Complex Variable
Part III

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Small print This is just a first draft of part of the course. The content of the course is what I say, not what these notes say. I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors. This document is written in LaTeX2e and stored in the file labelled `~twk/FTP/CV3.tex` on emu in (I hope) read permitted form. My e-mail address is `twk@dpmms`.

Contents

1 Simple connectedness and the logarithm

This is a second course in complex variable theory with an emphasis on technique rather than theory. None the less I intend to be rigorous and you should feel free to question any 'hand waving' that I indulge in.

But where should rigour start? It is neither necessary nor desirable to start by reproving all the results of a first course. Instead I shall proceed on the assumption that all the standard theorems (Cauchy's theorem, Taylor's theorem, Laurent's theorem and so on) have been proved rigourously for analytic functions[1] on an open disc and extend them as necessary.

Cambridge students are (or, at least ought to be) already familiar with one sort of extension.

Definition 1 *An open set* U *in* $\mathbb{C}$ *is called* disconnected *if we can find open sets* U_1 *and* U_2 *such that*

(i) $U_1 \cup U_2 = U$,

(ii) $U_1 \cap U_2 = \emptyset$,

(iii) $U_1, U_2 \neq \emptyset$.

An open set which is not disconnected is called connected.

Theorem 2 *If* U *is an open connected set in* $\mathbb{C}$ *and* $f : U \to \mathbb{C}$ *is analytic and not identically zero then all the zeros of* f *are isolated that is, given* $w \in U$ *with* $f(w) = 0$ *we can find a* $\delta > 0$ *such that* $D(w, \delta) \subseteq U$ *and* $f(z) \neq 0$ *whenever* $z \in D(w, \delta)$ *and* $z \neq w$.

Here and elsewhere

$$D(w, \delta) = \{z : |w - z| < \delta\}.$$

The hypothesis of connectedness is exactly what we need in Theorem 2.

Theorem 3 *If* U *is an open set then* U *is connected if and only if the zeros of every non-constant analytic function on* U *are isolated.*

[1] Analytic functions are sometimes called 'holomorphic functions'. We shall call a function which is 'analytic except for poles' a 'meromorphic function'.

If necessary, I shall quote results along the lines of Theorem 2 without proof but I will be happy to give proofs in supplementary lectures if requested.

Exercise 4 (Maximum principle) *(i) Suppose that $a, b \in \mathbb{C}$ with $b \neq 0$ and N is an integer with $N \geq 1$. Show that there is a $\theta \in \mathbb{R}$ such that*

$$|a + b(\delta \exp i\theta)^N| = |a| + |b|\delta^N$$

for all real δ with $\delta \geq 0$.

(ii) Suppose that $f : D(0,1) \to \mathbb{C}$ is analytic. Show that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where there is some constant M such that $|a_n| \leq M2^n$ (we can make much better estimates). Deduce that either f is constant or we can find $N \geq 1$ and $a_N \neq 0$ such that

$$f(z) = a_0 + (a_N + \eta(z))z^N$$

with $\eta_z \to 0$ as $z \to 0$.

(iii) If U is a connected open subset of $\mathbb{C}$ and f is a non-constant analytic function on U, show that $|f|$ has no maxima.

(iv) Does the result of (iii) mean that f is unbounded on U? Give reasons.

(v) Show that if is an open set which is not connected then there exists a non-constant analytic function f on U such that $|f|$ has a maximum.

Exercise 5 *(i) Suppose $f : D(0,1) \to \mathbb{C}$ is a non-constant analytic function with $f(0) = 0$. Show that we can find a δ with $0 < \delta < 1$ such that $f(z) \neq 0$ for all $|z| = \delta$ and an $\epsilon > 0$ such that $|f(z)| \geq \epsilon$ for all $|z| = \delta$. Use Rouché's theorem to deduce that $f(D(0,1)) \supseteq D(0,\epsilon)$.*

(ii) **(Open mapping theorem)** *If U is a connected open subset of $\mathbb{C}$ and f is a non-constant analytic function on U show that $f(U)$ is open.*

(iii) Deduce the result of Exercise 4. (Thus the maximum principle follows from the open mapping theorem.)

It can be argued that much of complex analysis reduces to the study of the logarithm and this course is no exception. We need a general condition on an open set which allows us to define a logarithm. Recall that we write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Definition 6 *An open set* U *in* $\mathbb{C}$ *is said to be* simply connected *if it is connected and given any continuous function* $\gamma : \mathbb{T} \to U$ *we can find a continuous function* $G : [0,1] \times \mathbb{T} \to U$ *such that*

$$G(0,t) = \gamma(t)$$
$$G(1,t) = G(1,0)$$

for all $t \in \mathbb{T}$.

In the language of elementary algebraic topology a connected open set is simply connected if every loop can be homotoped to a point.

Theorem 7 *If* U *is an open simply connected set in* $\mathbb{C}$ *that does not contain* 0 *we can find an analytic function* $\log : U \to \mathbb{C}$ *such that* $\exp(\log z) = z$ *for all* $z \in U$. *The function* $\log$ *is unique up to the addition of integer multiple of* $2\pi i$.

From an elementary viewpoint, the most direct way of proving Theorem 7 is to show that any piece wise smooth loop can be homotoped *through piecewise smooth loops* to a point and then use the integral definition of the logarithm. However, the proof is a little messy and we shall use a different approach which is longer but introduces some useful ideas.

Theorem 8 *(i) If* $0 < r < |w|$ *we can find an analytic function* $\log : D(w,r) \to \mathbb{C}$ *such that* $\exp(\log z) = z$ *for all* $z \in D(w,r)$. *The function* $\log$ *is unique up to the addition of integer multiple of* $2\pi i$.

(ii) If $\gamma : [a,b] \to \mathbb{C} \setminus \{0\}$ *is continuous we can find a continuous function* $\tilde{\gamma} : [a,b] \to \mathbb{C}$ *such that* $\exp \circ \tilde{\gamma} = \gamma$ *for all*

(iii) Under the hypotheses of (ii), if $\tilde{\tilde{\gamma}} : [a,b] \to \mathbb{C}$ *is a continuous function such that* $\exp \circ \tilde{\tilde{\gamma}} = \gamma$ *then we can find an integer* n *such that* $\tilde{\tilde{\gamma}} = \tilde{\gamma} + 2\pi i n$.

(iv) If U *is a simply connected open set not containing* 0 *then, if* $\gamma : [a,b] \to U$ *is continuous,* $\gamma(a) = \gamma(b)$, *and* $\tilde{\gamma} : [a,b] \to \mathbb{C}$ *is a continuous function such that* $\exp \circ \tilde{\gamma} = \gamma$ *then* $\tilde{\gamma}(a) = \tilde{\gamma}(b)$.

Theorem 7 is now relatively easy to prove.

It would be nice to show that simple connectedness is the correct condition here. The following result, although not the best possible, is hard enough and shows that this is effectively the case.

Lemma 9 *Suppose that* U *is a non-empty open connected set in* $\mathbb{C}$ *with non-empty complement. The following two conditions are equivalent.*

(i) The set U *is simply connected.*

(ii) If $f : U \to \mathbb{C}$ *is a non-constant analytic function with no zeros then we can find an analytic function* $\log : f(U) \to \mathbb{C}$ *with* $\exp(\log f(z)) = f(z)$ *for all* $z \in U$

(In looking at condition (ii), recall that the open mapping theorem given in Exercise 5 tells us that $f(U)$ is open.) The reader is invited to try and prove this result directly but we shall obtain it only after a long chain of arguments leading to the Riemann mapping theorem.

The following result is trivial but worth noting.

Lemma 10 *If U and V are open subsets of $\mathbb{C}$ such that there exists a homeomorphism $f : U \to V$ then if U is simply connected so is V.*

Exercise 11 *In the next two exercises we develop an alternative approach to Theorem 7 along the lines suggested above.*

(i) Suppose that U is an open set in $\mathbb{C}$ and that $G : [0,1] \times \mathbb{T} \to U$ is a continuous function. Show, by using compactness arguments or otherwise, that there exists an $\epsilon > 0$ such that $N(G(s,t),\epsilon) \subseteq U$ for all $(s,t) \in [0,1]\times\mathbb{T}$, and that we can find an integer $N \geq 1$ such that if

$$(s_1,t_1),(s_2,t_2) \in [0,1]\times\mathbb{T} \text{ and } |s_1-s_2| < 4N^{-1},\ |t_1-t_2| < 8\pi N^{-1}$$

then $|G(s_1,t_1) - G(s_2,t_2)| < \epsilon/4$.

(ii) Continuing with the notation and hypotheses of of (i) show that if $\gamma_1, \gamma_2 : \mathbb{T} \to \mathbb{C}$ are the piecewise linear functions[2] with

$$\gamma_0(2\pi r/N) = G(0, 2\pi r/N)$$
$$\gamma_1(2\pi r/N) = G(1, 2\pi r/N)$$

for all integers r with $0 \leq r \leq N$ then there exists a constant λ and a continuous function $H : [0,1]\times\mathbb{T} \to U$ with

$$H(0,t) = \gamma_0(0,t)$$
$$H(1,t) = \gamma_1(1,t)$$

for all $t \in [0,1]$, such that, for each fixed t, $H(s,t)$ is a piecewise linear function of s and the curve $H(\ ,t) : \mathbb{T} \to U$ is of length less than λ.

(iii) Continuing with the notation and hypotheses of of (i) show that if $G(s,1)$ and $G(s,0)$ are piecewise smooth functions of s then there exists a constant λ and a continuous function $F : [0,1]\times\mathbb{T} \to U$ with

$$F(0,t) = \gamma_0(0,t)$$
$$F(1,t) = \gamma_1(1,t)$$

for all $t \in [0,1]$, such that, for each fixed t, $F(s,t)$ is a piecewise smooth function of s and the curve $F(\ ,t) : \mathbb{T} \to U$ is of length less than λ.

[2]Strictly speaking the simplest piecewise linear functions.

(iv) Show that in any simply connected open set any piece wise smooth loop can be homotoped through piecewise smooth loops of bounded length to a point.

Exercise 12 *(i) Suppose that U is an open set in $\mathbb{C}$ and $F : [0,1] \times \mathbb{T} \to U$ is a continuous function such that, for each fixed t, $F(s,t)$ is a piecewise smooth function of s and the curve $F(\ ,t) : \mathbb{T} \to U$ is of length less than λ. We write Γ_s for the contour defined by $F(\ ,t)$. Show by a compactness argument, or otherwise, that if $f : U \to \mathbb{C}$ is continuous then $\int_{\Gamma_s} f(z)\,dz$ is a continuous function of s.*

(ii) If $0 < \delta < |w|$ show that if Γ is a contour lying entirely within $N(w,\delta)$ joining $z_1 = r_1 e^{i\theta_1}$ to $z_2 = r_2 e^{i\theta_2}$ [$r_1, r_2 > 0$, $\theta_1, \theta_2 \in \mathbb{R}$] show that

$$\int_\Gamma \frac{1}{z}\,dz = (\log r_2 - \log r_1) + i(\theta_1 - \theta_2) + 2n\pi i$$

for some integer n.

