# Topics in Fourier Analysis
# Part III, Autumn 2005

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**Small print** This is just a first draft for the course. The content of the course will be what I say, not what these notes say. (I may change my mind on the contents of the last part of the course.) Experience shows that skeleton notes (at least when I write them) are very error prone, so use these notes with care. I should **very much** appreciate being told of any corrections or possible improvements. My e-mail address is `twk@dpmms`. These notes are written in LaTeX $2_\varepsilon$ and should be available in tex, ps, pdf and dvi format from my home page

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## Contents

# 1 Some notes of explanation

Since the birth of the Lebesgue integral it has been clear that it is a much more powerful tool for studying Fourier analysis than the Riemann integral. However I shall try to make the course accessible to (and examinable for) those who have not done measure theory (though they may have to take the statement of certain results on trust). If either those who know Lebesgue integration or those who do not feel that this leads to any problems, they should raise them with me.

# 2 Radars and suchlike

This first section may also appear most opaque. However, it touches on themes which will reappear throughout the course and, in particular, in Section 8 when we discuss Heisenberg's inequality.

We work in the $(x, y)$ plane. Consider an array of $2N + 1$ radio transmitters broadcasting at frequency $\omega$. Let the $k$th transmitter be at $(0, kl)$ $[k = -N,\ -N + 1,\ \ldots,\ 0,\ 1,\ \ldots,\ N]$. It is reasonable to take the signal at $(x, y)$ due to the $k$th transmitter to be

$$A_k r_k^{-2} \exp(i(\omega t - \lambda^{-1} r_k - \phi_k))$$

where $\lambda$ is the wavelength (thus $\omega\lambda = c$ the speed of light) and $r_k^2 = x^2 + (y - kl)^2$. The total signal at $(x, y)$ is

$$S(x, y, t) = \sum_{k=-N}^{N} A_k r_k^{-2} \exp(i(\omega t - \lambda^{-1} r_k - \phi_k)).$$

**Lemma 2.1** *If $x = R\cos\theta$, $y = R\sin\theta$ where $R$ is very large then, to a very good approximation,*

$$S(R\cos\theta, R\sin\theta, t) = R^{-2} \exp(i(\omega t - \lambda R)Q(u)$$

*where*

$$Q(u) = \sum_{k=-N}^{N} A_k \exp(i(ku - \phi_k)),$$

*and* $u = \lambda^{-1} l \sin\theta$.

In the discussion that follows we use the notation of Lemma 2.1 and the discussion that preceded it. We set

$$P(u) = \sum_{k=-N}^{N} A_k \exp(iku),$$

that is $P = Q$ with $\phi_k = 0$ for all $k$. Since we could take the $A_k$ to be complex, there was no real increase in generality in allowing $\phi_k \neq 0$, but I wished to make the following points.

**Lemma 2.2** *(i) Given* $\theta_0$ *we can find* $\phi_k$ *such that*

$$Q(u) = P(u - u_0).$$

*(ii)* $S(-x, -y, t) = S(x, y, t)$.
*(iii) If* $l > \lambda\pi/2$ *then there exist* $0 < \theta_1 < \pi/2$ *such that*

$$S(R\cos\theta_1, R\sin\theta_1, t) = S(0, R, t).$$

Bearing in mind that the equations governing the reception of signal at $(x, y)$ transmitted from our array are essentially the same as those governing the reception of signal at our array, Lemma 2.2 (i) talks about electronic stearability of radar beams, and Lemma 2.2 (ii) and (iii) deal with ambiguity. It is worth noting that in practice the signal received by a radar corresponds to $|Q(u)|$.

Let us look at $P(u)$ in two interesting cases.

**Lemma 2.3** *(i) If* $A_k = (2N+1)^{-1}$ *then, writing* $P_N(u) = P(u)$,

$$P_N(u) = \frac{1}{2N+1} \frac{\sin((N + \frac{1}{2})u)}{\sin(\frac{1}{2}u)}.$$

*(ii) With the notation of (i)*

$$P_N(av/N) \to \frac{\sin av}{2av}$$

*as* $N \to \infty$.

Lemma 2.3 is usually interpreted as saying that a radar cannot discriminate between two targets if their angular distance is of the order of the size of $\lambda/a$ where $\lambda$ is the wave length used and $a$ is the length of the array. It is natural to ask if a cleverer choice of $A_k$ might enable us to avoid this problem. We shall see that the choice in Lemma 2.3 may not be the best there is no way of avoiding the $\lambda/a$ rule.

