Baire Category, Probabilistic Constructions and Convolution Squares

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Contents

1 Introduction

In the past few years I have written a number of papers using simple Baire category and probabilistic results. The object of this course is to give examples of the main theorems and the methods used to obtain them.

Although we shall obtain other results, our main concern will be the question. Knowing something about the measure μ, what can we say about its convolution with itself (that is to say, the convolution square) $\mu * \mu$? This question goes back at least as far as the paper of Wiener and Wintner [28] in which they show that the convolution square of a singular measure need not be singular.

Those who already know about such things may find it useful to see some of our main results. Those who do not, should be reassured that we will provide appropriate definitions and background in due course. We give a new proof of the following theorem of Besicovitch.

Theorem 5.4. *There exists a closed bounded set of Lebesgue measure containing lines of length at least* 1 *in every direction.*

We prove a quantitive verision of a theorem of Rudin.

Theorem 7.3. *Suppose that* $\phi : \mathbb{N} \to \mathbb{R}$ *is a sequence of strictly positive numbers with* $r^\alpha \phi(r) \to \infty$ *as* $r \to \infty$ *whenever* $\alpha > 0$. *Then there exists a probability measure* μ *such that* $\phi(|r|) \geq |\hat{\mu}(r)|$ *for all* $r \neq 0$, *but* $\operatorname{supp} \mu$ *is independent.*

We prove an extension (found independently by Matheron and Zelený) of a celebrated theorem of Debs and Saint Reymond.

Theorem 12.5. *Let* B *be a set of first category in* $\mathbb{T}$. *Then we can find a probability measure* μ *with* $\hat{\mu}(r) \to 0$ *as* $|r| \to \infty$ *such that* $\operatorname{supp} \mu$ *is* *independent and the subgroup G of $\mathbb{T}$ generated by* $\operatorname{supp}\mu$ *satisfies*

$$G \cap B \subseteq \{0\}.$$

We produce two substantial extensions of the theorem of Wiener and Wintner. The first is related to the theorem of Debs and Saint Reymond.

Theorem 15.1. *Let A be a set of first category in $\mathbb{T}$. Then we can find a probability measure μ such that* $\operatorname{supp}\mu \cap A = \varnothing$ *but $d(\mu * \mu)t = f(t)\,dt$ where f is a Lebesgue L^1 function.*

The second further extends a result of Saeki.

Theorem 17.4. *If $1 > \alpha > 1/2$, then there exists a probability measure μ such that the Hausdorff dimension of the support of μ is α and $d(\mu * \mu)(t) = f(t)\,dt$ where f is Lipschitz $\alpha - \frac{1}{2}$.*

We conclude with a result on Hausdorff dimension.

Theorem 20.1. *Given a sequence α_j with $0 \leq \alpha_j \leq \alpha_{j+1} < 1$, we can find a closed set E such that*

$$E_{[j]} = \underbrace{E + E + \ldots + E}_{j}$$

has Hausdorff dimension α_j for each $j \geq 1$.

Any course like this is is bound to be rather uneven in difficulty and I have chosen to accentuate this unevenness by spending a substantial amount of time discussing well known and relatively easy results. The beginner should concentrate on these discussions, leaving the technical (and, I am afraid still imperfectly digested) details of my own arguments to a hypothetical interested expert.

2 Baire's theorem

The study of complete metric spaces enables us to replace certain recurring arguments by general principles. One of the most important of the principles is given by Baire's category theorem.

Theorem 2.1. [Baire's category theorem] *Let (X, d) be a non-empty complete metric space. If E_1, E_2, … are closed subsets of X with dense complements, then $\bigcup_{j=1}^{\infty} E_j$ is non-empty.*

Baire's theorem may be restated as follows.

Theorem 2.2. *Let (X,d) be a non-empty complete metric space. Suppose that P_j is a property such that:-*

(i) The property of being P_j is stable *in the sense that, given $x \in X$ which has property P_j, we can find an $\epsilon > 0$ such that whenever $d(x,y) < \epsilon$ the point y has the property P_j.*

(ii) The property of not being P_j is unstable *in the sense that, given $x \in X$ and $\epsilon > 0$, we can find a $y \in X$ with $d(x,y) < \epsilon$ which does not have the property P_j.*

Then there is an $x_0 \in X$ which has all of the of the properties P_1, P_2, ...

Proof of equivalence of Theorems 2.1 and 2.2. Let x have the property P_j if and only if $x \notin E_j$. ■

It is unlikely that anyone who reads these notes is unfamiliar with Theorem 2.1 or its proof, but I include a proof for completeness. We shall prove a slightly stronger version of Baire's theorem.

Theorem 2.3. *Let (X,d) be a complete metric space. If E_1, E_2, ... are closed sets with empty interiors, then $X \setminus \bigcup_{j=1}^{\infty} E_j$ is dense in X.*

Proof. Suppose that $x_0 \in X$ and $\delta_0 > 0$. We shall show that there exists a $y \in B(x_0, \delta_0)$ such that $y \notin \bigcup_{j=1}^{\infty} E_j$.

To this end, we perform the following inductive construction. Given $x_{n-1} \in X$ and $\delta_n > 0$, we can find $x_n \in X$ such that $x_n \in B(x_{n-1}, \delta_n/4)$, but $x_n \notin E_n$. (For, if not, we would have $B(x_{n-1}, \delta_{n-1}/4) \subseteq E_n$ and E_n would have a non-empty interior.) Since E_n is closed and $x_n \notin E_n$ we can now find a with $\delta_{n-1}/2 > \delta_n > 0$ such that $B(x_n, \delta_n) \bigcap E_n = \varnothing$.

Now observe that

$$\delta_m \leq 2^{-1}\delta_{m-1} \leq 2^{-2}\delta_{m-2} \leq \dots 2^{n-m}\delta_n$$

for all $m \geq n \geq 0$. It follows that, if $r \geq s$

$$d(x_r, x_s) \leq \sum_{j=r}^{s-1} d(x_{j+1}, x_j) \leq \sum_{j=r}^{s-1} \delta_r/4 \leq 4^{-1}\delta_0 \sum_{j=r}^{s-1} 2^{-r} \leq 2^{-r-1}\delta_0.$$

Thus the x_r form a Cauchy sequence and converges in (X,d) to some point y.

The same kind of calculation as in the last paragraph gives

$$d(x_r, x_s) \leq \sum_{j=r}^{s-1} d(x_{j+1}, x_j) \leq \sum_{j=r}^{s-1} \delta_r/4 \leq 4^{-1}\delta_r \sum_{j=0}^{s-r-1} 2^{-r} \leq \delta_r/2,$$

whenever $s \geq r$ and so

$$d(x_r, y) \leq d(x_r, x_s) + d(x_s, y) \leq \delta_r/2 + d(x_s, y) \to \delta_{r-1}/2$$

as $s \to \infty$. We thus have $d(x_r, y) \leq \delta_r/2$ so $y \in B(x_0, \delta_r)$ and $y \notin E_r$ for each $r \geq 1$ as required. ■

For historical reasons Baire's category theorem is associated with some rather peculiar nomenclature.

Definition 2.4. *Let (X, d) be a metric space. We say that a a subset A of X is* of the first category *if it is a subset of the union of a countable collection of closed sets with empty interior*[1]. *We say that* quasi-all *points of X belong to the complement $X \setminus A$ of X.*

Theorem 2.3 thus states that the complement of a set of the first category in a complete metric space is dense in that space. The next exercise gives a simple but very useful property of sets of first category.

Exercise 2.5. *Show that the countable union of sets of the first category is itself a set of the first category.*

The reader will have met the following theorem before.

Theorem 2.6. $\mathbb{R}$ *is uncountable.*

Proof. If we give $\mathbb{R}$ the usual metric, then point sets $\{x\}$ are closed and have empty interior. It follows that, if E is a countable subset of $\mathbb{R}$, then $E = \bigcup_{e \in E} \{e\}$ is of first category and so $E \neq \mathbb{R}$. ■

The standard undergraduate proof involves decimal expansions but this proof avoids have to talk about the relation between real numbers and decimals. It is also much closer to Cantor's original proof.

Exercise 2.7. *If (X, d) is a metric space we say that a point $x \in X$ is* isolated *if we can find a $\delta > 0$ such that $B(x, \delta) = \{x\}$.*

(i) Show that a point $x \in X$ is isolated if and only if $\{x\}$ is open.

(ii) Show that any complete non-empty metric space without isolated points is uncountable.

(iii) Give an example of a complete infinite metric space which is countable.

(vi) Give an example of a uncountable complete metric space with every point isolated.

[1]Some authors, say that A of X is of the first category if it is the union of a countable collection of closed sets with empty interior. They have history, on their side, but not common usage.

There are several reasons for using the Baire category theorem when we seek examples of particular types of behaviour.

The first is practical. Although any Baire category argument can obviously be replaced by a direct argument, if there are several properties involved, each of which involves countably many conditions the direct argument may require quite a lot of notation and careful interlocking of several inductions. Such arguments are not hard to write (and indeed may give the author some pleasure), but are may be hard to read.

The second argument is that a property which holds quasi-always is, in some sense, generic. The next exercise shows that we must not press this argument too far.

Exercise 2.8. *The following is a well known procedure for constructing 'Cantor sets'. Let $E_0 = [0, 1]$ and let ζ_1, ζ_2, ... be a sequence of real numbers with $0 < \zeta_j < 1$. At the nth stage E_n is the union of 2^n disjoint closed intervals $I(r, n)$ all of the same length. We define E_n to be the union of the 2^{n+1} disjoint closed intervals formed by removing an open interval $J(r, n)$ of length ζ_n times the length of the initial interval $I(r, n)$ from the centre of $I(r, n)$. (Thus if $I(r, n) = [c_{r,n} - \delta_n, c_{r,n} + \delta]$ we take $J(r, n) = (c_{r,n} - \zeta_n\delta_n, c_{r,n} + \zeta_n\delta_n)$ and*

$$E_{n+1} = \bigcup_{r=1}^{2^n}(I(r, n) \setminus J(r, n).$$

(i) Explain why $\zeta = \prod_{n=1}^{\infty} \zeta_n$ is well defined. Show that ζ can take any value subject only to the condition $1 > \zeta \geq 0$.

(ii) Show that $E = \bigcap_{n=1}^{\infty} E_n$ is a closed nowhere dense set without isolated points. Show that E has Lebesgue measure ζ.

(iii) Construct a set $H \subseteq [0, 1]$ of first Baire category but of Lebesgue measure 1. Points in H are 'generic in the the sense of measure theory' (almost all points in $[0, 1]$ lie in H) but the points of $[0, 1] \setminus H$ are 'generic in the the sense of topology' (quasi-all points in $[0, 1]$ lie $[0, 1] \setminus H$).

(iv) Construct a set $P \subseteq \mathbb{R}$ of first Baire category such that $\mathbb{R} \setminus P$ has Lebesgue measure zero.

The third argument is that the act of seeking a Baire type proof may, by itself, suggest a new ways of looking at your problem.

3 The Hausdorff metric

The Hausdorff metric measures the difference between compact sets.

Definition 3.1. *Let (X,d) be a metric space and let $\mathcal{E}$ be the collection of non-empty compact subsets of X. We write*

$$d_{\mathcal{E}}(E,F) = \sup_{e\in E}\inf_{f\in F} d(e,f) + \sup_{e\in E}\inf_{f\in F} d(e,f)$$

for all E, $F \in \mathcal{E}$ and call $d_{\mathcal{E}}$ the Hausdorff metric on $\mathcal{E}$.

The proof that the Hausdorff metric is indeed a metric is easy, but not, I think, trivial. We use the following subsidiary lemma.

Lemma 3.2. *Let (X,d) be a metric space and let $\mathcal{E}$ be the collection of non-empty compact subsets of X. If we write*

$$\Delta(E,F) = \sup_{e\in E}\inf_{f\in F} d(e,f)$$

for E, $F \in \mathcal{E}$ then

$$\Delta(E,G) \leq \Delta(E,F) + \Delta(F,G)$$

for all E, $F\, G \in \mathcal{E}$.

Proof. Write $d(e,F) = \inf_{f\in F} d(e,f)$ for $e \in X$ and $F \in \mathcal{E}$. By the triangle inequality,

$$d(e,g) \leq d(e,f) + d(f,g)$$

so

$$d(e,G) \leq d(e,f) + d(f,g)$$

for all $g \in G$ whence

$$d(e,G) \leq d(e,f) + d(f,G) \leq d(e,f) + \Delta(F,G)$$

for all $f \in F$. Hence

$$d(e,G) \leq d(e,F) + \Delta(F,G)$$

for all $e \in E$ and

$$\Delta(E,G) \leq \Delta(E,F) + \Delta(F,G).$$

■

Exercise 3.3. *Use Lemma 3.2 to show that the Hausdorff metric is indeed a metric.*

Lemma 3.4. *(We use the notation of Definition 3.1.) If (X,d) is complete, then the Hausdorff metric is complete.*

Proof. It is sufficient to show that, if $E_n \in \mathcal{E}$ and $d_{\mathcal{E}}(E_n, E_{n+1}) \leq 2^{-n-1}$ for all $n \geq 1$, then E_n converges in the Hausdorff metric.

To this end, let E be the set of $e \in X$ such that there exist $e_n \in E_n$ with $d(e_n, e)) \to 0$ as $n \to \infty$. We observe that, if $e \in E$ then, given any m, we can find an $n \geq m+1$ such that $d(e, e_n) < 2^{-m}$. Since

$$d_{\mathcal{E}}(E_m, E_n) \leq \sum_{j=m}^{n-1} d_{\mathcal{E}}(E_j, E_{j+1}) \leq \sum_{j=m}^{n-1} 2^{-(j+1)} < 2^{-m}$$

we can find $x_m \in E_m$ such that $d(x_m, e_n) < 2^{-m}$ and so $d(x_m, e) < 2^{-m+1}$. Thus

$$E \supseteq \{x : d(x, x_m) < 2^{-m+1} \text{ for some } x \in E_m.$$

We next show that E is compact. Suppose that $y(j) \in E$ for $j \geq 1$. We construct infinite subsets A_n of $\mathbb{N}$ as follows. Set $A_0 = \mathbb{N}$. If A_{m-1} has been defined we obtain A_m as follows. Since E is covered by open balls $B(x, 2^{-m+1})$ with $x \in E_m$ and E is compact, E is covered by a finite set of such balls and one of those balls $B(x_m, 2^{-m+1})$ must contain an infinite subset of A_m. We observe that $d(x_m, x_{m+1}) < 2^{-m}$ so the x_m converge to some $y \in E$. Choose $n(j) \in A_j$ so that $n(j) \to \infty$. Then $d(y_{n(j)}, y) \to 0$ as $j \to \infty$. Thus E is compact.

The second paragraph of the proof shows that

$$\sup_{f \in E_n} \inf_{e \in E} d(e, f) \leq 2^{-n+1}.$$

If $x_n \in E_n$ then we can find $x_j \in E_j$ such that $d(x_j, x_{j+1}) < 2^{-j+1}$. Since the x_j are Cauchy, they converge to some x. We have $x \in E$ and

$$d(x, x_n) \leq \sum_{j=n}^{\infty} d(x_j, x_{j+1}) \leq 2^{-n+2}$$

so

$$\sup_{e \in E} \inf_{f \in E_n} d(e, f) \leq 2^{-n+2}.$$

Thus $d_{\mathcal{E}}(E_n, E) \to 0$ as $n \to \infty$. ■

Exercise 3.5. *(We use the notation of Definition 3.1.) Show that, if $(\mathcal{E}, d_{\mathcal{E}})$ is complete, then (X, d) is.*

Exercise 3.6. *In these notes, we are not interested in metric spaces in general but in spaces like $[0,1]^n$, $\mathbb{T}^n$ and $\mathbb{R}^n$ with the usual Euclidean metric. We can then give a simpler proof of the completeness of the Hausdorff metric.*

Let us work in $\mathbb{R}^n$ with the usual Euclidean norm. Suppose that $E_n \in \mathcal{E}$ and $d_{\mathcal{E}}(E_n, E_m) \leq 2^{-n}$ for all m, $n \geq 1$. Let

$$K_n = E_n + B(0, \bar{2}^{-n+1}) = \{\mathbf{e} + \mathbf{x} \,:\, \|\mathbf{x}\| \leq 2^{-n+1}\}.$$

Show that $K_n \in \mathcal{E}$ and $K_n \supseteq K_{n+1}$. Setting $E = \bigcap_{n=1}^{\infty} K_n$, show that $E \in \mathcal{E}$ and $d_{\mathcal{E}}(E_n, E) \to 0$ as $n \to \infty$.

As a first exercise let us show that if we work in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the usual metric quasi-all members of $\mathcal{E}$ are perfect (that is to say totally disconnected with no isolated points).

Lemma 3.7. *Let us work in $\mathbb{T}$ with usual metric.*
(i) Quasi-all members of $\mathcal{E}$ have no isolated points.
(ii) Quasi-all members of $\mathcal{E}$ are disconnected.
(iii) Quasi-all members of $\mathcal{E}$ are perfect.

Proof. (i) Let $\mathcal{E}_m$ consist of all those $E \in \mathcal{E}$ such that there exists an $x \in E$ with $E \cap (x - 1/m, x + 1/m) = \{x\}$.

We claim that $\mathcal{E}_m$ is closed. Suppose that $F_n \in \mathcal{E}_m$ and $d_{\mathcal{E}}(F_n, E) \to 0$. We can $x_n \in E$ with $E \cap (x_n - 1/m, x_n + 1/m) = \{x\}$. By compactness there exists an $x \in \mathbb{T}$ and $n(j) \to \infty$ such that $x_{n(j)} \to x$. By extracting a subsequence we may suppose that $x_n \to x$. Automatically $x \in E$. Suppose that $y \in E$ and $y \neq x$. We can find $y_n \in F_n$ with $y_n \to y$. When n is sufficiently large $x_n \neq y_n$ and so $|y_n - x_n| \geq 1/m$. Proceeding to the limit we obtain $|y - x| \geq 1/m$. Thus $E \in \mathcal{E}_m$ and we have shown that $\mathcal{E}_m$ is closed.

To show that $\mathcal{E} \setminus \mathcal{E}_m$ is open let $E \in \mathcal{E}$ and $\epsilon > 0$ be given. If we choose an integer $N \geq \epsilon^{-1} + m + 1$ and set

$$F = E \cup \{r/N \,:\, r \in \mathbb{Z} \text{ and there exists a } y \in E \text{ with } |y - r/N| \leq 1/N\},$$

then $F \in \mathcal{E}_m$ and $d_{\mathcal{E}}(E, F) < \epsilon$. We have shown that $\mathcal{E} \setminus \mathcal{E}_m$ is open. Thus $\bigcap_1^{\infty} E_m$ is of first category in $(\mathcal{E}, d_{\mathcal{E}})$. Since every compact set with an isolated point lies in $\bigcap_1^{\infty} E_m$ we are done.

(ii) We prove the stronger statement that quasi-all members of $\mathcal{E}$ do not intersect $\mathbb{Q}$. If q is rational set

$$\mathcal{E}_q = \{E \in \mathcal{E} \,:\, q \in E\}.$$

It is clear that $\mathcal{E}_q$ is closed. To see that $\mathcal{E} \setminus \mathcal{E}_m$ is open suppose that $E \in \mathcal{E}$ and $1 > \epsilon > 0$ are given. If we set

$$F = \{q + \epsilon/3\} \cup \big(E \setminus (q - \epsilon/3, q + \epsilon/3),$$

then $F \in \mathcal{E}_q$ and $d_{\mathcal{E}}(E, F) < \epsilon$. Thus $\bigcap_1^{\infty} E_q$ is of first category in $(\mathcal{E}, d_{\mathcal{E}})$ and we are done.

(iii) If $\mathcal{A}$ and $\mathcal{B}$ are of firs category so is $\mathcal{A} \cup \mathcal{B}$. ■

Exercise 3.8. *If we work in $\mathbb{R}^n$ with usual metric show that quasi-all members of $\mathcal{E}$ are perfect.*

4 Independence and Kronecker sets

If we do harmonic analysis on the circle $\mathbb{T}$ we find that independence plays an important role.

Definition 4.1. *We say that points $x_1, x_2, \dots, x_n \in \mathbb{T}$ are independent if the equation*

$$\sum_{j=1}^{n} n_j x_j = 0$$

has no non-trivial solution with $n_j \in \mathbb{Z}$.

Lemma 4.2. [Kronecker's lemma] *The points $x_1, x_2, \dots, x_n \in \mathbb{T}$ are independent if and only if the following statement is true.*

Given $y_j \in \mathbb{T}$ and $\epsilon > 0$ we can find $N \in \mathbb{Z}$ with

$$|Nx_j - y_j| < \epsilon$$

for $1 \leq j \leq n$.

The 'only if' part of Kronecker's lemma is immediate. The next exercise gives a proof of a stronger version of the 'if part'.

Exercise 4.3. *Show that the following statements about a point $\mathbf{x} \in \mathbb{T}^n$ are equivalent. (We write* card A *for the number of elements in a finite set A.)*

(A) If I_j is a closed interval in $\mathbb{T}$ of length $|I_j|$ then

$$\frac{1}{M} \operatorname{card} \left\{ 0 \leq m \leq M-1 \, : \, m\mathbf{x} \in \prod_{j=1}^{n} I_j \} \right| \to \prod_{j=1}^{m} |I_j|$$

as $N \to \infty$.

(B) If $f \in C(\mathbb{T}^n)$, then

$$\frac{1}{M} \sum_{m=0}^{N-1} f(m\mathbf{x}) \to \int_{\mathbb{T}^n} f(\mathbf{t}) \, d\mathbf{t}$$

as $N \to \infty$.

(C) If $P \in C(\mathbb{T}^n)$ is a trigonometric polynomial, then

$$\frac{1}{M} \sum_{m=0}^{M-1} P(m\mathbf{x}) \to \int_{\mathbb{T}^n} P(\mathbf{t}) \, d\mathbf{t}$$

as $N \to \infty$.

(D) If

$$\chi_{\mathbf{k}}(\mathbf{t}) = \exp\left(2\pi i \sum_{j=1}^{n} k_j t_j\right)$$

with $\mathbf{k} \in \mathbb{Z}^n$*, then*

$$\frac{1}{M}\sum_{m=0}^{M-1} \chi_{\mathbf{k}}(m\mathbf{x}) \to \int_{\mathbb{T}^n} \chi_{\mathbf{k}}(\mathbf{t})\, d\mathbf{t}$$

as $N \to \infty$.

(E) $x_1, x_2, \ldots, x_n$ *are independent.*

The equivalence of (A) and (E) is Weyl's equidistribution theorem. Show that Kronecker's lemma follows from Weyl's equidistribution theorem.

Our discussion suggests that we investigate two types of compact subsets of $\mathbb{T}$. We write $\chi_n(t) = \exp(2\pi i n t)$.

Definition 4.4. *A compact subset* E *of* $\mathbb{T}$ *is called independent if every finite subset is independent.*

Definition 4.5. *A compact subset* E *of* $\mathbb{T}$ *is called a Kronecker set if given* $f \in C(\mathbb{T})$ *and* $\epsilon > 0$ *we can find* n *such that*

$$|\chi_n(t) - f(t)| < \epsilon \text{ for all } t \in E.$$

Exercise 4.6. *(i) Show that the complement of an independent compact set is dense in* $\mathbb{T}$.

(ii) Show that every Kronecker set is independent. (We shall see that the converse is false.)

From the point of view of harmonic analysis Kronecker sets are very 'thin'. We shall show that quasi-all compact sets are Kronecker. We need the following result.

Exercise 4.7. *(i) Let* $S(\mathbb{T})$ *be the subset of* $C(\mathbb{T})$ *consisting of those* f *such that* $|f(t)| = 1$ *for all* $t \in \mathbb{T}$. *Show that* $S(\mathbb{T})$ *has a countable dense subset. (This is easily proved by all sorts of arguments. The reader should try to find at least three.)*

(ii) Let A *be a countable dense subset of* $S(\mathbb{T})$. *Show that a compact subset of* $\mathbb{T}$ *is Kronecker if and only if given* $f \in A$ *and* $\epsilon > 0$ *we can find* n *such that*

$$|\chi_n(t) - f(t)| < \epsilon \text{ for all } t \in E.$$

Lemma 4.8. *If we work in $\mathbb{T}$ with usual metric then quasi-all members of $\mathcal{E}$ are Kronecker.*

Proof. Let f_1, f_2, ... be a countable dense subset of the set $S(\mathbb{T})$ defined in Exercise 4.7. Let $\mathcal{U}_{n,m}$ be the set of $E \in \mathcal{E}$ such that there exists an N with

$$|f_n(t) - \chi_N(t)| < 1/m \text{ for all } t \in E.$$

We shall show that $\mathcal{U}_{n,m}$ is open and dense in $\mathcal{E}$.

Observe first that, if $E \in \mathcal{U}_{n,m}$ then, by definition, we can find an N with

$$|f_n(t) - \chi_N(t)| < 1/m \text{ for all } t \in E.$$

Since $|f_n - \chi_N|$ is continuous and a continuous function on a compact set attains its bounds we can find an $\eta > 0$ such that

$$|f_n(t) - \chi_N(t)| < 1/m - \eta \text{ for all } t \in E.$$

Since $f_n - \chi_N$ is uniformly continuous on $\mathbb{T}$ we can find a $\delta > 0$ such that

$$\big|\big(f_n(t) - \chi_N(t)\big) - \big(f_n(s) - \chi_N(s)\big)\big| < \eta \text{ whenever } |s - t| < \eta.$$

It follows that, if $F \in \mathcal{E}$ and $d_{\mathcal{E}}(E, F) < \eta$, then $F \in \mathcal{U}_{n,m}$. Thus $\mathcal{U}_{n,m}$ is open.

Now suppose that we are given $E \in \mathcal{E}$ and $\epsilon > 0$. Set

$$\tilde{E} = E + [-\epsilon/2, \epsilon/2] = \{e + x : e \in E,\ |x| \leq \epsilon/2\}.$$

Then $\tilde{E} \in \mathcal{E}$, $d_{\mathcal{E}}(E, \tilde{E}) \leq \epsilon/2$ and given any $e \in \tilde{E}$ we can find an interval I of length ϵ such that $e \in I \subseteq \tilde{E}$. Now let

$$F = \tilde{E} \cap \{t \in \mathbb{T} : f_n(t) - \chi_M(t) = 0\}.$$

Automatically $F \in \mathcal{U}_{n,m}$ and (since f_n is uniformly continuous) we have $d_{\mathcal{E}}(F, \tilde{E}) < \epsilon/2$ and so $d_{\mathcal{E}}(F, E) < \epsilon$, provided only that M is large enough. Thus $U_{n,m}$ is dense.

Since every element of $\bigcup_{n,m \geq 1}(\mathcal{E} \setminus \mathcal{U}_{n,m})$ is Kronecker we have shown that quasi-all compact sets are Kronecker. ∎

It is a useful slogan that quasi-all compact sets are as 'thin' as possible. (Note that a slogan does not have to be true or even meaningful to be useful.) At first sight this seems to rule out the study of 'thick' sets but if we consider only 'thick' sets then we may hope that quasi-all such sets will be be as 'thin' as possible with respect to an appropriate metric.

As an example let us show the existence of two Kronecker sets E_1 and E_2 such that $E_1 + E_2 = \mathbb{T}$. Consider the space $\mathcal{E}^2$ of ordered pairs of compact sets with the product metric

$$d_2\big((E_1, E_2), (F_1, F_2)\big) = d_{\mathcal{E}}(E_1, F_1) + d_{\mathcal{E}}(E_2, F_2).$$

Lemma 4.9. *The collection*

$$\mathcal{G} = \{(E_1, E_2) \in \mathcal{E} \,:\, E_1 + E_2 = \mathbb{T}\}$$

is a non-empty closed subset of $\mathcal{F}^2$ and so $(\mathcal{G}, d_2)$ is a complete metric space.

Proof. To see that $\mathcal{G}$ is non-empty, consider $(\mathbb{T}, \mathbb{T})$. To see that $\mathcal{G}$ is closed, we argue as follows. Suppose that $(F_1, F_2) \in \mathcal{E}^2$, $\big(E_1(n), E_2(n)\big) \in \mathcal{G}$ and

$$\big(E_1(n), E_2(n)\big) \underset{d_2}{\to} (F_1, F_2)$$

as $n \to \infty$. If $y \in \mathbb{T}$, then we can find $(x_{1,n}, x_{2,n}) \in E_1(n) \times E_2(n)$ such that $x_{1,n} + x_{2,n} = y$. By the compactness of $\mathbb{T}$, we can find $n(j) \to \infty$ and $(x_1, x_2) \in F_1 \times F_2$ such that $x_{1,n(j)} \to x_1$ and $x_{2,n(j)} \to x_2$ as $j \to \infty$. Automatically $(x_1, x_2) \in F_1 \times F_2$ and $x_1 + x_2 = y$. Thus $F_1 + F_2 = \mathbb{T}$ and we are done. ■

Theorem 4.10. *Let us work in the complete metric space $(\mathcal{G}, d_2)$ defined in Lemma 4.9. Quasi-all elements of $(\mathcal{G}, d_2)$ are pairs of Kronecker sets.*

Proof. Let f_1, f_2, … be a countable dense subset of the set $S(\mathbb{T})$ defined in Exercise 4.7. Let $\mathcal{U}_{q,n,m}$ be the set of $(E_1, E_2) \in \mathcal{G}$ such that there exists an N with

$$|f_n(t) - \chi_N(t)| < 1/m \text{ for all } t \in E_q$$

with $q = 1$, 2 and n, $m \geq 1$.

The proof that $\mathcal{U}_{q,n,m}$ is open in $\mathcal{G}$ follows the similar proof in Lemma 4.8. We now show that $\mathcal{U}_{q,n,m}$ is dense.

By symmetry it suffices to look at $\mathcal{U}_{1,n,m}$. Suppose, therefore, that we are given $(E_1, E_2) \in \mathcal{G}$ and $\epsilon > 0$. Set

$$\tilde{E}_j = E_j + [-\epsilon/4, \epsilon/4] = \{e + x \,:\, e \in E_j,\ |x| \leq \epsilon/4\}$$

Then $(\tilde{E}_1, \tilde{E}_2) \in \mathcal{G}$ and the following results hold.

(i) $d_2\big((E_1, E_2), (\tilde{E}_1, \tilde{E}_2)\big) \leq \epsilon/2$.

(ii) Each $e \in \tilde{E}_1$ belongs to some closed interval I of length at least $\epsilon/4$ lying entirely within $\tilde{E}_1$.