(iii) By using compactness arguments to split Γ into suitable bits, or otherwise, show that if U is any open set not containing 0 and Γ is any closed contour (i.e. loop) lying entirely within U then

$$\int_\Gamma \frac{1}{z}\,dz = 2N\pi i$$

for some integer N.

(iv) Use results from this exercise and its predecessor to show that if U is any simply connected open set not containing 0 and Γ is any closed contour lying entirely within U then

$$\int_\Gamma \frac{1}{z}\,dz = 0.$$

Hence, prove Theorem 7.

2 The Riemann mapping theorem

By using a very beautiful physical argument, Riemann obtained the following result.

Theorem 13 (Riemann mapping theorem) *If Ω is an non-empty, open, simply connected subset of $\mathbb{C}$ with non-empty complement then there exists a conformal map of Ω to the unit disc $D(0,1)$.*

Unfortunately his argument depended on the assumption of the existence of a function which minimises a certain energy. Since Riemann was an intellectual giant and his result is correct it is often suggested that all that was needed was a little rigour to be produced by pygmies. However, Riemann's argument actually fails in the related three dimensional case so (in the lecturer's opinion) although Riemann's argument certainly showed that a very wide class of sets could be conformally transformed into the unit disc the extreme generality of the final result could not reasonably have been expected from his argument alone.

In order to rescue the Riemann mapping theorem mathematicians embarked on two separate programmes. The first was to study conformal mapping in more detail and the second to find abstract principles to guarantee the existence of minima in a wide range of general circumstances (in modern terms, to find appropriate compact spaces). The contents of this section come from the first of these programmes, the contents of the next (on normal families) come from the second. (As a point of history, the first complete proof of the Riemann mapping theorem was given by Poincaré.)

Theorem 14 (Schwarz's inequality) *If* $f : D(0,1) \to D(0,1)$ *is analytic and* $f(0) = 0$ *then*

(i) $|f(z)| \leq |z|$ *for all* $|z| < 1$ *and* $|f'(0)| \leq 1$.

(ii) If $|f(w)| = |w|$ *for some* $|w| < 1$ *with* $w \neq 0$, *or if* $|f'(0)| = 1$, *then we can find a* $\theta \in \mathbb{R}$ *such that* $f(z) = e^{i\theta} z$ *for all* $|z| < 1$.

Schwarz's inequality enables us to classify the conformal maps of the unit disc into itself. If $a \in D(0,1)$ and $\theta \in \mathbb{R}$ let us write

$$T_a(z) = \frac{z - a}{1 - a^* z}$$
$$R_\theta(z) = e^{i\theta} z$$

Lemma 15 $a \in D(0,1)$ *and* $\theta \in \mathbb{R}$ *then* T_a *and* R_θ *map* $D(0,1)$ *conformally into itself. Further* $T_a^{-1} = T_a$.

Theorem 16 *(i) If* S *maps* $D(0,1)$ *conformally into itself then we can find* $a \in D(0,1)$ *and* $\theta \in \mathbb{R}$ *such that* $S = R_\theta T_a$. *If* $S = R_{\theta'} T_{a'}$ *with* $a' \in D(0,1)$ *and* $\theta' \in \mathbb{R}$ *then* $a = a'$ *and* $\theta - \theta' \in 2\pi\mathbb{Z}$.

(ii) Let U *be a simply connected open set and* $a \in U$. *If there exists a conformal map* $g : U \to D(0,1)$ *then there exists precisely one conformal map* $f : U \to D(0,1)$ *with* $f(a) = 0$ *and* $f'(a)$ *real and positive.*

Theorem 16 (ii) can be modified in various simple but useful ways.

We conclude this section with some results which are not needed for the proof of the Riemann mapping theorem but which show that the 'surrounding scenery' is also interesting and provide a useful revision of some results from earlier courses on complex variable. (If these results are strange to you, the lecturer can give a supplementary lecture.)

Example 17 *If* $a, b \in D(0,1)$ *then there exists a conformal map*

$$f : D(0,1) \setminus \{a\} \to D(0,1) \setminus \{b\}.$$

Example 18 *If* $a_1, a_2, b_1, b_2 \in D(0,1)$ *then there exists a conformal map*

$$f : D(0,1) \setminus \{a_1, a_2\} \to D(0,1) \setminus \{b_1, b_2\}$$

if and only if

$$\left| \frac{a_2 - a_1}{a_1^* a_2 - 1} \right| = \left| \frac{b_2 - b_1}{b_1^* b_2 - 1} \right|.$$

In Example 18 we see the the 'natural rigidity' of complex analysis reassert itself.

3 Normal families

Consider an open set U in $\mathbb{C}$ and the collection $\mathcal{F}$ of analytic functions $f : U \to \mathbb{C}$. What is the 'natural topology' on $\mathcal{F}$ or, in the more old fashioned language of this course, what is the 'natural mode of convergence' for $\mathcal{F}$? Looking at the convergence of power series and at results like Morera's theorem suggests the following approach.

Definition 19 *If* U *is an open set in* $\mathbb{C}$ *and* $f_n : U \to \mathbb{C}$ *we say that* $f_n \to f_0$ uniformly on compacta *if, whenever* K *is a compact subset of* U*,* $f_n|K \to f_0|K$ *uniformly on* K*.*

We shall prove the chain of equivalences in the next lemma, but the proof (and, sometimes, the explicit statement) of similar chains will be left to the reader. Here and elsewhere $\overline{E}$ is the closure of E.

Lemma 20 *Let* U *be an open set in* $\mathbb{C}$ *and* $f_n : U \to \mathbb{C}$ *a sequence of functions. The following four statements are equivalent.*

(i) $f_n \to f_0$ *uniformly on compacta.*

(ii) Whenever $\overline{D(w,\delta)} \subseteq U$ *then* $f_n|\overline{D(w,\delta)} \to f_0|\overline{D(w,\delta)}$ *uniformly on* $\overline{D(w,\delta)}$.

(iii) Whenever $D(w,\delta)$ *is an open disc with* $\overline{D(w,\delta)} \subseteq U$ *then* $f_n|D(w,\delta) \to f_0|D(w,\delta)$ *uniformly on* $D(w,\delta)$.

(iv) If $w \in U$ *we can find a* $\delta > 0$ *such that* $D(w,\delta) \subseteq U$ *and* $f_n|D(w,\delta) \to f_0|D(w,\delta)$ *uniformly on* $D(w,\delta)$.

We shall make use of the following result.

Lemma 21 *If* U *is an open subset of* $\mathbb{C}$ *we can find compact sets* $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$ *such that* $U = \bigcup_{j=1}^{\infty} \operatorname{Int} K_j = \bigcup_{j=1}^{\infty} K_j$.

Exercise 22 *(i) If* $d_1, d_2, \dots$ *are metrics on a space* X *show that*

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \min(1, d_j(x,y))$$

defines a metric on X*. Show that, if each* d_j *is a complete metric and each* d_j *defines the same topology (i.e. has the same open sets) then* d *is complete.*

(ii) If U *is an open set in* $\mathbb{C}$ *show that there is a complete metric* d *on the space* $C(U)$ *of continuous functions* $f : U \to \mathbb{C}$ *such that if* $f_n \in C(U)$ *then* $d(f_n, f_0) \to 0$ *if and only if* $f_n \to f_0$ *uniformly on compacta.*

We also note the following simple but important consequence of Morera's theorem.

Lemma 23 *Let* U *be an open set in* $\mathbb{C}$ *and* $f_n : U \to \mathbb{C}$ *a sequence of functions with the property that* $f_n \to f_0$ *uniformly on compacta. If* f_n *is analytic for each* $n \geq 1$ *then* f *is analytic.*

Exercise 24 *If* U *is an open set in* $\mathbb{C}$ *show that there is a complete metric* d *on the space* $A(U)$ *of analytic functions* $f : U \to \mathbb{C}$ *such that if* $f_n \in A(U)$ *then* $d(f_n, f_0) \to 0$ *if and only if* $f_n \to f_0$ *uniformly on compacta.*

We now talk about what in modern terms would be called compactness.

Definition 25 *Let* U *be an open set in* $\mathbb{C}$ *and let* $\mathcal{G}$ *be a collection of analytic functions on* U*. We say that* $\mathcal{G}$ *is* normal *if given any sequence* $f_n \in \mathcal{G}$ *we can find a subsequence* $f_{n(j)}$ *which converges uniformly on compacta to a limit.*

Note that we do not demand that the limit lies in $\mathcal{G}$.

Fortunately normal families have a simpler characterisation.

Definition 26 *Let U be an open set in $\mathbb{C}$ and let $\mathcal{G}$ be a collection of functions on U. We say that $\mathcal{G}$ is* uniformly bounded on compacta *if given any compact subset K of U we can find a constant C_K such that $|f(z)| \leq C_K$ for all $f \in \mathcal{G}$ and all $z \in K$.*

Theorem 27 *Let U be an open set in $\mathbb{C}$ and let $\mathcal{G}$ be a collection of analytic functions on U. Then $\mathcal{G}$ is normal if and only if it is uniformly bounded on compacta.*

We shall prove Theorem 27 via the following lemma.

Lemma 28 *Let $\mathcal{G}$ be a collection of analytic functions on the unit disc $D(0,1)$ with the property that $|f(z)| \leq 1$ for all $z \in D(0,1)$ and $f \in \mathcal{G}$. Then given any sequence $f_n \in \mathcal{G}$ we can find a subsequence $f_{n(j)}$ which converges uniformly on $D(0,1/2)$.*

Exercise 29 *The following is a slightly different treatment of Theorem 27. Recall that that we call a metric space (X,d)* sequentially compact *if given any sequence $x_n \in X$ we can find a convergent subsequence $x_{n(j)}$. (It can be shown that, for a metric space, sequential compactness is equivalent to compactness but we shall not need this.)*

(i) Show that if we adopt the notation of Exercise 24 a subset $\mathcal{G}$ of $A(U)$ is normal if and only if its closure $\overline{\mathcal{G}}$ is sequentially compact with respect to the metric d.

(ii) **(Arzeli-Ascoli theorem)** *Let K be a compact subset of $\mathbb{C}$. Consider the space $C(K)$ of continuous functions $f : K \to \mathbb{C}$ with the uniform norm. If $\mathcal{F}$ is a subset of $C(K)$ show that $\overline{\mathcal{F}}$ is sequentially compact if and only if*

(1) $\mathcal{F}$ is bounded, that is, there exists a constant λ such that $\|f\| \leq \lambda$ for all $f \in \mathcal{F}$.

(2) $\mathcal{F}$ is equicontinuous, that is, given any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that whenever $z, w \in U$, $|z-w| < \delta(\epsilon)$ and $f \in \mathcal{F}$ then $|f(z)-f(w)| < \epsilon$.

(iii) Let U be a compact subset of $\mathbb{C}$. Consider the space $C(U)$ of continuous functions $f : U \to \mathbb{C}$ under the metric d defined in Exercise 22. If $\mathcal{F}$ is a subset of $C(K)$ show that $\overline{\mathcal{F}}$ is sequentially compact if and only if

(1) $\mathcal{F}$ is bounded on compacta, that is, whenever K is a compact subset of U, there exists a constant λ_K such that $|f(z)| \leq \lambda_K$ for all $z \in K$ and $f \in \mathcal{F}$.