# 3 Fourier series on the circle

We work on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (that is on the interval $[0, 2\pi]$ with the two ends 0 and $2\pi$ identified). If $f : \mathbb{T} \to \mathbb{C}$ is integrable[1], we write

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t) \exp -int\, dt.$$

We shall see (Lemma 3.11) that $f$ is uniquely determined by its Fourier coefficients $\hat{f}(n)$. Indeed it is clear that there is a 'natural identification' (where natural is deliberately used in a vague sense)

$$f(t) \sim \sum_{r=-\infty}^{\infty} \hat{f}(r) \exp irt.$$

However we shall also see that, even when $f$ is continuous, $\sum_{r=-\infty}^{\infty} \hat{f}(r) \exp irt$ may fail to converge at some points $t$.

Fejér discovered that, although

$$S_n(f, t) = \sum_{r=-n}^{n} \hat{f}(r) \exp irt$$

may behave badly as $n \to \infty$, the average

$$\sigma_n(f, t) = (n+1)^{-1} \sum_{m=0}^{n} S_m(f, t) = \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \hat{f}(r) \exp irt$$

behaves much better. (We call $\sigma_n(f, t)$ the Fejér sum. We also write $S_n(f, t) = S_n(f)(t)$ and $\sigma_n(f, t) = \sigma_n(f)(t)$.)

**Exercise 3.1** *Let $a_1$, $a_2$, ... be a sequence of complex numbers.*

[1] That is Lebesgue integrable or Riemann integrable according to the reader's background.

*(i) Show that, if $a_n \to a$ then*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \to a$$

*as $n \to \infty$.*

*(ii) By taking an appropriate sequence of $0$s and $1$s or otherwise find a sequence $a_n$ such that $a_n$ does not tend to a limit as $n \to \infty$ but $(a_1 + a_2 + \cdots + a_n)/n$ does.*

*(iii) By taking an appropriate sequence of $0$s and $1$s or otherwise find a bounded sequence $a_n$ such that $(a_1 + a_2 + \cdots + a_n)/n$ does not tend to a limit as $n \to \infty$.*

**Lemma 3.2** *If $f$ is integrable, we have*

$$S(f) = f * D_n,$$
$$\sigma(f) = f * K_n$$

*where*

$$D_n(f,t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)},$$
$$K_n(f,t) = \frac{1}{n+1}\left(\frac{\sin(\frac{n+1}{2}t)}{\sin(\frac{1}{2}t)}\right)^2$$

*for $t \neq 0$.*

The key differences between the Dirichlet kernel $D_n$ and the Fejér kernel $K_n$ are illustrated by the next two lemmas.

**Lemma 3.3** *(i) $\int_{\mathbb{T}} D_n(t)\,dt = 1$.*

*(ii) If $t \neq \pi$ then $D_n(t)$ does not tend to a limit as $n \to \infty$.*

*(iii) There is a constant $A > 0$ such that $\int_{\mathbb{T}} |D_n(t)|\,dt \geq A \log n$ for $n \geq 1$.*

**Lemma 3.4** *(i) $\frac{1}{2\pi}\int_{\mathbb{T}} K_n(t)\,dt = 1$.*

*(ii) If $\eta > 0$ then $K_n \to 0$ uniformly for $|t| \geq \eta$ as $n \to \infty$.*

*(iii) $K_n(t) \geq 0$ for all $t$.*

The properties set out in Lemma 3.4 show why Fejér sums work so well. It is worth reflecting briefly on how we might use these ideas to shape radar beams in the manner of Section 1. Note however that we do not much improve angular discrimination.

**Exercise 3.5** *Justify the last sentence.*

**Theorem 3.6** *(i) If $f : \mathbb{T} \rightarrow \mathbb{T}$ is integrable and $f$ is continuous at $t$ then*

$$\sigma_n(f, t) \rightarrow f(t)$$

*as $n \rightarrow \infty$.*

*(ii) If $f : \mathbb{T} \rightarrow \mathbb{T}$ is continuous then*

$$\sigma_n(f) \rightarrow f$$

*uniformly as $n \rightarrow \infty$.*

**Exercise 3.7** *Suppose that $L_n : \mathbb{T} \rightarrow \mathbb{R}$ is continuous (if you know Lebesgue theory you merely need integrable) and*

*(A) $\frac{1}{2\pi} \int_{\mathbb{T}} L_n(t)\, dt = 1$,*

*(B) If $\eta > 0$ then $L_n \rightarrow 0$ uniformly for $|t| \geq \eta$ as $n \rightarrow \infty$,*

*(C) $L_n(t) \geq 0$ for all $t$.*

*(i) Show that if $f : \mathbb{T} \rightarrow \mathbb{R}$ is integrable and $f$ is continuous at $t$ then*

$$L_n * f(t) \rightarrow f(t)$$

*as $n \rightarrow \infty$.*

*(ii) Show that, if $f : \mathbb{T} \rightarrow \mathbb{C}$ is continuous, then*

$$L_n * f \rightarrow f$$

*uniformly as $n \rightarrow \infty$.*

*(iii) Show that condition (C) can be replaced by*

*(C') There exists a constant $A > 0$ such that*

$$\int_{\mathbb{T}} |L_n(t)|\, dt \leq A$$

*in parts (i) and (ii). [You need only give the proof in one case and say that the other is 'similar'.]*