(iii) If F is a compact subset of $\mathbb{T}$ with Hausdorff distance $d(F, E_1) \leq \epsilon/4$, then $F + \tilde{E}_2 = \mathbb{T}$.

By the uniform continuity of f_j and the intermediate value theorem, any sufficiently large M will have the property that the equation $\chi_M(t) = f_j(t)$ has at least one solution in any closed interval I of length $\epsilon/4$. Choosing such an M and setting

$$F_1 = \{t \in \tilde{E}_1 : \chi_M(t) = f_j(t)\},\ F_2 = \tilde{E}_2,$$

we see, using (iii), that $(F_1, F_2) \in \mathcal{U}_{1,n,m}$ and $d_2\big((\tilde{E}_1, \tilde{E}_2), (F_1, F_2)\big) \leq \epsilon/4$ so that $d_2\big((E_1, E_2), (F_1, F_2)\big) \leq 3\epsilon/4$. The rest of the proof runs on standard lines. ■

5 Besicovitch Sets

A Besicovitch set is a compact subset E of $\mathbb{R}^2$ of Lebesgue measure zero containing line segments of length 1 in every direction. (Formally, if $\mathbf{u}$ is a unit vector, there exists an $\mathbf{x}$ such that $\mathbf{x} + \lambda\mathbf{u} \in E$ for all $0 \leq \lambda \leq 1$.) The first example of such a set was given by Besicovitch [2] and several constructions appear in the literature. (The construction we give here was inspired by the one given in [10].)

Clearly we can construct Besicovitch sets from compact sets of Lebesgue measure zero containing line segments of length 1 in each direction making an angle less than $\pi/4$ with some fixed direction. We shall show the existence of such sets by a category argument.

In what follows $\mathcal{E}$ will be the collection of compact subsets of $\mathbb{R}^2$ and $d_{\mathcal{E}}$ the usual Hausdorff metric on $\mathcal{E}$.

Definition 5.1. *We take $\mathcal{P}$ to be the collection of all closed subsets P of the rectangle $[-2, 2] \times [0, 1]$ with the following properties*

(i) P is the union of line segments joining points of the form $(x_1, 0)$ to points of the form $(x_2, 1)$ with x_1, $x_2 \in [-2, 2]$.

(ii) If $|v| \leq 1$, then we can find x_1, $x_2 \in [-2, 2]$ with $x_2 - x_1 = v$ and such that the line segment joining $(x_1, 0)$ to $(x_2, 1)$ lies in P.

Lemma 5.2. *$\mathcal{P}$ is a non-empty closed subset of $(\mathcal{E}, d_{\mathcal{E}})$ and so $(\mathcal{P}, d_{\mathcal{E}})$ is complete and non-empty.*

Proof. Suppose $P_n \in \mathcal{P}$, $E \in \mathcal{E}$ and $d_{\mathcal{E}}(P_n, E) \to 0$. We first show that E satisfies property (i) in Definition 5.1. To this end, suppose that $\mathbf{k} \in E$. By definition, we can find $\mathbf{p}_n \in P_n$ with $\|\mathbf{p}_n - \mathbf{k}\| \to 0$ as $n \to \infty$. Since P_n has property (i), we can find $x_{1,n}$, $x_{2,n} \in [-2, 2]$ such that the line segment l_n joining $(x_{1,n}, 0)$ to $(x_{2,n}, 1)$ contains $\mathbf{p}_n$. By the compactness of $[-2, 2]^2$, we can find a integer sequence $n(j) \to \infty$ and x_1, $x_2 \in [-1, 1]$ such that $x_{1,n(j)} \to x_1$ and $x_{2,n(j)} \to x_2$ as $j \to \infty$. If we denote the line segment joining $(x_1, 0)$ to $(x_2, 0)$ by l, then $d_{\mathcal{E}}(l_{n(j)}, l) \to 0$ as $j \to \infty$. It follows that $l \subseteq E$ and $\mathbf{k} \in l$. We have established that E has property (i).

To see that E has property (ii) choose $|v| \leq 1$. Since P_n has property (ii), we can find $x_{1,n}$, $x_{2,n} \in [-2, 2]$ such that $x_{2,n} - x_{1,n} = v$ and the line segment l_n joining $(x_{1,n}, 0)$ to $(x_{2,n}, 1)$ lies in P. By the compactness of $[-2, 2]^2$, we can find a integer sequence $n(j) \to \infty$ and $x_1, x_2 \in [-1,1]$ such that $x_{1,n(j)} \to x_1$ and $x_{2,n(j)} \to x_2$ as $j \to \infty$. Automatically $x_2 - x_1 = v$ If we denote the line segment joining $(x_1, 0)$ to $(x_2, 0)$ by l, then $d_{\mathcal{E}}(l_{n(j)}, l) \to 0$ as $j \to \infty$. It follows that $l \subseteq E$.

To see that $\mathcal{P}$ is non-empty observe that $[-2,2] \times [0,1] \in \mathcal{E}$. ■

Theorem 5.3. *If we work in the complete metric space $(\mathcal{P}, d_{\mathcal{E}})$, then quasi-all $P \in \mathcal{P}$ have Lebesgue measure zero.*

The path from Theorem refT;Besicovitch is completely standard. Baire's category theorem tells us that if quasi-all sets have a property at least one does so there exists a set $P_0 \in \mathcal{P}$ of Lebesgue measure zero. By part (ii) of Definition 5.2, P_0 contains line segments of length at least 1 in every direction making an angle of absolute value less than or equal to $\pi/4$ with the y axis. If we take the union of of P_0 with a copy of P_0 rotated though $\pi/2$ the result will be a Besicovitch set.

Theorem 5.4. *There exists a closed bounded set of Lebesgue measure containing lines of length at least* 1 *in every direction.*

The key to our proof of Theorem 5.3 is the following lemma.

Lemma 5.5. *If $u \in [0,1]$ and $\epsilon > 0$, write $\mathcal{P}(u, \epsilon)$ for the set of $P \in \mathcal{P}$ with the following property.*

There exists an N and $\kappa > 0$ (both depending on ϵ and u) such that whenever $y \in [0,1] \cap [u - \epsilon, u + \epsilon]$, we can find N intervals of total length less than $100\epsilon - \kappa$ covering the set

$$\{x \in [-1,1] \,:\, (x,y) \in P\}.$$

Then $\mathcal{P}(u, \epsilon)$ is open and dense in $(\mathcal{P}, d_{\mathcal{E}})$.

Proof. It is easy to check that $\mathcal{P}(u, \epsilon)$ is open. Suppose that $P \in \mathcal{P}(u, \epsilon)$. By definition, we can find N and $\kappa > 0$ (both depending on ϵ and u) such that, whenever $y \in [0,1] \cap [u - \epsilon, u + \epsilon]$, we can find N intervals of total length less than $100\epsilon - \kappa$ covering the set

$$\{x \in [-1,1] \,:\, (x,y) \in P\}.$$

If we choose $\eta > 0$ so that $2N\eta < \kappa/2$, then writing $\kappa' = \kappa/2$ we see that, if $P' \in \mathcal{P}$ and $d(P, P') < \eta$, then, whenever $y \in [0,1] \cap [u - \epsilon, u + \epsilon]$, we can find N intervals of total length less than $100\epsilon - \kappa'$ covering the set

$$\{x \in [-1,1] \,:\, (x,y) \in P'\}.$$

(Informally, if we used intervals $[a_r, b_r]$ for the set P, we use interval $[a_r - \eta, b_{r+\eta}]$ for the set P'.) Thus $P' \in \mathcal{P}(u, \epsilon)$.

We need to show that $\mathcal{P}(u, \epsilon)$ is dense.

To this end, let us write $l(x, \theta)$ for the line segment through (x, u) which joins a point on the line $y = 0$ to a point on the line $y = 1$ and which is at angle θ to the y-axis. We start with a bit of technical tidying up. Observe that, if $P \in \mathcal{P}$ and $1 > \eta > 0$, then writing

$$P' = \bigcup \{l(x+\eta, \theta) \,:\, l(x, \theta) \subseteq P \text{ and } x \leq 0\} \\ \cup \bigcup \{l(x-\eta, \theta) \,:\, l(x, \theta) \subseteq P \text{ and } x \geq 0\},$$

we have $P' \in \mathcal{P}$, $d(P, P') \leq \eta$ and $P' \subseteq [-1+\eta, 1-\eta] \times [0, 1]$.

Thus, to show that $\mathcal{P}(u, \epsilon)$ is dense, it suffices to show that, given $\delta > 0$, $\eta > 0$ and $P \in \mathcal{P}$ with $P \subseteq [-1+\eta, 1-\eta] \times [0, 1]$, we can find a $P' \in \mathcal{P}(y, \epsilon)$ with $d(P, P') < \delta$. To this end, note that we can find a $\rho > 0$ such that, writing

$$Q = \bigcup \{l(x, \phi) \,:\, |\phi - \theta| \leq \rho \text{ and } l(x, \theta) \subseteq P\},$$

we have $Q \in \mathcal{P}$ and $d(P, Q) < \delta/2$. We observe that the set of open intervals $(\theta - \rho, \theta + \rho)$ with $l(x, \theta) \subseteq P$ is an open cover of $[-\pi/4, \pi/4]$ (by condition (ii) of Definition 5.1) and so, by compactness, we can find $x_1, x_2, \ldots, x_M$ and $\theta_1, \theta_2, \ldots, \theta_M$ such that $l(x_m, \theta_m) \subseteq P$ for all $1 \leq m \leq M$ and

$$\bigcup_{m=1}^{M} (\theta_m - \rho, \theta_m + \rho) \supseteq [-\pi/4, \pi/4]$$

We can now find ρ_m and ρ'_m such that $\rho \geq \rho_m$, $\rho'_m > 0$ for $1 \leq m \leq M$,

$$\bigcup_{n=1}^{M} (\theta_m - \rho'_m, \theta_m + \rho'_m) \supseteq [-\pi/4, \pi/4] \text{ and } \sum_{m=1}^{M} \rho_m + \rho'_m \leq \pi.$$

Setting

$$Q' = \bigcup_{m=1}^{M} \{l(x_m, \phi) \,:\, \phi \in (\theta_m - \rho'_m, \theta_m + \rho_m)\},$$

we observe that $Q' \subseteq Q$ and $Q' \in \mathcal{P}$.

A simple compactness argument shows that we can find $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\widetilde{M}}$ and $\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_{\widetilde{M}}$ such that $l(\tilde{x}_m, \tilde{\theta}_m) \subseteq P$ for all $1 \leq m \leq \widetilde{M}$ and, writing

$$Q'' = \bigcup_{m=1}^{\widetilde{M}} l(\tilde{x}_m, \tilde{\theta}_m),$$

we have $d_{\mathcal{E}}(P, Q'') \leq \delta/2$. If we now take $P' = Q' \cup Q''$, then $P' \in \mathcal{P}$ and $d_{\mathcal{E}}(P', P) < \delta$.

At this point it may be worth the reader's while to sketch P'. If $1 \geq y + \epsilon$ the set

$$P' \cap \{(x, v) \,:\, -1 \leq x \leq 1,\ v \leq y \leq v + \epsilon\}$$

consists of a finite set of lines and a finite set of triangles with vertices on on the line $y = v$ and bases on the line $y = v + \epsilon$ of total length less than $4\pi\epsilon$ (it is not necessary to make best estimates here). But it is trivial that a triangle of base $K\epsilon$ intersects any line parallel to the base in a segment of length at most $K\epsilon$, so we have shown that $P' \in \mathcal{P}(y, \epsilon)$. ■

Lemma 5.5 gives us a slightly stronger version of. Theorem 5.3.

Theorem 5.6. *If we work in the complete metric space $(\mathcal{P}, d_{\mathcal{E}})$, then quasi-all $P \in \mathcal{P}$ have the property that*

$$\{x \,:\, (x, y) \in E\} \text{ has Lebesgue measure zero}$$

for all x.

Proof. Set $\mathcal{P}_n = \bigcap_{r=0}^{n} \mathcal{P}(r/n, 1/n)$. By the defining property of $\mathcal{P}(r/n, 1/n)$, we know that, if $P \in \mathcal{P}_n$, then

$$\{x \,:\, (x, y) \in P\} \text{ has Lebesgue measure strictly less than } 100/n$$

for all $y \in [0, 1]$. By Lemma 5.5 quasi-all P lie in $\mathcal{P}_n$.

If follows that quasi-all P lie in $\cap_{n=1}^{\infty} \mathcal{P}_n$ and so have the property that

$$\{x \,:\, (x, y) \in E\} \text{ has Lebesgue measure zero}$$

for all x. ■

Fubini's theorem shows that Theorem 5.6 implies Theorem 5.3. The following easy exercise shows why we needed a little care in our proof.

Exercise 5.7. *We use the standard Hausdorff metric associated with $\mathbb{T}$. Show that we can find finite sets F_n such that $d_{\mathcal{E}}(F_n, \mathbb{T}) \to \infty$. What can we say about the Lebesgue measure of F_n and $\mathbb{T}$?*

I do not think the next exercise is very illuminating, but the reader may wish to tackle it for completeness.

Exercise 5.8. *We work in $\mathbb{R}^2$. Let $\mathcal{Q}$ to be the collection of all closed subsets Q of the disc centre $\mathbf{0}$ radius 2 with the following properties.*

(i) Q is the union of line segments of length at least 1 joining points on the boundary of the disc.

(ii) We can find a line segment of the type described in (i) in every direction.

Show that, for an appropriate complete metric space, quasi-all elements of $\mathcal{Q}$ have Lebesgue measure 0.

6 Measures

For the remainder of the lectures we shall be interested in measures, their supports and their Fourier transforms. This section is not intended to be complete, but merely intended to establish notation and to jog the reader's memory. Later, I shall use results on measures which include several not mentioned here

We shall consider the space $M(\mathbb{T})$ of Borel measures μ on $\mathbb{T}$. From our point of view, the two key properties of Borel measures are that, if $\mu \in M(\mathbb{T})$, then $\int_{\mathbb{T}} f\,d\mu$ is defined for all $f \in C(\mathbb{T})$ and, that, if μ, $\tau \in M(\mathbb{T})$ satisfy

$$\int_{\mathbb{T}} f\,d\mu = \int_{\mathbb{T}} f\,d\tau$$

for all $f \in C(\mathbb{T})$, then $\mu = \tau$.

We recall that $M(\mathbb{T})$ has a natural norm

$$\|\mu\| = \sup\left\{\left|\int_{\mathbb{T}} f\,d\mu\right| \,:\, f \in C(\mathbb{T}),\ \|f\|_\infty \leq 1\right\}.$$

Standard theorems tell us that the unit ball for this norm is weakly compact, that is to say, that, given $\mu_n \in M(\mathbb{T})$ with $\|\mu_n\| \leq 1$, we can find $n(j) \to \infty$ and $\mu \in M(\mathbb{T})$ such that

$$\int_{\mathbb{T}} f\,d\mu_{n(j)} \to \int_{\mathbb{T}} f\,d\mu$$

for all $f \in C(\mathbb{T})$.

We say that a measure μ is positive if $\int_{\mathbb{T}} f\,d\mu$ is real and positive for all $f \in C(\mathbb{T})$ with $f(t)$ real and positive for all $t \in \mathbb{T}$. A probability measure is a positive measure of norm 1.

Exercise 6.1. *Show that the set of probability measures is closed under the standard norm. Show that it is weakly compact.*

Every measure μ has a support with the property that it is the smallest compact set E such that

$$\int_{\mathbb{T}} f\,d\mu = 0$$

for all $f \in C(\mathbb{T})$ with $f(t) = 0$ for all $t \in E$.

Recall that we write $\chi_n = \exp(2\pi i t)$. We define the nth Fourier coefficient $\hat{\mu}(n)$ in the natural way by

$$\hat{\mu}(n) = \int_{\mathbb{T}} \chi_{-n}\,d\mu.$$

Exercise 6.2. *By using the fact that the trigonometric polynomials are uniformly dense, or otherwise, prove the following results.*

(i) If μ, $\tau \in M(\mathbb{T})$ and $\hat{\mu}(n) = \hat{\tau}(n)$ for all n, then $\mu = \tau$.

(ii) Suppose μ_j is in the unit ball of $M(\mathbb{T})$ for all $j \geq 1$ and $\mu \in M(\mathbb{T})$. Then $\mu_j \to \mu$ weakly if and only if $\hat{\mu_j}(n) \to \hat{\mu}(n)$ for all n.

Any two measures μ, $\tau \in M(\mathbb{T})$ can be convolved to produce $\mu * \tau \in M(\mathbb{T})$. Whichever definition the reader uses, she should find it easy to deduce the key facts that $\|\mu * \tau\| \leq \|\mu\|\|\tau\|$, $\operatorname{supp}(\mu * \tau) \subseteq \operatorname{supp}\mu + \operatorname{supp}\tau$ and $\widehat{\mu * \tau}(n) = \hat{\mu}(n)\hat{\tau}(n)$. The next exercise gives practice in the kind of ideas we use.

Exercise 6.3. *This exercise gives one way of defining convolution from scratch.*

Recall that δ_a is the Dirac measure defined by

$$\int_{\mathbb{T}} f\,d\delta_a = f(a)$$

for $f \in \mathbb{T}$.

(i) Verify that δ_a is a probability measure. Observe that $\hat{\delta}_a(n) = \chi_n(a)$.

(ii) Let $\lambda_j \in \mathbb{C}$ and suppose $a(1)$, $a(2)$, $\ldots$, $a(n)$ are distinct points of $\mathbb{T}$. If $\mu = \sum_{j=1}^{n} \lambda_j \delta_{a(j)}$, show that

$$\|\mu\| = \sum_{j=1}^{n} |\lambda_j|.$$

State with proof, necessary and sufficient conditions for μ to be a positive measure and for μ to be a probability measure.

(iii) Let $M_F(\mathbb{T})$ be the set of measures of the form given in (ii). By considering

$$\sum_{r=0}^{n-1} \mu\big([r/n, (r+1)/n)\big)\delta_{r/n},$$

or otherwise, show that every $\mu \in M(\mathbb{T})$ is the weak limit of a sequence of $\mu_n \in M_F(\mathbb{T})$ with $d_{\mathcal{E}}(\operatorname{supp}\mu, \operatorname{supp}\mu_n) \to 0$.

(iv) If $\mu = \sum_{j=1}^n \lambda_j \delta_{a(j)}$ and $\tau = \sum_{k=1}^m \mu_k \delta_{b(k)}$ we define

$$\mu * \tau = \sum_{j=1}^{n} \sum_{k=1}^{m} \lambda_j \mu_k \delta_{a(j)+b(k)}.$$

(A very cautious reader will check that different representations of μ and τ give the same $\mu * \tau$.) Show that

$$\|\mu * \tau\| \leq \|\mu\|\|\tau\|, \ \operatorname{supp}(\mu * \tau) \subseteq \operatorname{supp}\mu + \operatorname{supp}\tau \ \textit{and} \ \widehat{\mu * \tau}(n) = \hat{\mu}(n)\hat{\tau}(n).$$

(v) Suppose that μ_m, $\tau_m \in M_F(\mathbb{T})$, μ, $\tau \in M(\mathbb{T})$ and $\mu_m \to \mu$, $\tau_m \to \tau$ weakly. Show that we can find $m(j) \to \infty$ and $\sigma \in M(\mathbb{T})$ such that $\mu_{m(j)} * \tau_{m(j)} \to \sigma$ weakly. Show now that $\mu_m * \tau_m \to \sigma$ weakly. If μ'_m, $\tau'_m \in M_F(\mathbb{T})$, and $\mu'_m \to \mu$, $\tau'_m \to \tau$ weakly, show that $\mu'_m * \tau'_m \to \sigma$ weakly. We can thus define $\sigma = \tau * \sigma$ unambiguously.

(vi) Show that, if μ, $\tau \in M(\mathbb{T})$, then

$$\|\mu * \tau\| \leq \|\mu\|\|\tau\|, \ \operatorname{supp}(\mu * \tau) \subseteq \operatorname{supp}\mu + \operatorname{supp}\tau \ \textit{and} \ \widehat{\mu * \tau}(n) = \hat{\mu}(n)\hat{\tau}(n).$$

Show also that, if μ and τ are positive, so is $\mu * \tau$. By considering $\widehat{\mu * \tau}(0)$, or otherwise, show that, if μ and τ are probability measures, so is $\mu * \tau$.

The central objects of study in these notes are the relations between convolution, supports and Fourier series of measures.

7 A theorem of Rudin

We start with a simple result which establishes a link between the algebraic properties of the support of a measure μ and the speed with which its Fourier coefficients $\hat{\mu}(n)$ can tend to zero as $|n| \to \infty$.

Lemma 7.1. *Suppose that μ is a non-zero measure on $\mathbb{T}$ and q is a positive integer such that we can find an $\alpha > 1/q$ and an $A > 0$ with*

$$|\hat{\mu}(r)| \leq A|r|^{-\alpha}$$

for all $r \neq 0$. Then we can find distinct points $x_1, x_2, \ldots, x_q \in \operatorname{supp}\mu$ and $m_j \in \mathbb{Z}$, not all zero, such that

$$\sum_{j=1}^{q} m_j x_j = 0.$$

Proof. Let $\mu_q = \mu * \mu * \cdots * \mu$, the convolution of μ with itself q times. Then

$$|\hat{\mu}_q(r)| = |\hat{\mu}(r)|^q \leq A^q |r|^{-q\alpha}$$

for all $r \neq 0$. It follows that $\hat{\mu}_q \in l^1$ and so $d\mu_q(t) = f(t)\,dt$ for some continuous function f. Thus (since μ_q is non-zero) $\operatorname{supp}\mu_q$ contains a non-trivial interval and so a non-zero rational number y and so we can find $y_1, y_2, \ldots, y_q \in \operatorname{supp}\mu$ such that

$$\sum_{j=1}^{q} y_j = y.$$

Since we do not know that the y_j are distinct, we can only conclude that there exists a q' with $1 \leq q' \leq q$, distinct points $x_1, x_2, \ldots, x_{q'} \in \operatorname{supp}\mu$ and non-zero $n_j \in \mathbb{Z}$ such that

$$\sum_{j=1}^{q'} n_j x_j = y.$$

If we take $n_j = 0$ for $j > q'$, it now follows that there are distinct points $x_1, x_2, \ldots, x_q \in \operatorname{supp}\mu$ and $n_j \in \mathbb{Z}$, not all zero, such that

$$\sum_{j=1}^{q} n_j x_j = y.$$

The stated result follows if we choose a non-zero $M \in \mathbb{Z}$ such that $My = 0$ and set $m_j = Mn_j$. ■

In [25], Rudin proved the following famous result in the other direction.

Theorem 7.2. *There exists a probability measure μ such that $\hat{\mu}(r) \to 0$ as $|r| \to \infty$, but $\operatorname{supp}\mu$ is independent.*

Our object is to prove the following quantitative version of Rudin's result.

Theorem 7.3. *Suppose that $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence of strictly positive numbers with $r^\alpha \phi(r) \to \infty$ as $r \to \infty$ whenever $\alpha > 0$. Then there exists a probability measure μ such that $\phi(|r|) \geq |\hat{\mu}(r)|$ for all $r \neq 0$, but $\operatorname{supp}\mu$ is independent.*

In view of Lemma 7.1, this result is best possible.

We prove Theorem 7.3 by using a Baire category argument but in order to do this we must first introduce an appropriate metric space.

Lemma 7.4. *Let $\phi : \mathbb{N} \to \mathbb{R}$ be a bounded sequence of strictly positive numbers. The following results hold.*

(i) The space Λ_ϕ of sequences $\mathbf{a} : \mathbb{Z} \to \mathbb{C}$ with $\sup_{r\in\mathbb{Z}} \phi(|r|)^{-1}|a_r|$ finite is a complete normed space under the norm

$$\|\mathbf{a}\|_\phi = \sup_{r\in\mathbb{Z}} \phi(|r|)^{-1}|a_r|.$$

(ii) Consider the space $\mathcal{P}_\phi$ consisting of ordered pairs (E, μ) where E is a compact subset of $\mathbb{T}$ and μ is a probability measure with $\operatorname{supp}\mu \subseteq E$ and $\sup_{r\in\mathbb{Z}} \phi(|r|)^{-1}|\hat{\mu}(r)|$ finite. Then

$$d_\phi\big((E, \mu), (F, \sigma)\big) = d(E, F) + \|\hat{\mu} - \hat{\sigma}\|_\phi$$

is a complete metric on $\mathcal{P}_\phi$.

(iii) If

$$\mathcal{G}_\phi = \{(E, \mu) \in \mathcal{P}_\phi \,:\, \phi(|r|)^{-1}|\hat{\mu}(r)| \to 0 \text{ as } |r| \to \infty\},$$

then $\mathcal{G}_\phi$ is a non-empty closed subset of $\mathcal{P}_\phi$. Thus $(\mathcal{G}_\phi, d_\phi)$ is a complete metric space.

Proof. (i) The standard proof is left to the reader.

(ii) It is easy to check that d_ϕ is a metric on $\mathcal{P}_\phi$. To see that d_ϕ is complete, suppose that (E_n, μ_n) is a Cauchy sequence. Since E_j is a Cauchy sequence in $(\mathcal{E}, d_\mathcal{E})$ we can find an $E \in \mathcal{E}$ such that $d_\mathcal{E}(E_n, E) \to 0$.

Since every sequence of probability measures contains a weakly convergent subsequence we can find a probability measure μ and a sequence $n(j) \to \infty$ such that $\mu_{n(j)} \to \mu$ weakly. Since $\hat{\mu}_{n(j)}(r) \to \hat{\mu}(r)$ for each r and $\hat{\mu}_n(r)$ is a Cauchy sequence in $\mathbb{R}$, we have $\hat{\mu}_j(r) \to \hat{\mu}(r)$ for each r and so $\mu_n \to \mu$ weakly. If $\epsilon > 0$ then

$$\operatorname{supp}\mu_n \subseteq E_n \subseteq E + [-\epsilon, \epsilon]$$

for all n sufficiently large, so $\operatorname{supp}\mu \subseteq E$. Finally, part (i) (or a direct proof) shows that $\hat{\mu} \in \Lambda_\phi$ and $\|\hat{\mu}_n - \hat{\mu}\|_\phi \to 0$. Thus $(E, \mu) \in \mathcal{P}_\phi$ and $d_\phi\big((E_n, \mu_n), (E, \mu)\big) \to 0$ as $n \to \infty$.

(iii) The standard proof is left to the reader. ■

The next exercise may help explain why we defined $\mathcal{G}_\phi$ as we did.

Exercise 7.5. *Let $\phi(n) = 1$ for all n*

(i) Given an example of a sequence $(\mu_n, E_n) \in \mathcal{G}_\phi$ and a $(\mu, E) \in \mathcal{G}_\phi$ such that

$$\operatorname{supp}\mu_n = E_n, \ \text{and } (\mu_n, E_n) \underset{d_\phi}{\to} (\mu, E) \ \text{but } \operatorname{supp}\mu \neq E.$$

(ii) By considering sets of the form

$$\mathcal{F}_{n,r} = \big\{(\mu, E) \in \mathcal{F} \,:\, E \cap [(r-1)/n, r/n] \neq \varnothing,\ \mu\big(E \cap [(r-1)/n, r/n]\big) = 0\big\},$$

or otherwise, show that quasi-all sets (μ, E) in $(\mathcal{G}_\phi, d_\phi)$ have the property that $\operatorname{supp}\mu = E$.

Exercise 7.6. *(i) What can you say about $\mathcal{G}_\phi$ if $\sum_{n=1}^\infty \phi(n)$ converges?*

(ii) For the rest of the question we suppose that $n^2\phi(n) \to \infty$. Let $f : \mathbb{T} \to \mathbb{R}$ be a three times continuously differentiable positive function with $\int_\mathbb{T} f(t)\,dt = 1$ and let μ be the measure defined by $d\mu(t) = f(t)\,dt$. Show that $(\mu, \operatorname{supp}\mu) \in \mathcal{G}_\phi$.

(iii) Repeat Exercise 7.5 for the ϕ of part (ii).

We can now state our Baire category version of Theorem 7.3.

Theorem 7.7. *Suppose that $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence of strictly positive numbers with $r^\alpha\phi(r) \to \infty$ as $r \to \infty$ whenever $\alpha > 0$. Then quasi-all $(\mu, E) \in \mathcal{G}_\phi$ have the property that E is independent.*

We obtain Theorem 7.7 by studying the set $\mathcal{H}(q, p, \mathbf{m})$ defined as follows.

Definition 7.8. *Suppose that ϕ is as in Theorem 7.7, q and p are positive integers and $\mathbf{m} = (m_1, m_2, \ldots, m_q) \in \mathbb{Z}^q$ with*

$$M = \sum_{j=1}^q |m_j| \neq 0.$$

Then the the set $\mathcal{H}(q, p, \mathbf{m})$ consists of those $(E, \mu) \in \mathcal{G}_\phi$ such that $\sum_{j=1}^q m_j x_j \neq 0$ whenever $x_j \in E$ and $|x_i - x_j| \geq 1/p$ for $i \neq j$.

Since the set of finite sequences of integers is countable, Theorem 7.7 follows from the following lemma.

Lemma 7.9. *The set $\mathcal{H}(q, p, \mathbf{m})$ is open and dense in $(\mathcal{G}_\phi, d_\phi)$.*

We split the proof of Lemma 7.9 into two parts. The firs part follows a familiar pattern.

Lemma 7.10. *The set $\mathcal{H}(q, p, \mathbf{m})$ is open.*

Proof. We show that the complement of $\mathcal{H}(q, p, \mathbf{m})$ is closed. Suppose that $(E_r, \mu_r) \notin \mathcal{H}(q, p, \mathbf{m})$ and $(E_r, \mu_r) \to (E, \mu)$. We can find $x_j(r) \in E_r$ such that $|x_i(r) - x_j(r)| \geq 1/p$ for $i \neq j$ and

$$\sum_{j=1}^q m_j x_j(r) = 0.$$

By an appropriate form of the Bolzano–Weierstrass theorem, we can find $x_j \in \mathbb{T}$ and $r(k) \to \infty$ such that $x_j(r(k)) \to x_j$ for each $1 \leq j \leq q$. Automatically, $|x_i - x_j| \geq 1/p$ for $i \neq j$ and

$$\sum_{j=1}^{q} m_j x_j = 0.$$

Since $d_\phi(E_{r(k)}, E) \to 0$ it follows that $x_j \in E$ for $1 \leq j \leq q$ and so $(E, \mu) \notin \mathcal{H}(q, p, \mathbf{m})$ as required. ■

The proof that $\mathcal{H}(q, p, \mathbf{m})$ is dense forms the meat of the proof. We shall use the simple but powerful probabilistic ideas developed in the next section.