(2) $\mathcal{F}$ is equicontinuous on compacta, that is, given any K is a compact subset of U and any $\epsilon > 0$ there exists a $\delta(\epsilon, K) > 0$ such that whenever $z, w \in K$, $|z - w| < \delta(\epsilon, K)$ and $f \in \mathcal{F}$ then $|f(z) - f(w)| < \epsilon$.

(iv) Suppose that $\mathcal{F}$ is a collection of analytic functions $f : D(0,1) \to \mathbb{C}$ with $|f(z)| \leq \lambda$ for all $z \in D(0,1)$. Show, by using Cauchy's formula, or otherwise that $\mathcal{F}$ is equicontinuous on $\overline{D(0, 1-\epsilon)}$ for every ϵ with $1 > \epsilon > 0$.

(v) Prove Theorem 27 using the ideas of (iii) and (iv).

4 Proof of the Riemann mapping theorem

We now embark on a series of lemmas which together prove the Riemann mapping theorem stated in Theorem 13.

Lemma 30 *If Ω is an non-empty, open, simply connected subset of $\mathbb{C}$ with non-empty complement then there exists a conformal map of Ω to a set Ω' such that $\mathbb{C} \setminus \Omega'$ contains a disc $D(w, \delta)$ with $\delta > 0$.*

Lemma 31 *If Ω is an non-empty, open, simply connected subset of $\mathbb{C}$ with non-empty complement then there exists a conformal map of Ω to a set Ω'' such that $\Omega'' \subseteq D(0,1)$.*

It is worth remarking that the condition 'Ω has non-empty complement' cannot be removed.

Lemma 32 *There does not exist a conformal map $f : \mathbb{C} \to D(0,1)$.*

Thus the Riemann mapping theorem follows from the following slightly simpler version.

Lemma 33 *If Ω is an non-empty, open, simply connected subset of $D(0,1)$ then there exists a conformal map of Ω to the unit disc $D(0,1)$.*

We expect the proof of the Riemann mapping theorem to involve a maximisation argument and Theorem 16 (ii) suggests one possibility.

Lemma 34 *Suppose Ω is a open, non-empty, simply connected subset of $D(0,1)$. If $a \in \Omega$ then the set $\mathcal{F}$ of injective analytic functions $f : \Omega \to D(0,1)$ with $f(a) = 0$, $f'(a)$ real and $f'(a) \geq 0$ then $\mathcal{F}$ is a non-empty normal set.*

Lemma 35 *With the hypotheses and notation of Lemma 34 there exists a κ such that $f'(a) \leq \kappa$ for all $f \in \mathcal{F}$.*

Lemma 36 *Suppose that U is an open connected subset of $\mathbb{C}$ and we have a sequence of analytic functions f_n on U with $f_N \to f$ uniformly. If each f_n is injective then either f is constant or f is injective.*

Lemma 37 *With the hypotheses and notation of Lemma 34 there exists a* $g \in \mathcal{F}$ *such that* $g'(a) \geq f'(a)$ *for all* $f \in \mathcal{F}$.

It may be worth remarking that though g is, in fact, unique we have not yet proved this.

All we need now is a little ingenuity and this is supplied by an idea of Koebe.

Lemma 38 *(i) If* U *is an open simply connected subset of* $D(0,1)$ *containing* 0 *but with* $U \neq D(0,1)$ *then we can find a bijective analytic function* $h : U \to D(0,1)$ *such that* $h(0) = 0$, $h'(0)$ *is real and* $h'(0) > 1$.

(ii) With the hypotheses and notation of Lemma 34, if $g \in \mathcal{F}$ *and* $g(\Omega) \neq D(0,1)$ *we can find an* $f \in \mathcal{F}$ *with* $f'(0) > g'(0)$.

Theorem 13 now follows at once. Combining Theorem 13 with Theorem 16 (ii) we obtain the following mild sharpening without further work.

Theorem 39 *If* U *be a simply connected open set with* $U \neq \mathbb{C}$ *and* $a \in U$. *then there exists precisely one conformal map* $f : U \to D(0,1)$ *with* $f(a) = 0$ *and* $f'(a)$ *real and positive.*

By reviewing the proof of the Riemann mapping theorem that we have given it becomes clear that we have in fact proved Lemma 9.

Exercise 40 *Check that we can prove Lemma 9 by the method used to prove the Riemann mapping theorem.*

It is important to realise that the intuition we gather from the use of simple conformal transforms in physics and elsewhere may be an unreliable guide in the more general context of the Riemann mapping theorem.

Example 41 *There is a bounded non-empty simply connected open set* U *such that if we have conformal map* $f : D(0,1) \to U$ *there does not exist a continuous bijective map* $\tilde{f} : \overline{D(0,1)} \to \overline{U}$ *with* $\tilde{f}|D(0,1) = f$.

Riemann's mapping theorem is a beginning and not an end. Riemann stated his result in a more general context than we have done here and the continuation of Riemann's ideas leads to Klein's uniformisation theorem. On the other hand, if we continued the development suggested here we would look at topics like Green's functions, boundary behaviour and the Picard theorems. If time permits I shall look at some of these ideas later but I am anxious not to hurry through the topics from number theory which will be discussed in the next few lectures.

5 Infinite products

Our object in the next few lectures will be to prove the following remarkable theorem of Dirichlet on primes in arithmetic progression.

Theorem 42 (Dirichlet) *If a and d are strictly positive coprime integers then there are infinitely many primes of the form $a + nd$ with n a positive integer.*

(Obviously the result must fail if a and d are not coprime.)

There exist a variety of proofs of special cases when d has particular values but, so far as I know, Dirichlet's proof of his theorem remains, essentially, the only approachable one. In particular there is no known reasonable[3] elementary (in the technical sense of not using analysis) proof.

Dirichlet's method starts from an observation of Euler.

Lemma 43 *If s is real with $s > 1$ then*

$$\prod_{\substack{p \text{ prime} \\ p \leq N}} \left(1 - \frac{1}{p^s}\right)^{-1} \to \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Using this result, we get a new proof of the existence of an infinity of primes.

Theorem 44 (Euclid) *There exist an infinity of primes.*

This suggests that it may be worth investigating infinite products a bit more.

Definition 45 *Let $a_j \in \mathbb{C}$. If $\prod_{n=1}^{N}(1 + a_n)$ tends to a limit L as $N \to \infty$ we say that the* infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges *to a value L and write*

$$\prod_{n=1}^{\infty}(1 + a_n) = L.$$

If the infinite product $\prod_{n=1}^{\infty}(1+|a_n|)$ converges then we say that $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent.

The next result was removed from the first year of the Tripos a couple of years before I took it.

[3] In the sense that most reasonable people would call reasonable. Selberg produced a (technically) elementary proof which may be found in his collected works.

Lemma 46 *Let* $a_j \in \mathbb{C}$.

(i) $\prod_{n=1}^{\infty}(1+a_n)$ *is absolutely convergent if and only if* $\sum_{n=1}^{\infty} a_n$ *is.*

(ii) If $\prod_{n=1}^{\infty}(1+a_n)$ *is absolutely convergent and* $1+a_n \neq 0$ *for each* n *then*

$$\prod_{n=1}^{\infty}(1+a_n) \neq 0.$$

Exercise 47 *Find* $a_j \in \mathbb{C}$ *such that* $\prod_{n=1}^{\infty}(1+a_n)$ *is not absolutely convergent but is convergent to a non-zero value.*

We shall only make use of absolute convergent infinite products.

Exercise 48 *If* $\prod_{n=1}^{\infty}(1+a_n)$ *is absolutely convergent and* $\sigma : \mathbb{N} \to \mathbb{N}$ *is a bijection (that is,* σ *is a permutation of* $\mathbb{N}$*) show that* $\prod_{n=1}^{\infty}(1+a_{\sigma(n)})$ *is absolutely convergent and*

$$\prod_{n=1}^{\infty}(1+a_{\sigma(n)}) = \prod_{n=1}^{\infty}(1+a_n)$$

Whilst this is a useful result to know, we shall make no essential use of it. When we write $\sum_{p \text{ prime}}$ or $\prod_{p \text{ prime}}$ we mean the primes p to be taken in order of increasing size.

Using Lemma 46 we obtain the following strengthening of Euclid's theorem.

Theorem 49 (Euler) $\displaystyle\sum_{p \text{ prime}} \frac{1}{p} = \infty.$

Since we wish to consider infinite products of functions it is obvious that we shall need an analogue of the Weierstrass M-test for products, obvious what that analogue should be and obvious how to prove it.

Lemma 50 *Suppose* U *is an open subset of* $\mathbb{C}$ *and that we have a sequence of functions* $g_n : U \to \mathbb{C}$ *and a sequence of positive real numbers* M_n *such that* $M_n \geq |g_n(z)|$ *for all* $z \in U$*. If* $\sum_{n=1}^{\infty} M_n$ *converges then* $\prod_{n=1}^{N}(1+g_n(z))$ *converges uniformly on* U*.*

Later we shall need to consider $\sum n^{-s}$ with s complex. To avoid ambiguity, we shall take $n^{-s} = \exp(-s \log n)$ where $\log n$ is the real logarithm of n.

Lemma 51 *If* $\Re s > 1$ *we have*

$$\prod_{p\ prime} (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$$

both sides being absolutely convergent for each s *and uniformly convergent for* $\Re s > 1 + \epsilon$ *for each fixed* $\epsilon > 0$*.*

We now detour briefly from the main argument to show how infinite products can be used to answer a very natural question. 'Can we always find an analytic function with specified zeros?' (We count multiple zeros multiply in the usual way.) Naturally we need to take account of the following fact.

Lemma 52 *If* z_1*,* z_2*,* $\dots$ *are distinct zeros of an analytic function which is not identically zero then* $z_n \to \infty$ *as* $n \to \infty$*.*

A little thought suggests the path we ought to take though we may not see how to reach it. A way to reach the path is provided by the Weierstrass primary function $E(z, m)$.

Definition 53 *If* m *is a strictly positive integer*

$$E(z, m) = (1 - z)e^{z + z^2/2 + z^3/3 + \cdots + z^m/m}.$$

Lemma 54 *The function* $E(\ , m) : \mathbb{C} \to \mathbb{C}$ *is analytic with a unique zero at 1. If* $|z| \leq 1$ *then*

$$|1 - E(z, m)| \leq |z|^{m+1}.$$

(It is nice to have such a neat result but for our purposes $|1 - E(z, m)| \leq A|z|^{m+1}$ for $|z| \leq R$ with any A and R would be just as good.)

