results which it cannot be assumed that the reader will know).

I do not suggest that you should improve on the prevailing standards. I do, however, point out that these standards are so low that you should on no account fall below them now or in the future.

6.1 Form of acknowledgement

In addition to the kind of references you find in most mathematical papers the nature of a Part III essay requires that you state all the major sources that you have used and indicate how you have used them. One way of doing this is to include a statement at the end of your introduction along the following lines.

> I have used the book of Kahane and Salem [21] and Kahane's *Séries de Fourier Absolument Convergentes* [18] as general sources of information. Sections 1 to 3 are based on [14], Section 4 is based on the treatment of Malliavin's theorem in [15] whilst section 5 is a composite from both sources. The proof of Theorem 6 is taken from [16] but I have modified it substantially. The proofs of Lemmas 3 and 4 are my own, though they may well exist elsewhere in the literature.

6.2 Secondary references

Suppose that the paper of Y [17] you consult says 'X has shown [18] that all left principles are right principles'. The obvious thing to do is to repeat the reference. However the obvious thing to do may be the wrong thing. *Never cite a reference that you have not read.* It may not exist, it may not say what you think it says and, in any case, by citing it you claim to have done something you have not done. The best thing to do is to get hold of the reference, verify that it does what Y says it does and then cite it. If you cannot get hold of the reference then what you do depends on whether you know by other means that all left principles are right principles. If you know that the result is true you are simply acknowledging priority and you can write 'X (see citation in [17]) has shown that all left principles are right principles' where [17] is your reference to Y's paper. If you do not know whether the result is true you should consider very carefully whether to include any mention of it at all. If you do decide to refer to it you must proceed with the utmost caution. The words 'X claims to have proved ....' may annoy X. Perhaps honesty 'In [17] Y reports a proof of X that left principles are right principles but I have not been able to obtain the original paper [18]' is the best policy.

6.3 Data

A rose by any other name would smell as sweet, and, presumably, the *Air on a G String* would sound as sweet if we did not know that it was by Bach. The fact that Stokes's Theorem was first discovered by Kelvin (and first published as a Cambridge examination problem) does not affect its truth. We acknowledge the authorship of poems, music and proofs out of courtesy (and in the hope that others will extend the same courtesy to us).

Data, on the other hand, is like evidence in court, valueless without a provenance. Compare the effect of 'Someone said that the knave of hearts stole the tarts' with 'X said that the knave of hearts stole the tarts', 'X told me that he saw the knave of hearts steal the tarts', and 'I saw the knave of hearts steal the tarts'. For this reason *data must always be accompanied by references* such as 'The graph of wing span against speed is taken from [19].'

From the point of view of your essay points where you quote data may well be points where you have the opportunity to add something. At its simplest you may be able to gather data from other papers but if your essay is about a statistical technique you could try it out on data from another source or obtained by Monte Carlo simulation. If you are comparing two numerical methods then instead of using run times provided by others you could do your own testing on your own machine.

6.4 When found, make a note of

Whenever you use a reference for the first time, record it in some fixed place (a file card in a file box, a notebook or a file in your computer). Otherwise, you may find that, when you write up, you cannot remember where you found the result you need.

Since the time scale for the essay is so short and the number of references that you are likely to use is small the advice just given may not be necessary now but if you go on to write a PhD it is vital. In the 1930's many mathematicians carried notebooks in which they jotted down problems, ideas, references and so on as they occurred to them. I suspect that this custom died out because it is easy to carry a large notebook in a suit pocket and hard to carry one in the back pocket of a pair of jeans, ('Is that a Nachschrift in your pocket or are you just pleased to see me?') Perhaps the custom should be revived with filofaxes.

7 Further advice

7.1 Books on writing

There are several good books about writing. You should start with *The Elements of Style* by Strunk and White [12]. It is American and aimed at general journalism but has the virtue of being very short indeed. If you want something longer and British then *The Complete Plain Words* by Sir Ernest Gowers[11] [2] is worth reading.

7.2 And what they say

Here is some of the advice given by such authorities.

1) Omit needless words.

2) Keep your sentences short.

3) Keep your paragraphs short.

4) Use active rather than passive forms of verbs. (Replace 'It has been shown by Brown that the method works when $n < 6$.' by 'Brown showed that the method works when $n < 6$.') The imperative is also useful, particularly in proofs. ('Take $n = 6$.')

5) Prefer the familiar word to the far-fetched.

6) Prefer the concrete word to the abstract.

7) Prefer the single word to the circumlocution.

The final piece of advice is mine.