8 The poor man's central limit theorem

Every student learns the statement and a few students learn the proof of the central limit theorem.

Theorem 8.1. *If X_1, X_2, ... are independent real valued random variables with mean 0 and variance 1, then*

$$\Pr\left(\frac{X_1 + X_2 + \ldots + X_n}{n^{1/2}} \in [a, b]\right) \to \frac{1}{2\pi} \int_a^b \exp(-t^2/2)\, dt$$

as $n \to \infty$.

However, knowing the statement, or even the proof, of a theorem is not the same as understanding it[2].

Exercise 8.2. *(i) Quickly sketch the graph of* $\exp x$ *that you usually draw.*

(ii) Sketch the graph of $\exp x$ *as x runs from -10 to 10 paying attention to the scales involved.*

(iii) Sketch the graph of $\exp(-x^2/2)$ *as x runs from -10 to 10 paying attention to the scales involved.*

Exercise 8.2 reminds us that, if X is a random variable with a normal distribution mean 0 and variance σ^2, then $\Pr(|X| \geq K\sigma) \to 0$ very rapidly as $K \to \infty$.

[2]The present author knows for certain that he did not understand the central theorem when he was a student. He strongly suspects that he does not understand it now.

Unfortunately, the central limit theorem, in the form given above, is purely a limit theorem and does not enable us to make statements about

$$\Pr\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n^{1/2}}\right|\right) > K)$$

for some specific n. However, we can use an idea which, I believe goes back to Bernstein to obtain a very useful substitute. We develop the idea in one of its simplest forms.

Lemma 8.3. *(i) If X is a real valued random variable with $|X| \leq 1$ and $\mathbb{E}X = 0$, then*

$$\mathbb{E}\exp(tX) \leq \exp(t^2).$$

(ii) If $X_1, X_2, \ldots, X_n$ are independent real valued random variables with $|X_j| \leq 1$ and $\mathbb{E}X_j = 0$, then

$$\mathbb{E}\exp\big(t\sum_{j=1}^{n} X_j\big) \leq \exp(nt^2).$$

(iii) If $X_1, X_2, \ldots, X_n$ are independent real valued random variables with $|X_j| \leq 1$ and $\mathbb{E}X_j = 0$, then

$$\Pr(X_1 + X_2 + \ldots + X_n \geq K) \leq \exp\big(-K^2 n/4\big)$$

and

$$\Pr\big(|X_1 + X_2 + \ldots + X_n| \geq K\big) \leq 2\exp\big(-K^2 n/4\big)$$

for all $K > 0$.

(iii) If $Z_1, Z_2, \ldots, Z_n$ are independent complex valued random variables with $|Z_j| \leq 1$ and $\mathbb{E}Z_j = 0$, then

$$\Pr\big(|Z_1 + Z_2 + \ldots + Z_n| \geq K\big) \leq 4\exp\big(-K^2 n/8)\big)$$

for all $K > 0$.

Proof. (i) We consider two cases. If $|t| \geq 1$, then

$$\mathbb{E}\exp(tX) \leq \mathbb{E}\exp|t| = \exp|t| \leq \exp(t^2).$$

If $|t| \leq 1$, then

$$\begin{aligned}\mathbb{E}\exp(tX) &= \mathbb{E}\sum_{m=0}^{\infty}\frac{(tX)^m}{m!} = \sum_{m=0}^{\infty}\frac{t^m\mathbb{E}X^m}{m!}\\ &= 1 + \sum_{m=2}^{\infty}\frac{t^m\mathbb{E}X^m}{m!} \leq 1 + \sum_{m=2}^{\infty}\frac{|t|^m}{m!}\\ &\leq 1 + t^2\sum_{m=2}^{\infty}\frac{1}{m!} \leq 1 + t^2 \leq \exp(t^2)\end{aligned}$$

and we are done.

(ii) Since independent expectations multiply,

$$\mathbb{E}\exp\big(t\sum_{j=1}^{n} X_j\big) = \mathbb{E}\prod_{j=1}^{n}\exp(tX_j) = \prod_{j=1}^{n}\mathbb{E}\exp(tX_j) \leq \exp(nt^2).$$

(iii) Observe that

$$\Pr(X_1+X_2+\ldots+X_n \geq K)\exp(tK) \leq \mathbb{E}\exp\big(t(X_1+X_2+\ldots+X_n)\big) = \exp(nt^2)$$

and so

$$\Pr(X_1+X_2+\ldots+X_n \geq K) \leq \exp(nt^2 - Kt) = \exp\big(n(t-K/2)^2 - nK^2/4\big).$$

Setting $t = K/2$, we see that

$$\Pr(X_1 + X_2 + \ldots + X_n \geq K) \leq \exp(-nK^2/4).$$

Replacing X_j by $-X_j$ we have

$$\Pr(X_1 + X_2 + \ldots + X_n \leq -K) \leq \exp(-nK^2/4),$$

so, using the last two displayed formula,

$$\Pr\big(|X_1 + X_2 + \ldots + X_n| \geq K\big) \leq 2\exp\big(-K^2 n/4\big).$$

(iii) If we write $Z_j = X_j + iY_j$ with X_j and Y_j real, then X_j and Y_j satisfy the conditions of (ii). Since

$$|\Re z|,\ |\Im z| \leq 2^{-1/2}K \Rightarrow |Z| \leq K$$

we have

$$\begin{aligned}\Pr\left(\left|\sum_{j=1}^{n} Z_j\right| \geq K\right) &\leq \Pr\left(\left|\sum_{j=1}^{n} X_j\right| \geq 2^{-1/2}K\right) + \Pr\left(\left|\sum_{j=1}^{n} Y_j\right| \geq 2^{-1/2}K\right)\\ &\leq 4\exp\big(-K^2 n/8\big).\end{aligned}$$

as stated. ■

The next lemma, which forms the main step in our proof that $\mathcal{H}(q, p, \mathbf{m})$ is dense, gives a good example of how Lemma 8.3 is used.

Lemma 8.4. *Let q be a strictly positive integer and let m_1, m_2, $\dots$, m_q be non-zero integers. Then, provided only that n is large enough, we can find distinct points x_1, x_2, $\dots$, x_n with the following three properties.*

(i) If we write $\mu = n^{-1}\sum_{u=1}^{n}\delta_{x_u}$, we have

$$|\hat{\mu}(r)| \leq 8q^{1/2}n^{-1/2}(\log n)^{1/2}$$

for all $1 \leq |r| \leq n^{4q}$.

(ii) If $j(1)$, $j(2)$, $\dots j(q)$ are distinct integers with $1 \leq j(k) \leq n$, then

$$\left|\sum_{k=1}^{q} m_k x_{j(k)}\right| \geq 8^{-1}n^{-q}.$$

Proof. Consider the independent random variables Y_u where each Y_u is uniformly distributed over $\mathbb{T}$. We look at the random measure

$$\sigma = n^{-1}\sum_{u=1}^{n}\delta_{Y_u}.$$

We note that

$$\hat{\sigma}(r) = n^{-1}\sum_{u=1}^{n}\exp(2\pi i r Y_u).$$

If $r \neq 0$, we see that the $\exp(2\pi i r Y_u)$ are are independent identically distributed complex valued random variables with

$$|\exp(2\pi i r Y_u)| = 1 \text{ and } \mathbb{E}\exp(2\pi i r Y_u) = 0.$$

Thus, by Lemma 8.3 with $K = 8q^{1/2}n^{1/2}(\log n)^{1/2}$,

$$\begin{aligned}\Pr\left(|\hat{\sigma}(r)| \geq 4q^{1/2}n^{-1/2}(\log n)^{1/2}\right) &= \Pr\left(\left|\sum_{u=1}^{n}\exp(2\pi i r Y_u)\right| \geq 8q^{1/2}n^{1/2}(\log n)^{1/2}\right)\\ &\leq 4\exp(-8q\log n) = 4n^{-8q}.\end{aligned}$$

Thus, provided only that n is large enough,

$$\begin{aligned}&\Pr\left(|\hat{\sigma}(r)| \geq 8q^{1/2}n^{-1/2}(\log n)^{1/2} \text{ for some } 1 \leq |r| \leq n^{4q}\right)\\ &\leq \sum_{1\leq|r|\leq n^{4q}}\Pr\left(|\hat{\sigma}(r)| \geq 4q^{1/2}n^{-1/2}(\log n)^{1/2}\right)\\ &\leq (2n^{4q}+1)4n^{-8q} \leq 1/4.\end{aligned}$$

Now suppose that $j(1), j(2), \ldots, j(q)$ are distinct integers with $1 \leq j(k) \leq n$. By symmetry or direct calculation, the random variable

$$\sum_{k=1}^{q} m_k Y_{j(k)}$$

is uniformly distributed and so

$$\Pr\left(\sum_{k=1}^{q} m_k Y_{j(k)} \in [-8^{-1}n^{-q}, 8^{-1}n^{-q}]\right) = 4^{-1}n^{-q}.$$

There are no more than n^q different q-tuples $j(1), j(2), \ldots, j(q)$ of the type discussed, so, by the same kind of argument as we used in the previous paragraph, the probability that

$$\sum_{k=1}^{q} m_k Y_{j(k)} \in [-8^{-1}n^{-q}, 8^{-1}n^{-q}]$$

for any such q-tuple is no more than $1/4$.

Combining the results of our last two paragraphs, we see that, provided n is large enough, the probability that $x_j = Y_j$ will fail to satisfy the conditions of our lemma is at most $1/2$. Since there must be an instance of any event with positive probability, the required result follows. ∎

9 Completion of the construction

The process by which we move from Lemma 8.4 to showing that $\mathcal{H}(q, p, \mathbf{m})$ is dense looks complicated but is not. I suggest the reader concentrates on the ideas rather than the computations.

The next exercise merely serves to establish notation.

Exercise 9.1. *Let $K : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function with the following properties.*

(i′) $K(x) \geq 0$ for all $x \in \mathbb{R}$.

(ii′) $\int_{\mathbb{R}} K(x)\, dx = 1$.

(iii′) $K(x) = 0$ for $|x| \geq 1/4$.

If N is a positive integer and we define $K_N : \mathbb{T} \to \mathbb{R}$ by

$$K_N(t) = \begin{cases} NK(Nt) & \text{if } |t| \leq 1/(4N), \\ 0 & \text{otherwise,} \end{cases}$$

then K_N is an infinitely differentiable function having the following properties.

(i) $K_N(t) \geq 0$ for all $t \in \mathbb{T}$.

(ii) $\int_{\mathbb{T}} K_N(t)\,dt = 1$.

(iii) $K_N(t) = 0$ for $|t| \geq 1/(4N)$.

(iv) $|\hat{K}_N(r)| \leq 1$ for all r.

(v) There exists a constant A, independent of N, such that $|\hat{K}_N(r)| \leq A(N/r)^2$ for all $r \neq 0$.

We now 'spread out' the measure of Lemma 8.4 to obtain the measure used in our construction.

Lemma 9.2. *Suppose that $\psi : \mathbb{N} \to \mathbb{R}$ is a sequence of positive numbers such that $\psi(r) \to \infty$ as $r \to \infty$. Suppose that q is a positive integer, ϵ, $\delta > 0$ and $\mathbf{m} = (m_1, m_2, \ldots, m_q) \in \mathbb{Z}^q$ with $M = \sum_{k=1}^{q} |m_k| \neq 0$.*

Then we can find an infinitely differentiable function $f : \mathbb{T} \to \mathbb{R}$ with the following properties.

(i) $f(t) \geq 0$ for all t.

(ii) $\int_{\mathbb{T}} f(t)\,dt = 1$.

(iii) $|\hat{f}(r)| \leq \epsilon |r|^{-1/(2q)} (\log(1 + |r|))^{1/2} \psi(|r|)$ for all $r \neq 0$.

(iv) If $t_k \in \operatorname{supp} f$ for $1 \leq k \leq q$ and $|t_k - t_l| \geq \delta$ for $1 \leq k < l \leq q$, then

$$\sum_{k=1}^{q} m_k t_k \neq 0.$$

Proof. Provided only that n is large enough, we can find x_u and μ satisfying the conditions of Lemma 8.4.

Now take $N(n) = 4Mn^q$, let $K_{N(n)}$ be defined as in Exercise 9.1 and set $f = \mu * K_{N(n)}$. Conclusions (i) and (ii) are immediate.

Now suppose n sufficiently large that $n^{4q} > N(n)$ and $N(n) \geq \delta^{-1}$. If $t_k \in \operatorname{supp} f$ for $1 \leq k \leq q$ and $|t_k - t_l| \geq \delta$ for $1 \leq k < l \leq q$, then, since

$$\operatorname{supp} f \subseteq \bigcup_{u=1}^{n} [x_u - N(n)^{-1}/4, x_u + N(n)^{-1}/4],$$

it follows that we can find distinct integers $j(1), j(2), \ldots j(q)$ with $1 \leq j(k) \leq n$ such that

$$|x_{j(k)} - t_k| \leq N(n)^{-1}/4$$

for all $1 \leq k \leq q$. Condition (ii) of Lemma 8.4 now tells us that

$$\begin{aligned} \left| \sum_{k=1}^{q} m_k t_k \right| &\geq \left| \sum_{k=1}^{q} m_k x_{j(k)} \right| - \sum_{k=1}^{q} |m_k| |x_{j(k)} - t_k| \\ &\geq 8^{-1} n^{-q} - 4^{-1} M N(n)^{-1} = 16^{-1} n^{-q} > 0 \end{aligned}$$

and condition (iv) follows.

We bound $|\hat{f}(r)|$ using condition (i) of Lemma 8.4, Exercise 9.1 and the trivial bounds $|\hat{f}(r)|, |\hat{\mu}(r)| \leq 1$. If $1 \leq |r| \leq N(n)$, then

$$|\hat{f}(r)| \leq |\hat{\mu}(r)| \leq 4q^{1/2}n^{-1/2}(\log n)^{1/2} \leq C_1 N(n)^{-1/(2q)}(\log N(n))^{1/2}$$

for some constant C_1 independent of n. If $N(n) \leq |r| \leq n^{4q}$, then

$$\begin{aligned}
|\hat{f}(r)| &\leq |\hat{\mu}(r)||\hat{K}_{N(n)}(r)| \leq \big(4q^{1/2}n^{-1/2}(\log n)^{1/2}\big)\big(A(N(n)/r)^2\big) \\
&\leq C_2 N(n)^{-1/(2q)}(\log N(n))^{1/2}(N(n)/r)^2 \\
&= C_2(N(n)/r)^{2-1/(2q)}r^{-(1/2q)}(\log N(n))^{1/2} \\
&\leq C_3|r|^{-1/(2q)}(\log |r|)^{1/2}
\end{aligned}$$

for some constants C_2 and C_3 independent of n. If $|r| \geq n^{4q}$, then

$$|\hat{f}(r)| \leq |\hat{K}_{N(n)}(r)| \leq A(N(n)/r)^2 = A|r|^{-1}(N(n)^2/|r|) \leq C_4|r|^{-1/(2q)}(\log |r|)^{1/2}$$

for some constant C_4 independent of n.

Since $\psi(r) \to \infty$ as $r \to \infty$, it follows that, provided only that n is large enough,

$$|\hat{f}(r)| \leq \epsilon |r|^{-1/(2q)}(\log(1+|r|))^{1/2}\psi(|r|)$$

for all $r \neq 0$ and condition (iv) holds. ■

We make a further observation.

Lemma 9.3. *Given $\epsilon > 0$, we can find an $\eta > 0$ such that, if μ is a probability measure with $|\hat{\mu}(r)| \leq \eta$ for $r \neq 0$, we know that* $\operatorname{supp}\mu$ *intersects every interval of length ϵ.*

Proof. By translation, it suffices to show that $\operatorname{supp}\mu$ intersects $(-\epsilon/2, \epsilon/2)$. Choose an integer N with $N \geq \epsilon^{-1}$. If $\operatorname{supp}\mu$ does not intersect $(-\epsilon/2, \epsilon/2)$, then

$$\begin{aligned}
0 = \left|\int_{\mathbb{T}} K_N(t)\, d\mu(t)\right| &= \left|\sum_{r=-\infty}^{\infty} \hat{K}_N(-r)\hat{\mu}(r)\right| \\
&\geq |\hat{K}_N(0)||\hat{\mu}(0)| - \sum_{r\neq 0}|\hat{K}_N(-r)\hat{\mu}(r)| \geq 1 - 2\eta A_2 N^2 \sum_{r=1}^{\infty} r^{-2}
\end{aligned}$$

which is impossible if η is sufficiently small. ■

Exercise 9.4. *Instead of using Lemma 9.3, we could have added an extra condition to Lemma 8.4. We suppose that we are also given some integer $Q \geq 1$.*

(iii) We have

$$[u/Q, (u+1)/Q] \cap \{x_1, x_2, \ldots, x_n\} \neq \varnothing$$

for all integers u with $0 \leq u \leq Q-1$.

Show how to modify the proof of Lemma 8.4 to add this condition. What condition does this addition enable us to add to Lemma 9.2?

Our next lemma is another 'spreading lemma' but rather simpler.

Lemma 9.5. *Given $(E, \mu) \in \mathcal{G}_\phi$ and $\epsilon > 0$, we can find an $(F, \sigma) \in \mathcal{G}_\phi$ with $d_\phi\big((E, \mu), (F, \sigma)\big) < \epsilon$ having the following properties.*

(i) $d\sigma(x) = g(x)\, dm(x)$, where g is infinitely differentiable and m is Lebesgue measure.

(ii) There exists an $\alpha > 0$ such that, whenever $x \in F$, we can find an interval $I = [y - \alpha, y + \alpha]$ with $x \in I \subseteq F$.

Proof. Choose $u_n : \mathbb{T} \to \mathbb{R}$ a non-negative, infinitely differentiable function, such that $\operatorname{supp} u_n \subseteq [-1/n, 1/n]$ and $\int_{\mathbb{T}} u_n(t)\, dt = 1$. Provided that n is large enough, standard theorems show that $g = u_n * \sigma$, $d\sigma(x) = g(x)\, dm(x)$, and $F = E + [-1/n, 1/n]$ satisfy the conclusions of the lemma. ∎

We also need the following calculation.

Lemma 9.6. *There exists a constant A with the following property. Suppose that $\omega : \mathbb{N} \to \mathbb{R}$ is a sequence of positive numbers with $\omega(0) = 1$,*

$$K^{-1} n^{-1} \leq \omega(n)$$

for all $n \neq 0$ and

$$K^{-1}\omega(n) \leq \omega(r) \leq K\omega(n)$$

for all $1 \leq n \leq r \leq 2n$ and some constant $K > 1$. Suppose also that f and g are continuous functions with $\hat{f}(0) = 1$ and

$$|\hat{g}(r)| \leq Br^{-2}, |\hat{f}(r)| \leq C\omega(|r|)$$

for all $r \neq 0$. Then

$$|\widehat{f \times g}(r) - \hat{g}(r)| \leq ABCK\omega(r)$$

for all r.

Proof. We have

$$\begin{aligned}
|\widehat{f \times g}(r) - \hat{g}(r)| &= \left|\sum_{u\neq 0}\hat{f}(r-u)\hat{g}(u)\right| \leq \sum_{u\neq 0}|\hat{f}(r-u)\hat{g}(u)| \\
&\leq BC\sum_{u\neq 0}\frac{\omega(|r-u|)}{u^2} \\
&= BC\sum_{0<|u|\leq|r|/2}\frac{\omega(|r-u|)}{u^2} + BC\sum_{|u|>r/2}\frac{\omega(|r-u|)}{u^2} \\
&\leq BC\sum_{|u|\leq|r|/2}\frac{K\omega(|r|)}{u^2} + BC\sum_{|u|>|r|/2}\frac{1}{u^2} \\
&\leq A_1BCK\omega(|r|) + A_2BC\frac{1}{|r|} \leq ABCK\omega(|r|)
\end{aligned}$$

for appropriate constants A_1, A_2 and A and all $r \neq 0$. A similar calculation works for $r = 0$. ■

We can now complete the proof of Lemma 7.9.

Lemma 9.7. *The set $\mathcal{H}(q, p, \mathbf{m})$ is dense in $(\mathcal{G}_\phi, d_\phi)$.*

Proof. We wish to show that, given any η with $1/10 > \eta > 0$ and any $(E, \mu) \in \mathcal{G}_\phi$, we can find an $(F, \sigma) \in \mathcal{H}(q, p, \mathbf{m})$ with

$$d_\phi\big((E, \mu), (F, \sigma)\big) < \eta.$$

In view of Lemma 9.5, we may suppose that $d\mu(x) = g(x)\,dm(x)$ where g is infinitely differentiable and there exists an $\alpha > 0$ such that every point of $\operatorname{supp} g$ lies in an interval $I \subseteq \operatorname{supp} g$ of length at least $\alpha > 0$. Since g is smooth, there exists a constant B such that

$$|\hat{g}(r)| \leq B|r|^{-2}$$

for all $r \neq 0$.

Lemma 9.2 tells us that, if $\epsilon > 0$, we can find an infinitely differentiable function $f_\epsilon : \mathbb{T} \to \mathbb{R}$ with the following properties.

(i) $f_\epsilon(t) \geq 0$ for all t.

(ii) $\int_{\mathbb{T}} f_\epsilon(t)\,dt = 1$.

(iii) $|\hat{f}_\epsilon(r)| \leq \epsilon|r|^{-1/(4q)}$ for all $r \neq 0$.

(iv) If $x_j \in \operatorname{supp} f_\epsilon$ for $1 \leq j \leq q$, and $|x_j - x_k| \geq 1/p$ for $1 \leq j < k \leq q$, then

$$\sum_{j=1}^{q} m_j x_j \neq 0.$$

(v) If I is an interval of length ϵ, then $\operatorname{supp} f_\epsilon \cap I \neq \varnothing$.

If we set $g_\epsilon(t) = g(t)f_\epsilon(t)$, and $E_\epsilon = E \cap \operatorname{supp} f_\epsilon$, then, automatically

(i′) $f_\epsilon(t) \geq 0$ for all t,

and, since $\operatorname{supp} g_\epsilon = \operatorname{supp} f_\epsilon \cap \operatorname{supp} g$, it follows that $\operatorname{supp} g_\epsilon \subseteq E_\epsilon$ and

(iv′) if $x_j \in \operatorname{supp} g_\epsilon$ for $1 \leq j \leq q$, and $|x_j - x_k| \geq 1/p$ for $1 \leq j < k \leq q$, then

$$\sum_{j=1}^{q} m_j x_j \neq 0.$$

On the other hand, condition (v) tells us, that, provided only ϵ is small enough,

$$d(E_\epsilon, E) < \eta/2.$$

If we take $\omega(0) = 1$, $\omega(r) = r^{-1/4q}$ for $r \geq 1$ and $C = \epsilon$ in Lemma 9.6, the inequality proved there shows that, if $\gamma > 0$ is fixed, $|\hat{g}_\epsilon(0) - \hat{g}(0)| \leq \gamma$ and

$$|\hat{g}_\epsilon(r) - \hat{g}(r)| \leq \gamma r^{-1/4q}$$

for all $r \neq 0$ and all sufficiently small ϵ. Since $r^{1/4q}\phi(r) \to \infty$ as $r \to \infty$, it follows that, if $\beta > 0$ is fixed with $1/10 > \beta > 0$,

$$\|\hat{g}_\epsilon - \hat{g}\|_\phi < \beta$$

for all ϵ sufficiently small. In particular, we know that

$$|\hat{g}_\epsilon(0) - 1| = |\hat{g}_\epsilon(0) - \hat{g}(0)| < \beta$$

for ϵ sufficiently small. Writing $G_\epsilon = \hat{g}_\epsilon(0)^{-1} g_\epsilon$, we have

$$\begin{aligned}
\|\hat{G}_\epsilon - \hat{g}\|_\phi &= \left\| \frac{\hat{g}_\epsilon}{\hat{g}(0)} - \hat{g} \right\|_\phi \\
&\leq \|\hat{g}_\epsilon - \hat{g}\|_\phi + \left(1 - \frac{1}{\hat{g}(0)}\right)(\|\hat{g}\|_\phi + \|\hat{g}_\epsilon - \hat{g}\|_\phi) \\
&\leq \beta + 2\beta(\|\hat{g}\|_\phi + \beta).
\end{aligned}$$

It follows that $G_\epsilon \in \mathcal{G}_\phi$ and, provided only that β (and so ϵ) is small enough,

$$\|\hat{G}_\epsilon - \hat{g}\|_\phi < \eta/2.$$

Thus, provided only that ϵ is small enough, $F = E_\epsilon$ and $d\sigma(x) = G_\epsilon(x)\, dm(x)$ satisfy the conclusions required by the first sentence of this proof. ∎

We have thus proved Theorem 7.7 and so Theorem 7.3.

The reader should do as much or as little of the exercises which conclude this section as she pleases. They will not be referred to again.

Exercise 9.8. *Using the same kind of methods as we used to establish Theorem 7.3, establish the following result.*

If q is an integer with $q \geq 1$, then, given any $\alpha > 1/(2q)$, there exists a probability measure μ such that

$$|\hat{\mu}(r)| \leq |r|^{\alpha}$$

for all $r \neq 0$, but, given distinct points $x_1, x_2, \dots, x_q \in \operatorname{supp} \mu$, the only solution to the equation

$$\sum_{j=1}^{q} m_j x_j = 0$$

with $m_j \in \mathbb{Z}$ is the trivial solution $m_1 = m_2 = \cdots = m_q = 0$.

Notice that there is a very big gap between the result of Exercise 9.8 and the result of Lemma 7.1.

Exercise 9.9. *Consider the independent random variables Y_u and the random measure*

$$\sigma = n^{-1} \sum_{u=1}^{n} \delta_{Y_u}$$

introduced in Lemma 8.4.

Show that, provided n is large enough, the probability that more than $n^{1/2}(\log n)^{1/2}\ n^q$ of different q-tuples $j(1), j(2), \dots, j(q)$ satisfy

$$\sum_{k=1}^{q} m_k Y_{j(k)} \in [n^{-q+1/2}, n^{-q+1/2}] \tag{$\bigstar$}$$

is very small indeed. By removing one element $Y_{j(1)}$ corresponding to every q-tuple which satisfies $\bigstar$, show that with high probability, the set $Y_1, Y_2, \dots, Y_n$ contains a subset $\{W_1, W_2, \dots, W_v\}$ with $v \geq n - n^{1/2}(\log n)^{-1/2}$ with the following property. If $j(1), j(2), \dots, j(q)$ are distinct integers with $1 \leq j(k) \leq v$ then

$$\sum_{k=1}^{q} m_k Y_{j(k)} \notin [n^{-q+1/2}, n^{-q+1/2}].$$

Let $\tau = v^{-1} \sum_{u=1}^{v} \delta_{Y_u}$. By comparing $\hat{\tau(r)}$ and $\hat{\sigma}(r)$ show that there exists an A depending only on q such that, if n is large enough, then, with high probability

$$|\hat{\tau}(r)| \leq A n^{-1/2} (\log n)^{1/2}$$

for all $1 \leq |r| \leq n^{4q}$.

Hence show that we can replace the condition $\alpha > 1/(2q)$ in Exercise 9.8 by the condition $\alpha > 1/(2q + \frac{1}{2})$.

A substantially more complicated construction, given in [23], shows that we can replace the condition $\alpha > 1/(2q)$ in Exercise 9.8 by the condition $\alpha > 1/(2q+1)$ but this still leaves a very large gap.

10 Sets of uniqueness and multiplicity

The contents of the next two sections are intended to provide general background to our next results. The reader who misses out these sections will loose nothing except this background.

We are used to the idea of studying the Fourier sum

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\chi_n$$

where f is some appropriate function. What happens if we study general trigonometric sums

$$\sum_{n=-\infty}^{\infty} a_n\chi_n$$

with $a_n \in \mathbb{C}$? One of the first questions about such sums is the problem of uniqueness. If

$$\sum_{n=-N}^{N} a_n\chi_n(t) \to 0$$

as $N \to \infty$ for all $t \in \mathbb{T}$, does it follow that $a_n = 0$ for all n?

Exercise 10.1. *(Easy.) Show that, if* $\sum_{n=-N}^{N} a_n\chi_n(t) \to 0$ *as* $N \to \infty$ *for all* $t \in \mathbb{T}$ *then* $a_n \to 0$ *as* $|n| \to \infty$.

Riemann had the happy idea of considering the effect of formally integrating twice to obtain

$$F(t) = A + Bt + \frac{a_0t^2}{2} - \sum_{n=-\infty}^{\infty} \frac{a_n}{n^2}\chi_n(t).$$

Exercise 10.2. *(Easy.) Suppose that* $a_n \to 0$ *as* $|n| \to \infty$. *Explain why* F *is a well defined continuous function.*

When $\sum_{n=-N}^{N} a_n\chi_n(t)$ converges to a certain value, we can recover that value by looking at the 'generalised second derivative'

$$\lim_{h\to 0} \frac{F(+h) - 2F(t) + F(t-h)}{4h^2}.$$

Exercise 10.3. *If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable at 0 with $f(0) = f'(0) = f''(0) = 0$, use the mean value theorem to show that*

$$\frac{f(h) - 2f(0) + f(-h)}{4h^2} \to 0$$

as $h \to 0$.

Deduce that if $g : \mathbb{R} \to \mathbb{R}$ is twice differentiable at 0, then

$$\frac{g(h) - 2g(0) + g(-h)}{4h^2} \to g''(0)$$

as $h \to 0$.

Exercise 10.4. *Suppose that $a_n \in \mathbb{C}$ and $a_n \to 0$ as $|n| \to \infty$. If*

$$F(t) = A + Bt + \frac{a_0 t^2}{2} - \sum_{n=-\infty}^{\infty} \frac{a_n}{n^2}\chi_n(t),$$

show that

$$\frac{F(x+h) - 2F(x) + F(x-h)}{4h^2} = a_0 + \sum_{n \neq 0} a_n \chi_n(x) \left(\frac{\sin 2\pi nh}{nh}\right)^2.$$

Our next task, which will take a little time is to prove the 'Riemann summation' result given in the next lemma.

Lemma 10.5. *If $\sum_{n=0}^{\infty} b_n$ converges then*

$$b_0 + \sum_{n=1}^{\infty} b_n \left(\frac{\sin nh}{nh}\right)^2 \to \sum_{n=0}^{\infty} b_n$$

as $h \to 0$.