Theorem 55 (Weierstrass) *If* k *is a positive integer and* $z_1, z_2, \dots$ *is a sequence of non-zero complex numbers with* $z_n \to \infty$ *then*

$$F(z) = z^k \prod_{j=1}^{\infty} E(z/z_j, j)$$

is a well defined analytic function with a zero of order k *at* 0*, and zeros at the* z_j *(multiple zeros counted multiply) and no others.*

Lemma 56 *If* f_1 *and* f_2 *are analytic functions on* $\mathbb{C}$ *with the same zeros (multiple zeros counted multiply) then there exists an analytic function* g *such that*

$$f_1(z) = e^{g(z)} f_2(z).$$

Lemma 57 *If $z_1, z_2, \ldots$ and $w_1, w_2, \ldots$ are sequences of complex numbers with $z_j, w_j \to \infty$ as $j \to \infty$ and $z_j \neq w_k$ for all j, k then there exists a meromorphic function with zeros at the z_j and poles at the w_k (observing the usual multiplicity conventions).*

Exercise 58 *(It may be helpful to attack parts of this question non-rigourously first and then tighten up the argument second.)*

(i) If C_N is the contour consisting of the square with vertices $\pm(N+1/2)\pm(N+1/2)i$ described anti-clockwise show that there is a constant K such that

$$|\cot \pi z| \leq K$$

for all $z \in C_N$ and all integers $N \geq 1$.

(ii) By integrating an appropriate function round the contour C_N, or otherwise, show that, if $w \notin \mathbb{Z}$,

$$\sum_{n=-N}^{n=N} \frac{1}{w-n} \to \pi \cot \pi w.$$

(iii) Is it true that, if $w \notin \mathbb{Z}$,

$$\sum_{n=-M}^{n=N} \frac{1}{w-n} \to \pi \cot \pi w,$$

as $M, N \to \infty$? Give reasons.

(iv) Show that

$$P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

is a well defined analytic function and that there exists an analytic function g such that

$$\sin \pi z = e^{g(z)} P(z).$$

(v) Find a simple expression for $P'(z)/P(z)$. [Hint: If $p(z) = \prod_{j=1}^{N}(z - \alpha_j)$, what is $p'(z)/p(z)$?] Find a related expression for $\frac{d}{dz}\sin \pi z / \sin \pi z$.

(vi) Show that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(vii) Find a similar expression for $\cos \pi z$. (These results are due to Euler.)

Exercise 59 *(This makes use of some of the techniques of the previous exercise.) (i) Show that the infinite product*

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)$$

exists and is analytic on the whole complex plane.

(ii) Show that

$$g'(z) = g(z) \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right).$$

Explain why $\sum_{n=1}^{\infty}(\frac{1}{z-n} + \frac{1}{n})$ *is indeed a well defined everywhere analytic function.*

(iii) By using (ii), or otherwise, show that

$$g(z+1) = -Azg(z) \qquad (*)$$

for some constant A.

(iv) By considering a particular value of z, or otherwise, show that A is real and positive and

$$\sum_{n=1}^{N} \frac{1}{n} - \log N \to \log A$$

as $N \to \infty$. *Deduce the existence of Euler's constant* $\gamma = i(\lim_{N\to\infty} \sum_{n=1}^{N} n^{-1} - \log N)$ *and rewrite* $(*)$ *as*

$$g(z+1) = -e^{\gamma} zg(z)$$

(v) Find a simple expression for $zg(z)g(-z)$. *Use* $(*)$ *to show that* $\sin \pi z$ *is periodic.*

6 Fourier analysis on finite Abelian groups

One of Dirichlet's main ideas is a clever extension of Fourier analysis from its classical frame. Recall that classical Fourier analysis deals with formulae like

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n(t)$$

where $e_n(t) = \exp(int)$. The clue to further extension lies in the following observation.

Lemma 60 *Consider the Abelian group $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and the subgroup $S = \{z : |z| = 1\}$ of $(\mathbb{C} \setminus \{0\}, \times)$. The continuous homomorphisms $\theta : \mathbb{T} \to S$ are precisely the functions $e_n : \mathbb{T} \to S$ given by $e_n(t) = \exp(int)$ with $n \in \mathbb{Z}$.*

Exercise 61 *(i) Find (with proof) all the continuous homomorphisms $\theta : (\mathbb{R}, +) \to (S, \times)$. What is the connection with Fourier transforms?*

(ii) (Only for those who know Zorn's lemma[4].) Assuming Zorn's lemma show that any linearly independent set in a vector space can be extended to a basis. If we consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ show that there exists a linear map $T : \mathbb{R} \to \mathbb{R}$ such that $T(1) = 1$, $T(\sqrt{2}) = 0$. Deduce the existence of a function $T : \mathbb{R} \to \mathbb{R}$ such that $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}$ which is not continuous (with respect to the usual metric). Show that, if we accept Zorn's lemma, there exist discontinuous homomorphisms $\theta : (\mathbb{R}, +) \to (S, \times)$.

This suggests the following definition.

Definition 62 *If G is a finite Abelian group we say that a homomorphism $\chi : G \to S$ is a* character*. We write $\hat{G}$ for the collection of such characters.*

In this section we shall accumulate a substantial amount of information about $\hat{G}$ by a succession of small steps.

Lemma 63 *Let G be a finite Abelian group.*

(i) If $x \in G$ has order m and $\chi \in \hat{G}$ then $\chi(x)$ is an mth root of unity.

(ii) $\hat{G}$ is a finite Abelian group under pointwise multiplication.

To go further we consider for each finite Abelian group G the collection $C(G)$ of functions $f : G \to \mathbb{C}$. If G has order $|G|$ then $C(G)$ is a vector space of dimension N which can be made into a complex inner product space by means of the inner product

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x) g(x)^*.$$

Exercise 64 *Verify the statements just made.*

Lemma 65 *Let G be a finite Abelian group. The elements of $\hat{G}$ form an orthonormal system in $C(G)$.*

Does $\hat{G}$ form an orthonormal basis of $C(G)$? The next lemma tells us how we may hope to resolve this question.

[4] And, particularly, those who only know Zorn's lemma.

Lemma 66 *Let G be a finite Abelian group. The elements of $\hat{G}$ form an orthonormal basis if an only if given an element $x \in G$ which is not the identity we can find a character χ with $\chi(x) \neq 1$.*

The way forward is now clear.

Lemma 67 *Suppose that H is a subgroup of a finite Abelian group G and that $\chi \in \hat{H}$. If K is a subgroup of G generated by H and an element $a \in G$ then we can find a $\tilde{\chi} \in \hat{K}$ such that $\tilde{\chi}|H = \chi$.*

Lemma 68 *Let G be a finite Abelian group and x an element of G of order m. Then we can find a $\chi \in \hat{G}$ with $\chi(x) = \exp 2\pi i/m$.*

Theorem 69 *If G is a finite Abelian group then $\hat{G}$ has the same number of elements as G and they form an orthonormal basis for $C(G)$.*

Lemma 70 *If G is a finite Abelian group and $f \in C(G)$ then*

$$f = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$$

where $\hat{f}(\chi) = \langle f, \chi \rangle$.

Exercise 71 *Suppose that G is a finite Abelian group. Show that if we define $\theta_x : \hat{G} \to \mathbb{C}$ by $\theta_x(\chi) = \chi(x)$ for $\chi \in \hat{G}$, $x \in G$ then the map $\Theta : G \to \hat{\hat{G}}$ given by $\Theta(x) = \theta_x$ is an isomorphism.*

If we now identify x with θ_x (and, so, G with $\hat{\hat{G}}$) show that

$$\hat{\hat{f}}(x) = |G| f(x^{-1})$$

for all $f \in C(G)$ and $x \in G$.

We have now done all that that is required to understand Dirichlet's motivation. However, it seems worthwhile to make a slight detour to put 'computational' bones on this section by exhibiting the structure of G and $\hat{G}$.

Lemma 72 *Let $(G, \times)$ be an Abelian group.*

(i) Suppose that $x, y \in G$ have order r and s with r and s coprime. Then xy has order rs.

(ii) If G contains elements of order n and m then G contains an element of order the least common multiple of n and m.

Lemma 73 *Let $(G, \times)$ be a finite Abelian group. Then there exists an integer N and an element k such that k has order N and, whenever $x \in G$ we have $x^N = e$.*

Lemma 74 *With the hypotheses and notation of Lemma 73 we can write $G = K \times H$ where K is the cyclic group generated by x and H is another subgroup of K.*

As usual we write C_n for the cyclic group of order n.

Theorem 75 *If G is a finite Abelian group we can find $n(1)$, $n(2)$, ... $n(m)$ with $n(j+1)|n(j)$ such that G is isomorphic to*

$$C_{n(1)} \times C_{n(2)} \times \dots C_{n(m)}.$$

Lemma 76 *If we have two sequences $n(1)$, $n(2)$, ... $n(m)$ with $n(j+1)|n(j)$ and $n'(1)$, $n'(2)$, ... $n'(m')$ with $n'(j+1)|n'(j)$ then*

$$C_{n(1)} \times C_{n(2)} \times \dots C_{n(m)} \text{ is isomorphic to } C_{n'(1)} \times C_{n'(2)} \times \dots C_{n'(m')}$$

if and only if $m = m'$ and $n(j) = n'(j)$ for each $1 \leq j \leq m$.

It is easy to identify $\hat{G}$.

Lemma 77 *Suppose that*

$$G = C_{n(1)} \times C_{n(2)} \times \dots C_{n(m)}$$

with $C_{n(j)}$ a cyclic group of order $n(j)$ generated by x_j. Then the elements of $\hat{G}$ have the form $\chi_{\omega_{n(1)}^{r(1)}, \omega_{n(2)}^{r(2)}, \dots \omega_{n(m)}^{r(m)}}$ with $\omega_{n(j)} = \exp(2\pi i/n(j)$ and

$$\chi_{\omega_{n(1)}^{r(1)}, \omega_{n(2)}^{r(2)}, \dots \omega_{n(m)}^{r(m)}}(x_1^{s(1)} x_2^{s(2)} \dots x_m^{s(m)}) = \omega_{n(1)}^{r(1)s(1)} \omega_{n(2)}^{r(2)s(2)} \dots \omega_{n(m)}^{r(m)s(m)}.$$

My readers will see that $\hat{G}$ is isomorphic to G but the more sophisticated algebraists will also see that this is *not a natural isomorphism* (whereas G and $\hat{\hat{G}}$ are *naturally isomorphic*). Fortunately such matters are of no importance for the present course.

7 The Euler-Dirichlet formula

Dirichlet was interested in a particular group. If d is a positive integer consider $\mathbb{Z}/(n)$ the set of equivalence classes

$$[m] = \{r : r \equiv m \mod d\}$$

under the usual multiplication modulo n. We set

$$G_d = \{[m] : m \text{ and } d \text{ coprime}\}$$

and write $\phi(d)$ for the order of G_d (ϕ is called Euler's totient function).

Lemma 78 *The set G_d forms a finite Abelian group under standard multiplication.*

The results of the previous section show that, if $[a] \in G_n$ and we define $\delta_a : G_d \to \mathbb{C}$ by

$$\begin{aligned} \delta_a([a]) &= 1 \\ \delta_a([m]) &= 0 \qquad \text{if } [m] \neq [a], \end{aligned}$$

then

$$\delta_a = \phi(d)^{-1} \sum_{\chi \in G_d} \chi([a])^* \chi$$

We now take up the proof of Dirichlet's theorem in earnest. We shall operate under the standing assumption that a and d are positive coprime integers and our object is to show that the sequence

$$a,\ a+d,\ a+2d,\ \ldots, a+nd,\ \ldots$$

contains infinitely many primes. Following Euler's proof that there exist infinitely many primes we shall seek to prove this by showing that

$$\sum_{\substack{p \text{ prime} \\ p=a+nd \text{ for some } n}} \frac{1}{p} = \infty.$$

Henceforward, at least in the number theory part of the course p will be a prime, $\sum_p$ will mean the sum over all primes and so on.