**Exercise 3.8** *Suppose that $L_n : \mathbb{T} \rightarrow \mathbb{R}$ is continuous but that*

$$\sup_n \int_{\mathbb{T}} |L_n(t)|\, dt = \infty.$$

*Show that we can find a sequence of continuous functions $g_n : \mathbb{T} \rightarrow \mathbb{R}$ with $|f_n(t)| \leq 1$ for all $t$, $L_n * g_n(0) \geq 0$ for all $n$ and*

$$\sup_n L_n * g_n(0) = \infty.$$

*(i) If you know some functional analysis deduce the existence of a continuous function $f$ such that*

$$\sup_n L_n * f(0) = \infty.$$

*(ii) Even if you can obtain the result of (i) by slick functional analysis there is some point in obtaining the result directly.*

*(a) Suppose that we have defined positive integers $n(1) < n(2) < \cdots < n(k)$, a continuous function $g_k$ and a real number $\epsilon(k)$ with $2^{-k} > \epsilon(k) > 0$. Show that there is an $\epsilon(k+1)$ with $\epsilon(k)/2 > \epsilon(k+1) > 0$ such that whenever $g$ is a continuous function with $\|g - g_k\|_\infty < 2\epsilon(k+1)$ we have $|L_{n(j)} * g - L_{n(j)} * g_k(0)| \leq 1$. for $1 \leq j \leq k$.*

*(b) If $\sup |L_m * g_k(0)| = \infty$ we have proved our theorem, so we may suppose $|L_m * g_k(0)| \leq A_k$ for some $A_k$ and all $m$. Continuing with the notation of (a), show that there exists an $n(k+1) > n(k)$ and a continuous function $g_{k+1}$ with $\|g_{k+1} - g_k\|_\infty \leq \epsilon(k+1)$ such that $|L_{n(k+1)} * g_{k+1}(0)| > 2^{k+1}$.*

*(c) By carrying out the appropriate induction and considering the uniform limit of $g_k$ obtain (i).*

*(iii) Show that there exists a continuous function $f$ such that $S_n(f, 0)$ fails to converge as $n \to \infty$. (We shall obtain a stronger result later in Theorem 4.2.)*

Theorem 3.6 has several very useful consequences.

**Theorem 3.9 (Density of trigonometric polynomials)** *The trigonometric polynomials are uniformly dense in the continuous functions on $\mathbb{T}$.*

**Lemma 3.10 (Riemann-Lebesgue lemma)** *If $f$ is an integrable function on $\mathbb{T}$, then $\hat{f}(n) \to 0$ as $|n| \to \infty$.*

**Theorem 3.11 (Uniqueness)** *If $f$ and $g$ are integrable functions on $\mathbb{T}$ with $\hat{f}(n) = \hat{g}(n)$ for all $n$, then $f = g$.*

**Lemma 3.12** *If $f$ is an integrable function on $\mathbb{T}$ and $\sum_j |\hat{f}(j)|$ converges, then $f$ is continuous and $f(t) = \sum_j \hat{f}(j) \exp ijt$.*

As a preliminary to the next couple of results we need the following temporary lemma (which will be immediately superseded by Theorem 3.15).

**Lemma 3.13 (Bessel's inequality)** *If $f$ is a continuous function on $\mathbb{T}$, then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)|^2 \, dt.$$

**Theorem 3.14 (Mean square convergence)** *If $f$ is a continuous function on $\mathbb{T}$, then*

$$\frac{1}{2\pi}\int_{\mathbb{T}} |f(t) - S_n(f,t)|^2\, dt \to 0$$

*as $n \to \infty$.*

**Theorem 3.15 (Parseval's Theorem)** *If $f$ is a continuous function on $\mathbb{T}$, then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^2\, dt.$$

*More generally, if $f$ and $g$ are continuous,*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)^* = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(t)^*\, dt.$$

(The extension to all $L^2$ functions of Theorems 3.14 and 3.15 uses easy measure theory.)

**Exercise 3.16** *If you use Lebesgue integration state and prove Theorems 3.14 and 3.15 for $(L^2(\mathbb{T}), \| \ \|_2)$.*

*If you use Riemann integration extend and prove Theorems 3.14 and 3.15 for all Riemann integrable function.*

Note the following complement to the Riemann-Lebesgue lemma.

**Lemma 3.17** *If $\kappa(n) \to \infty$ as $n \to \infty$, then we can find a continuous function $f$ such that $\limsup_{n\to\infty} \kappa(n)\hat{f}(n) = \infty$.*

The proof of the next result is perhaps more interesting than the result itself.