Exercise 10.6. *Deduce from Lemma 10.5 that, if*

$$\sum_{n=-N}^{N} a_n \chi_n(t) \to 0$$

as $N \to \infty$ for all $t \in \mathbb{T}$ and we set

$$F(t) = \frac{a_0 t^2}{2} - \sum_{n=-\infty}^{\infty} \frac{a_n}{n^2}\chi_n(t),$$

then

$$\frac{F(t+h) - 2F(t) + F(t-h)}{4h^2} \to 0$$

as $h \to 0$ for all $t \in \mathbb{T}$

Part of the proof of Lemma 10.5 rests on ideas which are now familiar from elementary functional analysis.

Exercise 10.7. *(i) Suppose that $\gamma_n(h) \in \mathbb{C}$ satisfies the following two conditions.*

(A) $\gamma_n(h) \to 0$ as $h \to 0$.

(B) There exists a C such that

$$\sum_{n=0}^{\infty} |\gamma_n(h)| \leq C$$

for all h.

Then, if $t_n \to 0$ as $n \to \infty$, it follows that

$$\sum_{n=0}^{\infty} \gamma_n(h) t_n \to 0$$

as $h \to 0$.

(ii) Suppose, in addition, that

(C) $\displaystyle\sum_{n=1}^{\infty} \gamma_n(h) \to 1$ as $h \to 0$.

Then, if $s_n \to t$ as $n \to \infty$, it follows that

$$\sum_{n=0}^{\infty} \gamma_n(h) s_n \to t$$

as $h \to 0$.

Proof of Lemma 10.5. If we write $s_n = \sum_{r=0}^{\infty} b_r$

$$\gamma_0(h) = 1 - \left(\frac{\sin 2\pi h}{h}\right)^2, \ \gamma_n(h) = \left(\frac{\sin 2\pi nh}{nh}\right)^2 - \left(\frac{\sin 2\pi(n+1)h}{(n+1)h}\right)^2$$

for $n \geq 1$, Abel summation (that is to say, summation by parts) yields

$$b_0 + \sum_{n=1}^{\infty} b_n \left(\frac{2\pi \sin nh}{nh}\right)^2 = \sum_{n=1}^{\infty} \gamma_0(h) s_n.$$

We wish to estimate $\sum_{n=0}^{\infty} |\gamma_n(h)|$.

To this end, observe that, writing $u(t) = \big((\sin 2\pi t)/t\big)^2$, we have

$$u'(t) = -2\frac{\sin 2\pi t}{t} \times \frac{2\pi t \cos 2\pi t - \sin 2\pi t}{t^2}$$

so $u'(t) \to 0$ as $t \to 0$ and
$$|u'(t)| \leq \frac{20}{t^2}$$
for $t \geq 1$. Thus
$$\sum_{n=0}^{\infty} |\gamma_n(h)| = \sum_{n=0}^{\infty} \left| \int_{nh}^{(n+1)h} u'(t)\,dt \right| \leq \int_0^{\infty} |u'(t)|\,dt,$$
with the convergence of the integral guaranteed by the estimates in the previous sentence.

Since $\sum_{n=0}^{\infty} \gamma_n(h) = 1$ and $\gamma_n(h) \to 0$ as $h \to 0$ for each fixed n, Exercise 10.7 gives the required result. ■

We combine the result of Lemma 10.6 with a very neat result of Schwarz.

Lemma 10.8. *Let $f : [a,b] \to \mathbb{R}$ be continuous.*

(i) Suppose that $f(a) = f(b)$ and
$$\limsup_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{4h^2} > 0$$
for all $x \in (a,b)$. Then $f(x) \leq 0$ for all $x \in [a,b]$.

(ii) Suppose that $f(a) = f(b)$ and
$$\limsup_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{4h^2} \geq 0$$
for all $x \in (a,b)$. Then $f(x) \leq 0$ for all $x \in [a,b]$

(iii) Suppose that $f(a) = f(b)$ and
$$\frac{f(x+h) - 2f(x) + f(x-h)}{4h^2} \to 0$$
as $h \to 0$ for all $x \in (a,b)$. Then $f(x) = 0$ for all $x \in [a,b]$.

(iv) Suppose that
$$\frac{f(x+h) - 2f(x) + f(x-h)}{4h^2} \to 0$$
as $h \to 0$ for all $x \in (a,b)$. Then there exist constants A and B such that $f(x) = Ax + B$ for all $x \in [a,b]$.

Proof. (i) Since f is continuous on the closed interval $[a,b]$, it is bounded and attains its bounds. Suppose that f attains its maximum at x_0. If $x_0 \in (a,b)$ then

$$\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{4h^2} \leq 0$$

for all $x_0+h,\ x_0-h \in [a,b]$ so

$$\limsup_{h\to 0} \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{4h^2} > 0.$$

Since this is excluded, $x_0 \in \{a,b\}$ and $f(x) \leq f(x_0) = 0$ for all $x \in [a,b]$.

(ii) If we set $g(x) = f(x) - \epsilon\big((x-(a+b)/2)^2 - (b-a)^2/4\big)/2$ with $\epsilon > 0$, then $g(a) = g(b) = 0$ and

$$\frac{g(x+h)-2g(x)+g(x-h)}{4h^2} = -\epsilon + \frac{f(x+h)-2f(x)+f(x-h)}{4h^2},$$

so part (i) tells us that $g(x) \leq 0$ for all $x \in [a,b]$. Allowing $\epsilon \to 0$, we get $f(x) \leq 0$.

(iii) Apply part (ii) to f and $-f$.

(iv) Choose A and B so that, writing $g(x) = f(x) - A - Bx$, we have $g(a) = g(b) = 0$ and apply part (iii) to the function g. ■

Putting our results together, we obtain the following uniqueness theorem.

Theorem 10.9. *If*

$$\sum_{n=-N}^{N} a_n \chi_n(t) \to 0$$

as $N \to \infty$ for all $t \in \mathbb{T}$, then $a_n = 0$ for all n.

Proof. Consider the continuous function

$$F(t) = \frac{a_0 t^2}{2} - \sum_{n=-\infty}^{\infty} \frac{a_n}{n^2}\chi_n(t)$$

on the subset $[-\pi/4, 3\pi/2]$ of $\mathbb{T}$. By Exercise10.6,

$$\frac{F(t+h)-2F(t)+F(t-h)}{4h^2} \to 0$$

as $h \to 0$ for all $t \in \mathbb{T}$ so, by Lemma 10.8 (iv), we can find A, B such that $F(t) = At + B$ and so

$$\sum_{n\neq 0} \frac{a_n}{n^2}\chi_n(t) = A + Bt + \frac{a_0 t^2}{2}$$

for all $-\pi/4 \leq t \leq 3\pi/2$. Exactly the same argument shows that we can find constants A' and B' such that

$$\sum_{n\neq 0} \frac{a_n}{n^2}\chi_n(t) = A + Bt + \frac{a_0 t^2}{2}$$

for $-3\pi/2 \leq t \leq \pi/4$. For these statements to be consistent we must have $A = A' = B = B' = 0$, $a_0 = 0$ and

$$\sum_{n\neq 0} \frac{a_n}{n^2}\chi_n(t) = 0$$

for all $t \in \mathbb{T}$.

By the uniqueness of Fourier coefficients for continuous functions, we have $a_n = 0$ for all n and we are done. ∎

Cantor realised that this result could be extended.

Definition 10.10. *We say that a subset E of $\mathbb{T}$ is of* uniqueness *if*

$$\sum_{n=-N}^{N} a_n \chi_n(t) \to 0$$

as $N \to \infty$ for all $t \notin E$ and $a_n \to 0$ as $|n| \to \infty$ implies $a_n = 0$ for all n. If E is not a set of uniqueness we say that E is of multiplicity.

Cantor showed that every finite set is of uniqueness, every closed set with a finite set of limit points is of uniqueness, every closed set with whose set of limit points has a finite set of limit points is of uniqueness and so on. This line of research led him naturally in to the study of ordinals. Later it was shown that every countable closed set is of uniqueness and Young showed that every countable set is of uniqueness. It is not hard (once we understand Lebesgue measure) to show that no set of strictly positive Lebesgue measure can be of uniqueness and it must have been plausible to suppose that all sets of Lebesgue measure zero would turn out to be of uniqueness. Menšov showed that this is not the case.

To show this we need a version of the Riemann localisation lemma.

Lemma 10.11. *Suppose that μ is a measure such that $\hat{\mu}(n) \to 0$ as $|n| \to \infty$ and $f : \mathbb{T} \to \mathbb{C}$ is an infinitely differentiable function. If we set $d\nu(t) = f(t)\,d\mu(t)$ then*

$$\left| f(t) \sum_{n=-N}^{N} \hat{\mu}(n)\chi_n(t) - \sum_{n=-N}^{N} \hat{\nu}(n)\chi_n(t) \right| \to 0$$

as $N \to \infty$.

Proof. Using the fact that $|\hat{f}(n)| \leq A(1+|n|)^{-4}$ to justify various interchanges of summation,

$$\begin{aligned}
&\left| f(t) \sum_{n=-N}^{N} \hat{\mu}(n)\chi_n(t) - \sum_{n=-N}^{N} \hat{\nu}(n)\chi_n(t) \right| \\
&\qquad = \left| \sum_{m=-\infty}^{\infty} \hat{f}(m)\chi_m(t) \sum_{n=-N}^{N} \hat{\mu}(n)\chi_n(t) - \sum_{n=-N}^{N} \sum_{q=-\infty}^{\infty} \hat{\nu}(n-q)\hat{f}(q)\chi_n(t) \right| \\
&\qquad = \left| \sum_{(n,m)\in A(N)} - \sum_{(n,m)\in B(N)} \hat{f}(m)\hat{\mu}(n)\chi_{m+n}(t) \right| \\
&\qquad \leq \sum_{A(N)\triangle B(N)} |\hat{f}(m)\hat{\mu}(n)|
\end{aligned}$$

with

$$A(N) = \{(m,n) : m \in \mathbb{Z}, |n| \leq N\} \text{ and } B(N) = \{(m,n) : |m-n| \leq N\}.$$

Now observe that

$$\sum_{A(N)\triangle B(N)} (1+|n|)^{-3} \leq 4 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (1+n)^{-3} \leq C' \sum_{m=0}^{\infty} (1+m)^{-2} \leq C$$

for appropriate constants C' and C. so, if M is fixed,

$$\begin{aligned}
&\left| f(t) \sum_{n=-N}^{N} \hat{\mu}(n)\chi_n(t) - \sum_{n=-N}^{N} \hat{\nu}(n)\chi_n(t) \right| \\
&\qquad \leq \sum_{A(N)\triangle B(N)} (1+|n|)^{-3} \sup_{|r|\geq M} |\hat{\mu}(r)| + \sum_{|m|\leq M, |n-r|\geq N} (1+|n|)^{-3} \sup_{|r\in\mathbb{Z}} |\hat{\mu}(r)| \\
&\qquad \leq C \sup_{|r|\geq M} |\hat{\mu}(r)| + \sum_{|m|\leq M, |n-r|\geq N} (1+|n|)^{-3} \sup_{|r|\in\mathbb{Z}} |\hat{\mu}(r)| \\
&\qquad \to C \sup_{|r|\geq M} |\hat{\mu}(r)|
\end{aligned}$$

as $N \to \infty$. Allowing $M \to \infty$, gives the required result. ■

Theorem 10.12. *If μ is a non-zero measure with $\hat{\mu}(n) \to 0$ as $|n| \to 0$ then* $\operatorname{supp} \mu$ *is a set of multiplicity.*

Menšov then exhibited a non-zero measure with support on a compact set E of Lebesgue measure zero thus showing that there existed compact sets of multiplicity with Lebesgue measure zero. During the first half of the 20th century all such constructions involved sets E with many arithmetic relations. Rudin's theorem (Theorem 7.2) showed that there was no simple arithmetic condition which could characterise sets of multiplicity.

11 Distributions

Even if $a_n \to 0$ as $|n| \to \infty$, it is not true every trigonometric sum

$$\sum_{n=-\infty}^{\infty} a_n \chi_n$$

can be made to correspond to a measure. To get round this problem classical analysts resorted to a series of tricks which allowed them to act as though the formal series was a measure.

Schwartz showed that these ideas could be linked with other formal tricks from the study of partial differential equations to give the theory of distributions. If the reader is not familiar with that theory, she may omit the rest of this section without loss. If she is familiar with the theory, she may be interested to see how the study of sets of uniqueness fits in.

When we talk of a distribution we shall mean a member of the dual $\mathcal{D}'(\mathbb{T})$ of the space $\mathcal{D}(\mathbb{T})$ of infinitely differentiable functions.

It is easy to check that if a_n is bounded then the relation

$$\langle S, f \rangle = \sum_{n=-\infty}^{\infty} a_n \hat{f}(n)$$

with $f \in \mathcal{D}(\mathbb{T})$ defines an element $S \in \mathcal{D}'(\mathbb{T})$. It is natural to define

$$\hat{S}(n) = a_n = \langle S, chi_{-n} \rangle.$$

We know that every distribution T has a support defined to be the smallest closed set E with the property that, if the support of $f \in \mathcal{D}(\mathbb{T})$ is disjoint from E then $\langle T, f \rangle = 0$. The calculations which gave Lemma 10.11 go through unchanged to give us the following characterisation of a closed set of multiplicity.

Lemma 11.1. *A compact set E is of multiplicity if and only if we can find a non-zero distribution S with $\hat{S}(n) \to 0$ as $|n| \to \infty$ such that* $\operatorname{supp} S \subseteq E$.

Let us see how this idea can be used.

Lemma 11.2. *(i) If S is a non-zero distribution whose support is a single point then $\hat{S}(n) \nrightarrow 0$ as $n \to \infty$.*

(ii) If S is distribution with $\hat{S}(n) \to 0$ as $n \to \infty$ then the support of S can not contain an isolated point.

(iii) A countable closed set must be of uniqueness.

Proof. (i) If S is a distribution with support a, then it can be shown that

$$S = \sum_{r=0}^{m} \lambda_r \delta_a^{(r)}$$

with not all the λ_r zero. Thus

$$\hat{S}(n) = \sum_{r=0}^{m} \lambda_r (-in)^r \chi(a) \nrightarrow 0$$

as $|n| \to 0$.

(ii) Suppose that the support of S contains an isolated point a. We can find an $f \in \mathcal{D}(\mathbb{T})$ such that $f = 1$ in a neighbourhood of a and $\operatorname{supp} f \cap \operatorname{supp} S = a$. Automatically, T is a non-zero distribution and

$$\hat{T}(r) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{S}(k+r) \to 0$$

as $|r| \to \infty$. This contradicts part (i) so the result follows by reductio ad absurdum.

(iii) If E is a countable closed set, T a distribution, $E \supseteq \operatorname{supp} T$ and $T \neq 0$ then $\operatorname{supp} T$ is a non-empty countable closed set and so contains an isolated point. By part (ii), $\hat{T}(n) \nrightarrow 0$ as $n \to \infty$. ■

12 Debs and Saint-Reymond

Just as it was possible to hope that closed sets of multiplicity might be characterised by arithmetic properties, so one might hope that they could be characterised by 'metric properties' such as Hausdorff h-measure.

Definition 12.1. *Let $h : [0, \infty) \to \mathbb{R}$ be a strictly increasing function. We say that a set E has* Hausdorff h-measure zero *if, given any $\epsilon > 0$, we can find a sequence I_j of intervals of length $|I_j|$ such that*

$$\sum_{j=1}^{\infty} h(|I_j|) < \epsilon \text{ but } \bigcup_{j=1}^{\infty} I_j \supseteq E.$$

However, Lusin, who seems to had a remarkable instinct in such matters, conjectured that every complement of a set of first Baire category would turn out to be a set of multiplicity.

That this is the case is shown by the famous theorem of Debs and Saint-Reymond.

Theorem 12.2. *Let B be a set of first category in $\mathbb{T}$. Then we can find a probability measure μ with $\hat{\mu}(r) \to 0$ as $|r| \to \infty$ such that*

$$\operatorname{supp} \mu \cap B = \varnothing.$$

As an indication of the power of this result we derive a theorem of Ivašev-Musatov [8].

Theorem 12.3. *If $h : [0, \infty) \to \mathbb{R}$ is a strictly increasing continuous function with $h(0) = 0$, then we can find a probability measure μ with $\hat{\mu}(r) \to 0$ as $|r| \to \infty$ such that* $\operatorname{supp} \mu$ *has Hausdorff h-measure zero.*

Proof. Enumerate the rationals as y_1, y_2, y_3, … and choose $\epsilon_n > 0$ so that $\sum_{n=1}^{\infty} h(2\epsilon_n)$ converges. Then

$$B = \bigcup_{j=1}^{\infty} \bigcap_{n=1}^{\infty} \big(\mathbb{T} \setminus (y_n - \epsilon_{n+j}, y_n + \epsilon_{n+j})\big)$$

is a set of first category whose complement has Hausdorff h-measure zero. ■

We shall prove the following result which includes both the theorem of Rudin and that of Debs and Saint Raymond as corollaries. (A similar result to the one presented here was obtained independently by Matheron and Zelený in [24].)

Theorem 12.4. *Let A_q be a set of first category in $\mathbb{T}^q$ $[q \geq 1]$ and let*

$$A = \bigcup_{q=1}^{\infty} \{\mathbf{x} \in \mathbb{T}^{\mathbb{N}^+} : (x_1, x_2, \dots, x_q) \in A_q\}.$$

Then we can find a probability measure μ with $\hat{\mu}(r) \to 0$ as $|r| \to \infty$ such that, whenever x_1, x_2, … are distinct points of $\operatorname{supp} \mu$, $\mathbf{x} \notin A$.

The following corollary makes the connection with Rudin's theorem explicit.

Theorem 12.5. *Let B be a set of first category in $\mathbb{T}$. Then we can find a probability measure μ with $\hat{\mu}(r) \to 0$ as $|r| \to \infty$ such that* $\operatorname{supp} \mu$ *is independent and the subgroup G of $\mathbb{T}$ generated by* $\operatorname{supp} \mu$ *satisfies*

$$G \cap B \subseteq \{0\}.$$

Theorem 12.5 can be restated as follows. (Note that, if B is of first category, so is $B \cup \{0\}$, so there is no loss of generality in supposing $0 \in B$.)

Theorem 12.6. *Let B be a set of first category in $\mathbb{T}$. Then we can find a probability measure μ with $\hat{\mu}(r) \to 0$ as $|r| \to \infty$ such that whenever $q \geq 1$ and $x_1, x_2, \ldots, x_q$ are distinct points of* $\operatorname{supp} \mu$ *and $m_1, m_2, \ldots, m_q$ are integers not all zero, then*

$$m_1 x_1 + m_2 x_2 + \cdots + m_q x_q \notin B.$$

Proof. Suppose that K is a closed subset of $\mathbb{T}$ with empty interior, q is an integer with $q \geq 1$ and $m_1, m_2, \ldots, m_q$ are integers, not all zero. Then

$$\{\mathbf{x} \in \mathbb{T}^q : m_1 x_1 + m_2 x_2 + \cdots + m_q x_q \in K\}$$

is closed with empty interior.

Since the countable union of sets of the first category is of the first category, the set $\tilde{K}$ of $\mathbf{x} \in \mathbb{T}^q$ such that the there exists a $q \geq 1$ and integers m_1, $m_2, \ldots, m_q$, not all zero, with

$$m_1 x_1 + m_2 x_2 + \cdots + m_q x_q \in K$$

is of first category.

We now observe that B is a subset of a countable union of closed sets with empty interior, so, since the countable union of sets of the first category is of the first category, the set A_q of $\mathbf{x} \in \mathbb{T}^q$ such that there exists a $q \geq 1$ and integers $m_1, m_2, \ldots, m_q$ not all zero with

$$m_1 x_1 + m_2 x_2 + \cdots + m_q x_q \in B$$

is of first category in $\mathbb{T}^q$ and we may apply Theorem 12.4. ∎

The following trivial remark explains why we stated Theorem 12.4 in the way we did.

Example 12.7. *The set*

$$A = \{\mathbf{x} \in \mathbb{T}^4 : x_1 + x_2 = x_3 + x_4\}$$

is closed and has empty interior. None the less, if E is any non-empty set in $\mathbb{T}$, we have

$$E^4 \cap A \neq \varnothing.$$

However, we are interested in the statement that

$$x_1 + x_2 \neq x_3 + x_4$$

whenever x_1, x_2, x_3 and x_4 are *distinct* points of E.

The original proof of their theorem by Debs and Saint Raymond employed descriptive set theory and other sophisticated tools. Later Kechris and Louveau produced a much simpler proof, If readers consider our proof of Theorem 12.4 in the case when

$$A_q = \{\mathbf{x} \in \mathbb{T}^q \,:\, x_1 \in B\},$$

they will recover a lowbrow version of the the proof of Kechris and Louveau.

Definition 12.8. *We set*

$$(\mathcal{P}, d_{\mathcal{P}}) = (\mathcal{G}_\phi, d_\phi)$$

where $\psi(n) = 1$ *for all* n *and* $\mathcal{G}_\psi, d_\psi)$ *is defined as in Lemma 7.4.*

Exercise 12.9. *Write down the definition of* $(\mathcal{P}, d_{\mathcal{P}})$ *explicitly (ie without using Lemma 7.4).*

We can now state a Baire category version of Theorem 12.4.

Theorem 12.10. *Let* A_q *be a set of first category in* $\mathbb{T}^q$ $[q \geq 1]$ *and let*

$$A = \bigcup_{q=1}^{\infty} \{\mathbf{x} \in \mathbb{T}^{\mathbb{N}^+} \,:\, (x_1, x_2, \dots, x_q) \in A_q\}.$$

Consider the set $\mathcal{E}$ *of* $(E, \mu) \in \mathcal{P}$ *such that, whenever* x_1, x_2, ... *are distinct points of* $\operatorname{supp}\mu$*, it follows that* $\mathbf{x} \notin A$*. Then the complement of* $\mathcal{E}$ *is of first category in* $(\mathcal{P}, d_{\mathcal{P}})$*.*

Theorem 12.10 follows from a slightly simpler result.

Lemma 12.11. *Suppose that* q *is a strictly positive integer,* $\alpha > 0$ *and* K *is a closed subset of* $\mathbb{T}^q$ *with empty interior. Consider the set* $\mathcal{E}_\alpha$ *of* $(E, \mu) \in \mathcal{P}$ *such that, whenever* x_1, x_2, ..., $x_q \in \operatorname{supp}\mu$ *and* $|x_k - x_l| \geq \alpha$ *for* $k \neq l$*, it follows that* $\mathbf{x} \notin K$*. Then the complement of* $\mathcal{E}_\alpha$ *is closed with empty interior.*

Proof of Theorem 12.10 from Lemma 12.11. By standard Baire category arguments, Theorem 12.10 follows from the special case in which

$$A = \{\mathbf{x} \in \mathbb{T}^{\mathbb{N}^+} \,:\, (x_1, x_2, \dots, x_q) \in K\}$$

and K is a closed set in $\mathbb{T}^q$ with empty interior. By Lemma 12.11 we know that the complement of $\mathcal{E}_{1/n}$ is closed with empty interior. Since $\mathcal{E} = \bigcap_{n=1}^{\infty} \mathcal{E}_{1/n}$, it follows that the complement of $\mathcal{E}$ is of first category. ■

A further slight simplification reduces our proof to the following core lemma.

Lemma 12.12. *Suppose that q is a strictly positive integer, $\alpha > 0$ and K is a closed subset of $\mathbb{T}^q$ with empty interior. Consider the set $\mathcal{E}$ of $(H, \rho) \in \mathcal{P}$ such that, whenever $x_1, x_2, \dots, x_q \in \operatorname{supp} \rho$ and $|x_k - x_l| \geq \alpha$ for $k \neq l$, it follows that $\mathbf{x} \notin A$. Then, given any $\epsilon > 0$ and any $(E, \mu) \in \mathcal{P}$, we can find an $(F, \sigma) \in \mathcal{E}$ with $d_{\mathcal{P}}\big((E, \mu), (F, \sigma)\big) < \epsilon$.*

Proof of Lemma 12.11 from Lemma 12.12. This follows a familiar pattern.

Lemma 12.12 states that the complement of $\mathcal{E}$ contains no non-empty open set. Thus we need only show that the complement of $\mathcal{E}$ is closed. To this end, suppose that $(E_n, \mu_n) \notin \mathcal{E}$ and $(E_n, \mu_n) \underset{d_{\mathcal{P}}}{\rightarrow} (E, \mu)$ as $n \to \infty$.

By definition, we can find $x_j(n) \in E_n$ such that $|x_k(n) - x_l(n)| \geq \alpha$ for $k \neq l$ and $\mathbf{x}(n) \in A$. By applying the Theorem of Bolzano–Weierstrass and extracting a subsequence, we may suppose that $x_j(n) \to x_j$ for all $1 \leq j \leq q$. Automatically, $x_j \in E$, $|x_k - x_l| \geq \alpha$ for $k \neq l$ and, since K is closed, $\mathbf{x} \in K$. Thus $(E, \mu) \notin \mathcal{E}$ and we are done. ■

13 The perturbation argument

The proof of Lemma 12.12 depends on a simple but very useful observation.

Lemma 13.1. *Suppose that q is a strictly positive integer, that K is a closed subset of $\mathbb{T}^q$ with empty interior, and that $\{e_1, e_2, \dots, e_n\}$ is a finite subset of $\mathbb{T}$. Then, given any $\epsilon > 0$, we can find $f_k \in \mathbb{T}$ and $\eta > 0$ such that*

(i) $|f_k - e_k| < \epsilon$ for $1 \leq k \leq n$,

(ii) $|f_j - f_k| > 2\eta$ for all $1 \leq j < k \leq n$, and

(iii) if we write $F = \{f_k : 1 \leq k \leq n\}$, we know that, whenever y_1, y_2, $\dots y_q$ are distinct points of F and $|y_j - x_j| < \eta$ for $1 \leq j \leq q$, then $\mathbf{x} \notin K$.

Proof. Let Θ be the set of maps

$$\theta : \{1, 2, \dots, q\} \to \{1, 2, \dots, n\}.$$

Observe that the sets

$$H_\theta = \{\mathbf{x} \in \mathbb{T}^n : (x_{\theta(1)}, x_{\theta(2)}, \dots, x_{\theta(q)}) \in K\}$$

with $\theta \in \Theta$ and the sets

$$L_{jk} = \{\mathbf{x} \in \mathbb{T}^n : x_j = x_k\}$$

with $1 \leq j < k \leq n$ are all closed with empty interior. We now use a 'miniature' Baire category argument. ■

The following exercise repeats ideas from Exercise 6.3 (particularly part (iii)).

Exercise 13.2. *Suppose that E is a closed set and μ a probability measure with $\operatorname{supp}\mu \subseteq E$. Then, given any integer $N \geq 0$ and any $\epsilon > 0$, we can find an $M \geq 1$, $\eta > 0$, points $y_p \in \mathbb{T}$ real numbers $\lambda_p \geq 0$ $[1 \leq p \leq M]$ with $\sum_{p=1}^{M} \lambda_p = 1$ having the following properties.*

Whenever $|f_p - y_p| < \eta$ $[1 \leq p \leq M]$ and we write

$$F = \{f_1, f_2, \dots, f_M\} \text{ and } \sigma = \sum_{p=1}^{M} \lambda_p \delta_{f_p}$$

we have

(i) $d_{\mathcal{H}}(E, F) < \epsilon$ and

(ii) $|\hat{\mu}(r) - \hat{\sigma}(r)| < \epsilon$ for all $|r| \leq N$.

Our proof uses little beyond Lemma 13.1 and Exercise 6.3.

Lemma 13.3. *Suppose that q is a strictly positive integer, $\alpha > 0$ and K is a closed subset of $\mathbb{T}^q$ with empty interior. Then, given any $\epsilon > 0$ and any $(E_j, \mu_j) \in \mathcal{P}$ $[1 \leq j \leq q]$, we can find $N' \geq N$, $\gamma > 0$ and $(F_j, \sigma_j) \in \mathcal{E}$ with the following properties.*

(i) $d_{\mathcal{H}}(E_j, F_j) < \epsilon/2$ for all $1 \leq j \leq q$.

(ii) If $d_{\mathcal{H}}(F_j, G_j) < \gamma$, then, whenever

$$x_1,\, x_2,\, \dots,\, x_q \in \bigcup_{j=1}^{q} G_j \text{ and } |x_k - x_l| \geq \alpha \text{ for } k \neq l$$

it follows that $\mathbf{x} \notin K$.

(iii) $|\hat{\mu}_j(r) - \hat{\sigma}_j(r)| < \epsilon$ for all $|r| \leq N$ and for all $|r| \geq N'$ $[1 \leq j \leq q]$.

Proof. By Exercise 13.2, we can find $\eta > 0$, integers $M(j) \geq 1$, points $y_{p,j} \in \mathbb{T}$, and real numbers $\lambda_{p,j} \geq 0$ $[1 \leq p \leq M(j)]$ with $\sum_{p=1}^{M(j)} \lambda_{p,j} = 1$ having the following properties.

Whenever $|x_{p,j} - y_{p,j}| < \eta$ $[1 \leq p \leq M(j)]$ and we write

$$L_j = \{x_{1,j}, x_{2,j}, \dots, x_{M(j),j}\} \text{ and } \rho_j = \sum_{p=1}^{M(j)} \lambda_p \delta_{x_{p,j}}$$

we have

(i)′ $d_{\mathcal{H}}(E_j, L_j) < \epsilon$ and

(iii)′ $|\hat{\mu}_j(r) - \hat{\rho}_j(r)| < \epsilon$ for all $|r| \leq N$.

By Lemma 13.1, we can find $t_{p,j}$ and a $\gamma > 0$ with $\gamma < \min(\eta, \epsilon)/4$ such that

(i)$''$ $|t_{p,j} - y_{p,j}| < \epsilon/4$ for $1 \leq p \leq M(j)$ and $1 \leq j \leq q$,

(iv)$''$ $|t_{p,j} - t_{p',j'}| > 4\gamma$ for all $(p, j) \neq (p', j')$, and

(ii)$''$ if we write $L = \bigcup_{j=1}^{q} \bigcup_{p=1}^{M(j)} \{t_{p,j}\}$, we know that, whenever a_1, a_2, $\dots a_q$ are distinct points of L and $|a_j - b_j| < 2\gamma$ for $1 \leq j \leq q$, then $\mathbf{b} \notin K$.