In order to simplify our notation it will also be convenient to modify the definition of a character. From now on, we say that χ is a character if χ is a map from $\mathbb{N}$ to $\mathbb{C}$ such that there exists a character (in the old sense) $\tilde{\chi} \in \hat{G}_d$ such that

$$\begin{aligned} \chi(m) &= \tilde{\chi}([m]) \qquad \text{if } m \text{ and } d \text{ are coprime} \\ \chi(m) &= 0 \qquad \text{otherwise.} \end{aligned}$$

We write $\sum_\chi$ to mean the sum over all characters and take χ_0 to be the character with $\chi_0([m]) = 1$ whenever m and d are coprime.

Lemma 79 *(i) If χ is a character then $\chi(m_1 m_2) = \chi(m_1)\chi(m_2)$ for all $m_1, m_2 \geq 0$.*

(ii) If $\chi \neq \chi_0$ then $\sum_{m=k+1}^{k+d} \chi(m) = 0$.

(iii) If $\delta_a(m) = \phi(d)^{-1} \sum_\chi \chi(a)^* \chi(m)$ then $\delta_a(m) = 1$ when $m = a + nd$ and $\delta_a(m) = 0$ otherwise.

(iv) $\displaystyle\sum_{p=a+nd} p^{-s} = \phi(d)^{-1} \sum_\chi \chi(a)^* \sum_p \chi(p) p^{-s}.$

Lemma 80 *The sum $\sum_{p=a+nd} p^{-1}$ diverges if $\sum_p \chi(p) p^{-s}$ remains bounded as s tends to 1 through real values of $s > 1$ for all $\chi \neq \chi_0$.*

We now prove a new version of Euler's formula.

Theorem 81 (Euler-Dirichlet formula) *With the notation of this section,*

$$\prod_{n=1}^{\infty} (1 - \chi(p) p^{-s})^{-1} = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

both sides being absolutely convergent for $\Re s > 1$.

To link $\prod_{n=1}^{\infty} (1 - \chi(p) p^{-s})^{-1}$ with $\sum_p \chi(p) p^{-s}$ we use logarithms. (If you go back to our discussion of infinite products you will see that this is not unexpected.) However, we must, as usual, use care when choosing our logarithm function. For the rest of the argument log will be the function on

$$\mathbb{C} \setminus \{x : x \text{ real and } x \leq 0\}$$

defined by $\log(re^{i\theta}) = \log r + i\theta$ $[r > 0,\ -\pi < \theta < \pi]$.

Lemma 82 *(i) If $|z| \leq 1/2$ then $|\log(1 - z) + z| \leq |z|^2$.*

(ii) If $\epsilon > 0$ then $\sum_p \log(1 - \chi(p) p^{-s})$ and $\sum_p \chi(p) p^{-s}$ converge uniformly in $\Re s \geq 1 + \epsilon$, whilst

$$\left| \sum_p \log(1 - \chi(p) p^{-s}) + \sum_p \chi(p) p^{-s} \right| \leq \sum_{n=1}^{\infty} n^{-2}.$$

We have thus shown that if $\sum_p \log(1-\chi(p)p^{-s})$ remains bounded as $s \to 1+$ then $\sum_p \chi(p)p^{-s}$ does. Unfortunately we can not equate $\sum_p \log(1-\chi(p)p^{-s})$ with $\log(\prod_{n=1}^{\infty}(1-\chi(p)p^{-s})^{-1})$.

However we can refresh our spirits by proving Dirichlet's theorem in some special cases.

Example 83 *There are an infinity of primes of the form* $3n+1$ *and* $3n+2$*.*

Exercise 84 *Use the same techniques to show that there are an infinity of primes of the form* $4n+1$ *and* $4n+3$*.*

8 Analytic continuation of the Dirichlet functions

Dirichlet completed his argument without having to consider $\sum_{n=1}^{\infty}\chi(n)n^{-s}$ for anything other than real s with $s>1$. However, as we have already seen, $\sum_{n=1}^{\infty}\chi(n)n^{-s} = L(s,\chi)$ is defined and well behaved in $\Re s > 1$. Riemann showed that it is advantageous to extend the definition of analytic functions like $L(s,\chi)$ to larger domains.

There are many ways of obtaining such *analytic continuations*. Here is one.

Lemma 85 *If* $f:\mathbb{R}\to\mathbb{R}$ *is bounded on* $\mathbb{R}$ *and locally integrable*[5] *then*

$$F(s) = \int_1^{\infty} f(x)x^{-s}\,dx$$

is a well defined analytic function on the set of s *with* $\Re s > 1$*.*

Lemma 86 *(i) If* $\chi \neq \chi_0$ *and* $S(x) = \sum_{1\leq m\leq x}\chi(m)$ *then* $S:\mathbb{R}\to\mathbb{R}$ *is bounded and locally integrable. We have*

$$\sum_{n=1}^{N}\chi(n)n^{-s} \to s\int_1^{\infty} S(x)x^{-s-1}\,dx$$

as $N\to\infty$ *for all* s *with* $\Re s > 1$*.*

(ii) If $S_0(x)=0$ *for* $x\leq 0$ *and* $S_0(x) = \sum_{1\leq m\leq x}\chi_0(m)$ *then, writing* $T_0(x) = S_0(x) - d^{-1}\phi(d)x$, $T_0:\mathbb{R}\to\mathbb{R}$ *is bounded and locally integrable. We have*

$$\sum_{n=1}^{N}\chi(n)n^{-s} \to s\int_1^{\infty} T_0(x)x^{-s-1}\,dx + \frac{\phi(d)s}{d(s-1)}$$

as $N\to\infty$ *for all* s *with* $\Re s > 1$*.*

[5] Riemann or Lebesgue at the reader's choice

Lemma 87 *(i) If* $\chi \neq \chi_0$ *then* $\sum_{n=1}^{\infty} \chi(n)n^{-s}$ *converges to an analytic function* $L(s,\chi)$*, say, on* $\{s \in \mathbb{C} : \Re s > 0\}$*.*

(ii) There exists an meromorphic function $L(s,\chi_0)$ *analytic on* $\{s \in \mathbb{C} : \Re s > 0\}$ *except for a simple pole, residue* $\phi(d)/d$ *at* 1 *such that* $\sum_{n=1}^{\infty} \chi_0(n)n^{-s}$ *converges to* $L(s,\chi)$ *for* $\Re s > 1$*.*

Exercise 88 *(i) Explain carefully why* $L(\ ,\chi_0)$ *is defined uniquely by the conditions given.*

(ii) Show that $\sum_{n=1}^{\infty} \chi_0(n)n^{-s}$ *diverges for* s *real and* $1 \geq s > 0$*.*

We now take up from where we left off at the end of the previous section.

Lemma 89 *(i) If* $\Re s > 1$ *then* $\exp(-\sum_p \log(1-\chi(p)p^{-s}) = L(s,\chi)$*.*

(ii) If $\Re s > 1$ *then* $L(s,\chi) \neq 0$*.*

(iii) There exists a function $\operatorname{Log} L(s,\chi)$ *analytic on* $\{s : \Re s > 1\}$ *such that* $\exp(\operatorname{Log} L(s,\chi)) = L(s,\chi)$ *for all* s *with* $\Re s > 1$*.*

(iv) If $\chi \neq \chi_0$ *and* $L(1,\chi) \neq 0$ *then* $\operatorname{Log} L(s,\chi))$ *tends to a finite limit as* $s \to 1$ *through real values with* $s > 1$*.*

(v) There is a fixed integer M_χ *such that*

$$\operatorname{Log} L(s,\chi) + \sum_p \log(1-\chi(p)p^{-s}) = 2\pi M_\chi$$

for all $\Re s > 1$*.*

(vi) If $\chi \neq \chi_0$ *and* $L(1,\chi) \neq 0$ *then* $\sum_p \chi(p)p^{-s}$ *remains bounded as* $s \to 1$ *through real values with* $s > 1$*.*

We mark our progress with a theorem.

Theorem 90 *If* $L(1,\chi) \neq 0$ *for all* $\chi \neq \chi_0$ *then there are an infinity of primes of the form* $a + nd$*.*

Since it is easy to find the characters χ in any given case and since it is then easy to compute $\sum_{n=1}^{N} \chi(n)n^{-1}$ and to estimate the error $\sum_{n=N+1}^{\infty} \chi(n)n^{-1}$ to sufficient accuracy to prove that $L(1,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-1} \neq 0$, it now becomes possible to prove Dirichlet's theorem for any particular coprime a and d.

Exercise 91 *Choose* a *and* d *and carry out the program just suggested.*

However, we still need to show that the theorem holds in all cases.

9 $L(1,\chi)$ is not zero

Our first steps are easy.

Lemma 92 *(i) If s is real and $s>1$ then*

$$\prod_{\chi} L(s,\chi) = \exp(-\sum_{p}\sum_{\chi}\log(1-\chi(p)p^{-s}).$$

(ii) If s is real and $s>1$ then $\prod_\chi L(s,\chi)$ is real and $\prod_\chi L(s,\chi)\geq 1$.
(iii) $\prod_\chi L(s,\chi)\nrightarrow 0$ as $s\rightarrow 1$.

Lemma 93 *(i) There can be at most one character χ with $L(1,\chi)=0$.*
(ii) If a character χ takes non-real values then $L(1,\chi)\neq 0$.

We have thus reduced the proof of Dirichlet's theorem to showing that if χ is a character with $\chi\neq\chi_0$ which only takes the values 1, -1 and 0 then $L(1,\chi)\neq 0$. There are several approaches to this problem but none are short and transparent. We use a proof of de la Vallée Poussin which is quite short but not, I think, transparent.

Lemma 94 *Suppose that the character $\chi\neq\chi_0$ and only takes the values 1, -1 and 0. Set*

$$\psi(s)=\frac{L(s,\chi)L(s,\chi_0)}{L(2s,\chi_0)}.$$

(i) The function ψ is well defined and meromorphic for $\Re s>\frac{1}{2}$. It analytic except, possibly for a simple pole at 1.
(ii) If $L(1,\chi)=0$ then 1 is a removable singularity and ψ is analytic everywhere on $\{s:\Re s>\frac{1}{2}\}$.
(iii) We have $\psi(s)\rightarrow 0$ as $s\rightarrow\frac{1}{2}$ through real values of s with $s\geq\frac{1}{2}$.