**Lemma 3.18** *Suppose that $f$ is an integrable function on $\mathbb{T}$ such that there exists an $A$ with $|\hat{f}(n)| \leq A|n|^{-1}$ for all $n \neq 0$. If $f$ is continuous at $t$, then $S_n(f,t) \to f(t)$ as $n \to \infty$.*

**Exercise 3.19** *Suppose that $a_n \in \mathbb{C}$ and there exists an $A$ with*

$$|a_{n+1} - a_n| \leq A|n|^{-1} \text{ for all } n \geq 1.$$

*Show that if*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \to a$$

*as $n \to \infty$, then $a_n \to a$ as $n \to \infty$. (Results like this are called Tauberian theorems.)*

**Exercise 3.20** *(i) Suppose that* $f : [-\pi, \pi) \to \mathbb{R}$ *is increasing and bounded. Write* $f(\pi) = \lim_{t\to 0} f(\pi - t)$*. Show that*

$$\int_{-\pi}^{\pi} f(t) \exp it \, dt = \int_{0}^{\pi} (f(t) - f(t - \pi)) \exp it \, dt$$

*and deduce that* $|\hat{f}(1)| \leq (f(\pi) - f(-\pi))/2 \leq (f(\pi) - f(-\pi))$*.*

*(ii) Under the assumptions of (i) show that*

$$|\hat{f}(n)| \leq (f(\pi) - f(-\pi))/|n|$$

*for all* $n \neq 0$*.*

*(iii) (Dirichlet's theorem) Suppose that* $g = f_1 - f_2$ *where* $f_k : [-\pi, \pi) \to \mathbb{R}$ *is increasing and bounded* $[k = 1, 2]$*. (It can be shown that functions* $g$ *of this form are the so called functions of bounded variation.) Show that, if* $g$ *is continuous at* $t$*, then* $S_n(g, t) \to f(t)$ *as* $n \to \infty$*.*

Most readers will already be aware of the next fact.

**Lemma 3.21** *If* $f : \mathbb{T} \to \mathbb{C}$ *is continuously differentiable, then*

$$(f')\hat{}(n) = in\hat{f}(n).$$

This means that Lemma 3.18 applies, but we can do better.

**Lemma 3.22** *If* $f : \mathbb{T} \to \mathbb{C}$ *is continuously differentiable, then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Here is a beautiful application due to Weyl of Theorem 3.9. If $x$ is real let us write $\langle x \rangle$ for the fractional part of $x$, that is, let us write

$$\langle x \rangle = x - [x].$$

**Theorem 3.23** *If* $\alpha$ *is an irrational number and* $0 \leq a \leq b \leq 1$*, then*

$$\frac{\operatorname{card}\{1 \leq n \leq N \mid \langle n\alpha \rangle \in [a, b]\}}{N} \to b - a$$

*as* $N \to \infty$*. The result is false if* $\alpha$ *is rational.*

(Of course this result may be deduced from the ergodic theorem and Theorem 3.9 itself can be deduced from the Stone-Weierstrass theorem but the techniques used can be extended in directions not covered by the more general theorems.)

**Exercise 3.24** *(i) Suppose that*

$$\tau = \frac{1+\sqrt{5}}{2} \text{ and } \sigma = \frac{1-\sqrt{5}}{2}.$$

*Show that, if* $u_n = \tau^n + \sigma^n$, *then* $u_0$, $u_1 \in \mathbb{Z}$ *and*

$$u_{n+1} = u_n + u_{n-1}$$

*for all* $n$. *Conclude that* $u_n \in \mathbb{Z}$ *for all* $\mathbb{Z}$. *Explain why* $\sigma^n \to 0$ *as* $n \to \infty$ *and conclude that, in general,*

$$\frac{\operatorname{card}\{1 \le n \le N \mid \langle \tau^n \rangle \in [a,b]\}}{N} \not\to b-a$$

*as* $N \to \infty$.

*(ii) (This just an aside unrelated to the course.) Use the ideas of (i) to show that, if* $F_n$ *is the* $n$*th Fibonacci number*[2], *then* $F_n$ *is the nearest integer to*

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

*for all* $n \ge 0$.

Hurwitz used Parseval's theorem in a neat proof of the isoperimetric inequality.

**Theorem 3.25** *Among all smooth closed non-self-intersecting curves of given length the one which encloses greatest area is the circle.*

(Reasonably simple arguments show that requirement of smoothness can be dropped.)

# 4 A Theorem of Kahane and Katznelson

We need to recall (or learn) the following definition.

**Definition 4.1** *A subset* $E$ *of* $\mathbb{T}$ *has (Lebesgue) measure zero if, given* $\epsilon > 0$ *we can find intervals* $I_j$ *of length* $|I_j|$ *such that* $\bigcup_{j=1}^{\infty} I_j \supseteq E$ *but* $\sum_{j=1}^{\infty} |I_j| < \epsilon$.

[2] Recall $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$.