Now choose h an infinitely differentiable positive function with $\operatorname{supp} h \subseteq [-\gamma/2, \gamma/2]$ and $\int_{\mathbb{T}} h(t)\, dt = 1$. If we take

$$F_j = \bigcup_{p=1}^{M(j)} [t_{p,j} - \gamma/2, t_{p,j} + \gamma/2] \text{ and } \sigma_j = \left(\sum_{p=1}^{M(j)} \lambda_{p,j} \delta_{t_{p,j}} \right) * g$$

then all the conclusions of the lemma (with the possible exception of that involving N') are satisfied.

Since $\hat{\mu}_j(r), \hat{\sigma}_j(r) \to 0$ as $|r| \to \infty$ we can choose N' so that $|\hat{\mu}_j(r)|, |\hat{\sigma}_j(r)| < \epsilon/2$ for all $1 \leq j \leq q$ and $|r| \geq N'$. Automatically, $|\hat{\mu}_j(r) - \hat{\sigma}_j(r)| < \epsilon$ for all $|r| \geq N'$, so we are done. ■

Lemma 13.4. *Suppose that q and m are strictly positive integers with $m \geq q$, $\alpha > 0$ and K is a closed subset of $\mathbb{T}^q$ with empty interior. Then, given any $\epsilon > 0$ and any $(E_k, \mu_k) \in \mathcal{P}$ $[1 \leq k \leq m]$, we can find $(F_k, \sigma_k) \in \mathcal{E}$ $[1 \leq k \leq m]$ with the following properties.*

(i) $d_{\mathcal{H}}(E_k, F_k) < \epsilon$ for all $1 \leq k \leq m$

(ii) Whenever x_1, x_2, $\dots$, $x_q \in \bigcup_{k=1}^{m} F_k$ and $|x_j - x_l| \geq \alpha$ for $j \neq l$ it follows that $\mathbf{x} \notin K$.

(iii) $\sum_{k=1}^{m} |\hat{\mu}_k(r) - \hat{\sigma}_k(r)| < 2q + 1$ for all r.

Proof. Let Φ be the collection of subsets of $\{1, 2, \dots, m\}$ containing exactly q elements. Let $M = \binom{m}{q}$, the number of elements of Φ, and let

$$\theta : \{1,\, 2,\, \dots,\, M\} \to \Phi$$

be a bijection. Set $N(0) = 0$, $\gamma_0 = \epsilon/4$, $E_{k,0} = E_k$ and $\mu_{k,0} = \mu_k$. By repeated use of Lemma 13.3, we can find $N(w)$, $E_{k,w}$, $\mu_{k,w}$, γ_w with $N(w) > N(w-1)$, $\gamma_{w-1}/4 > \gamma_w > 0$ and $(E_{k,w}, \mu_{k,w}) \in \mathcal{P}$ for $w = 1$, 2, $\dots$, M with the following properties.

(i)$_w$ $d_{\mathcal{H}}(E_{k,w}, E_{k,w-1}) < \gamma_{w-1}/2$ for all $k \in \theta(w)$.

(ii)$_w$ If $d_{\mathcal{H}}(E_{k,w}, G_{k,w}) < \gamma$, then, whenever x_1, x_2, $\dots$, $x_q \in \bigcup_{k \in \theta(w)} G_{k,w}$ and $|x_j - x_l| \geq \alpha$ for $j \neq l$, it follows that $\mathbf{x} \notin K$.

(iii)$_w$ Whenever $k \in \theta(w)$, $|\hat{\mu}_{k,(w-1)}(r) - \hat{\mu}_{k,w}(r)| < 1/(2Mq)$ for all $|r| \leq N(k-1)$ and for all $|r| \geq N(k)$.

(iv)$_w$ If $k \notin \theta(w)$, then $E_{k,w} = E_{k,w-1}$ and $\mu_{k,w} = \mu_{k,w-1}$.

We now set $F_k = E_{k,M}$ and $\mu_k = \mu_{k,M}$. By construction, $(F_k, \sigma_k) \in \mathcal{P}$ and (using (ii)$_w$ and (iv)$_w$)

$$d_{\mathcal{H}}(E_k, F_k) \leq \sum_{w=1}^{M} d_{\mathcal{H}}(E_{k,(w-1)}, E_{k,w}) \leq \sum_{w=1}^{M} \gamma_w < \epsilon.$$

Thus (i) holds.

Now suppose that $x_j \in \bigcup_{k=1}^{m} F_k$ for $1 \leq j \leq q$ and $|x_j - x_l| \geq \alpha$ for $j \neq l$. By the definition of θ, we can find a $1 \leq w \leq M$ such that $x_j \in \bigcup_{k\in\theta(w)} F_k$. Arguing as in the previous paragraph, we can find $y_j \in \bigcup_{k\in\theta(w)} E_{k,p}$ such that $|x_j - y_j| \leq \gamma_p$ $[1 \leq j \leq q]$ and so, by (ii)$_w$, $\mathbf{x} \notin K$. Thus (ii) holds.

Now suppose $N(a-1) \leq |r| \leq N(a)$ for some integer a with $1 \leq a \leq M$. By (iii)$_w$ and (iv)$_w$,

$$\begin{aligned}
\sum_{k=1}^{m} |\hat{\mu}_k(r) - \hat{\sigma}_k(r)| &\leq \sum_{k=1}^{m} \sum_{w=1}^{M} |\hat{\mu}_{k,(w-1)}(r) - \hat{\mu}_{w,p}(r)| \\
&= \sum_{w=1}^{M} \sum_{k=1}^{m} |\hat{\mu}_{k,(w-1)}(r) - \hat{\mu}_{k,w}(r)| \\
&= \sum_{w=1}^{M} \sum_{k\in\theta(w)} |\hat{\mu}_{k,(w-1)}(r) - \hat{\mu}_{k,w}(r)| \\
&= \sum_{w\neq a} \sum_{k\in\theta(w)|} \hat{\mu}_{k,(w-1)}(r) - \hat{\mu}_{w,p}(r)| \\
&\qquad\qquad + \sum_{k\in\theta(a)} |\hat{\mu}_{k,(a-1)}(r) - \hat{\mu}_{k,a}(r)| \\
&\leq \sum_{w\neq a} \sum_{k\in\theta(w)} 1/(2Mq) + \sum_{k\in\theta(a)} 2 \\
&< 2q+1.
\end{aligned}$$

Less complicated estimates work if $|r| \geq N(M)$. Thus (iii) holds. ■

We can now complete the proof of Theorem 12.4 by proving Lemma 12.12.

Proof of Lemma 12.12. We are given $\epsilon > 0$ and an $(E, \mu) \in \mathcal{P}$. Choose an integer m such that $(2q+1)/m < \epsilon/2$ and write $(E_k, \mu_k) = (E, \mu)$ for $1 \leq k \leq m$. By Lemma 13.4 we can find $(F_k, \sigma_k) \in \mathcal{P}$, with the following properties.

(i) $d_{\mathcal{H}}(E, F_k) < \epsilon/2$ for all $1 \leq k \leq m$.

(ii) Whenever $x_1, x_2, \dots, x_q \in \bigcup_{k=1}^m F_k$ and $|x_j - x_l| \geq \alpha$ for $j \neq l$, it follows that $\mathbf{x} \notin K$.

(iii) $\sum_{j=1}^m |\hat{\mu}(r) - \hat{\sigma}_j(r)| < 2q+1$ for all r.

If we now set $F = \bigcup_{k=1}^m F_k$ and $\sigma = m^{-1} \sum_{k=1}^m \sigma_k$. Then, automatically, $(F,\sigma) \in \mathcal{P}$ and statement (ii) tells us that $(F,\sigma) \in \mathcal{E}$.

Statement (i) tells us that $d_{\mathcal{H}}(E,F) < \epsilon/2$ and statement (iii) tells us that

$$|\hat{\mu}(r) - \hat{\sigma}(r)| \leq m^{-1} \sum_{k=1}^m |\hat{\mu}_k(r) - \hat{\sigma}_k(r)| \leq (2q+1)/m < \epsilon/2$$

for all r. Thus $d_{\mathcal{P}}\big((E,\mu),(F,\sigma)\big) < \epsilon$ and we are done. ■

14 Convolution of distinct measures

Except for this preliminary section the rest of these notes deal with 'convolution squares' that is to say convolutions $\mu * \mu$ of a measure with itself. However, as a warm up exercise, we shall deal with an easier theorem involving the convolution of two measures.

Theorem 14.1. *Let B be a set of first category in $\mathbb{T}$. Then there exist Kronecker sets E_1 and E_2 and probability measures μ_1 and μ_2 with* $\operatorname{supp} \mu_j \subseteq E_j \subseteq \mathbb{T} \setminus B$ *such that $\mu_1 * \mu_2$ is an infinitely differentiable nowhere zero function.*

As usual we seek a Baire category proof and this requires an appropriate metric.

Exercise 14.2. *(i) Consider the space $C^\infty(\mathbb{T})$ of infinitely differentiable functions $f : \mathbb{T} \to \mathbb{C}$. Show, by using theorems on uniform convergence, or otherwise, that*

$$\rho(f,g) = \sum_{k=0}^\infty \frac{2^{-k}\|f^{(k)} - g^{(k)}\|_\infty}{1 + \|f^{(k)} - g^{(k)}\|_\infty}$$

defines a complete metric on $C^\infty(\mathbb{T})$.

(ii) Consider the space $\mathcal{P}$ of probability measures on $\mathbb{T}$. Show, by using theorems on weak convergence or otherwise, that

$$d_P(\mu,\tau) = \sum_{k=-\infty}^\infty 2^{-|k|}|\hat{\mu}(k) - \hat{\tau}(k)|$$

defines a complete metric on $\mathcal{P}$.

(iii) Consider the space $\mathcal{K}$ with elements $(E_1, E_2, \mu_1, \mu_2, f)$ where E_i is a compact set, μ_j is a probability measure with $\operatorname{supp}\mu_j \supseteq E_j$ $[j = 1, 2]$, *$f \in C^\infty(\mathbb{T})$ and $f = \mu_1 * \mu_2$. Why is $\mathcal{K}$ non-empty? Show that*

$$\begin{aligned} &d_{\mathcal{K}}\big((E_1, E_2, \mu_1, \mu_2, f), (F_1, F_2, \tau_1, \tau_2, g)\big) \\ &\qquad = d_{\mathcal{H}}(E_1, F_1) + d_{\mathcal{H}}(E_2, F_2) + d_P(\mu_1, \tau_1) + d_P(\mu_2, \tau_2) + \rho(f, g) \end{aligned}$$

(where $d_{\mathcal{H}}$ is the usual Hausdorff metric) defines a complete metric on $\mathcal{K}$.

We can now state our Baire category theorem.

Theorem 14.3. *Let B be a set of first category in $\mathbb{T}$. Then quasi-all $(E_1, E_2, \mu_1, \mu_2, f) \in \mathcal{K}$ have the property that E_j is Kronecker and $E_j \cap B = \varnothing$*

Exercise 14.4. *Deduce Theorem 14.1 from Theorem 14.3.*

We now perform our standard reductions.

Lemma 14.5. *(i) Let K be a compact subset of $\mathbb{T}$ whose complement is dense. Then the set of $(E_1, E_2, \mu_1, \mu_2, f) \in \mathcal{K}$ with the property that $E_1 \cap K = \varnothing$ is open and dense.*

(ii) Let $u \in S(\mathbb{T})$ and $n \geq 1$. Then the set of $(E_1, E_2, \mu_1, \mu_2, f) \in \mathcal{K}$ with the property that there exists an integer Q such that

$$|u(t) - \chi_Q(t)| \leq 1/n \text{ for all } t \in E_1$$

is open and dense.

Exercise 14.6. *Deduce Theorem 14.3 from Lemma 14.5.*

Exercise 14.7. *(i) Let L be a compact subset of $\mathbb{T}$. Show that the set of $(E_1, E_2, \mu_1, \mu_2, f) \in \mathcal{K}$ with the property that $E_1 \cap K = \varnothing$ is open and dense.*

(ii) Let $u \in S(\mathbb{T})$ and $n \geq 1$. Show that the set of $(E_1, E_2, \mu_1, \mu_2, f) \in \mathcal{K}$ with the property that there exists an integer Q such that

$$|u(t) - \chi_Q(t)| \leq 1/n \text{ for all } t \in E_1$$

is open and dense.

The proof of Theorem 14.3 thus reduces to the proof of the two parts of the following lemma.

Lemma 14.8. *(i) Let L be a compact subset of $\mathbb{T}$ whose complement is dense. Given $(F_1, F_2, \tau_1, \tau_2, g) \in \mathcal{K}$ and $\epsilon > 0$, we can find $(E_1, E_2, \mu_1, \mu_2, f) \in \mathcal{K}$ with*

$$d_{\mathcal{K}}\big((E_1, E_2, \mu_1, \mu_2, f), (F_1, F_2, \tau_1, \tau_2, g)\big) < \epsilon$$

such that $E_1 \cap K = \varnothing$.

(ii) Let $u \in S(\mathbb{T})$ and $n \geq 1$. Then, given $(F_1, F_2, \tau_1, \tau_2, g) \in \mathcal{K}$, and $\epsilon > 0$ we can find $(E_1, E_2, \mu_1, \mu_2, f) \in \mathcal{K}$ with

$$d_{\mathcal{K}}\big((E_1, E_2, \mu_1, \mu_2, f), (F_1, F_2, \tau_1, \tau_2, g)\big) < \epsilon$$

and an integer Q such that

$$|u(t) - \chi_Q(t)| \leq 1/n \text{ for all } t \in E_1.$$

The proofs of the two parts are very similar. We make use of the following well known result.

Exercise 14.9. *If μ is a measure and K is n times continuously differentiable show that $K*\mu$ is n times continuously differentiable with $(K*\mu)^{(n)} = K^{(n)}*\mu$. (If the reader finds our statement too informal then she should formalise it.)*

Exercise 14.10. *Use Exercise 14.9 and the kind of ideas used in Exercise 13.2 to prove the following result. Suppose that F is a closed set and $h : \mathbb{T} \to \mathbb{C}$ is an N times continuously differentiable function. Then, given any $\epsilon > 0$ and any positive integer N', we can find an $M \geq 1$, $\eta > 0$, points $y_p \in \mathbb{T}$ real numbers $\lambda_p \geq 0$ $[1 \leq p \leq M]$ with $\sum_{p=1}^{M} \lambda_p = 1$ having the following properties.*

Whenever $|e_p - y_p| < \eta$ $[1 \leq p \leq M]$ and we write

$$E = \{e_1, e_2, \dots, e_M\} \text{ and } \sigma = \sum_{p=1}^{M} \lambda_p \delta_{e_p},$$

we have

(i) $d_{\mathcal{H}}(E, F) < \epsilon$,

(ii) $|(\sigma * h)^{(q)}(t) - (\tau * h)^{(q)}(t)| < \epsilon$ for all $q \leq N$ and all $t \in \mathbb{T}$ and

(iii) $|\hat{\sigma}(r) - \hat{\tau}(r)| < \epsilon$ for all $|r| \leq N$.

Our first step is to 'spread out' the measure τ_2.

Lemma 14.11. *Given $(F_1, F_2, \tau_1, \tau_2, g) \in \mathcal{K}$ and $\epsilon > 0$ we can find $(E_1, E_2, \mu_1, \mu_2, f) \in \mathcal{K}$ with*

$$d_{\mathcal{K}}\big((E_1, E_2, \mu_1, \mu_2, f), (F_1, F_2, \tau_1, \tau_2, g)\big) < \epsilon$$

such that $d\mu_2(t) = h(t)\,dt$ where h is an infinitely differentiable function.

Proof. Let K_N be the function discussed in Exercise 9.1. If we set $\mu_{1,N} = \tau_1$, $E_{1,N} = F_1$, $\mu_{2,N} = \tau_2 * K_N$, $E_{2,N}(N) = F_2 + \operatorname{supp} K_N$ and $f_N = g * K_N$, then $(E_{1,N}, E_{2,N}, \mu_{1,N}, \mu_{2,N}, f_N) \in \mathcal{K}$ and $d\mu_{2,N}(N)(t) = h_N(t)\,dt$ for some infinitely differentiable function h_N. We have

$$d_{\mathcal{H}}(E_{2,N}, F_1) \to 0,\ d_P(\mu_{2,N}, \tau_2) \to 0 \text{ and } \rho(f_N, g) \to 0$$

as $N \to \infty$, so the required result follows on taking

$$(E_1, E_2, \mu_1, \mu_2, f) = (E_{1,N}, E_{2,N}, \mu_{1,N}, \mu_{2,N}, f_N)$$

with N sufficiently large. ■

Proof of Lemma 14.8. (i) By Lemma 14.11, it suffices to consider the case when $d\tau_2(t) = h(t)\,dt$, where h is an infinitely differentiable function. We can now use Exercise 14.10 to tell us that we can find an $M \geq 1$, $\eta > 0$, points $y_p \in \mathbb{T}$ real numbers $\lambda_p \geq 0$ $[1 \leq p \leq M]$ with $\sum_{p=1}^M \lambda_p = 1$ having the following properties.

Whenever $|e_p - y_p| < \eta$ $[1 \leq p \leq M]$ and we write

$$E = \{e_1, e_2, \dots, e_M\} \text{ and } \sigma = \sum_{p=1}^{M} \lambda_p \delta_{e_p}$$

we have

(i) $d_{\mathcal{H}}(E, F_1) < \epsilon/3$,

(ii) $\rho(\sigma * \tau_2, \tau_1 * \tau_2) < \epsilon/3$, and

(iii) $d_P(\sigma, \tau_1) < \epsilon/3$.

Since the complement of L is dense we can certainly choose the $e_p \notin L$. Taking

$$E_1 = E,\ E_2 = F_2,\ \mu_1 = \sigma,\ \mu_2 = \tau_2,\ f = \mu_1 * \mu_2$$

we are done.

(ii) The argument is the same as for (i) but the last but one sentence must be replaced by 'Provided M is large enough we can choose the e_p so that $\chi_M(e_p) = f(e_p)$'. ■

15 The Wiener–Wintner theorem

In a famous paper Wiener and Wintner showed that there exists a singular measure μ (that is to say a measure whose support has Lebesgue measure zero) such that $\mu * \mu$ is absolutely continuous (that is to say $d(\mu * \mu)t = f(t)\,dt$ where f is a Lebesgue L^1 function). The measure of Wiener and

Wintner is very thick (for example high convolution powers of μ correspond to continuous functions). We shall produce other examples of such thick measures later. First we shall produce examples measures μ with extremely thin support such that $\mu * \mu$ is absolutely continuous.

Theorem 15.1. *Let A be a set of first category in $\mathbb{T}$. Then we can find a probability measure μ such that* $\operatorname{supp}\mu \cap A = \varnothing$ *but* $d(\mu * \mu)t = f(t)\,dt$ *where f is a Lebesgue L^1 function.*

It is easy to be lulled by the easy rhythm of Baire category proof into a feeling that one proof is very much like another. You should note that Theorem 15.1 implies the theorem of Debs and Saint-Raymond (since $\hat{\mu}(n)^2 = \widehat{\mu * \mu}(n) = \hat{f}(n) \to 0$ as $|n| \to \infty$) so it can not be trivial.

Exercise 15.2. *We have just used the Riemann–Lebesgue lemma that if $f \in L^1$ (for Lebesgue measure), then $\hat{f}(n) \to 0$ as $|n| \to \infty$. Prove this (for example, by noting that the trigonometric polynomials are L^1 dense).*

Exercise 15.3. *(i) Show that if E is a Kronecker set and* $\operatorname{supp}\mu \subseteq E$*, then*

$$\sup |\hat{\mu}(n)| = \|\mu\|.$$

(ii) If $f \in S(\mathbb{T})$, show that there exist $n(j)$ with $|n(j)| \to \infty$ such that

$$\sup_{t \in E} |\chi_{n(j)}(t) - f(t)| \to 0$$

as $j \to \infty$.

(iii) Show that if E is a Kronecker set and $\operatorname{supp}\mu \subseteq E$ *then*

$$\limsup_{|n| \to \infty} |\hat{\mu}(n)| = \|\mu\|.$$

(i) If μ_1 and μ_2 are the measures appearing in Theorem 14.1, show that

$$\big((\mu_1 + \mu_2) * (\mu_1 + \mu_2)\big)\hat{}\,(n) = (\hat{\mu}_1(n))^2 + 2\hat{\mu}_1(n)\hat{\mu}_2(n) + (\hat{\mu}_2(n))^2 \nrightarrow 0.$$

Conclude that $(\mu_1 + \mu_2) * (\mu_1 + \mu_2)$ is not absolutely continuous.

Having issued this warning we continue along a standard path. We first define a suitable metric space.

Exercise 15.4. *Let $\mathcal{Q}$ be the space consisting of ordered pairs (E, μ) where E is a compact subset of $\mathbb{T}$ and μ is a probability measure with* $\operatorname{supp}\mu \subseteq E$ *and*

$$\mu * \mu = f_\mu$$

with $f_\mu \in L^1$. If we set

$$d_{\mathcal{Q}}\big((E,\mu),(F,\sigma)\big) = d_{\mathcal{H}}(E,F) + \sup_{r\in\mathbb{Z}} 2^{-|r|}|\hat{\mu}(r) - \hat{\sigma}(r)| + \|f_\mu - f_\sigma\|_1$$

for all (E,μ), $(F,\sigma) \in \mathcal{Q}$, show that $(\mathcal{Q}, d_{\mathcal{Q}})$ is a complete non-empty metric space.

We now state the Baire category version of our theorem.

Theorem 15.5. *Let A be a set of first category in $\mathbb{T}$. Then quasi-all $(E,\mu) \in \mathcal{Q}$ have the property that $E \cap A = \varnothing$.*

As usual, we deduce Theorem 15.5 from a simpler result.

Lemma 15.6. *Let L be a compact set in $\mathbb{T}$ with dense complement. Then the set $\mathcal{Q}_L$ of $(E,\mu) \in \mathcal{Q}$ with the property that $E \cap L = \varnothing$ is open and dense.*

Exercise 15.7. *(i) Show that Theorem 15.5 follows from Lemma 15.6.*
(ii) Show that the set $\mathcal{Q}_L$ in Lemma 15.6 is indeed open.

Exercise 15.7 shows that the proof of Theorem 15.5 reduces to the following lemma.

Lemma 15.8. *Let L be a compact set in $\mathbb{T}$ with dense complement. Given $(F,\sigma) \in \mathcal{Q}$ and $\epsilon > 0$, we can find an $(E,\mu) \in \mathcal{Q}$ with*

$$E \cap L = \varnothing \text{ and } d_{\mathcal{Q}}\big((E,\mu),(F,\sigma)\big) \le \epsilon.$$

We can make life somewhat easier for ourselves by spreading out measures in the standard manner. Let us write $\mathcal{Q}_S$ for the set of $(E,\mu) \in \mathcal{Q}$ such that $d\mu(t) = h_\mu(t)\,dt$ with h_μ an infinitely differentiable function.

Exercise 15.9. *Show, by convolving with a function K_n of the type considered in Exercise 9.1, or otherwise, that given $(F,\sigma) \in \mathcal{Q}$ and $\epsilon > 0$ we can find an $(E,\mu) \in \mathcal{Q}_S$ with $d_{\mathcal{Q}}\big((E,\mu),(F,\sigma)\big) \le \epsilon$.*

Thus Lemma 15.8 will follow from the following modified version

Lemma 15.10. *Let L be a compact set in $\mathbb{T}$ with dense complement. Given $(F,\sigma) \in \mathcal{Q}_S$ and $\epsilon > 0$ we can find an $(E,\mu) \in \mathcal{Q}_S$ with*

$$E \cap L = \varnothing \text{ and } d_{\mathcal{Q}}\big((E,\mu),(F,\sigma)\big) \le \epsilon.$$

It is at this point that the proof requires thought. We obtain Lemma 15.10 from the following result.

Lemma 15.11. *Let L be a compact set in $\mathbb{T}$ with dense complement and let $P \geq 2$ Given*

$$(F_1, \sigma_1),\ (F_2, \sigma_2),\ \dots,\ (F_P, \sigma_P) \in \mathcal{Q}_S$$

and $\eta > 0$, we can find an $(E_1, \mu_1) \in \mathcal{Q}_S$ such that $E_1 \cap L = \varnothing$ with the following property.

If we write

$$F = \bigcup_{j=1}^{P} F_j,\ \ \sigma = P^{-1} \sum_{j=1}^{P} \mu_j,\quad E = E_1 \cup \bigcup_{j=2}^{P} F_j,\ \ \mu = P^{-1}\left(\mu_1 + \sum_{j=2}^{P} \sigma_j\right)$$

then $d_{\mathcal{Q}}\big((E, \mu), (F, \sigma)\big) \leq 2P^{-2} + \eta$.

Proof. Let $d\sigma_j(t) = h_j(t)\,dt$ with h_j an infinitely differentiable function. We use Exercise 14.10 to tell us that we can find an $M \geq 1$, $\kappa > 0$, points $y_m \in \mathbb{T}$ and real numbers $\lambda_m \geq 0$ $[1 \leq m \leq M]$ with $\sum_{m=1}^{M} \lambda_p = 1$ having the following properties.

Whenever $|e_m - y_m| < \eta$ $[1 \leq m \leq M]$ and we write

$$E' = \{e_1, e_2, \dots, e_M\} \text{ and } \tau = \sum_{m=1}^{M} \lambda_m \kappa_{e_m}$$

we have

(i) $d_{\mathcal{H}}(E', F_1) < \eta/6$,

(ii) $\sup_{r\in\mathbb{Z}} 2^{-|r|}|\hat{\tau}(r) - \hat{\sigma}_1(r)| < \eta/6$,

(iii) $\|\tau * g_j - g_1 * g_j\|_1 \leq \eta/4$ for $2 \leq j \leq P$.

Since the complement of L is dense, we can choose choose the $e_p \notin L$. We now take $\mu_1(t) = \tau * K_n$ $E_1 = E' + \operatorname{supp} K_n$ where K_n is the function considered in Exercise 9.1. Provided that n is large enough, we have

(i) $d_{\mathcal{H}}(E_1, F_1) < \eta/3$,

(ii) $\sup_{r\in\mathbb{Z}} 2^{-|r|}|\hat{\mu}_1(r) - \hat{\sigma}_1(r)| < \eta/3$,

(iii) $\|\sigma_1 * g_j - g_1 * g_j\|_1 \leq \eta/3$ for $2 \leq j \leq P$.

Condition (iii) tells us that

$$\begin{aligned}
\|\sigma * \sigma - \mu * \mu\|_1 &= P^{-2}\left\|2\sum_{j=2}^{P}(\sigma_1 - \mu_1) * \mu_j + \sigma_1 * \sigma_1 - \mu_1 * \mu_1\right\|_1 \\
&\leq 2P^{-2}\sum_{j=2}^{P}\|\sigma_1 * g_j - g_1 * g_j\|_1 + P^{-2}(\|\sigma_1 * \sigma_1\|_1 + \|\mu_1 * \mu_1\|_1) \\
&\leq 2P^{-2} + \eta/3.
\end{aligned}$$

Combining this result with (i) and (ii), we get

$$d_{\mathcal{Q}}\big((E,\mu),(F,\sigma)\big) \leq 2P^{-2} + \eta$$

as required. ■

Proof of Lemma 15.10 from Lemma 15.11. Set

$$(F_1,\sigma_1) = (F_2,\sigma_2) = \ldots = (F_P,\sigma_P) = (F,\sigma).$$

By using Lemma 15.11 P times we can find $(E_j,\mu_j) \in \mathcal{Q}_S$ such that $E_j \cap L = \varnothing$ for $1 \leq j \leq P$ with the following property.

If we write

$$E = \bigcup_{j=1}^{P} E_j,\ \ \mu = P^{-1}\sum_{j=1}^{P}\mu_j,$$

then $d_{\mathcal{Q}}\big((E,\mu),(F,\sigma)\big) \leq P(P^{-2}+\eta) = P^{-1} + P\eta$.

If we choose P and η so that $P^{-1} + P\eta < \epsilon$, then the required result follows. ■

16 Hausdorff dimension and measures

The Hausdorff dimension provides a useful measure of the thiness of of a set.

Definition 16.1. *If $0 < \kappa \leq 1$, write $h_\kappa(t) = t^\kappa$. We say that a set E has* Hausdorff dimension α *if Edoes not have zero Hausdorff h_κ-measur measure zero for all $\alpha < \kappa$ but does not have Hausdorff h_κ-measure zero for any $\alpha < \kappa$.*

Exercise 16.2. *(i) Let $1 \geq \alpha > \beta > 0$. Show that, if a set E has has Hausdorff h_α-measure zero, then it has Hausdorff h_β-measure zero.*

(ii) If $F \supseteq E$ show that the Hausdorff dimension of F is at least as large as that of E.

(iii) If E does not have Hausdorff h_α-measure zero, show that there is a $\gamma > 0$ such that, for any sequence I_j of intervals,

$$\bigcup_{j=1}^{\infty} I_j \supseteq E \Rightarrow \sum_{j=1}^{\infty} |I_j|^\alpha \geq \gamma.$$

When we construct sets 'by hand', it is often easy to prove upper bounds for the Hausdorff dimension of a set by providing a suitable cover of intervals, but not so simple to prove lower bounds. We shall obtain lower bounds by using the following well known result (the easy part of of theorem of Frostman).

Theorem 16.3. *Let E be a closed set in $\mathbb{T}$ and $1 > \alpha \geq 0$. If we can find a probability measure μ with support contained in E such that*

$$\iint_{\mathbb{T}^2} \frac{d\mu(x)\,d\mu(y)}{|x-y|^\alpha} < \infty,$$

then the Hausdorff dimension of E is at least α.

Proof. Let $t > 0$ and let E_t be the set of $y \in E$ such that

$$\int_{\mathbb{T}} \frac{d\mu(x)}{|x-y|^\alpha} \leq t.$$

We fix t sufficiently large that E_t has strictly positive μ measure.

Consider a covering of E_t by intervals I_j of length $|Ij|$. By choosing a subsequence if necessary, we may suppose that $I_j \cap E_t \neq \varnothing$ for each j. Picking $y_j \in I_j$, we obtain

$$\mu(I_j) = \int_{I_j} d\mu \leq \int_{I_j} \frac{|I_j|^\alpha}{|x-y_j|^\alpha}\,d\mu \leq t|I_j|^\alpha$$

whence

$$t\sum_{j=1}^{\infty} |I_j|^\alpha \geq \sum_{j=1}^{\infty} \mu(I_j) \geq \mu(E_t)$$

and $\sum_{j=1}^{\infty} |I_j|^\alpha \geq t^{-1}\mu(E_t)$. Thus E_t must have Hausdorff dimension at least α (see Exercise [E;easy dimension]) and so (since $E \supseteq E_t$) E must have dimension at least α. ■

Although we shall not make any use of the hard part of Frostman's theorem, it seems a pity not to give it here. We return to the main argument at the end of the proof.