Lemma 95 *We adopt the hypotheses and notation of Lemma 94. If $\Re s>1$ then the following is true.*
(i) $\psi(s)=\prod_{\chi(p)=1}\frac{1+p^{-s}}{1-p^{-s}}$.
(ii) We can find subsets Q_1 and Q_2 of $\mathbb{Z}$ such that

$$\prod_{\chi(p)=1}(1+p^{-s})=\sum_{n\in Q_1}n^{-s}$$
$$\prod_{\chi(p)=1}(1-p^{-s})^{-1}=\sum_{n\in Q_2}n^{-s}.$$

(iii) There is a sequence of real positive numbers a_n with $a_1 = 1$ such that

$$\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Lemma 96 *We adopt the hypotheses and notation of Lemmas 94 and 95.*
(i) If $\Re s > 1$ then

$$\psi^{(m)}(s) = \sum_{n=1}^{\infty} a_n (-\log n)^m n^{-s}.$$

(ii) If $\Re s > 1$ then $(-1)^m \psi^{(m)}(s) > 0$.
(iii) If ψ has no pole at 1 then if $\Re s_0 > 1$ and $|s - s_0| < \Re s_0 - 1/2$ we have

$$\psi(s) = \sum_{m=0}^{\infty} \frac{\psi^{(m)}(s_0)}{m!}(s - s_0)^m.$$

(iv) If ψ has no pole at 1 then $\psi(s) \nrightarrow 0$ as $s \to \frac{1}{2}$ through real values of s with $s \geq \frac{1}{2}$.

We have proved the result we set out to obtain.

Lemma 97 *If a character $\chi \neq \chi_0$ only takes real values then $L(1, \chi) \neq 0$.*

Theorem 98 *If $\chi \neq \chi_0$ then $L(1, \chi) \neq 0$.*

We have thus proved Theorem 42, If a and d are strictly positive coprime integers then there are infinitely many primes of the form $a + nd$ with n a positive integer.

10 Natural boundaries

This section is included partly for light relief between two long and tough topics and partly to remind the reader that analytic continuation is not quite as simple as it looks.

Lemma 99 *If*

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

for $|z| < 1$ then $f : D(0,1) \to \mathbb{C}$ is analytic but if U is any connected open set with $U \cap D(0,1) \neq \emptyset$ and $U \setminus D(0,1) \neq \emptyset$ then there does not exist an analytic function $g : U \to \mathbb{C}$ with $g(z) = f(z)$ for all $z \in U \cap D(0,1)$.

More briefly we say that the unit circle is a natural boundary for f.

Here is a much more subtle result whose central idea goes back to Borel.

Theorem 100 *Suppose $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1 and suppose $Z_1, Z_2, \dots$ is a sequence of independent random variables with each Z_j uniformly distributed over $\{z \in \mathbb{C} : |z| = 1\}$. Then, with probability 1, the unit circle is a natural boundary for*

$$\sum_{n=1}^{\infty} Z_n a_n z^n.$$

Our proof will require the following result due to Kolmogorov.

Theorem 101 (Kolmogorov zero-one law) *Let P be a reasonable property which any sequence of complex numbers w_n either does or does not have and such that any two sequences w_n and w_n' with $w_n = w_n'$ for n sufficiently large either both have or both do not have. Then if W_n is a sequence of independent complex valued random variables, it follows that W_n has property P with probability 0 or has property P with probability 1.*

I will try to explain why this result is plausible but for the purposes of the exam this result may be assumed without discussion.

We remark the following simple consequence.

Lemma 102 *There exists a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with the unit circle as natural boundary having the property that f and all its derivatives can be extended to continuous functions on $\overline{D(0,1)}$.*

Exercise 103 *Obtain Lemma 102 by non-probabilistic means. One possibility is to consider $f(z) = \sum_{n=1}^{\infty} \epsilon_n g((1-n^{-1})z^{n!})$ where $g(z) = (1-z)^{-1}$ and ϵ_n is a very rapidly decreasing sequence of positive numbers.*

Finally we remark that whilst it is easy to define a natural boundary for a power series the notion does not easily extend.

Lemma 104 *Let $\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\}$ We can find an analytic function $f : \Omega \to \mathbb{C}$ with the following properties.*

(i) There exists an analytic function $F : \{z : \Re z > 0\} \to \mathbb{C}$ such that $F(z) = f(z)$ whenever $\Re z > 0$, $\Im z < 0$.

(2) If D is an open disc which contains z_1, z_2 with $\Re z_1 > 0$, $\Im z_1 > 0$ and $\Re z_2 > 0$, $\Im z_2 < 0$ there exists no analytic function g on D with $g(z) = f(z)$ for all $z \in D$ with $\Re z > 0$, $\Im z > 0$.

11 Chebychev and the distribution of primes

On the strength of numerical evidence, Gauss was lead to conjecture that the number $\pi(n)$ of primes less than n was approximately $n/\log n$. The theorem which confirms this conjecture is known as the prime number theorem. The first real progress in this direction was due to Chebychev[6] We give his results, not out of historical piety, but because we shall make use of them in our proof of the prime number theorem. (Note the obvious conventions that n is an integer with $n \geq 1$, $\prod_{n<p\leq 2n}$ means the product over all primes p with $n < p \leq 2n$ and so on. It is sometimes useful to exclude small values of n.)

Lemma 105 *(i)* $2^n < \binom{2n}{n} < 2^{2n}$.

(ii) $\binom{2n}{n}$ *divides* $\prod_{p<2n} p^{[(\log 2n)/(\log p)]}$ *and* $\prod_{n<p\leq 2n} p$ *divides* $\binom{2n}{n}$.

(iii) We have $\pi(2n) > (\log 2)n/(\log 2n)$.

(iv) There exists a constant $A > 0$ *such that* $\pi(n) \geq An(\log n)^{-1}$.

(v) There exists a constant B' *such that* $\sum_{p\leq n} \log p \leq B'n$.

(vi) There exists a constant B *such that* $\pi(n) \leq Bn(\log n)^{-1}$.

We restate the main conclusions of Lemma 105.

Theorem 106 (Chebychev) *There exist constants* A *and* B *with* $0 < A \leq B$ *such that*

$$An(\log n)^{-1} \leq \pi(n) \leq Bn(\log n)^{-1}.$$

Riemann's approach to the prime number theorem involves considering $\theta(n) = \sum_{p\leq n} \log p$ rather than $\pi(n)$.

Lemma 107 *Let* Q *be a set of positive integers and write* $\alpha(n) = \sum_{q\in Q, q\leq n} 1$ *and* $\beta(n) = \sum_{q\in Q, q\leq n} \log q$. *If there exist constants* A *and* B *with* $0 < A \leq B$ *such that*

$$An(\log n)^{-1} \leq \alpha(n) \leq Bn(\log n)^{-1},$$

then $n^{-1}(\log n)\alpha(n) \to 1$ *as* $n \to \infty$ *if and only if* $n^{-1}\beta(n) \to 1$ *as* $n \to \infty$.

Lemma 108 *If* $n^{-1}\theta(n) \to 1$ *as* $n \to \infty$ *then* $n^{-1}(\log n)\pi(n) \to 1$ *as* $n \to \infty$.

[6]His prefered transliteration seems to have been Tchebycheff, but he has been overruled.

12 The prime number theorem

We start by recalling various facts about the Laplace transform.

Exercise 109 *If a is a real number let us write $\mathcal{E}_a$ for the collection of locally integrable*[7] *functions $F : \mathbb{R} \to \mathbb{C}$ such that $F(t) = 0$ for all $t < 0$ and $F(t)e^{-at} \to 0$ as $t \to \infty$.*

(i) If $F \in \mathcal{E}_a$ explain why the Laplace transform

$$f(z) = \mathcal{L}F(z) = \int_{-\infty}^{\infty} F(t)\exp(-zt)\,dt$$

is well defined and analytic on $\{z \in \mathbb{C} : \Re z > a\}$.

(ii) We define the Heaviside function H by writing $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geq 0$. If $a \in \mathbb{R}$ and $b \geq 0$ consider $H_{a,b}(t) = H(t-b)e^{at}$. Show that $H_{a,b} \in \mathcal{E}_a$ and that $\mathcal{L}H_{a,b}(z)$ can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at a.

Engineers are convinced that the converse to Exercise 109 (i) holds in the sense that if $F \in \mathcal{E}_a$ has a Laplace transform f which can be extended to a function $\tilde{f}$ analytic on $\{z \in \mathbb{C} : \Re z > b\}$ [a, b real, $a \geq b$] then $F \in \mathcal{E}_b$. Unfortunately, this is not true but it represents a good heuristic principle to bear in mind in what follows. Number theorists use the Mellin transform

$$\mathcal{M}F(z) = \int_0^{\infty} F(t)t^{z-1}\,dt$$

in preference to the Laplace transform but the two transforms are simply related.

Exercise 110 *Give the relation explicitly.*

Riemann considered the two functions

$$\Phi(s) = \sum_p p^{-s}\log p$$

and the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Both of these functions are defined for $\Re s > 1$ but Riemann saw that they could be extended to analytic functions over a larger domain.

The next lemma is essentially a repeat of Lemmas 86 (ii) and 87 (ii).

[7] Use your favourite definition of this or replace by 'well behaved'.

Lemma 111 *(i) Let $S_0(x) = 0$ for $x \leq 0$ and $S_0(x) = \sum_{1\leq m\leq x} 1$. If $S_0 : \mathbb{R} \to \mathbb{R}$ and $T_0(x) = S_0(x) - x$ then T_0 is bounded and locally integrable. We have*

$$\sum_{n=1}^{N} n^{-s} \to s\int_1^\infty T_0(x)x^{-s-1}\,dx + \frac{s}{s-1}$$

as $N \to \infty$ for all s with $\Re s > 1$.

(ii) There exists an meromorphic function ζ analytic on $\{s \in \mathbb{C} : \Re s > 0\}$ except for a simple pole, residue 1 at 1 such that $\sum_{n=1}^\infty n^{-s}$ converges to $\zeta(s)$ for $\Re s > 1$.

The use of s rather than z goes back to Riemann. Riemann showed that ζ can be extended to a meromorphic function over $\mathbb{C}$ but we shall not need this.

How does this help us study Φ?

Lemma 112 *(i) We have $\prod p < N(1-p^s)^{-1} \to \zeta(s)$ uniformly for $\Re s > 1 + \delta$ whenever $\delta > 0$.*

(ii) We have

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log p}{p^s - 1}$$

for all $\Re s > 1$.

(iii) We have

$$\Phi(s) = -\frac{\zeta'(s)}{\zeta(s)} - \sum_p \frac{\log p}{(p^s-1)p^s}$$

for all $\Re s > 1$.

(iv) The function Φ can be analytically extended to a meromorphic function on $\{s : \Re s > \frac{1}{2}\}$. It has a simple pole at 1 with residue 1 and simple poles at the zeros of ζ but nowhere else.

The next exercise is long and will not be used later but is, I think, instructive.

Exercise 113 *(i) Show by grouping in pairs that $\sum_{n=1}^\infty (-1)^{n-1}n^{-s}$ converges to an analytic function $g(s)$ in the region $\{s : \Re s > 0\}$.*

(ii) Find A and B such that $g(s) = A\zeta(s) + B2^{-s}\zeta(s)$ for all $\Re s > 1$. Why does this give another proof that ζ can be extended to an analytic function on $\{s : \Re s > 0\}$.

(iii) Show that $g(1/2) \neq 0$ *and deduce that* $\zeta(1/2) \neq 0$.