There is a deep and difficult theorem of Carleson which tells us that if $f : \mathbb{T} \to \mathbb{C}$ is continuous (or even $L^2$), then the set

$$E = \{t \in \mathbb{T} \mid S_n(f,t) \nrightarrow f(t) \text{ as } n \to \infty\}$$

has measure 0. (We shall neither prove nor make use of this result which is included for information only.) Kahane and Katznelson proved a converse which though much easier to prove is still remarkable.

**Theorem 4.2 (Kahane and Katznelson)** *Given any subset $E$ of $\mathbb{T}$ with measure zero we can find a continuous function $f$ such that*

$$\limsup_{n\to\infty} |S_n(f,t)| \to \infty$$

*for all $t \in E$.*

The theorem follows relatively simply from its 'finite version'.

**Lemma 4.3** *Given any $K > 0$ we can find a $\delta(K) > 0$ such that if $J_1$, $J_2$, $\dots J_N$ is any finite collection of intervals with $\sum_{r=1}^{N} |J_r| < \delta(K)$ we can find a trigonometric polynomial $P$ such that $\|P\|_\infty \leq 1$ but*

$$\sup_n |S_n(P,t)| \geq K$$

*for all $t \in \bigcup_{r=1}^{N} J_r$.*

It is the proof of Lemma 4.3 which contains the key idea. This is given in the next lemma.

**Lemma 4.4** *Let us define $\log z$ on $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$ so that $\log x$ is real when $x$ is real and positive. Suppose that $1 > \delta > 0$ and that $\theta_1,\ \theta_2,\ \dots,\ \theta_N \in \mathbb{R}$. If we set*

$$\phi(z) = \log\left(N^{-1}\sum_{n=1}^{N}\frac{1+\delta}{1+\delta - ze^{-i\theta_n}}\right)$$

*then $\phi$ is a well defined analytic function on $\{z \mid |z| < 1+\delta/2\}$ such that*

*(i) $|\Im\phi(z)| < \pi$ for all $|z| < 1+\delta/2$,*

*(ii) $\phi(0) = 0$,*

*(iii) $|\Re\phi(e^{i\theta})| \geq \log(\delta^{-1}/4N)$ for all $|\theta - \theta_n| \leq \delta/2$ and $1 \leq n \leq N$.*

## 5 Many Dimensions

The extension of these ideas to higher dimensions can be either trivial or very hard. If $f : \mathbb{T}^m \to \mathbb{C}$, we define

$$\hat{f}(\mathbf{n}) = \int_{\mathbb{T}} \dots \int_{\mathbb{T}} f(\mathbf{t}) \exp(-i\mathbf{n}.\mathbf{t})\, dt_1 \dots dt_m.$$

Very little is known about the convergence of

$$\sum_{u^2+v^2 \le N} \hat{f}(u,v) \exp(i(ux+vy))$$

as $N \to \infty$, even when $f$ is continuous. (Of course, under stronger conditions, such as those in Exercise 5.7 below the matter becomes much easier.)

However the treatment of the sums of type

$$\sum_{|u|,|v| \le N} \hat{f}(u,v) \exp(i(ux+vy))$$

is a straightforward. The following results are part of the course but will be left as exercises. (Of course, if you have trouble with them you can ask the lecturer to do them.)

**Lemma 5.1** *If we define $\tilde{K} : \mathbb{T}^m \to \mathbb{R}$ by $\tilde{K}(t_1, t_2, \dots, t_m) = \prod_{j=1}^m K(t_j)$ then we have the following results.*

*(i)* $\frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \tilde{K}_n(\mathbf{t})\, d\mathbf{t} = 1.$

*(ii) If* $\eta > 0$, *then* $\int_{|\mathbf{t}| \ge \eta} \tilde{K}_n(\mathbf{t})\, d\mathbf{t} \to 0$ *as* $n \to \infty$.

*(iii)* $\tilde{K}_n(\mathbf{t}) \ge 0$ *for all* $\mathbf{t}$.

*(iv)* $\tilde{K}_n$ *is a (multidimensional) trigonometric polynomial.*

**Lemma 5.2** *If $f : \mathbb{T}^m \to \mathbb{C}$ is integrable and $P : \mathbb{T}^m \to \mathbb{C}$ is a trigonometric polynomial, then*

$$P * f(\mathbf{x}) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} P(\mathbf{x} - \mathbf{t}) f(\mathbf{t})\, d\mathbf{t}$$

*is a trigonometric polynomial in* $\mathbf{x}$.