8) Use 'we' when you are doing something with the reader. 'We see that F is closed.' 'We now check that G is Abelian.' Use 'I' when expressing an opinion or (in later life) talking about something you have done. 'I think method A is more transparent.' 'In an earlier paper [13] I showed that all primes (with at most one exception) are odd.'

7.3 A digression

It is clear that mathematics papers meant for an international audience should use simple English with as little decoration as possible. What is more interesting is that, for the last 200 years, this has been the preferred

[11] Great-grandfather of our own Professor W. T. Gowers.

style when the English write for the English. Good written English is modelled on good spoken English. Good English style should be invisible as a pane of glass leaving the reader free to concentrate on content.

The fact that there is no 'high style' in English has many causes but it is worth noting that many of those who successfully advocated the simple style linked it with an inclusive rather than an exclusive view of society. 400 years ago Ascham wrote (quoted in [2])

> He that will write well in any tongue, must follow the counsel of Aristotle, to speak as the common people do, to think as the wise do; and so should every man understand him, and the judgement of wise men allow him.

7.4 Mathematical writing

Few mathematicians have had the courage to give advice on writing mathematics. I have already quoted from Halmos's marvellous essay *How To Write Mathematics*. This appeared in an excellent collection of essays on mathematical writing [11] and is reprinted along with his essay *How To Talk Mathematics* in Volume 2 of his selected works [4]. Yet more good advice will be found in his 'automathography' *I Want to be a Mathematician* [3]. All young mathematicians should read Littlewood's *The Mathematician's Art of Work* reprinted in the second edition of his *A Mathematician's Miscellany* [9] and in his collected works [10].

The book *Writing Mathematics Well* by Gillman [1] is good on the nuts and bolts but lacks the zest of my previous recommendations. The book *Mathematical Writing* [6] in the MAA Notes ought to have been a masterpiece since it was based on lectures by some of the greatest mathematical expositors of the present time. Unfortunately the lectures were reported second hand, translated into Californian mellow speak, diluted by student comments, homogenised without being edited and then issued without index, bibliography or helpful table of contents. (This is only my opinion, you may disagree and, in any case, there are some useful pieces of advice buried here and there.)

The recent book of S. G. Krantz entitled *A Primer of Mathematical Writing* [8] is like his previous *How to Teach Mathematics* [7] full of excellent advice not only on the subject in hand but also on many related topics. Taken together they form something like a *Rough Guide to a Life in Mathematics* dealing with some of the matters which more up-market guides prefer to ignore. N. J. Higham's *Handbook of Writing for the Mathematical Sciences* [5] is another good book and (unlike both Krantz's book and the present essay) is written from the standpoint of the applied mathematician.

7.5 Valediction

The problem with advice, however well meant, is that it can make a task seem harder than it is. Most of those who do Part III take the essay option and, as far as can be judged, most of those who take the option do it well, find it useful and enjoy it.

A good essay will show that you have read some papers, understood some hard mathematics and can communicate what you have learnt clearly and enthusiastically. I wish you good fortune in your enterprise.

References

[1] L. Gillman *Writing Mathematics Well.* AMS (1987).

[2] E. Gowers *The Complete Plain Words.* (3rd Edition, revised by S. Greenbaum and J. Whitcut.) Penguin (1987).

[3] P.R. Halmos *I Want to be a Mathematician.* Springer (1985).

[4] P. R. Halmos *Selecta.* (Two volumes.) Springer (1983).

[5] N. J. Higham *Handbook of Writing for the Mathematical Sciences.* SIAM (Philadelphia), 1993.

[6] D. E. Knuth, T. Larrabee, P. M. Roberts *Mathematical Writing.* MAA Notes, **14**, Washington (1989).

[7] S. G. Krantz *How to Teach Mathematics.* AMS (Providence, R. I.), 1993.

[8] S. G. Krantz *A Primer of Mathematical Writing.* AMS (Providence R. I.),1996.

[9] J. E. Littlewood *A Mathematician's Miscellany.* (2nd Edition, Editor B. Bollobás.) CUP (1986)

[10] J. E. Littlewood *Collected Works.* (Two volumes.) OUP (1982).

[11] I. N. Steenrod, P. R. Halmos and others *How to Write Mathematics.* AMS (1973).

[12] W. Strunk *The Elements of Style.* (3rd Edition, revised by E. B. White.) MacMillan (1979).

[Printed out November 11, 2009. These notes are written in LaTeX2e and can be accessed via my web home page

http://www.dpmms.cam.ac.uk/~twk/.

Also available:

'Dr Körner's Helpful Guide For Mathematicians Seeking A Cambridge Research Fellowship',

'In Praise of Lectures' (how to listen to a mathematics lecture),

'An Unofficial Guide To Part III',

'A Supervisor's Primer'.]