Theorem 16.4. *(i) Let E be a closed set in $\mathbb{T}$ and $1 > \alpha \geq 0$. If E does not have zero Hausdorff-h_α measure (where $h_\alpha(t) = t^\alpha$) then we can find a probability measure μ with support contained in E and a constant $C > 0$ such that*

$$\mu(I) \leq C|I|^\alpha$$

for every interval I.

(ii) Let E be a closed set in $\mathbb{T}$ and $1 > \beta > \alpha \geq 0$. If E has Hausdorff dimension α we can find a probability measure μ with support contained in E such that

$$\iint_{\mathbb{T}^2} \frac{d\mu(x)\,d\mu(y)}{|x-y|^\beta} < \infty.$$

Proof. (i) Since E does not have zero, Hausdorff h_α-measure, Exercise 16.2 (iii) tells us that there exists a $\gamma > 0$ such that

$$\bigcup_{j=1}^{\infty} I_j \supseteq E \Rightarrow \sum_{j=1}^{\infty} |I_j|^\alpha \geq \gamma.$$

Let $\mathcal{I}_n$ be the collection of dyadic intervals $[2\pi r 2^{-n}, 2\pi(r+1)2^{-n})$. If $m \geq 1$ define measures $\tau_{m,r}$ with $0 \leq r \leq m$ as follows. If $I \in \mathcal{I}_m$, then $\tau_{m,0}|_I$, the restriction of $\tau_{m,0}$ to I, is the zero measure if $E \cap I =$ and the uniform measure on I with total mass $2^{-m\alpha}$ if $E \cap I \neq$. Once $\tau_{m,r-1}$ has been defined with $1 \leq r \leq m$, we define $\tau_{m,r}$ as follows. If $I \in \mathcal{I}_{m-r}$

$$\tau_{m,r}|_I = \begin{cases} \tau_{m,r-1}|_I & \text{if } \tau_{m,r-1}(I) \leq 2^{-(m-r)\alpha}, \\ 2^{-(m-r)\alpha}\big(\tau_{m,r-1}(I)\big)^{-1}\tau_{m,r-1}|_I & \text{otherwise.} \end{cases}$$

Finally we set $\tau_m = \tau_{m,m}$.

We observe that if I is a dyadic interval with $I \cap E = \varnothing$, then $\tau_m(I) = 0$. By construction $\tau_m(I) \leq (2\pi)^{-\alpha}|I|^\alpha$ for every dyadic interval of length at least $2\pi\ 2^{-m}$ and each $x \in E$ lies in some dyadic interval I of length at least $2\pi\ 2^{-m}$ such that

$$\tau_m(I) \leq (2\pi)^{-\alpha}|I|^\alpha.$$

and so each $x \in E$ lies in some dyadic interval I_x of greatest length such that

$$\tau_m(I_x) \leq (2\pi)^{-\alpha}|I_x|^\alpha.$$

Let

$$\mathcal{J}_m = \{I_x : x \in E\}.$$

Then $\mathcal{J}_m$ consists of a finite set of disjoint intervals covering E and esatisfying $\tau_m(J) \leq (2\pi)^{-\alpha}|J|^\alpha$ for each $J \in \mathcal{J}_m$. Automatically

$$\|\tau_m\| = \tau_m\left(\bigcup_{J\in\mathcal{J}_m} J\right) = \sum_{J\in\mathcal{J}_m} \tau(J) = (2\pi)^{-\alpha}\sum_{J\in\mathcal{J}} |J|^\alpha \geq (2\pi)^{-\alpha}\gamma.$$

If we now set $\mu_m = \|\tau_m\|^{-1}\tau_m$ we see that μ_m is a probabilty measure such that

$$\operatorname{supp}\mu_m \subseteq E + [2\pi 2^{-m}, 2\pi 2^{-m}]$$

and

$$\operatorname{supp}\mu_m(I) = \|\tau_m\|^{-1}\tau_m(I) \leq \|\tau_m\|^{-1}(2\pi)^{-\alpha}|I|^\alpha \leq \gamma|I|^\alpha$$

for every dyadic interval I of length at least $2\pi\, 2^{-m}$. By weak compactness we can extract a subsequence $\mu_{m(r)}$ tending weakly to some probabilty measure μ. Automatically

$$\operatorname{supp}\mu \subseteq E$$

and

$$\mu(I) \leq \gamma|I|^{\alpha}$$

for every dyadic inteval I.

We now remark that every interval I can be covered by two dyadic intervals I_1 and I_2 of length no greater than, $2|I|$ so

$$\mu(I) \leq \mu(I_1) + \mu(I_2) \leq \gamma(|I_1|^{\alpha} + |I_2|^{\alpha}) \leq 2^{1+\alpha}\gamma|I|^{\alpha}$$

and we are done.

(ii) By part (i) we we can find a probability measure μ with support contained in E and a constant $C > 0$ such that

$$\mu(I) \leq C|I|^{\alpha}$$

for every interval I. By applying the measure theoretic version of integration by parts,

$$\begin{aligned}
\int_{\mathbb{T}} \frac{d\mu(x)}{|x-y|^{\beta}} &= -\int_0^{\pi} \mu\big([y-t, y+t)\big) \frac{\partial}{\partial x}\frac{1}{(x-y)^{\beta}}\bigg|_{x=y+t} dt \\
&= \int_0^{\pi} \mu\big([y-t, y+t)\big) \frac{\beta}{t^{1+\beta}}\, dt \\
&\leq \int_0^{\pi} C(2t)^{\alpha} \frac{\beta}{t^{1+\beta}} = 2^{\alpha} C\beta \int_0^{\pi} \frac{1}{t^{1+\beta-\alpha}}\, dt = A
\end{aligned}$$

for some constant A. Since $\int_{\mathbb{T}} |x-y|^{-\beta}\, d\mu(x)$ is uniformly bounded as a function of y, we have

$$\iint_{\mathbb{T}^2} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{\beta}} < \infty$$

so we are done. ■

The link to Fourier series is established in a rather pretty way. If $0 < x \leq \pi$, let us define the triangle function $\Delta_x : \mathbb{T} \to \mathbb{R}$ by

$$\Delta_x(t) = \begin{cases} 1 - x^{-1}|t| & \text{for } |t| \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

Direct calculation shows that all the Fourier coefficients of Δ_x are positive.

Exercise 16.5. *Show, by direct computation, or otherwise, that*

$$\hat{\Delta_x}(n) = x\left(\frac{2}{nx}\sin\frac{nx}{2}\right)^2 \geq 0$$

for all $n \in \mathbb{Z}$ *and all* $0 < x \leq \pi$.

The observation of Exercise 16.5 becomes more important when combined with the following remark.

Lemma 16.6. *(i) Suppose that* $f : \mathbb{T} \setminus \{0\} \to \mathbb{R}$ *is even (that is to say,* $f(t) = f(-t)$*), twice differentiable and convex (that is to say,* $f''(t) \geq 0$*) on* $(0, \pi]$ *with* $f(\pi) = f'(\pi) = 0$*. Then there exists a continuous positive function* $g : (0, \pi] \to \mathbb{R}$ *such that*

$$f(t) = \int_0^\pi g(x)\Delta_x(t)\,dt$$

for all $0 < |t| \leq \pi$.

(ii) Suppose that $F : \mathbb{T} \setminus \{0\} \to \mathbb{R}$ *is even, twice differentiable on* $(0, \pi]$ *(using one sided derivatives at* π*) and convex on* $(0, \pi]$ *with* $F(\pi) \geq 0$ *and* $F'(\pi-) \leq 0$*. Then there are positive numbers* A *and* B *and a continuous positive function* $g : (0, \pi] \to \mathbb{R}$ *such that*

$$F(t) = A + B\Delta_\pi(x) + \int_0^\pi g(x)\Delta_x(t)\,dt$$

for all $0 < |t| \leq \pi$.

Proof. (i) Set $g(x) = xf''(x)$ and observe that, by integrating by parts,

$$\begin{aligned}\int_0^\pi g(x)\Delta_x(t)\,dx &= \int_t^\pi f''(x)(x-t)\,dx\\ &= [f'(x)(x-t)]_t^\pi - \int_t^\pi f'(x)\,dt = f(t).\end{aligned}$$

(ii) Observe that

$$f = F - F(\pi) - \pi F'(\pi-)\triangle_\pi$$

satisfies the conditions of (i). ■

If the reader draws a few diagrams she will see that that the smoothness conditions on f are irrelevant and we should expect any even, positive, convex on $(0.\pi]$ function to be a weighted integral of triangle functions. This is true but requires more thought about the nature of convexity than we shall give here.

Exercise 16.7. *Suppose that the function f described in Lemma 16.6 is such that g is bounded. (The proof of Lemma 16.6 shows that this will certainly be the case if f'' is.) By using Fubini's theorem and Exercise 16.5, deduce that*

$$\hat{f}(n) \geq 0$$

for all n.

Exercise 16.8. *(i) Suppose that μ is a probability measure. If P is a trigonometric polynomial show that*

$$\iint_{\mathbb{T}^2} P(x-y)\,d\mu(x)\,d\mu(y) = \sum_{r=-\infty}^{\infty} \hat{P}(r)|\hat{\mu}(r)|^2.$$

Show that

$$\iint_{\mathbb{T}^2} f(x-y)\,d\mu(x)\,d\mu(y) = \sum_{r=-\infty}^{\infty} \hat{f}(r)|\hat{\mu}(r)|^2$$

for all functions $f : \mathbb{T} \to \mathbb{C}$ such that $\sum_{r=-\infty}^{\infty} |\hat{f}(r)|$ converges.

(ii) Suppose that $g : \mathbb{T} \to \mathbb{C}$ is a continuous function with $\hat{g}(r)$ real and positive for all $r \in \mathbb{Z}$. By considering the value of the Féjer sums at 0, show that $\sum_{r=-\infty}^{\infty} \hat{g}(r)$ converges. Deduce that, if μ is a probability measure,

$$\iint_{\mathbb{T}^2} g(x-y)\,d\mu(x)\,d\mu(y) = \sum_{r=-\infty}^{\infty} \hat{g}(r)|\hat{\mu}(r)|^2$$

where we observe the convention that if one side of the equation diverges, then the other side must.

Lemma 16.9. *Suppose that $1 > \alpha > 0$ and we define $k : \mathbb{T} \to \mathbb{R}$ by $k(t) = |t|^{-\alpha}$ for $t \neq 0$ $k(0) = 0$. Then, if μ is a probability measure,*

$$\iint_{\mathbb{T}^2} k(x-y)\,d\mu(x)\,d\mu(y) = \sum_{r=-\infty}^{\infty} \hat{k}(r)|\hat{\mu}(r)|^2.$$

Proof. By Lemma 16.6, we can find a continuous positive function $g : (0, \pi] \to \mathbb{R}$ such that

$$f(t) = A + B\Delta_\pi(x) + \int_0^\pi g(x)\Delta_x(t)\,dx$$

for all $0 < |t| \leq \pi$. Define $g_n : [0, \pi] \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} g(t) & \text{if } x \geq n^{-1}, \\ g(1/n) & \text{otherwise.} \end{cases}$$

and set

$$k_n(t) = A + B\Delta_\pi(x) + \int_0^\pi g_n(x)\Delta_x(t)\,dt.$$

We observe that, for each fixed $t \neq 0$, $k_n(t)$ is an increasing sequence with $k_n(t) \to k(t)$ as $n \to \infty$. Using Fubini's theorem

$$\hat{k}_n(r) = B\hat{\Delta}_\pi(r) + \int_0^\pi g_n(x)\hat{\Delta}_x(r)\,dx$$

for $r \neq 0$ and

$$\hat{k}_n(0) = A + B\hat{\Delta}_\pi(0) + \int_0^\pi g_n(x)\hat{\Delta}_x(0)\,dx$$

so $\hat{k}_n(r)$ is a positive increasing sequence. By dominated convergence, $\hat{k}_n(r) \to \hat{k}(r)$.

By Exercise 16.8, we know that

$$\iint_{\mathbb{T}^2} k_n(x-y)\,d\mu(x)\,d\mu(y) = \sum_{r=-\infty}^{\infty} \hat{k}_n(r)|\hat{\mu}(r)|^2$$

so, allowing $n \to \infty$ and applying the monotone convergence theorem to both sides of the equation, we have

$$\iint_{\mathbb{T}^2} k(x-y)\,d\mu(x)\,d\mu(y) = \sum_{r=-\infty}^{\infty} \hat{k}(r)|\hat{\mu}(r)|^2$$

as required. ■

In order to use Theorem 16.3, we need to know something about the behaviour of $\hat{k}(r)$ as $|r| \to \infty$.

Lemma 16.10. *(i) If $1 > \alpha > 0$ and we set*

$$A = \frac{1}{\pi}\int_0^\infty \frac{\cos t}{t^\alpha}\,dt,$$

then $A > 0$.

(ii) Suppose that $1 > \alpha > 0$ and we define $k : \mathbb{T} \to \mathbb{R}$ by $k(t) = |t|^{-\alpha}$ for $t \neq 0$, $k(0) = 0$. Then

$$|r|^{1-\alpha}\hat{k}(r) \to A$$

as $|r| \to \infty$

Proof. (i) We can write

$$A = \sum_{j=0}^{\infty} (-1)^j \frac{1}{\pi} \int_{j\pi}^{(j+1)\pi} \frac{|\cos t|}{t^\alpha}\, dt$$

so, since the error in evaluating an alternating sum (of terms which decrease in absolute size) is no greater than the modulus of the first term neglected,

$$A \geq \frac{1}{\pi} \int_0^{\pi} \frac{|\cos t|}{t^\alpha}\, dt - \frac{1}{\pi} \int_{\pi}^{2\pi} \frac{|\cos t|}{t^\alpha}\, dt > 0.$$

(ii) Observe that

$$\begin{aligned} |r|^{1-\alpha} \hat{k}(r) &= |r|^{1-\alpha} \frac{1}{2\pi} \int_{\mathbb{T}} e^{-irt} k(t)\, dt = |r|^{1-\alpha} \frac{1}{\pi} \int_0^{\pi} \frac{\cos rt}{t^\alpha}\, dt \\ &= \frac{1}{\pi} \int_0^{|r|\pi} \frac{\cos s}{s^\alpha}\, dt \to A \end{aligned}$$

as $|r| \to \infty$. ∎

Putting our results together, we obtain the following key theorem

Theorem 16.11. *Let E be a bounded closed set and $1 > \alpha \geq 0$. If we can find a probability measure μ with support contained in E such that*

$$\sum_{r \neq 0} \frac{|\hat{\mu}(r)|^2}{|r|^{1-\alpha}} < \infty,$$

then the Hausdorff dimension of E is at least α.

Proof. Set $k(t) = t^\alpha$. If

$$\sum_{r \neq 0} \frac{|\hat{\mu}(r)|^2}{|r|^{1-\alpha}} < \infty,$$

then Lemma 16.10 tells us that

$$\sum_{r=-\infty}^{\infty} \hat{k}(r) |\hat{\mu}(r)|^2 < \infty,$$

so Lemma 16.9 gives us

$$\iint_{\mathbb{T}^2} k(x-y)\, d\mu(x)\, d\mu(y) < \infty$$

and Theorem 16.3 tells us that the Hausdorff dimension of E is at least α. ∎

Theorem 16.11 immediately yields the following result of Salem [27]

Theorem 16.12. *If μ is a probability measure whose support has Hausdorff dimension α then*

$$\limsup_{n\to\infty} |n|^{\beta/2}|\hat{\mu}(n)| = \infty$$

for all $\beta > \alpha$.

In particular if μ is a probability measure whose support has Hausdorff dimension α, then

$$\limsup_{n\to\infty} |n|^{\alpha}|\hat{\mu}(n)| = \infty$$

for all $\alpha > 0$.

Exercise 16.13. *It is an easy but rather lengthy exercise to modify the proof of Theorem 7.7 to obtain the following result.*

Suppose that $\phi : \mathbb{N} \to \mathbb{R}$ is a sequence of strictly positive numbers with $r^{\alpha}\phi(r) \to \infty$ as $r \to \infty$ whenever $\alpha > 0$. Then quasi-all $(\mu, E) \in \mathcal{G}_{\phi}$ have the property that E is independent and has Hausdorff dimension zero.

We can make the following observation about Theorem 15.1. (You should recall Theorem 12.3, taking $h(t) = -\log t$.)

Lemma 16.14. *Let A be a set in $\mathbb{T}$ whose complement has Hausdorff dimension zero. Then if μ is a probability measure such that $\operatorname{supp}\mu \cap A = \varnothing$ and $d(\mu * \mu)t = f(t)\,dt$ where f is a Lebesgue L^1 function then*

$$f_m = \underbrace{f * f * \ldots * f}_{m}$$

cannot be a Lebesgue L^2 function for any m.

Proof. Suppose that $f_m \in L^2$. By Hölder's inequality,

$$\begin{aligned}
\sum_{r\neq 0} \frac{|\hat{\mu}(r)|^2}{|r|^{1-m^{-1}/4}} &= \sum_{r\neq 0} \frac{|\hat{f}(r)|}{|r|^{1-m^{-1}/4}} \\
&\leq \left(\sum_{r\neq 0} |\hat{f}(r)|^{2m}\right)^{1/2m} \left(\sum_{r\neq 0} \frac{1}{(|r|^{1-m^{-1}/4})^{2m/(2m-1)}}\right)^{(2m-1)/(2m)} \\
&= \left(\sum_{r\neq 0} |\hat{f}_m(r)|^{m}\right)^{1/2m} \left(\sum_{r\neq 0} \frac{1}{(|r|^{1-m^{-1}/4})^{2m/(2m-1)}}\right)^{(2m-1)/(2m)} \\
&= \|f_m\|_2^{1/m} \left(\sum_{r\neq 0} \frac{1}{|r|^{(2m-1)/(4m-1)}}\right)^{(2m-1)/(2m)} \qquad < \infty
\end{aligned}$$

By Theorem 16.11, it follows that $\operatorname{supp}\mu$ has Hausdorff dimension at least $1/(4m)$. ■

Thus we may think of the measures in Theorem 15.1 as rather thin.

17 Thick Wiener–Wintner measures

In [28], Wiener and Wintner constructed a singular measure μ whose convolution square $\mu * \mu$ was an L^1 function. In [26], Saeki constructed a singular measure μ whose convolution square $\mu * \mu$ was continuous. However, there are strong constraints on how smooth $\mu * \mu$ can be depending on the nature of the support of μ.

Definition 17.1. *If $0 < \alpha \leq 1$, we say that function $f : \mathbb{T} \to \mathbb{C}$ lies in Λ_α the space of* Lipschitz *(or Hölder) α functions if*

$$\sup_{t,h\in\mathbb{T},h\neq 0} |h|^{-\alpha}|f(t+h) - f(t)| < \infty.$$

Exercise 17.2. *Show that*

$$\|f\|_\alpha = \|f\|_\infty + \sup_{t,h\in\mathbb{T},h\neq 0} |h|^{-\alpha}|f(t+h) - f(t)|$$

defines a complete norm on Λ_α.

Using Theorem 16.11 and another result from Fourier analysis, we can relate the Hausdorff dimension of $\operatorname{supp}\mu$ with the possible Lipschitz smoothness of $\mu * \mu$.

Lemma 17.3. *(i) There is a constant C with the following property. If $f : \mathbb{T} \to \mathbb{C}$ is Lischitz β, then*

$$\sum_{n\leq|k|\leq 2n-1} |\hat{f}(k)| \leq C\|f\|_\beta n^{(1-2\beta)/2}$$

(ii) If μ is a measure whose support has Hausdorff dimension α and $d(\mu * \mu)(t) = f(t)\,dt$ where f is Lipschitz β, then $\alpha - \frac{1}{2} \geq \beta$.

(iii) If μ is a measure whose support has Hausdorff dimension α and $d(\mu * \mu)(t) = f(t)\,dt$ where f is continuous, then $\alpha \geq \frac{1}{2}$.

Proof. (i) If $h \in \mathbb{R}$, set

$$g_h(t) = f(t+h) - f(t-h).$$

We have

$$\hat{g}_h(k) = (2\sin kh)\hat{f}(k),$$

so, by Parseval's equality,

$$\begin{aligned}
4\sum_{k=-\infty}^{\infty} |\sin kh|^2 |\hat{f}(k)|^2 = \sum_{k=-\infty}^{\infty} |\hat{g}_h(k)|^2 &= \frac{1}{2\pi}\int_{\mathbb{T}} |g_h(t)|^2\,dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} |f(t+h) - f(t-h)|^2\,dt \\
&\leq \frac{1}{2\pi}\int_{\mathbb{T}} \|f\|_\beta^2 (2h)^\beta\,dt = \|f\|_\beta^2 (2h)^\beta
\end{aligned}$$

If we set $h = 4\pi/n$ and observe that, with this choice,

$$|\sin kh|^2 \geq 1/2$$

for all $n \leq |k| \leq 2n-1$, we obtain

$$2\sum_{n\leq|k|\leq 2n-1} |\hat{f}(k)|^2 \leq \|f\|_\beta^2 (2h)^{2\beta} = \|f\|_\beta^2 (4\pi)^{2\beta} n^{-2\beta}$$

Schwarz's inequality now gives

$$\sum_{n\leq|k|\leq 2n-1} |\hat{f}(k)| \leq \left(\sum_{n\leq|k|\leq 2n-1} 1^2\right)^{1/2} \left(\sum_{n\leq|k|\leq 2n-1} |\hat{f}(k)|^2\right)^{1/2} \leq C\|f\|_\beta n^{(1-2\beta)/2}$$

for an appropriate constant C.

(ii) Since f is Lipschitz β, it follows from part (i)

$$\sum_{n\leq|k|\leq 2n-1} |\hat{f}(k)| \leq C_1 n^{(1-2\beta)/2}$$

for some constant C_1 depending on f. Since $|\hat{f}(k)| = |\hat{\mu}(k)|^2$, we have

$$\sum_{n\leq|k|\leq 2n-1} |\hat{\mu}(k)|^2 \leq C_1 n^{(1-2\beta)/2}$$

and so, if $\eta > 0$,

$$\sum_{k=n}^{2n-1} \frac{|\hat{\mu}(k)|^2}{|k|^{1-\eta}} \leq C_2 n^{-(1+2\beta-2\eta)/2}$$

for all $n \geq 1$ and some constant C_2. By Cauchy's condensation test,

$$\sum_{k \neq 0} \frac{|\hat{\mu}(k)|^2}{|k|^{1-\eta}} \text{ converges}$$

whenever $(1 + 2\beta)/2 > \eta$.

We know, by Theorem 16.11, that if σ is a positive, non-zero measure with

$$\sum_{k \neq 0} \frac{|\hat{\sigma}(k)|^2}{|k|^{1-\eta}} \text{ convergent}$$

for some $0 < \eta < 1$, it follows that the Hausdorff dimension of $\operatorname{supp} \mu$ must be at least η. Thus the Hausdorff dimension of $\operatorname{supp} \mu$ must be at least η for each η with $(1 + 2\beta)/2 > \eta$. We conclude that the Hausdorff dimension of $\operatorname{supp} \mu$ must be at least $(1 + 2\beta)/2$.

(ii) This follows the proof of (i) with $\beta = 0$. ■

We shall show that these constraints are best possible.

Theorem 17.4. *If $1 > \alpha > 1/2$, then there exists a probability measure μ such that the Hausdorff dimension of the support of μ is α and $d(\mu * \mu)(t) = f(t)\, dt$ where f is Lipschitz $\alpha - \frac{1}{2}$.*

(In [19] I give appropriate versions of this result for $\alpha = 1$ and $\alpha = 1/2$.)

The reader will probably be able to guess the framework of our proof.

Exercise 17.5. *If $1/2 > \beta > 0$, let $\mathcal{L}_\beta$ consist of the pairs (E, μ) where E is compact and μ is a probability measure with* $\operatorname{supp} \mu \subseteq E$ *and $d(\mu * \mu)(t) = f_\mu(t)\, dt$ where $f_\mu \in \Lambda_\beta$. If (E, μ), $(F, \sigma) \in \Lambda_\beta$ with $d(\mu * \mu)(t) = f_\mu(t)\, dt$ and $d(\sigma * \sigma)(t) = f_\mu(t)\, dt$, let*

$$d_\beta\big((E, \mu), (F, \sigma)\big) = d_{\mathcal{H}}(E, F) + \sup_{r \in \mathbb{Z}} |\hat{\mu}(r) - \hat{\sigma}(r)| + \|f - g\|_\beta.$$

Show that $(\mathcal{L}_\beta, d_\beta)$ is a complete metric space.

We shall prove the Baire category version of Theorem 17.4 for the space $(\mathcal{L}_\beta, d_\beta)$ defined in Exercise 17.5.

Theorem 17.6. *If $1 > \alpha > 1/2$ and $\beta = \alpha - \frac{1}{2}$, then quasi-all $(E, \mu) \in \mathcal{L}_\beta$ are such that E has Hausdorff dimension α.*

As usual, we can reduce this result to a simpler one.

Lemma 17.7. *Let $\mathcal{H}_{\alpha,n}$ be the subset of consisting of those $(E,\mu) \in \mathcal{L}_\beta$ such that we can find a finite collection of intervals $\mathcal{I}$ with*

$$\bigcup_{I\in\mathcal{I}} I \supseteq E \text{ and } \sum_{I\in\mathcal{I}} |I|^{\alpha+1/n} < 1/n.$$

Then $\mathcal{H}_{\alpha,n}$ is open and dense in $(\mathcal{L}_\beta, d_\beta)$.

Exercise 17.8. *(i) Deduce Theorem 17.6 from Lemma 17.7.*

(ii) Show that, using the notation of Lemma 17.7, $\mathcal{H}_{\alpha,n}$ is open in $(\mathcal{L}_\beta, d_\beta)$.

Let us write $\mathcal{L}_{S,\beta}$ for the set of $(E,\mu) \in \mathcal{L}_\beta$ with f_μ infinitely differentiable.

Exercise 17.9. *Show, by our usual method of convolving with a suitable K_n, or otherwise, that, given $(F,\sigma) \in \mathcal{L}_\beta$ and $\epsilon > 0$, we can find an $(E,\mu) \in \mathcal{L}_{S,\beta}$ with $d_\beta\big((F,\sigma),(E,\mu)\big), \epsilon$.*

Exercise 17.10. *Explain why Exercises 17.8 (ii) and 17.9 enable us to reduce the proof of Lemma 17.7 to the proof of the next lemma (Lemma 17.11).*

Lemma 17.11. *Let Let $1 > \alpha > 1/2$ and $\beta = \alpha - \frac{1}{2}$, Given $(F,\sigma) \in \mathcal{L}_{S,\beta}$ and $\epsilon > 0$, we can find an $(E,\mu) \in H_{\alpha,n}$ with $d_\beta\big((F,\sigma),(E,\mu)\big) \leq \epsilon$.*

Of course, Lemma 17.11 is the heart of the matter. The next two sections are devoted to its proof.

18 More probability

The proof of Lemma 17.11 depends on the following central step.

Lemma 18.1. *If $1 > \gamma > \kappa > 0$ and $\epsilon > 0$, there exist an $M(\alpha,\gamma)$ and $n_0(\kappa,\gamma) \geq 1$ with the following property. If $n \geq n_0$, n is odd and $n^\kappa \geq N > n^\kappa - 1$ we can find N points*

$$x_j \in \{r/n : r \in \mathbb{Z}\}$$

(not necessarily distinct) such that, writing

$$\mu = N^{-1}\sum_{j=1}^{N} \delta_{x_j}$$

we have

$$|\mu * \mu(\{k/n\}) - n^{-1}| \leq n^{\gamma-1/2}$$

and

$$\mu(\{k/n\}) \leq \frac{M(\kappa)}{N}$$

for all $1 \leq k \leq n$.

Since any event with positive probability must have at least one instance, Lemma 18.1 follows from its probabilistic version.

Lemma 18.2. *If* $1 > \gamma > \kappa > 0$, *there exist an* $M(\kappa)$ *and* $n_0(\kappa, \gamma) \geq 1$ *with the following property. Suppose* $n \geq n_0$, n *is odd,* $n^\kappa \geq N > n^\kappa - 1$ *and* X_1, X_2, $\dots$, X_N *are independent random variables each uniformly distributed on*

$$\Gamma_n = \{r/n \in \mathbb{T} : 1 \leq r \leq n\}.$$

Then, if we write, $\sigma = N^{-1} \sum_{j=1}^{N} \delta_{X_j}$, *we have*

$$|\sigma * \sigma(\{k/n\}) - n^{-1}| \leq n^{\gamma - 1/2}$$

and

$$\sigma(\{k/n\}) \leq \frac{M(\kappa)}{N}$$

for all $1 \leq k \leq n$ *with probability at least* $1/2$

Exercise 18.3. *How do you expect* $\sigma * \sigma(\{k/n\}$ *to behave, assuming that the random variables* $X_j + X_k$ $[j \leq k]$ *behave as though they are independent? Are they in fact independent? Why?*

The reader may be inclined to ask three questions.

(1) Why do we take n odd? This is merely a technical convenience. It will be helpful to know that $(k/n) + (k/n) = 0$ only if $k/n = 0$.

(2) Why do we distribute the X_j uniformly on the nth roots of unity rather than uniformly over $\mathbb{T}$? I think (though I have not checked the details) that the arguments would transfer but (at least for me) the details seem messier.

(3) Can we strengthen Lemma 18.1 somewhat? Yes we can (see [19]), but (at least in my argument) we lose any extra sharpness when we come to the proof of Lemma 19.6.

We start our proof of Lemma 18.2 with a simple observation.