**Theorem 5.3 (Density of trigonometric polynomials)** *The trigonometric polynomials are uniformly dense in the continuous functions on $\mathbb{T}^m$.*

**Lemma 5.4 (Riemann-Lebesgue lemma)** *If $f$ is an integrable function on $\mathbb{T}^m$, then $\hat{f}(\mathbf{n}) \to 0$ as $|\mathbf{n}| \to \infty$.*

**Theorem 5.5 (Uniqueness)** *If $f$ and $g$ are integrable functions on $\mathbb{T}^m$ with $\hat{f}(\mathbf{n}) = \hat{g}(\mathbf{n})$ for all $\mathbf{n}$, then $f = g$.*

**Lemma 5.6** *If $f$ is an integrable function on $\mathbb{T}$ and $\sum_{\mathbf{j}} |\hat{f}(\mathbf{j})|$ converges then $f$ is continuous and $f(\mathbf{t}) = \sum_{\mathbf{j}} \hat{f}(\mathbf{j}) \exp i\mathbf{j}.\mathbf{t}$.*

**Exercise 5.7** *Suppose that $f : \mathbb{T}^m \to \mathbb{C}$ is integrable and $\sum_{(u,v)\in\mathbb{Z}^2} |\hat{f}(u,v)| < \infty$. Show that*

$$\sum_{u^2+v^2\leq N} \hat{f}(u,v)\exp(i(ux+vy)) \to f(x,y)$$

*uniformly as $N \to \infty$.*

**Theorem 5.8 (Parseval's Theorem)** *If $f$ is a continuous function on $\mathbb{T}^m$, then*

$$\sum_{n\in\mathbb{Z}^m} |\hat{f}(\mathbf{n})|^2 = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} |f(\mathbf{t})|^2\,d\mathbf{t}.$$

*More generally, if $f$ and $g$ are continuous,*

$$\sum_{n\in\mathbb{Z}^m} \hat{f}(\mathbf{n})\hat{g}(\mathbf{n})^* = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} f(\mathbf{t})g(\mathbf{t})^*\,d\mathbf{t}.$$

**Exercise 5.9** *Prove the results from Lemma 5.1 to Theorem 5.8*

**Exercise 5.10** *The extension of Lemma 3.18 to many dimensions is not required for the course but makes a nice exercise. The proof follows the one dimensional proof but is not quite word for word.*

*(i) Suppose that $f$ is a bounded integrable function on $\mathbb{T}^2$ such that there exists an $A$ with $|\hat{f}(u,v)| \leq A(u^2+v^2)^{-1}$ for all $(u,v) \neq (0,0)$. Show that, if $f$ is continuous at $(s,t)$,, then*

$$\sum_{|u|,|v|\leq n} \hat{f}(u,v)\exp(i(us+vt)) \to f(s,t)$$

*(ii) (This generalises Lemma 3.22.) Suppose that $f$ is an twice differentiable function on $\mathbb{T}^2$ with $\dfrac{\partial^2 f(x,y)}{\partial x \partial y}$ continuous. Show that $\sum_{(u,v)\in\mathbb{Z}^2} |\hat{f}(u,v)| < \infty$.*

*(iii) State the correct generalisations of parts (i) and (ii) to higher dimensions.*

We immediately obtain a striking generalisation of Weyl's theorem (Theorem 3.23).

**Theorem 5.11** *Suppose that* $\alpha_1, \alpha_2, \ldots, \alpha_M$ *are real numbers. A necessary and sufficient condition that*

$$\frac{\operatorname{card}\{1 \leq n \leq N \mid (\langle n\alpha_1\rangle, \langle n\alpha_2\rangle, \ldots, \langle n\alpha_M\rangle \in \prod_{j=1}^{M}[a_j, b_j]\}}{N} \to \prod_{j=1}^{M}(b_j - a_j)$$

*as* $N \to \infty$ *whenever* $0 \leq a_j \leq b_j \leq 1$ *is that*

$$\sum_{j=1}^{M} n_j\alpha_j \notin \mathbb{Z} \text{ for integer } n_j \text{ not all zero.} \qquad \bigstar$$

If $\alpha_1, \alpha_2, \ldots, \alpha_M$ satisfy $\bigstar$ we say that they are independent. The multidimensional version of Weyl's theorem has an important corollary.

**Theorem 5.12 (Kronecker's theorem)** *Suppose that* $\alpha_1, \alpha_2, \ldots, \alpha_M$ *are independent real numbers. Then given real numbers* $\beta_1, \beta_2, \ldots, \beta_M$ *and* $\epsilon > 0$ *we can find integers* $N, m_1, m_2, \ldots, m_M$ *such that*

$$|N\alpha_j - \beta_j - m_j| < \epsilon$$

*for each* $1 \leq j \leq M$.

*The result is false if* $\alpha_1, \alpha_2, \ldots, \alpha_M$ *are not independent.*

We use this to obtain a theorem of Kolmogorov.

**Theorem 5.13** *There exists a Lebesgue integrable (that is an* $L^1$*) function* $f : \mathbb{T} \to \mathbb{C}$ *such that*

$$\limsup_{N\to\infty} |S_n(f, t)| = \infty$$

*for all* $t \in \mathbb{T}$.