(iv) By imitating the arguments of Lemma 112 show that we we can find an analytic function G *defined on* $\{s : \Re s > 1/3\}$ *such that*

$$\Phi(s) = -\frac{\zeta'(s)}{\zeta(s)} - \Phi(2s) - G(s).$$

Deduce that Φ *can be extended to a meromorphic function on* $\{s : \Re s > 1/3\}$.

(v) Show, using (iii), that Φ *has a pole at* $1/2$.

(vi) Show that the assumption that $|\sum_{p<N} \log p - N| \leq AN^{1/2-\epsilon}$ *for some* $\epsilon > 0$ *and* $A > 0$ *and all large enough* N *leads to the conclusion that* Φ *can be analytically extended from* $\{s : \Re s > 1\}$ *to an everywhere analytic function on* $\{s : \Re s > 1/2 - \epsilon\}$.

(vii) Deduce that if $\epsilon > 0$ *and* $A > 0$

$$|\sum_{p<N} \log p - N| \geq AN^{1/2-\epsilon} \text{ for infinitely many values of } N.$$

It is well known that Riemann conjectured that ζ has no zeros in $\{s : \Re s > 1/2\}$ and that his conjecture is the most famous open problem in mathematics. The best we can do is to follow Hadamard and de la Vallée Poussin and show that ζ has no zero on $\{s : \Re s = 1\}$. Our proof makes use of the slightly unconventional convention that if h and g are analytic in a neighbourhood of w, $g(w) \neq 0$ and and $h(z) = (z - w)^k g(z)$ then we say that h has a zero of order k at w. (The mild unconventionality arises when $k = 0$.)

Lemma 114 *Suppose that* ζ *has a zero of order* μ *at* $1 + i\alpha$ *and a zero of order* ν *at* $1 + 2i\alpha$ *with* α *real and* $\alpha > 0$. *Then*

(i) ζ *has a zero of order* μ *at* $1 - i\alpha$ *and a zero of order* ν *at* $1 - 2i\alpha$.

(ii) As $\epsilon \to 0$ *through real positive values of* ϵ

$$\begin{aligned}\epsilon\Phi(1 + \epsilon \pm i\alpha) &\to -\mu \\ \epsilon\Phi(1 + \epsilon \pm 2i\alpha) &\to -\nu \\ \epsilon\Phi(1 + \epsilon) &\to 1.\end{aligned}$$

(iii) If $s = 1 + \epsilon$ *with* ϵ *real and positive then*

$$\begin{aligned}0 &\leq \sum_p p^{-s} \log p (e^{(i\alpha \log p)/2} + e^{-(i\alpha \log p)/2})^4 \\ &= \Phi(s + 2i\alpha) + \Phi(s - 2i\alpha) + 4(\Phi(s + i\alpha) + \Phi(s - i\alpha)) + 6\Phi(s).\end{aligned}$$

(iv) We have $0 \leq -2\nu - 8\mu + 6$.

Theorem 115 *If* $\Re s = 1$ *then* $\zeta(s) \neq 0$.

We note the following trivial consequence.

Lemma 116 *If we write*

$$T(s) = \frac{\zeta'(s)}{\zeta(s)} - (s-1)^{-1},$$

then given any $R > 0$ *we can find a* $\delta(R)$ *such that* T *has no poles in* $\{z : \Re z \geq 1 - \delta(R),\ Imz \leq R\}$

We shall show that the results we have obtained on the behaviour of ζ suffice to show that

$$\int_1^X \frac{\theta(x) - x}{x^2}\, dx$$

tends to a finite limit as $X \to \infty$. The next lemma shows that this is sufficient to give the prime number theorem.

Lemma 117 *Suppose that* $\beta : [1, \infty) \to \mathbb{R}$ *is an increasing (so integrable) function.*

(i) If $\lambda > 1$, $y > 1$ *and* $y^{-1}\beta(y) > \lambda$ *then*

$$\int_y^{\lambda y} \frac{\beta(x) - x}{x^2}\, dx \geq A(\lambda)$$

where $A(\lambda)$ *is a strictly positive number depending only on* λ.

(ii) If $\int_1^X \frac{\beta(x)-x}{x^2}\, dx$ *tends to limit as* $X \to \infty$ *then* $x^{-1}\beta(x) \to 1$ *as* $x \to \infty$.

We need a couple of further preliminaries. First we note a simple consequence of the Chebychev estimates (Theorem 106).

Lemma 118 *There exists a constant* K *such that*

$$\frac{|\theta(x) - x|}{x} \leq K$$

for all $x \geq 1$.

Our second step is to translate our results into the language of Laplace transforms. (It is just a matter of taste whether to work with Laplace transforms or Mellin transforms.)

Lemma 119 *Let $f(t) = \theta(e^t)e^{-t} - 1$ for $t \geq 0$ and $f(t) = 0$ otherwise. Then*

$$\mathcal{L}f(z) = \int_{-\infty}^{\infty} f(t)e^{-tz}\,dt$$

is well defined and

$$\mathcal{L}f(z) = \frac{\Phi(z-1)}{z} - \frac{1}{z}$$

for all $\Re z > 0$.

The statement $\int_1 \infty(\theta(x) - x)/x^2\,dx$ convergent is equivalent to the statement that $\int_{-\infty}^{\infty} f(t)\,dt$ converges.

We have reduced the proof of the prime number theorem to the proof of the following lemma.

Lemma 120 *Suppose Ω is an open set with $\Omega \supseteq \{z : \Re z \geq 0\}$, $F : \Omega \to \mathbb{C}$ is an analytic function and $f : [0, \infty] \to \mathbb{R}$ is bounded locally integrable function such that*

$$F(z) = \mathcal{L}f(z) = \int_0^{\infty} f(t)e^{-tz}\,dt$$

for $\Re z > 0$. Then $\int_0^{\infty} f(t)\,dt$ converges.

This lemma and its use to prove the prime number theorem are due to D. Newman. (A version will be found in [1].)

13 Boundary behaviour of conformal maps

We now return to the boundary behaviour of the Riemann mapping. (Strictly speaking we should say, a Riemann mapping but we have seen that it is 'essentially unique'. We saw in Example 41 that there is no general theorem but the following result is very satisfactory.

Theorem 121 *If Ω is a simply connected open set in $\mathbb{C}$ with boundary a Jordan curve then any bijective analytic map $f : D(0,1) \to \Omega$ can be extended to a bijective continuous map from $\overline{D(0,1)} \to \overline{\Omega}$.*

Recall[8] that a *Jordan curve* is a continuous injective map $\gamma : \mathbb{T} \to \mathbb{C}$. We say that γ is the boundary of Ω if the image of γ is $\overline{\Omega} \setminus \Omega$.

[8]In the normal weasel-worded mathematical sense.

I shall use the proof in Zygmund's magnificent treatise [10] (see Theorem 10.9 of Chapter VII) which has the advantage of minimising the topology but the minor disadvantage of using measure theory (students who do not know measure theory may take the results on trust) and the slightly greater disadvantage of using an idea from Fourier analysis (the *conjugate* trigonometric sum $\tilde{S}_N(f,t)$) which can not be properly placed in context here.

Definition 122 *If $f : \mathbb{T} \to \mathbb{C}$ is an integrable*[9] *function we define*

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\exp(-int)\,dt.$$

We set

$$S_N(f,t) = \sum_{n=-N}^{N} \hat{f}(n)\exp(int)$$

$$\tilde{S}_N(f,t) = -i\sum_{n=-N}^{N} \operatorname{sgn}(n)\hat{f}(n)\exp(int)$$

$$\sigma_N(f,t) = (N+1)^{-1}\sum_{n=0}^{N} S_n(f,t)$$

Recall that

$$f * g(x) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(x-t)\,dt.$$

Lemma 123 *We have*

$$S_N(f) = D_N * f,\ \tilde{S}_N(f) = \tilde{D}_N * f, \sigma_N(f) = K_N * f,$$

with

$$D_N(t) = \sum_{n=-N}^{N} \exp int = \frac{\sin(N+\frac{1}{2})t}{\frac{t}{2}}$$

$$\tilde{D}_N(t) = 2\sum_{n=1}^{N} \sin nt = \frac{\cos\frac{1}{2}t - \cos(N+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

$$K_N(t) = \frac{1}{N+1}\left(\frac{\sin\frac{N+1}{2}t}{\sin\frac{1}{2}t}\right)^2.$$

[9] Use whichever integral you are happiest with.

(Formally speaking, we have not defined $D_N(t)$, $K_N(t)$ and $\tilde{D}_N(t)$ when $t = 0$. By inspection $D_N(0) = 2N + 1$, $K_N(0) = N + 1$, $\tilde{D}_N(0) = 0$.)

By looking at the properties of the *kernels* $K_N(t)$ and $\tilde{D}_N(t)$ we obtain results about the associated sums.

Lemma 124 *We have*

$$\begin{aligned} K_N(t) &> 0 && \text{for all } t \\ K_N(t) &\to 0 && \text{uniformly for } |t| \geq \delta \text{ whenever } \delta > 0 \\ \frac{1}{2\pi}\int_{\mathbb{T}} K_N(t)\,dt &= 1. \end{aligned}$$

Theorem 125 (Féjer) *If $f : \mathbb{T} \to \mathbb{C}$ is integrable and f is continuous at x then*

$$\sigma_N(f, x) \to f(x) \text{ as } N \to \infty.$$

We shall only use the following simple consequence.

Lemma 126 *If $f : \mathbb{T} \to \mathbb{R}$ is integrable but $S_N(f, x) \to \infty$ as $N \to \infty$ then f can not be continuous at x.*

Exercise 127 *If $f : \mathbb{T} \to \mathbb{R}$ is integrable and there exist $\delta > 0$ and $M > 0$ such that $|f(t)| \leq M$ for all $|t| < \delta$ show that it is not possible to have $S_N(f, 0) \to \infty$ as $N \to \infty$.*

Lemma 128 *(i) If $f : \mathbb{T} \to \mathbb{C}$ is integrable and f is continuous at x then*

$$\frac{\tilde{S}_N(f, x)}{\log N} \to 0$$

as $N \to 0$.

(ii) If $h(t) = \operatorname{sgn}(t) - t/\pi$ then there is a non-zero constant L such that

$$\frac{\tilde{S}_N(h, 0)}{\log N} \to L$$

as $N \to \infty$.

(iii) If $f : \mathbb{T} \to \mathbb{C}$ is integrable and $f(x + \eta) \to f(x+)$, $f(x - \eta) \to f(x-)$ as $\eta \to 0$ through positive values then

$$\frac{\tilde{S}_N(f, x)}{\log N} \to \frac{L(f(x+) - f(x-))}{2}$$

as $N \to 0$.

We now come to the object of our Fourier analysis.

Lemma 129 *If $f : \mathbb{T} \to \mathbb{C}$ is integrable with $\hat{f}(n) = 0$ for $n < 0$. If $f(x+\eta) \to f(x+)$, $f(x-\eta) \to f(x-)$ as $\eta \to 0$ through positive values then $f(x+) = f(x-)$.*

In other words, power series cannot have 'discontinuities of the first kind'.