Lemma 18.4. *Suppose that* $0 < Np \leq 1$ *and* $m \geq 2$. *Then, if* Y_1, Y_2, $\dots$, Y_N *are independent random variables with*

$$\Pr(Y_j = 1) = p, \ \Pr(Y_j = 0) = 1 - p,$$

it follows that

$$\Pr\left(\sum_{j=1}^{N} Y_j \geq m\right) \leq \frac{2(Np)^m}{m!}.$$

Proof. If we set

$$u_k = \binom{N}{k} p^k,$$

then

$$\Pr\left(\sum_{j=1}^{N} Y_j \geq m\right) = \sum_{k=m}^{N} \Pr\left(\sum_{j=1}^{n} Y_j = k\right) \leq \sum_{k=m}^{N} u_k.$$

But

$$\frac{u_{k+1}}{u_k} = \frac{(N-k)p}{k} \leq \frac{1}{k} \leq \frac{1}{2}$$

for all $N \geq k \geq m$ and so

$$\Pr\left(\sum_{j=1}^{N} Y_j \geq m\right) \leq \sum_{k=m}^{N} u_k \leq 2u_m \leq \frac{2(Np)^m}{m!}.$$

■

Lemma 18.5. *If* $1 > \kappa > 0$, *there exists an* $M(\kappa)$ *with the following property. If* $n^\kappa \geq N > n^\kappa - 1$ *and* $X_1, X_2, \ldots, X_N$ *are independent random variables each uniformly distributed on*

$$\Gamma_n = \{r/n \in \mathbb{T} : 1 \leq r \leq n\}.$$

Then, if we write $\sigma = N^{-1} \sum_{j=1}^{N} \delta_{X_j}$, *we have*

$$\sigma(\{k/n\}) \leq \frac{M(\kappa)}{N}$$

for all $1 \leq k \leq n$. *with probability at least* $1 - (4n^2)^{-1}$

Proof. Take $M(\kappa) = 3(1-\kappa)^{-1}$. It is sufficient to look at the case when $n^\kappa \geq 8$ and so $N \geq 2$. Fix r for the time being and set

$$Y_j = \delta_{X_j}(\{r/n\}).$$

We observe that $Y_1, Y_2, \ldots, Y_N$ are independent random variables with

$$\Pr(Y_j = 1) = n^{-1},\ \Pr(Y_j = 0) = 1 - n^{-1}.$$

By Lemma 18.4, it follows that

$$\Pr\left(\sum_{j=1}^{N} \delta_{X_j}(\{r/n\}) \geq M(\kappa)\right) = \Pr\left(\sum_{j=1}^{N} Y_j \geq M(\kappa)\right)$$

$$\leq \frac{2(Nn^{-1})^{M(\kappa)}}{M(\kappa)!} \leq 2n^{-(1-\kappa)M(\kappa)} \leq 2n^{-3} < \frac{1}{4n^2}.$$

Thus

$$\Pr\left(\sum_{j=1}^{N}\delta_{X_j}(\{r/n\}) \geq M(\kappa) \text{ for some } 0 \leq r \leq n-1\right) < n \times \frac{1}{4n} = \frac{1}{4}$$

as required. ∎

We will need to deal with random variables which are not independent, so we will need the following standard extension of Bernstein's idea (Lemma 8.3).

Definition 18.6. *A sequence W_r is said to be a martingale with respect to a sequence X_r of random variables if*
(i) $\mathbb{E}|W_r| < \infty$,
(ii) $\mathbb{E}(W_{r+1} \,|\, X_0, X_1, \ldots X_r) = W_r$.

Lemma 18.7. *(i) Let $\delta > 0$ and let W_r be a martingale with respect to a sequence X_r of random variables. Write $Y_{r+1} = W_{r+1} - W_r$. Suppose that*

$$\mathbb{E}(e^{\lambda|Y_{r+1}|} \,|\, X_0, X_1, \ldots X_r) \leq e^{a_{r+1}\lambda^2/2}$$

for all $0 < \lambda < \delta$ and some $a_{r+1} \geq 0$. Then

$$\mathbb{E}(e^{\lambda(W_N - W_0)}) \leq e^{A\lambda^2/2}$$

where $A = \sum_{r=1}^{N} a_r$.
(ii) Suppose W is a a random variable with

$$\mathbb{E}(e^{\lambda W}) \leq e^{A\lambda^2/2}$$

for all $0 < \lambda < \delta$. Then, provided that $0 \leq x < \delta A$, we have

$$\Pr\big(|Y| \geq x\big) \leq 2\exp(-x^2/A).$$

Proof. (i) Observe that, if $0 < \lambda < \delta$,

$$\mathbb{E}(e^{\lambda(W_N - W_0)}) = \mathbb{E}(e^{\lambda(Y_1+Y_2+\cdots+Y_N)}) \leq \mathbb{E}\prod_{r=1}^{N}(e^{\lambda|Y_r|}) \leq \prod_{r=1}^{N} e^{a_r\lambda^2/2} = e^{A\lambda^2/2}$$

where $A = \sum_{r=1}^{N} a_r$.
(ii) Use the argument of Lemma 8.3. ∎

We can now embark on the proof of Lemma 18.2.

Proof of Lemma 18.2. Let $M(\kappa)$ be as in Lemma 18.5. Fix r for the time being and define $Y_1, Y_2, \dots, Y_N$ as follows. If $\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\}) < M(\kappa)$ for all u with $1 \le u \le n$, set

$$Y_j = -\frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1}\delta_{X_v+X_j}(\{r/n\}).$$

Otherwise set $Y_j = 0$. We take $W_0 = 0$ and

$$W_j = \sum_{m=1}^{j} Y_m.$$

We now find $\mathbb{E}Y_j$, given that the values of $X_1, X_2, \dots X_{j-1}$ are known. To this end, observe that, if s is a fixed integer, $X_j + s/n$ and $2X_j$ are uniformly distributed over Γ_n. Thus, if $\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\}) < M(\kappa)$ for all u with $1 \le u \le n$, then

$$\begin{aligned}\mathbb{E}Y_j &= -\frac{2j-1}{n} + \mathbb{E}\delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1}\mathbb{E}\delta_{X_v+X_j}(\{r/n\}) \\ &= -\frac{2j-1}{n} + \frac{1}{n} + 2\sum_{v=1}^{j-1}\frac{1}{n} = 0.\end{aligned}$$

If it is not true that $\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\}) < M(\kappa)$ for all u, then, automatically, $\mathbb{E}Y_j = \mathbb{E}0 = 0$. Thus the sequence W_j is a martingale.

In order to apply Lemma 18.7 we must estimate $\mathbb{E}(e^{\lambda Y_j})$, given that the values of $X_1, X_2, \dots X_{j-1}$ are known. First suppose that

$$\sum_{v=1}^{j-1}\delta_{X_v}(\{u/n\}) < M(\kappa)$$

for all u with $1 \le u \le n$. Observe that $Y_j \le 2M(\kappa)$ and, if we set $Z_j = Y_j + (2j-1)/n$, then $\Pr(Z_j \ne 0) \le j/n$. Thus, provided only that n is large enough and $0 \le \lambda \le 1/\bigl(2M(\kappa)\bigr)$,

$$\begin{aligned}
\mathbb{E}(e^{\lambda Y_j}) &= \sum_{k=0}^{\infty} \frac{\lambda \mathbb{E} Y_j^k}{k!} = 1 + \sum_{k=2}^{\infty} \frac{\lambda^k \mathbb{E} Y_j^k}{k!} \le 1 + \sum_{k=2}^{\infty} \frac{|\lambda|^k \mathbb{E}|Y_j|^k}{k!} \\
&= 1 + \Pr(Z_j = 0) \sum_{k=2}^{\infty} \frac{|\lambda|^k \mathbb{E}(|Y_j|^k | Z_j = 0)}{k!} \\
&\qquad + \Pr(Z_j \ne 0) \sum_{k=2}^{\infty} \frac{|\lambda|^k \mathbb{E}(|Y_j|^k | Z_j \ne 0)}{k!} \\
&\le 1 + \sum_{k=2}^{\infty} \frac{\bigl(|\lambda|(2j-1)/n\bigr)^k}{k!} + \frac{j}{n} \sum_{k=2}^{\infty} \frac{\bigl(2|\lambda| M(\gamma)\bigr)^k}{k!} \\
&\le 1 + \sum_{k=2}^{\infty} \frac{\bigl(|\lambda|(2N-1)/n\bigr)^k}{k!} + \frac{N}{n} \sum_{k=2}^{\infty} \frac{\bigl(2|\lambda| M(\gamma)\bigr)^k}{k!} \\
&\le 1 + \frac{\bigl(\lambda(2N-1)/n\bigr)^2}{2} \sum_{k=2}^{\infty} 2^{2-k} + \frac{N}{n} \frac{\bigl(2\lambda M(\gamma)\bigr)^2}{2} \sum_{k=2}^{\infty} 2^{2-k} \\
&= 1 + \left(\left(\frac{2N-1}{N} \right)^2 + \frac{4N}{n} M(\kappa)^2 \right) \lambda^2 \\
&\le 1 + (4M(\kappa)^2 + 1) \frac{N}{n} \lambda^2 \le \exp\left(8(M(\kappa)^2 + 1) \frac{N}{n} \lambda^2 \right).
\end{aligned}$$

If it is not true that $\sum_{v=1}^{j-1} \delta_{X_v}(\{u/n\}) < M(\kappa)$ for all u, then, automatically

$$\mathbb{E}(e^{\lambda Y_j}) = \mathbb{E}(1) = 1 \le \exp\left(8(M(\kappa)^2 + 1) \frac{N}{n} \lambda^2 \right).$$

Combining the two cases we obtain

$$\mathbb{E}(e^{\lambda Y_r} | X_0, X_1, \dots X_{r-1}) \le \exp\left((8M(\kappa)^2 + 1) \frac{N}{n} \lambda^2 \right).$$

Applying Lemma 18.7 (ii) with

$$A = 16(M(\kappa)^2 + 1) \frac{N^2}{n} \text{ and } x = n^{\gamma - 1/2} N^2,$$

we see that (since $n^{\kappa - \gamma} \to \infty$ as $n \to \infty$) we can choose $n_0(\kappa, \gamma)$ so that

$$\Pr(|W_N - W_0| \ge N^2 n^{\gamma - 1/2}) \le 1/(4n)$$

for all $n \geq n_0(\kappa, \gamma)$. For the rest of the proof we assume that n satisfies this condition.

By Lemma 18.5, we know that, with probability at least $1 - 1/(4n)$,

$$\sum_{v=1}^{N} \delta_{X_v}(\{r/n\}) \leq M(\kappa)$$

for all $1 \leq r \leq n$, whence

$$\sum_{v=1}^{j-1} \delta_{X_v}(\{r/n\}) \leq M(\gamma)$$

for all r with $1 \leq r \leq n$ and all $1 \leq j \leq n$ so that

$$Y_j = -\frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1} \delta_{X_v + X_j}(\{r/n\})$$

for all $1 \leq j \leq n$ and so

$$\begin{aligned} W_N - W_0 &= \sum_{j=1}^{N} \left(-\frac{2j-1}{n} + \delta_{2X_j}(\{r/n\}) + 2\sum_{v=1}^{j-1} \delta_{X_v + X_j}(\{r/n\}) \right) \\ &= -\frac{N^2}{n} + \sum_{v=1}^{n} \sum_{j=1}^{n} \delta_{X_v + X_j}(\{r/n\}). \end{aligned}$$

Combining the results of the last two paragraphs, we see that, with probability at least $1 - 1/(2n)$, we have

$$\sum_{v=1}^{j-1} \delta_{X_v}(\{r/n\}) \leq M(\kappa)$$

for all r with $1 \leq r \leq n$ and

$$\left| \sum_{v=1}^{n} \sum_{j=1}^{n} \delta_{X_v + X_j}(\{r/n\}) - \frac{N^2}{n} \right| < N^2 n^{-\gamma - 1/2}.$$

If we write $\sigma = N^{-1} \sum_{j=1}^{N} \delta_{X_j}$, then this inequality can be written

$$|\sigma * \sigma(\{r/n\}) - n^{-1}| \leq n^{-\gamma - 1/2}.$$

If we now allow r to take the values 1 to n, the result follows. ■

19 Point masses to smooth functions

In this section we convert Lemma 18.1 to a more usable form.

We write $\mathbb{I}_A$ for the indicator function of the set A (so that $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ otherwise).

Lemma 19.1. *If $1 > \gamma > \kappa > 0$, there exist an $M(\kappa)$ and $n_0(\kappa, \gamma) \geq 1$ with the following property. Suppose $n \geq n_0$, n is odd and $n^\kappa \geq N > n^\kappa - 1$. Then we can find*

$$x_j \in \{r/n : r \in \mathbb{Z}\}$$

(not necessarily distinct) such that, writing

$$g = \frac{n}{N} \sum_{j=1}^{N} \mathbb{I}_{[x_j-(2n)^{-1}, x_j+(2n)^{-1})},$$

we have $g * g$ continuous and

(i) $\|g * g - 1\|_\infty \leq n^{1/2-\gamma}$,

(ii) $|h|^{-1}|g * g(t+h) - g * g(t)| \leq 2n^{3/2-\gamma}$ for all $t, h \in \mathbb{T}$, $h \neq 0$,

(iii) $|g(t)| \leq M(\gamma)n^{1-\kappa} + 1$ for all $t \in \mathbb{T}$.

Proof. Let x_j and μ be as in Lemma 18.1. Then $g = \mu * n\mathbb{I}_{[-(2n)^{-1},(2n)^{-1})}$ and so

$$g * g = \mu * \mu * \left(n\mathbb{I}_{[-(2n)^{-1},(2n)^{-1})} * n\mathbb{I}_{[-(2n)^{-1},(2n)^{-1})}\right) = \mu * \mu * n^2 \triangle_n$$

where

$$\triangle_n = \max(0, 1 - n|x|).$$

Thus $g * g$ is the simplest piecewise linear function with

$$g * g(r/n) = n\mu * \mu(\{r/n\}).$$

By inspection, $g * g$ is continuous everywhere and linear on each interval $[r/n, (r+1)/n]$. Since

$$|g * g(r/n) - 1| \leq n^{-\gamma+1/2}.$$

Conclusions (i) to (iii) follow at once. ■

Condition (iii) is not very important, but it is helpful to have some bound on $\|g\|_\infty$.

We now smooth g by convolving with a suitable function.

Lemma 19.2. *If $1 > \gamma > \kappa > 0$, there exist an $M_1(\kappa)$ and $n_0(\kappa, \gamma) \geq 1$ with the following property. Suppose $n \geq n_0$ and $n^\kappa \geq N > n^\kappa - 1$. If $n \geq n_0(\kappa, \beta)$ and $n^\gamma \geq N$, we can find a positive infinitely differentiable function f such that*

(i) $\|f * f - 1\|_\infty \leq n^{1/2-\gamma}$.
(ii) $\|(f * f)'\|_\infty \leq n^{3/2-\gamma}$.
(iii) $\|f\|_\infty \leq M_1 n^{1-\kappa}$.
(iv) $\|f'\|_\infty \leq M_1 n^{2-\kappa}$.
(v) $\int_{\mathbb{T}} f(t)\, dt = 1$.
(vi) $\operatorname{supp} f$ *can be covered by n^κ intervals of length $2/n$.*

Proof. (By considering $1 > \gamma > \gamma' > \kappa' > \kappa > 0$ if necessary we can now drop the restriction n odd.) The result follows by considering $g * K_n$ where g is chosen as in Lemma 19.1 and K_n as in Exercise 9.1. ■

Lemma 19.3. *Suppose that $\alpha - \frac{1}{2} > \beta > 0$. If $\epsilon > 0$ there exists an $n_1(\alpha, \beta, \epsilon) \geq 1$ with the following property. If $n > n_1(\alpha, \beta, \epsilon)$ we can find a positive infinitely differentiable function f with the following properties.*

(i) $\|f * f - 1\|_\infty \leq \epsilon$.
(ii) $\|(f * f)'\| \leq \epsilon n^{1-\beta}$.
(iii) $\|f\|_\infty \leq \epsilon n$.
(iv) $\|f'\|_\infty \leq \epsilon n^2$.
(v) $\int_{\mathbb{T}} f(t)\, dt = 1$.
(vi) $\operatorname{supp} f$ *can be covered by less than $\epsilon n^\alpha/2$ intervals of length $2/n$.*
(vii) $|h|^\beta |f * f(t + h) - f * f(t)| \leq \epsilon$ *for all $t, h \in \mathbb{T}$ with $h \neq 0$.*
(viii) $|\hat{f}(r)| \leq \epsilon$ *for all $r \neq 0$.*

Proof. Choose $\kappa = (2/3)\alpha + (1/3)(\beta + 1/2)$ and $\gamma = (1/3)\alpha + (2/3)(\beta + 1/2)$ and take $N = [n^\kappa]$. Provided that n is large enough, Lemma 19.2, tells us that we can find a positive infinitely differentiable function f with the following properties.

(i) $\|f * f - 1\|_\infty \leq \epsilon n^{-\beta}/2$.
(ii) $\|(f * f)'\| \leq \epsilon n^{1-\beta}$.
(iii) $\|f\|_\infty \leq \epsilon n$.
(iv) $\|f'\|_\infty \leq \epsilon n^2$.
(v) $\int_{\mathbb{T}} f(t)\, dt = 1$.
(vi) $\operatorname{supp} f$ can be covered by less than $\epsilon n^\alpha/2$ intervals of length $2n^{-1}$.

By the mean value theorem, condition (ii) gives

$$|h|^{-1} |f * f(t + h) - f * f(t)| \leq \epsilon n^{1-\beta}$$

for all $t, h \in \mathbb{T}$ with $h \neq 0$. In particular,

$$\begin{aligned}|h|^{-\beta}|f * f(t+h) - f * f(t)| &= |h|^{1-\beta}|h|^{-1}|f * f(t+h) - f * f(t)| \\ &\leq \epsilon|h|^{1-\beta}n^{1-\beta} \leq \epsilon\end{aligned}$$

for $|h| \leq n^{-1}$. However, if $|h| \geq n^{-1}$, then using (i),

$$|h|^{-\beta}|f * f(t+h) - f * f(t)| \leq |h|^{-\beta}2\|f * f\|_\infty \leq \epsilon|h|^{-\beta}n^{-\beta} \leq \epsilon.$$

■

Lemma 19.4. *Suppose that $\frac{1}{2} - \alpha > \beta > 0$. Then there there exists an integer $k_0(\alpha, \beta)$ such that, given any ϵ, there exists an $m_1(k, \alpha, \beta, \epsilon) \geq 1$ with the following property. If $m > m_1(k, \alpha, \beta, \epsilon)$, we can find a positive infinitely differentiable function F which is periodic with period $1/m$ and obeys the following conditions.*

(i) $\|F * F - 1\|_\infty \leq \epsilon$.
(ii) $\|(F * F)'\| \leq m^{k(1-\beta)}$.
(iii) $\|F\|_\infty \leq m^k$.
(iv) $\|F'\|_\infty \leq m^{2k+1}$.
(v) $\int_\mathbb{T} F(t) = 1$.
(vi) We can find a finite collection of intervals $\mathcal{I}$ such that

$$\bigcup_{I \in \mathcal{I}} I \supseteq \operatorname{supp} F \text{ and } \sum_{I \in \mathcal{I}} |I|^\alpha < \epsilon.$$

(vii) $|h|^\beta|F * F(t+h) - F * F(t)| \leq \epsilon$ for all t, $h \in \mathbb{T}$ with $h \neq 0$.
(viii) $|\hat{F}(r)| \leq \epsilon$ for all $r \neq 0$.

Proof. Let

$$\alpha_1 = \frac{3}{4}\alpha + \frac{1}{4}\left(\beta + \frac{1}{2}\right), \ \beta_1 = \frac{1}{4}\left(\alpha - \frac{1}{2}\right) + \frac{3}{4}\beta.$$

By Lemma 19.3 with $n = m^k$ we know that, provided only that m is large enough, we can find a positive infinitely differentiable function f with the following properties

(i′) $\|f * f - 1\|_\infty \leq \epsilon$.
(ii′) $\|(f * f)'\| \leq \epsilon m^{k(1-\beta_1)}$
(iii′) $\|f\|_\infty \leq \epsilon m^k$,
(iv′) $\|f'\|_\infty \leq \epsilon m^{2k}$
(v′) $\int_\mathbb{T} f(t)\, dt = 1$.
(vi′) $\operatorname{supp} f$ can be covered by less than $m^{k\alpha}/2$ intervals of length $2m^{-k}$.

(vii′) $|h|^{\beta_1}|f*f(t+h)-f*f(t)|\leq\epsilon$ for all $t,h\in\mathbb{T}$ with $h\neq 0$.

(viii′) $|\hat{f}(r)|\leq\epsilon$ for all $r\neq 0$.

If we set $F(t)=f(mt)$, we see, at once, that F is positive infinitely differentiable function such that

(i) $\|F*F-1\|_\infty\leq\epsilon$.

(ii) $\|(F*F)'\|\leq\epsilon m^{k(1-\beta_1)+1}$

(iii) $\|F\|_\infty\leq\epsilon m^k$.

(iv) $\|F'\|_\infty\leq\epsilon m^{2k+1}$.

(v) $\int_{\mathbb{T}}F(t)\,dt=1$.

(vi) We can find a collection $\mathcal{I}$ of at most $m^{1+k\alpha_1}/2$ intervals each of of length m^{-k-1} such that

$$\bigcup_{I\in\mathcal{I}}I\supseteq\operatorname{supp}f.$$

Provided that k is large enough (depending only on α and β) the result follows. ■

We shall need some results on Λ_β which are obtained in much the same way as the corresponding results on differentiation.

Exercise 19.5. *If $f\in\Lambda_\beta$ let us write*

$$\omega_\beta(f)=\sup_{t,h\in\mathbb{T},h\neq 0}|h|^{-\beta}|f(t+h)-f(t)|$$

so that $\|f\|_\beta=\|f\|_\infty+\omega_\beta(f)$. Prove the following results.

(i) If f, $g\Lambda_\beta$, then $f+g\in\Lambda_\beta$ and

$$\omega_\beta(fg)\leq\omega_\beta(f)\|g\|_\infty+\omega_\beta(g)\|f\|_\infty.$$

(ii) If $f\in\Lambda_\beta$ and $g\in L^1(\mathbb{T})$, then $f*g\in\Lambda_\beta$ and

$$\omega_\beta(f*g)\leq\omega_\beta(f)\|g\|_1.$$

(iii) If $f:\mathbb{T}\to\mathbb{C}$ has continuous derivative, then $f\in\Lambda_\beta$ and

$$\omega_\beta(f)\leq\|f'\|_\infty.$$

We now prove Lemma 17.11 and conclude the proof. In fact we shall prove the result a slightly more concrete form.

Lemma 19.6. *Let Let $1>\alpha>1/2$ and $\beta=\alpha-\frac{1}{2}$. Suppose that $g:\mathbb{T}\to\mathbb{R}$ is an infinitely positive differentiable function with*

$$\int_{\mathbb{T}}g(t)\,dt=1$$

and H is closed set with $H \supseteq \operatorname{supp} g$. Then, given $\epsilon > 0$, we can find an infinitely differentiable positive function $f : \mathbb{T} \to \mathbb{R}$ with

$$\int_{\mathbb{T}} f(t)\, dt = 1$$

and a closed set $E \supseteq \operatorname{supp} f$ such that, writing $d\sigma(t) = g(t)\, dt$, $d\mu(t) = f(t)\, dt$, we have $(E, \mu) \in \mathcal{H}_n$ and

$$d_\beta\big((E,\mu),(H,\sigma)\big) < \epsilon.$$

Proof. Since $\mathcal{H}_n \supseteq \mathcal{H}_{n+1}$, we may restrict ourselves to the case when $\alpha + 1/n < 1$. Lemma 19.4 tells us that we can find a positive integer k with the property described in the next sentence. Let $\eta > 0$, then, when m is sufficiently large, we can find a positive infinitely differentiable function F_m which is periodic with period $1/(2m+1)$ satisfying the following conditions. (The result corresponding to $(\mathrm{ii})_m$ is not required.)

$(\mathrm{i})_m$ $\|F_m * F_m - 1\|_\infty \leq \eta$.

$(\mathrm{iii})_m$ $\|F_m\|_\infty \leq 4^{2k} m^{2k}$.

$(\mathrm{iv})_m$ $\|F_m'\|_\infty \leq 4^{2k+1} m^{2k+1}$.

$(\mathrm{v})_m$ $\int_{\mathbb{T}} F_m(t)\, dt = 1$.

$(\mathrm{vi})_m$ We can find a finite collection of intervals $\mathcal{I}_m$ such that

$$\bigcup_{I \in \mathcal{I}_m} I \supseteq \operatorname{supp} F_m \text{ and } \sum_{I \in \mathcal{I}_m} |I|^{\alpha + 1/n} < \frac{1}{n}.$$

$(\mathrm{vii})_m$ $\omega_\psi(F_m * F_m) \leq \eta$.

$(\mathrm{viii})_m$ $|\hat{F}_m(r)| \leq \eta$ for all $r \neq 0$.

Since g is infinitely differentiable, repeated integration by parts shows that there exists a constant C_1 such that

$$|\hat{g}(r)| \leq C_1 |r|^{-(2k+4)}$$

for $r \neq 0$ and so there exists a constant C such that

$$\sum_{|r| \geq m} |r||\hat{g}(r)| \leq C|m|^{-(2k+2)} \qquad \bigstar$$

for all $m \geq 1$.

If we set $G_m(t) = g(t) F_m(t)$ and

$$f(t) = \left(\int_{\mathbb{T}} G_m(s)\, ds\right)^{-1} G_m(t),$$

then, automatically,

$$\operatorname{supp} f \subseteq E \cap \operatorname{supp} F_m.$$

Thus, by choosing an appropriate finite set A and setting $H = A \cup \operatorname{supp} f$, we can ensure that $(E, \mu) \in \mathcal{L}_\beta$,

$$d_\beta\big((E, \mu), (H, \sigma)\big) < \epsilon/4.$$

and we can find a finite collection of intervals $\mathcal{I}$ such that

$$\bigcup_{I \in \mathcal{I}_m} I \supseteq E \text{ and } \sum_{I \in \mathcal{I}_m} |I|^{\alpha + 1/n} < \frac{1}{n}.$$

We have shown that (setting $d\mu(t) = f(t)\,dt$) $(E, \mu) \in \mathcal{H}_n$ and all we need to do is to show that, for appropriate choices of η and m we have

$$\sup_{r \in \mathbb{Z}} |\hat{f}(r) - \hat{g}(r)| < \epsilon/4,\ \|f * f - g * g\|_\infty < \epsilon/4 \text{ and } \omega_\beta(f * f - g * g) < \epsilon/4.$$

Without loss of generality we may suppose $\epsilon < 1$, so simple calculations show that it is sufficient to prove

$$\sup_{r \in \mathbb{Z}} |\hat{g}(r) - \hat{G}_m(r)| < \epsilon/8,\ \|f * f - G_m * G_m\|_\infty < \epsilon/8 \text{ and } \omega_\beta(f * f - g * g) < \epsilon/8.$$

Using $(vii)_m$, we have

$$\begin{aligned} |\hat{g}(r) - \hat{G}_m(r)| &= \left|\hat{g}(r) - \sum_{j=-\infty}^{\infty} \hat{g}(r-j)\hat{F}_m(j)\right| \\ &= \left|\sum_{u \neq 0} \hat{g}(r-u)\hat{F}_m(u)\right| \leq \sum_{u \neq 0} |\hat{g}(r-u)||\hat{F}_m(u)| \\ &\leq \sum_{u \neq 0} |\hat{g}(r-u)|\eta \leq \eta \sum_{j=-\infty}^{\infty} |\hat{g}(j)| < \epsilon/8 \end{aligned}$$

for all r provided only that η is small enough. We now fix η once and for all so that the inequality just stated holds and

$$\eta\big((1 + \|g\|_\infty)^2 + \omega_\beta(g * g) + 2\big) < \epsilon/12$$

but leave m free.

We have now arrived at the central estimates of the proof which show that

$$\|g * g - G_m * G_m\|_\infty < \epsilon/8 \text{ and } \omega_\beta(g * g - G_m * G_m) < \epsilon/8,$$

provided only that m is large enough. The proofs of the two inequalities are similar. We start with the first which is slightly easier.

We write

$$P_m(t) = \sum_{|r|\leq m} \hat{g}(r)\exp(irt) \text{ and } g_m(t) = g(t) - P_m(t).$$

By $\bigstar$, we see that, if $m \geq 1$,

$$\|g - P_m\|_\infty, \|g' - P'_m\|_\infty \leq C|m|^{-(2k+2)}. \qquad \bigstar\bigstar$$

We shall take m sufficiently large that

$$\|g - P_m\|_\infty, \|g' - P'_m\|_\infty \leq 1.$$

Now since F_m is periodic with period $1/(2m+1)$ and P_m is a trigonometric polynomial of degree at most m,

$$\widehat{P_m F_m}\big((2m+1)u + v\big) = \hat{F}_m((2m+1)u\big)\hat{P}_j(v)$$

for all u and v so that

$$\begin{aligned}\big((P_mF_m) * (P_mF_m)\big)\hat{}\big((2m+1)u + v\big) &= \big(\hat{F}_m\big((2m+1)u\big)\hat{P}_j(v)\big)^2 \\ &= \bigg((P_mF_m)\hat{}\big((2m+1)u + v\big)\bigg)^2 \\ &= \big((P_m * P_m)(F_m * F_m)\big)\hat{}((2m+1)u + v\big)\end{aligned}$$

and

$$(P_mF_m) * (P_mF_m)(t) = (P_m * P_m)(t)(F_m * F_m)(t).$$

Using this equality, we obtain

$$\begin{aligned}\|g * g - G_m * G_m\|_\infty &= \|g * g - (gF_m) * (gF_m)\|_\infty \\ &\leq \|g * g - P_m * P_m\|_\infty + \|P_m * P_m - (P_mF_m) * (P_mF_m)\|_\infty \\ &\qquad + \|(P_mF_m) * (P_mF_m) - (gF_m) * (gF_m)\|_\infty \\ &= \|g * g - P_m * P_m\|_\infty + \|P_m * P_m - (P_m * P_m)(F_m * F_m)\|_\infty \\ &\qquad + \|(P_mF_m) * (P_mF_m) - (gF_m) * (gF_m)\|_\infty.\end{aligned}$$

We estimate the three terms separately.

First we observe that

$$\begin{aligned}\|g * g - P_m * P_m\|_\infty &= \|(g - P_m) * (g - P_m) + 2(g - P_m) * P_m\|_\infty \\ &\leq \|(g - P_m) * (g - P_m)\|_\infty + 2\|(g - P_m) * P_m\|_\infty \\ &\leq \|g - P_m\|_\infty^2 + 2\|g - P_m\|\|P_m\|_\infty \\ &\leq \|g - P_m\|_\infty^2 + 2\|g - P_m\|(1 + \|g\|_\infty) < \epsilon/12,\end{aligned}$$

provided only that m is large enough.