Although this result is genuinely one of Lebesgue integration it can be obtained by simple (Lebesgue measure) arguments from a result not involving Lebesgue integration.

**Lemma 5.14** *Given any* $K > 0$ *we can find a trigonometric polynomial* $P$ *such that*

*(i)* $\int_{\mathbb{T}} |P(t)|\, dt \leq 1$,

*(ii)* $\max_{n\geq 0} |S_n(P, t)| \geq K$ *for all* $t \in \mathbb{T}$.

In our discussion of Kronecker's theorem (Theorem 5.12) we worked modulo 1. In what follows it is easier to work modulo $2\pi$. The readier will readily check that the definition and theorem that follow give the appropriate restatement of Kronecker's theorem.

**Definition 5.15** *We work in $\mathbb{T}$. If $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_M$ satisfy*

$$\sum_{j=1}^{M} n_j\alpha_j \neq 0 \text{ for integer } n_j \text{ not all zero.} \qquad \bigstar$$

*we say that they are independent.*

**Theorem 5.16 (Kronecker's theorem (alternative statement))** *Suppose that $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_M$ are independent points in $\mathbb{T}$. Then given complex numbers $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_M$ with $|\lambda_j| = 1$ $[j = 1,\ 2,\ \ldots,\ M]$ and $\epsilon > 0$ we can find an integer $N$ $N$ such that*

$$|\exp(iN\alpha_j) - \lambda_j| < \epsilon$$

*for each $1 \leq j \leq M$.*

Our construction requires some preliminary results.

**Lemma 5.17** *If $x_1,\ x_2,\ \ldots,\ x_M$ are independent points in $\mathbb{T}$ and $t \in \mathbb{T}$ and*

$$\sum_{j=1}^{M} n_j(x_j - t) = 0 \text{ for integer } n_j \text{ not all zero}$$
$$\sum_{j=1}^{M} m_j(x_j - t) = 0 \text{ for integer } m_j \text{ not all zero}$$

*then there exist $P, Q \in \{n : n \in \mathbb{Z},\ n \neq 0\}$ such that $Pn_j = Qm_j$ for $1 \leq j \leq M$.*

**Lemma 5.18** *If $x_1,\ x_2,\ \ldots,\ x_M$ are independent points in $\mathbb{T}$ and $t \in \mathbb{T}$, then one of the following must hold:-*

*(a) There exists an $i \neq 1$ such that the points $x_j - t$ with $j \neq i$ are independent.*

*(b) $x_1 - t$ is a rational multiple of $2\pi$ and the points $x_j - t$ with $j \neq 1$ are independent.*

**Lemma 5.19** *(i) If $I$ is an open interval in $\mathbb{T}$ and $x_1, x_2, \ldots, x_m$ are independent we can find $x_{m+1} \in I$ such that $x_1, x_2, \ldots, x_{m+1}$ are independent.*

*(ii) Given an integer $M \geq 1$, we can find independent points $x_1, x_2, \ldots, x_M$ in $\mathbb{T}$ such that*

$$|x_j - 2\pi j/M| \leq 10^{-3}M^{-1}$$

**Lemma 5.20** *If $M$ and $x_1, x_2, \ldots, x_M$ in $\mathbb{T}$ are as in Lemma 5.19 (ii), then setting*

$$\mu = M^{-1}\sum_{j=1}^{M}\delta_{x_j}$$

*we have the following two results.*

*(i)* $\max_{n\geq 0}|S_n(\mu,t)| \geq 100^{-1}\log M$ *for each $t \in \mathbb{T}$,*

*(ii) There exists an $N$ such that* $\max_{N\geq n\geq 0}|S_n(\mu,t)| \geq 200^{-1}\log M$ *for each $t \in \mathbb{T}$.*

*Remark 1* If you wish you may treat $S_n(\mu,t)$ as a purely formal object. However, it is better for later work to think what it actually is.
*Remark 2* Factors like $10^{-3}$ and $100^{-1}$ in Lemmas 5.19 (ii) and 5.20 are more or less chosen at random and are not 'best possible'.

A simple argument using Lemma 5.20 now gives Lemma 5.14 and we are done.

**Exercise 5.21** *(i) Show, by considering Fejér sums or otherwise, that we cannot find a continuous function $f$ such that $S_n(f,t) \to \infty$ uniformly as $n \to \infty$.*

*(ii) (Requires measure theory and a little bit more thought.) Show that we cannot find an integrable (i.e. $L^1$) function $f$ such that $S_n(f,t) \to \infty$ for almost all $t$ as $n \to \infty$.*

# 6 Some simple geometry of numbers

We need the following extension of Theorem 5.8.