Exercise 130 *Give an example of a discontinuous function with no discontinuities of the first kind.*

Once Lemma 129 has been got out of the way we can return to the proof of Theorem 121 on the boundary behaviour of the Riemann mapping. The proof turns out to be long but reasonably clear. We start with a very general result.

Lemma 131 *If Ω is a simply connected open set in $\mathbb{C}$ and $f : D(0,1) \to \Omega$ is a bijective bicontinuous map then given any compact subset K of Ω we can find an $1 > r_K > 0$ such that, whenever $1 > |z| > r_K$, $f(z) \notin K$.*

Any bounded open set Ω has an area $|\Omega|$ and a simple application of the Cauchy-Riemann equations yields the following result.

Lemma 132 *Suppose that Ω is a simply connected bounded open set in $\mathbb{C}$ and $f : D(0,1) \to \Omega$ is a bijective analytic map. Then*

$$|\Omega| = \int_{0 \le r < 1} \int_0^{2\pi} |f'(re^{i\theta})|^2 r \, d\theta \, dr.$$

Lemma 133 *Suppose that Ω is a simply connected bounded open set in $\mathbb{C}$ and $f : D(0,1) \to \Omega$ is a bijective analytic map. The set X of $\theta[0, 2\pi)$ such that $f(re^{i\theta})$ tends to a limit as $r \to 1$ from below has complement of Lebesgue measure 0.*

From now on until the end of the section we operate under the standing hypothesis that Ω is a simply connected open set in $\mathbb{C}$ with boundary a Jordan curve. This means that Ω is bounded (we shall accept this as a topological fact). We take X as in Lemma 133 and write $f(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$ whenever $\theta \in X$. We shall assume (as we may without loss of generality) that $0 \in X$.

Lemma 134 *Under our standing hypotheses we can find a continuous bijective map $g : \mathbb{T} \to \mathbb{C}$ such that $g(0) = f(1)$ and such that, if $x_1, x_2 \in X$ with $0 \le x_1 \le x_2 < 2\pi$ and t_1, t_2 satisfy $g(t_1) = x_1$, $g(t_2) = x_2$ and $0 \le t_1, t_2 < 2\pi$ then $t_1 \le t_2$.*

(The reader will, I hope, either excuse or correct the slight abuse of notation.)

We now need a simple lemma.

Lemma 135 *Suppose $G : D(0,1) \to \mathbb{C}$ is a bounded analytic function such that $G(re^{i\theta}) \to 0$ as $r \to 1-$ for all $|\theta| < \delta$ and some $\delta > 0$. Then $G = 0$.*

Using this we can strengthen Lemma 134

Lemma 136 *Under our standing hypotheses we can find a continuous bijective map $\gamma : \mathbb{T} \to \mathbb{C}$ such that $\gamma(0) = f(1)$ and such that, if x_1, $x_2 \in X$ with $0 \leq x_1 < x_2 < 2\pi$ and t_1, t_2 satisfy $g(t_1) = x_1$, $g(t_2) = x_2$ and $0 \leq t_1, t_2 < 2\pi$ then $t_1 < t_2$.*

From now on we add to our standing hypotheses the condition that γ satisfies the conclusions of Lemma 136.

We now 'fill in the gaps'.

Lemma 137 *We can find a strictly increasing function $w : [0, 2\pi] \to [0, 2\pi]$ with $w(0) = 0$ and $w(2\pi) = 2\pi$, such that $\gamma(w(\theta)) = f(e^{i\theta})$ for all $\theta \in X$.*

We now set $f(e^{i\theta}) = \gamma(w(\theta))$ and $F(\theta) = f(e^{i\theta})$ for all θ. A simple use of dominated convergence gives us the next lemma.

Lemma 138 *If $f(z) = \sum_{n=1}^{\infty} c_n z^n$ for $|z| < 1$ then, we have $\hat{F}(n) = c_n$ for $n \geq 0$ and $\hat{\gamma}(n) = 0$ for $n < 0$.*

However increasing functions can only have discontinuities of the first kind. Thus w and so F can only have discontinuities of the first kind. But, using our investment in Fourier analysis (Lemma 129) we see that F can have no discontinuities of the first kind..

Lemma 139 *The function $F : \mathbb{T} \to \mathbb{C}$ is continuous.*

Using the density of X in $\mathbb{T}$ we have the required result.

Lemma 140 *The function $f : \overline{D(0,1)} \to \overline{\Omega}$ is continuous and bijective.*

This completes the proof of Theorem121.

Using a little analytic topology we may restate Theorem121 as follows. is very satisfactory.

Theorem 141 *If Ω is a simply connected open set in $\mathbb{C}$ with boundary a Jordan curve then any bijective analytic map $f : D(0,1) \to \Omega$ can be extended to a bijective continuous map from $\overline{D(0,1)} \to \overline{\Omega}$. The map $f^{-1}\overline{\Omega} \to \overline{D(0,1)}$ is continuous on $\overline{\Omega}$.*

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14 Picard's little theorem

The object of this section is to prove the following remarkable result.

Theorem 142 (Picard's little theorem) *If $f : \mathbb{C} \to \mathbb{C}$ is analytic then $\mathbb{C} \setminus f(\mathbb{C})$ contains at most one point.*

The example of exp shows that $\mathbb{C} \setminus f(\mathbb{C})$ may contain one point.

The key to Picard's theorem is the following result.

Theorem 143 *There exists an analytic map $\lambda : D(0,1) \to \mathbb{C} \setminus \{0,1\}$ with the property that given $z_0 \in \mathbb{C} \setminus \{0,1\}$, $w_0 \in D(0,1)$ and $\delta > 0$ such that $\lambda(w_0) = z_0$ and $D(z_0, \delta) \subseteq \mathbb{C} \setminus \{0,1\}$ we can find an analytic function $g : D(z_0, \delta) \to D(0,1)$ such that $\lambda(g(z)) = z$ for all $z \in D(z_0, \delta)$.*

We combine this with a result whose proof differs hardly at all from that of Theorem 7.

Lemma 144 *Suppose that U and V are open sets and that $\tau : U \to V$ is a analytic map with the following property. Given $u_0 \in U$ and $v_0 \in V$ such that $\tau(u_0) = v_0$ then, given any $\delta > 0$ with $D(v_0, \delta) \subseteq V$, we can find an analytic function $g : D(v_0, \delta) \to U$ such that $\lambda(g(z)) = z$ for all $z \in D(v_0, \delta)$. Then if W is an open simply connected set and $f : W \to U$ is analytic we can find an analytic function $F : W \to U$ such that $\tau(F(z)) = f(z)$ for all $zinW$.*

(The key words here are 'lifting' and 'monodromy'. It is at points like this that the resolutely 'practical' nature of the presentation shows its weaknesses. A little more theory about analytic continuation for its own sake would turn a 'technique' into a theorem.)

In the case that we require, Lemma 144 gives the following result.

Lemma 145 *If λ is as in Theorem 143 and $f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\}$ is analytic we can find $F : \mathbb{C} \to D(0,1)$ such that $\lambda(F(z)) = f(z)$.*

Picard's little theorem follows on considering Louiville's theorem that a bounded analytic function on $\mathbb{C}$ is constant. The proof of Picard's theorem thus reduces to the construction of the function λ of Theorem 143. We make use ideas concerning reflection which I assume the reader has already met.

Definition 146 *(i) Let $\mathbf{p}$ and $\mathbf{q}$ are orthonormal vectors in $\mathbb{R}^2$. If $\mathbf{a}$ is a vector in $\mathbb{R}^2$ and $x, y \in \mathbb{R}$ the reflection of $\mathbf{a} + x\mathbf{p} + y\mathbf{q}$ in the line through $\mathbf{a}$ parallel to $\mathbf{p}$ is $\mathbf{a} + x\mathbf{p} - y\mathbf{q}$.*

(ii) If $\mathbf{a}$ and are vectors in $\mathbb{R}^2$ and $R, r > 0$ then the reflection of $\mathbf{a} + r\mathbf{b}$ in the circle centre $\mathbf{a}$ and radius R is $\mathbf{a} + r^{-1}R^2\mathbf{b}$.

Lemma 147 (Schwarz reflection principle) *Let Σ_1 and Σ_2 be two circles (or straight lines). Suppose G is an open set which is taken to itself by reflection in Σ_1. Write G_+ for that part of G on one side*[10] *of Σ_1 and $G_0 = G \cap \Sigma_1$. If $f : G_+ \cup G_0$ is a continuous function, analytic on G_+ with $f(G_0) \subseteq \Sigma_2$ then we can find an analytic function $\tilde{f} : G \to \mathbb{C}$ with $\tilde{f}(z) = f(z)$ for all $z \in G_+ \cup G_0$. If $f(G_+)$ lies on one side of Σ_2 then we can ensure that $\tilde{f}(G_-)$ lies on the other.*

We first prove the result when Σ_1 and Σ_2 are the real axis and then use Möbius transforms to get the full result.

We now use the work of section 13 on boundary behaviour.

Lemma 148 *Let $\mathcal{H}$ be the upper half plane $\{z : \Im z > 0$ and V the region bounded by the lines $C_1 = \{iy : y \geq 0\}$, $C_3 = \{1 + iy : y \geq 0\}$, and the arc $C_2 = \{z : |z - \frac{1}{2}| = \frac{1}{2},\ \Im z \geq 0\}$ and containing the point $\frac{1}{2} + i$. There is a continuous bijective map $f : \overline{V} \to \overline{\mathcal{H}}$ which is analytic on V, takes 0 to 0, 1 to 1, C_1 to $\{x : x \leq 0\}$, C_2 to $\{x : 0 \leq x \leq 1\}$, and C_3 to $\{x : x \geq 1\}$.*

By repeated use of the Schwarz reflection principle we continue f analytically to the whole of $\mathcal{H}$.

Lemma 149 *Let $\mathcal{H}$ be the upper half plane. There exists an analytic map $\tau : \mathcal{H} \to \mathbb{C} \setminus \{0, 1\}$ with the property that given $z_0 \in \mathbb{C} \setminus \{0, 1\}$ and $w_0 \in \mathcal{H}$ such that $\tau(w_0) = z_0$ we can find $\delta > 0$ with $D(z_0, \delta) \subseteq \mathbb{C} \setminus \{0, 1\}$ and an analytic function $g : D(z_0, \delta) \to \mathcal{H}$ such that $\tau(g(z)) = z$ for all $z \in D(z_0, \delta)$.*

Since $\mathcal{H}$ can be mapped conformally to $D(0, 1)$ Theorem 143 follows at once and we have proved Picard's little theorem.

15 References and further reading

There exist many good books on advanced classical complex variable theory which cover what is in this course and much more. I particularly like [9] and [3]. For those who wish to study from the masters there are Hille's two volumes [4] and the elegant text of Nevanlinna [7]. There is an excellent treatment of Dirichlet's theorem and much more in Davenport's *Multiplicative Number Theory* [2] [The changes between the first and second editions are substantial but do not affect that part which deals with material in this course.] If you wish to know more about the Riemann zeta-function you can start with [8]. In preparing this course I have also used [5] and [6] since I find the author sympathetic.

[10] There are no topological difficulties here. The two sides of $|z - a| = r$ are $\{z : |z - a| < r\}$ and $\{z : |z - a| > r\}$.

References

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