Next we observe that

$$\begin{aligned}
&\|(P_mF_m) * (P_mF_m) - (gF_m) * (gF_m)\|_\infty \\
&\quad \le \big\|\big((g-P_m)F_m\big) * \big((g-P_m)F_m\big)\|_\infty + 2\big\|\big((g-P_m)F_m\big) * (gP_m)\|_\infty \\
&\quad \le \|(g-P_m)F_m\|_\infty^2 + \|(g-P_m)F_m\|_\infty\|gP_m\|_\infty \\
&\quad \le (\|(g-P_m)\|_\infty\|F_m\|_\infty)^2 + \|g-P_m\|_\infty\|F_m\|_\infty\|g\|_\infty\|P_m\|_\infty \\
&\quad \le (\|(g-P_m)\|_\infty\|F_m\|_\infty)^2 + \|g-P_m\|_\infty\|F_m\|_\infty\|g\|_\infty(1+\|g\|_\infty) \\
&\quad \le \left(\frac{C}{m^{2k+2}}4^km^k\right)^2 + \frac{C}{m^{2k+2}}4^km^k\|g\|_\infty(1+\|g\|_\infty) < \epsilon/12,
\end{aligned}$$

provided only that m is large enough.

Finally we note that

$$\begin{aligned}
\|P_m * P_m - (P_m * P_m)(F_m * F_m)\|_\infty &= \|(P_m * P_m)(1-(F_m * F_m)\|_\infty \\
&= \|P_m * P_m\|_\infty\|(1-(F_m * F_m)\|_\infty \\
&\le \|P_m\|_\infty^2\|(1-(F_m * F_m)\|_\infty \\
&\le (1+\|g\|_\infty)^2\eta < \epsilon/12.
\end{aligned}$$

Combining our estimates we obtain

$$\|g * g - G_m * G_m\|_\infty < \epsilon/4$$

as required.

We turn now to the second inequality. Much as before,

$$\begin{aligned}
&\omega_\beta(g * g - G_m * G_m) \\
&\qquad = \omega_\beta(g * g - P_m * P_m) + \omega_\beta\big(P_m * P_m - (P_m * P_m)(F_m * F_m)\big) \\
&\qquad\qquad + \omega_\beta\big(P_mF_m) * (P_mF_m) - (gF_m) * (gF_m)\big).
\end{aligned}$$

We bound the first term.

$$\begin{aligned}
\omega_\beta(g * g - P_m * P_m) &\le \omega_\beta\big((g-P_m) * (g-P_m)\big) + 2\omega_\beta\big((g-P_m) * P_m\big) \\
&\le \big\|\big(((g-P_m) * (g-P_m)\big)'\big\|_\infty + 2\big\|\big((g-P_m) * P_m\big)'\big\|_\infty \\
&= \|(g-P_m)' * (g-P_m)\|_\infty + 2\|(g-P_m)' * P_m\|_\infty \\
&\le \|(g-P_m)'\|_\infty\|g-P_m\|_\infty + 2\|(g-P_m)'\|_\infty\|P_m\|_\infty \\
&\le \|(g-P_m)'\|_\infty\|g-P_m\|_\infty + 2\|(g-P_m)'\|_\infty(1+\|g\|_\infty) < \epsilon/12,
\end{aligned}$$

provided only that m is large enough.

Next we bound the third term.

$$
\begin{aligned}
&\omega_\beta\big((P_mF_m) * (P_mF_m) - (gF_m) * (gF_m)\big) \\
&\le \omega_\beta\bigg(\big((g-P_m)F_m\big) * \big((g-P_m)F_m\big)\bigg) + 2\omega_\beta\bigg(\big((g-P_m)F_m\big) * (gP_m)\bigg) \\
&\le \big\|\big(\big((g-P_m)F_m\big) * \big((g-P_m)F_m\big)\big)'\big\|_\infty + 2\big\|\big((g-P_m)F_m\big) * (gP_m)\big)'\big\|_\infty \\
&= \|\big((g-P_m)F_m\big)' * \big((g-P_m)F_m\big)\| + 2\big\|\big((g-P_m)F_m\big)' * (gP_m)\big\|_\infty \\
&\le \|((g-P_m)F_m)'\|_\infty \|(g-P_m)F_m\|_\infty + 2\|((g-P_m)F_m)'\|_\infty \|gP_m\|_\infty \\
&\le (\|(g-P_m)'\|_\infty \|F_m\|_\infty + \|g-P_m\|_\infty \|F_m'\|_\infty)\|g-P_m\|_\infty \|F_m\|_\infty \\
&\qquad + 2(\|(g-P_m)'\|_\infty \|F_m\|_\infty + \|g-P_m\|_\infty \|F_m'\|_\infty)\|g\|_\infty \|P_m\|_\infty \\
&\le \left(\frac{C}{m^{2k+2}}4^{2k}m^{2k} + \frac{C}{m^{2k+2}}4^{2k+1}m^{2k+1}\right)\frac{C}{m^{2k+2}}4^{2k}m^{2k} \\
&\qquad\qquad + 2\left(\frac{C}{m^{2k+2}}4^{2k}m^{2k} + \frac{C}{m^{2k+2}}4^{2k+1}m^{2k+1}\right)\|g\|_\infty(1+\|g\|_\infty) \\
&< \frac{\epsilon}{12},
\end{aligned}
$$

provided only that m is large enough.

Finally we estimate the second term

$$
\begin{aligned}
\omega_\beta\big(P_m * P_m - (P_m * P_m)(F_m * F_m)\big) &= (\omega_\beta\big(P_m * P_m)(1 - (F_m * F_m)\big) \\
&\le \|P_m * P_m\|_\infty \omega_\beta\big((1-(F_m * F_m)\big) + \omega_\beta(P_m * P_m)\|1 - F_m * F_m\|_\infty \\
&\le \|P_m * P_m\|_\infty \omega_\beta(F_m * F_m) + \omega_\beta(P_m * P_m)\|1 - F_m * F_m\|_\infty \\
&\le \|P_m * P_m\|_\infty \eta + \omega_\beta(P_m * P_m)\eta.
\end{aligned}
$$

Estimates of a familiar kind show that

$$
\begin{aligned}
\omega_\beta(P_m * P_m) &\le \omega_\beta(g * g) + \omega_\beta(P_m * P_m - g * g) \\
&\le \omega_\beta(g * g) + \omega_\beta\big((P_m - g) * (P_m - g)\big) + 2\omega_\beta\big((P_m - g) * g\big) \\
&\le \omega_\beta(g * g) + \|(g-P_m)'\|_\infty \|g - P_m\|_\infty + \|(g-P_m)'\|_\infty \|g\|_\infty \\
&\le \omega_\beta(g * g) + 1
\end{aligned}
$$

and, similarly,

$$
\|P_m * P_m\|_\infty \le \|g * g\|_\infty + 1 \le \|g\|_\infty^2 + 1,
$$

provided only that m is large enough. Thus

$$
\omega_\beta\big(P_m * P_m - (P_m * P_m)(F_m * F_m)\big) \quad \le (\omega_\beta(g * g) + \|g\|_\infty^2 + 2)\eta < \epsilon/12,
$$

provided only that m is large enough.

Combining our estimates we obtain

$$\omega_\beta(g * g - G_m * G_m) < \epsilon/4$$

and this completes the proof. ∎

The estimates which occupy the last two pages of the previous proof are nowhere delicate and I strongly suspect that there is a much shorter direct argument which does not use Fourier series.

20 Hausdorff dimension and sums

We have concluded the main business of these notes, but I cannot resist including one further result on sums and Hausdorff dimension.

Theorem 20.1. *Given a sequence α_j with $0 \leq \alpha_j \leq \alpha_{j+1} < 1$, we can find a closed set E such that*

$$E_{[j]} = \underbrace{E + E + \ldots + E}_{j}$$

has Hausdorff dimension α_j for each $j \geq 1$.

(See also Exercise 21.5.)

We shall use Theorem 16.3 which the reader is invited to reread together with a couple of elementary observation.

Lemma 20.2. *Let $1 > \beta > \alpha \geq 0$. Let g be a piecewise continuous positive function. If we define g_n, for $n^{1-(1/\beta)} \geq 2$, by the conditions*

$$g_n(x) = \begin{cases} a_{r,n} & \text{if } |x - rn^{-1}| \leq n^{-1/\beta},\ r \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$a_{r,n} = \int_{(r-1/2)/n}^{(r+1/2)/n} g(x)\,dx,$$

then

$$\iint_{\mathbb{T}^2} \frac{g_n(x)g_n(y)}{|x-y|^\alpha}\,dx\,dy \to \iint_{\mathbb{T}^2} \frac{g(x)g(y)}{|x-y|^\alpha}\,dx\,dy$$

as $n \to \infty$.

Proof. We show that, in fact,

$$\int_{\mathbb{T}} \frac{g_n(x)}{|x-y|^\alpha}\,dx \to \int_{\mathbb{T}} \frac{g(x)}{|x-y|^\alpha}\,dx,$$

uniformly in y. To this end, observe that, if $10^{-1} > \delta > 0$,

$$\int_{|x-y|\geq\delta} \frac{g_n(x)}{|x-y|^\alpha}\,dx \to \int_{|x-y|\geq\delta} \frac{g(x)}{|x-y|^\alpha}\,dx$$

uniformly as $n \to \infty$. Next note that

$$\int_{|x-y|\leq\delta} \frac{g(x)}{|x-y|^\alpha}\,dx \leq \|g\|_\infty \int_{|x|\leq\delta} |x|^\alpha\,dx = \frac{2\|g\|_\infty}{1-\alpha}\delta^{1-\alpha} \to 0$$

as $\delta \to 0$. Finally observe that simple estimates give $|a_{r,n}| \leq 2n^{(1/\beta)-1}\|g\|_\infty$ and

$$\begin{aligned}
\int_{|x-y|\geq\delta} \frac{g_n(x)}{|x-y|^\alpha}\,dx &\leq 2\|g\|_\infty n^{(1/\beta)-1}\int_{|x|\leq 8n^{-1/\beta}} \frac{1}{|x|^\alpha}\,dx + 2\|g\|_\infty \sum_{1\leq r\leq n\delta} \frac{n^\alpha}{|r|^\alpha} \\
&\leq 2\|g\|_\infty n^{(1/\beta)-1}\frac{8^{1-\alpha}}{1-\alpha}n^{-(1-\alpha)/\beta} + \frac{4\|g\|_\infty}{1-\alpha}\delta^{1-\alpha} \\
&\leq \frac{16\|g\|_\infty}{1-\alpha}n^{(\alpha/\beta)-1} + \frac{4\|g\|_\infty}{1-\alpha}\delta^{1-\alpha} \to 0
\end{aligned}$$

as $\delta \to 0$ and $n \to \infty$. ∎

Lemma 20.3. *Let j be a strictly positive integer and let $K > 0$. Suppose that $E(n)$ is a closed subset of $\mathbb{T}$ such that there exists a probability measure μ_n with*

$$\operatorname{supp}\mu_n \subseteq E(n)_{[j]} \text{ and } \iint_{\mathbb{T}^2} \frac{d\mu_n(x)\,d\mu_n(y)}{|x-y|^\alpha} \leq K.$$

Then, if $E \in \mathcal{F}$ and $d_{\mathcal{F}}(E(n), E) \to 0$ as $n \to \infty$, there exists a probability measure μ with

$$\operatorname{supp}\mu \subseteq E_{[j]} \text{ and } \iint_{\mathbb{T}^2} \frac{d\mu(x)\,d\mu(y)}{|x-y|^\alpha} \leq K.$$

Proof. Since the set of probability measures is weak-star compact, we may suppose, by extracting a subsequence, that $\mu_n \to \mu$ weakly as $n \to \infty$. Since $d_{\mathcal{F}}(E(n), E) \to 0$ we have $d_{\mathcal{F}}(E(n)_{[j]}, E_{[j]}) \to 0$ and $\operatorname{supp}\mu \subseteq E_{[j]}$. Since

$$\iint_{\mathbb{T}^2} \frac{d\mu(x)\,d\mu(y)}{|x-y|^\alpha} \leq \liminf_{n\to\infty} \iint_{\mathbb{T}^2} \frac{d\mu_n(x)\,d\mu_n(y)}{|x-y|^\alpha} \leq K$$

we are done. ∎

Lemma 20.4. *We work in $(\mathcal{E}, d_{\mathcal{E}})$ the space of compact subsets of $\mathbb{T}$ with the usual Hausdorff metric $d_{\mathcal{E}}$. Let $0 \leq \alpha_j \leq \alpha_{j+1} < 1$ and $K_j > 0$. Let $\mathcal{G}$ be the collection of compact sets E such that, for each $j \geq 1$, there exists a probability measure μ_j with*

$$\operatorname{supp} \mu_j \subseteq E_{[j]} \text{ and } \iint_{\mathbb{T}^2} \frac{d\mu_j(x)\, d\mu_j(y)}{|x-y|^{\alpha_j}} \leq K_j.$$

Then $\mathcal{G}$ is a closed subset of $(\mathcal{E}, d_{\mathcal{E}})$.

As matters stand, $\mathcal{G}$ could be empty. However, if E is the union of a finite collection of closed intervals (for example if $E = \mathbb{T}$), then, if we take tτ to be the uniform probability measure on E and set

$$K_j = 1 + \iint_{\mathbb{T}^2} \frac{d\tau(x)\, d\tau(y)}{|x-y|^{\alpha_j}},$$

we will have $E \in \mathcal{G}$.

From now on, the α_j will form a fixed sequence satisfying the conditions of Lemma 20.4 and the K_j will be fixed sequence chosen so that

$$K_j > \iint_{\mathbb{T}^2} \frac{1}{|x-y|^{\alpha_j}} dx\, dy.$$

If $d_{\mathcal{G}}$ is the restriction of the metric $d_{\mathcal{E}}$ to the space $\mathcal{G}$, we now know that $(\mathcal{G}, d_{\mathcal{G}})$ is complete and non-empty. Theorem 20.1 thus follows from its Baire category version.

Theorem 20.5. *The set of $E \in \mathcal{G}$ such that $E_{[j]}$ has Hausdorff dimension α_j for all $j \geq 1$ is of second category in $(\mathcal{G}, d_{\mathcal{G}})$.*

We can now reduce the proof of Theorem 20.5 to the following lemma in our usual manner.

Lemma 20.6. *Let $\eta > 0$ and $n \geq 1$. Then the set $\mathcal{J}$ of $E \in \mathcal{G}$ such that there exist a finite collection $\mathcal{I}$ of closed intervals with*

$$\bigcup_{I \in \mathcal{I}} I \supseteq E_{[n]} \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha_n + \eta} < \eta$$

is dense in $(\mathcal{G}, d_{\mathcal{G}})$.

Proof of Theorem 20.5 from Lemma 20.6. We first observe that if

$$\bigcup_{I \in \mathcal{I}} I \supseteq E_{[n]} \text{ and } \sum_{I \in \mathcal{I}} |I|^{\alpha_n + \eta} < \eta$$

then, if $\theta > 0$ is small enough,

$$\sum_{I \in \mathcal{I}} \left|(I + [-\theta, \theta])\right|^{\alpha_n + \eta} < \eta$$

and

$$\bigcup_{I \in \mathcal{I}} (I + [-\theta, \theta]) \supseteq F_{[n]}$$

whenever $d(F, E) < \theta/n$. Thus $\mathcal{E}$ is open.

Let us write $\mathcal{E}(j, m)$ for the set of $E \in \mathcal{G}$ such that there exist a finite collection $\mathcal{I}(j, m)$ of closed intervals with

$$\bigcup_{I \in \mathcal{I}(j,m)} I \supseteq E_{[j]} \text{ and } \sum_{I \in \mathcal{I}(j,m)} |I|^{\alpha_j + 1/m} < 1/m. \qquad \bigstar$$

By the first paragraph and the conclusion of Lemma 20.6 $\mathcal{I}(j, m)$ is open and dense, so the complement of

$$\mathcal{H} = \bigcap_{j=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{E}(j, m)$$

is of first category in in $(\mathcal{G}, d_{\mathcal{G}})$.

If $E \in \mathcal{H}$ and $j \geq 1$, then the definition of $\mathcal{G}$ together with Theorem 16.3 tells us that $E_{[j]}$ has Hausdorff dimension at least α_j. However, $E_{[j]}$ also obeys the conditions given in $\bigstar$ so $E_{[j]}$ has Hausdorff dimension at most α_j and we are done. ■

21 The final construction

The central step in our proof of Lemma 20.6 is laid out in the next lemma.

Lemma 21.1. *Let δ, $\eta > 0$, and $n, m \geq 1$. Write*

$$\Lambda = \{r : n + m \geq r \geq 1\}$$

Suppose that E_1, E_2, …, E_{n+m} are each the finite union of non-trivial closed intervals such that, whenever $L \subseteq \Lambda$, and L contains j elements with $n \geq j \geq 1$, there exists a piecewise continuous positive $g_L : \mathbb{T} \to \mathbb{R}$ with

$$\operatorname{supp} g_L \subseteq \left(\bigcup_{r \in L} E_r\right)_{[j]}, \quad \int_{\mathbb{T}} g_L(x)\,dx = 1 \text{ and } \iint_{\mathbb{T}^2} \frac{g_L(x) g_L(y)}{|x - y|^{\alpha_j}}\,dx\,dy < K_j.$$

Then, given any subset P of Λ containing exactly n members, we can find $\tilde{E}_1$, $\tilde{E}_2$, ..., $\tilde{E}_{n+m}$ each the finite union of non-trivial closed intervals together with piecewise continuous positive functions $\tilde{g}_L : \mathbb{T} \to \mathbb{R}$ corresponding to every $L \subseteq \Lambda$ containing at least one and at most n elements having the following properties.

(i) $d_{\mathcal{F}}(E_r, \tilde{E}_r) < \delta$ for all $1 \leq r \leq n+m$.

(ii) $\sum_{r\in Q} \tilde{E}_r \supseteq \sum_{r\in Q} E_r$ whenever $Q \subseteq \Lambda$ contains at least $n+1$ members.

(iii) We can find a finite collection $\mathcal{I}(P)$ of intervals such that

$$\bigcup_{I\in\mathcal{I}(P)} I \supseteq \left(\bigcup_{r\in P} \tilde{E}_r\right)_{[n]} \quad \textit{and} \quad \sum_{I\in\mathcal{I}(P)} |I|^{\alpha_n+\eta} < \eta \binom{n+m}{n}^{-1}.$$

(iv) If $L \subseteq \Lambda$ contains j points with $n \geq j \geq 1$, then

$$\operatorname{supp} \tilde{g}_L \subseteq \left(\bigcup_{r\in L} \tilde{E}_r\right)_{[j]}, \quad \int_{\mathbb{T}} \tilde{g}_L(x)\,dx = 1 \ \textit{and} \ \iint_{\mathbb{T}^2} \frac{\tilde{g}_L(x)\tilde{g}_L(y)}{|x-y|^{\alpha_j}}\,dx\,dy < K_j.$$

Proof. Let $d(x,E) = \inf_{e\in E}|x-e|$. We set $\beta_n = \alpha_n + \eta/2$, take

$$\tilde{E}_r = \begin{cases} E_r + [-\delta/2, \delta/2] & \text{if } r \notin P, \\ \bigcup_{d(m/N,E_r)\leq\delta/2}[(m/N) - N^{-1/\beta_n}, (m/N) + N^{-1/\beta_n}] & \text{if } r \in P \end{cases}$$

and define $\tilde{g}_L$ by the conditions

$$\tilde{g}_L(x) = \begin{cases} a_{L,m,N} & \text{if } |x - m/N| \leq N^{-1/\beta_n}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$a_{L,m,N} = \int_{(m-1/2)/N}^{(m+1/2)/N} g_L(x)\,dx.$$

Provided the integer N is large enough, conclusions (i) and (ii) hold automatically whilst (iv) follows from Lemma 20.2. Finally we observe that

$$\bigcup_{m=1}^{N} [(m/N) - nN^{-1/\beta_n}, (m/N) + nN^{-1/\beta_n}] \supseteq \left(\bigcup_{r\neq p} \tilde{E}_r\right)_{[n]}$$

and

$$\begin{aligned}\sum_{m=1}^{N} \left|[(m/N) - nN^{-1/\beta_n}, (m/N) + nN^{-1/\beta_n}]\right|^{\alpha_n+\eta} &= N \times (2nN^{-1/\beta_n})^{\alpha_n} \\ &= (2n)^{\alpha_n+\eta} N^{-\eta/(2\beta_n)} < \eta,\end{aligned}$$

provided that N is large enough. ■

It is easy to deduce a very slightly stronger result.

Lemma 21.2. *Let $\delta, \eta > 0$, and $n, m \geq 1$. Write*

$$\Lambda = \{r \,:\, n+m \geq r \geq 1\}$$

Suppose E_1, E_2, ..., E_{n+m} are each the finite union of non-trivial closed intervals such that, whenever $L \subseteq \Lambda$, and L contains j elements with $n \geq j \geq 1$, there exists a piecewise continuous positive $g_L : \mathbb{T} \to \mathbb{R}$ with

$$\operatorname{supp} g_L \subseteq \left(\bigcup_{r \in L} E_r\right)_{[j]}, \ \int_{\mathbb{T}} g_L(x)\,dx = 1 \text{ and } \iint_{\mathbb{T}^2} \frac{g_L(x)g_L(y)}{|x-y|^{\alpha_j}}\,dx\,dy < K_j.$$

Then we can find $\tilde{E}_1$, $\tilde{E}_2$, ..., $\tilde{E}_{n+m}$ each the finite union of non-trivial closed intervals together with piecewise continuous positive functions $\tilde{g}_L : \mathbb{T} \to \mathbb{R}$ corresponding to every $L \subseteq \Lambda$ containing at least one and at most n elements having the following properties.

(i) $d_{\mathcal{F}}(E_r, \tilde{E}_r) < \delta$ for all $1 \leq r \leq n+m$.

(ii) $\sum_{r \in Q} \tilde{E}_r \supseteq \sum_{r \in Q} E_r$ whenever $Q \subseteq \Lambda$ contains at least $n+1$ members.

(iii) Whenever $P \subseteq \Lambda$ contains exactly n members there exists a finite collection $\mathcal{I}(P)$ of intervals such that

$$\bigcup_{I \in \mathcal{I}(P)} I \supseteq \left(\bigcup_{r \in P} \tilde{E}_r\right)_{[n]} \text{ and } \sum_{I \in \mathcal{I}(P)} |I|^{\alpha_n + \eta} < \delta \binom{n+m}{m}^{-1}.$$

(iv) If $L \subseteq \Lambda$ contains j points with $n \geq j \geq 1$, then

$$\operatorname{supp} \tilde{g}_L \subseteq \left(\bigcup_{r \in L} \tilde{E}_r\right)_{[j]}, \ \int_{\mathbb{T}} \tilde{g}_L(x)\,dx = 1 \text{ and } \iint_{\mathbb{T}^2} \frac{\tilde{g}_L(x)\tilde{g}_L(y)}{|x-y|^{\alpha_j}}\,dx\,dy < K_j.$$

Proof. Apply Lemma 21.1 repeatedly with P every possible subset of Λ with n elements. ■

We need one further remark.

Lemma 21.3. *Suppose $1 > \delta > 0$ and $E \in \mathcal{G}$. Then we can find $F \in \mathcal{G}$ with $d_{\mathcal{G}}(E, F) < \delta$ such that E is the finite union of non-trivial closed intervals and there exist piecewise continuous positive $g_j : \mathbb{T} \to \mathbb{R}$ such that*

$$\int_{\mathbb{T}} g_j(x)\,dx = 1, \ \operatorname{supp} g_j \subseteq E_{[j]} \text{ and } \iint_{\mathbb{T}^2} \frac{g_j(x)g_j(y)}{|x-y|^{\alpha_j}} < K_j$$

for all $j \geq 1$.

Proof. Let

$$\Delta(x) = \max\Big(0, 2\delta^{-1}\big(1 - 2\delta^{-1}|x|\big)\Big).$$

We know that there exist probability measures μ_j with

$$\operatorname{supp}\mu_j \subseteq E_{[j]} \text{ and } \iint_{\mathbb{T}^2} \frac{d\mu_j(x)\,d\mu_j(y)}{|x-y|^{\alpha_j}} \leq K_j,$$

and we have chosen

$$K_j > \iint_{\mathbb{T}^2} \frac{1}{|x-y|^{\alpha_j}} dx\,dy.$$

Thus, if we set $F = E + [\delta/2, -\delta/2]$ and $g_j = \Delta * \mu_j$, we have the required result. ■

Proof of Lemma 20.6. By Lemma 21.3, it suffices to show that, given $n \geq 1$, δ, $\eta > 0$ and E satisfying the conclusions of Lemma 21.3, we can find an $F \in \mathcal{E}$ with $d(F, E) < \delta$.

Since E contains non-trivial intervals, we can find an $m \geq 1$ such that $E_{[n+m]} = \mathbb{T}$. Write $E_r = E$ for $1 \leq r \leq n+m$,

$$\Lambda = \{r \,:\, n+m \geq r \geq 1\}$$

and, if $L \subseteq \Lambda$ contains j elements with $n \geq j \geq 1$, set $g_L = g_j$.

Now choose $\tilde{E}_r$ and $\tilde{g}_L$ so that the conclusions of Lemma 21.2 hold. We set $F = \bigcup_{r=1}^{n+m} \tilde{E}_r$. By Lemma 21.2 (i),

$$d_{\mathcal{F}}(E, \tilde{E}_r) < \delta$$

for all r and so $d_{\mathcal{F}}(E, F) < \delta$. If we write Γ for the collection of subsets of Λ with exactly n elements, then

$$\bigcup_{P\in\Gamma}\left(\bigcup_{r\in P}\tilde{E}_r\right)_{[n]} = \left(\bigcup_{r=1}^{n+m}\tilde{E}_r\right)_{[n]} = F$$

so, by Lemma 21.2 (iii),

$$\bigcup_{P\in\Gamma}\bigcup_{I\in\mathcal{I}(P)} I \supseteq F$$

and

$$\sum_{P\in\Gamma}\sum_{I\in\mathcal{I}(P)} |I|^{\alpha_n+\eta} < \delta.$$

Thus, if $F \in \mathcal{G}$, then $F \in \mathcal{E}$.

In order to show that $F \in \mathcal{G}$ we shall find piecewise continuous positive functions $f_j : \mathbb{T} \to \mathbb{R}$ such that

$$\int_{\mathbb{T}} f_j(x)\,dx = 1,\ \operatorname{supp} f_j \subseteq F_{[j]} \text{ and } \iint_{\mathbb{T}^2} \frac{f_j(x)f_j(y)}{|x-y|^{\alpha_j}}\,dx,\,dy < K_j$$

for all $j \geq 1$. We split our task into three parts.

If $1 \leq j \leq n$, we set $f_j = \tilde{g}_{\{1,2,\ldots,j\}}$ and use Lemma 21.2 (iv), together with the observation that

$$\left(\bigcup_{r=1}^{j} \tilde{E}_r\right)_{[j]} \subseteq \left(\bigcup_{r=1}^{j} F\right)_{[j]} = F_{[j]}.$$

If $n+1 \leq j \leq n+m$, we set $f_j = g_j$ and use Lemma 21.2 (ii) to show that

$$\operatorname{supp} f_j = \operatorname{supp} g_j \subseteq E_{[j]} = \sum_{r=1}^{j} E = \sum_{r=1}^{j} E_r \subseteq \sum_{r=1}^{j} \tilde{E}_r \subseteq F_{[j]}.$$

If $j \geq n+m+1$, we observe that the same calculation shows that

$$\mathbb{T} = E_{[n+m]} \subseteq F_{[n+m]}$$

so $F_{[n+m]} = \mathbb{T}$ and $F_{[j]} = \mathbb{T}$. We set $f_j = 1$. ■

Exercise 21.4. *We work in $(\mathcal{E}, d_{\mathcal{E}})$, the space of compact subsets of $\mathbb{T}$ with the usual Hausdorff metric $d_{\mathcal{E}}$. Let $0 < \alpha \leq 1$ and*

$$K > \iint_{\mathbb{T}^2} \frac{dx\,dy}{|x-y|^{\alpha}}.$$

Let $\mathcal{G}(\alpha)$ be the collection of compact sets E such that there exists a probability measure μ with

$$\operatorname{supp} \mu \subseteq E \text{ and } \iint_{\mathbb{T}^2} \frac{d\mu(x)\,d\mu(y)}{|x-y|^{\alpha}} \leq K.$$

Show that $\mathcal{G}(\alpha)$ is a closed subset of $(\mathcal{E}, d_{\mathcal{E}})$ Show that, if we use the restriction metric, quasi-all subsets of $\mathcal{G}(\alpha)$ are Kronecker sets with Hausdorff dimension exactly α. (The existence of Kronecker sets with specified Hausdorff dimension was proved, in a much neater manner, by Kaufman in [14].)

Exercise 21.5. *(This is a very long exercise and is really just included for the reader's information.) Given a sequence α_j with $0 \leq \alpha_j \leq \alpha_{j+1} \leq 1$, show that we can find a closed set E such that $E_{[j]}$ has Hausdorff dimension α_j for each $j \geq 1$.*

If $\alpha_{k+1} = 1$, show that we can choose E so that, in addition, $E_{[k+1]} = \mathbb{T}$ but $E_{[k]}$ has Lebesgue measure zero or we can choose E so that $E_{[j]}$ has Lebesgue measure zero for all j.

22 Remarks

The history of the use of probabilistic methods to prove results outside probability theory remains to be written, but I suspect that the diligent historian will be able to trace an uninterrupted path back to Borel. (This does not exclude the possibility of isolated examples before Borel and repeated independent discoveries afterwards.) I discovered the usefulness of throwing delta measures down at random from [14]. The reader in search of further inspiration cannot do better than read Kahane's beautiful book [9]. The kind of coin tossing estimates we have used are pretty crude (but, correspondingly robust). The first chapter of Bollobás's book shows what clever and determined mathematicians can do with coin tossing.

Baire category arguments go back to Baire (and as a set of related tools much earlier). They were powerfully exploited by Banach and his school. Kaufman introduced category methods into harmonic analysis in [13] and they were further exploited by Kahane in [11]. It must be said that, whilst category methods are a useful tool, probabilistic methods constitute an entire programme.

Debs and Saint-Reymond obtained their famous theorem in [5] by the methods of descriptive set theory. The book [15] discusses this and other applications of descriptive set theory. In particular, as we noted earlier, Matheron and Zelený have used these methods to obtain Theorem 12.5 independently.(see [24]).

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