**Lemma 6.1** *If $A$ is a well behaved set and $f$ is the characteristic function of $A$ (that is $f(x) = 1$ if $x \in A$, $f(x) = 0$ otherwise), then*

$$\sum_{n\in\mathbb{Z}^m}|\hat{f}(\mathbf{n})|^2 = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m}|f(\mathbf{t})|^2\,d\mathbf{t}.$$

If you know Lebesgue measure, then this is obvious (for bounded measurable sets, say) since a simple density argument shows that Parseval's Theorem (Theorem 5.8) holds for every $f \in L^1 \cap L^2$. If we restrict ourselves to Riemann integration, it is obvious what sort of approximation argument we should use but the technical details are a typically painful.

**Exercise 6.2** *EITHER (i) Give the detailed proof of Lemma 6.1 in terms of Lebesgue measure.*

*OR (ii) Give the detailed proof of Lemma 6.1 in terms of Riemann integration in the special case when $A$ is a sphere.*

We use Parseval's Theorem (in the form of Lemma 6.1) to give Siegel's proof of Minkowski's theorem.

**Theorem 6.3 (Minkowski)** *Let $\Gamma$ be an open symmetric convex set in $\mathbb{R}^m$ with volume $V$. If $V > 2^n$, then $\Gamma \cap \mathbb{Z}^m$ contains at least two points.*

The reader will recall that $\Gamma$ is convex if

$$\mathbf{x}, \mathbf{y} \in \Gamma \text{ and } 0 \leq \lambda \leq 1 \Rightarrow \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in \Gamma$$

and that $\Gamma$ is symmetric if

$$\mathbf{x} \in \Gamma \Rightarrow -\mathbf{x} \in \Gamma.$$

It is not entirely obvious obvious (though it is true) that every open convex set has a (possibly infinite) volume in the sense of Riemann. Readers who wish to use Riemann integration may add the words 'well behaved' to the statement of Minkowski's theorem.

Before beginning the proof we observe that Minkowski's result is best possible.

**Lemma 6.4** *If $V \leq 2^m$ there exists an open symmetric convex set $\Gamma$ in $\mathbb{R}^m$ with volume $V$ such that $\Gamma \cap \mathbb{Z}^m = \{\mathbf{0}\}$.*

To prove Minkowski's theorem (Theorem 6.3) it suffices to prove an essentially equivalent result.

**Theorem 6.5** *Let $\Gamma$ be a bounded open symmetric convex set in $\mathbb{R}^m$ with volume $V$. If $V > 2^n(2\pi)^n$, then $\Gamma \cap (2\pi\mathbb{Z})^m$ contains at least two points.*

We need the following simple result.

**Lemma 6.6** *If $\Gamma$ is symmetric convex set and $\mathbf{x}, \mathbf{x} - 2\mathbf{y} \in \Gamma$, then $\mathbf{y} \in \Gamma$.*

By applying Parseval's theorem to $f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^m} \phi(2\mathbf{x}-2\pi\mathbf{k})$ we obtain the following results.

**Lemma 6.7** *Let $\Gamma$ be a bounded open symmetric convex set in $\mathbb{R}^m$ with volume $V$ such that $\Gamma \cap (2\pi\mathbb{Z})^m$ only contains $\mathbf{0}$. Then*

$$2^{-m} \sum_{\mathbf{k}\in\mathbb{Z})^m} \left| \int \phi(\mathbf{x}) e^{i\mathbf{k}.\mathbf{x}}\, d\mathbf{x} \right|^2 = (2\pi)^m V. \qquad \bigstar$$

Minkowski's theorem follows at once by considering the term with $\mathbf{k} = \mathbf{0}$ in equation $\bigstar$.

Here is a simple application of Minkowski's theorem.

**Lemma 6.8** *Suppose that $a$, $b$, $c$, $d$ are real numbers with $ad-bc = 1$. Given $l > 0$ and $\epsilon > 0$ we can find integers $m$ and $n$ such that*

$$|an + bm| \leq (1+\epsilon)l, \ \ |cn + bm| \leq (1+\epsilon)l^{-1}.$$

Taking $c = x$, $a = 1$, $b = 0$, $d = 1$ and thinking carefully we obtain in quick succession.

**Lemma 6.9** *If $x$ is real there exist $n$ and $m$ integers with $m \neq 0$ such that*

$$\left| x - \frac{n}{m} \right| \leq \frac{1}{m^2}.$$

**Lemma 6.10** *If $x$ is real there exist infinitely many pairs of integers $n$ and $m$ with $m \neq 0$ such that*

$$\left| x - \frac{n}{m} \right| \leq \frac{1}{m^2}.$$

Here is another simple consequence.

**Lemma 6.11 (Quantitative version of Dirichlet's theorem)** *If $\mathbf{x} \in \mathbb{R}^m$, then given $l > 0$ and $\epsilon > 0$ we can find $n,\ n_1,\ n_2,\ \ldots,\ n_m \in \mathbb{Z}$ such that*

$$|n\mathbf{x} - n_j| \leq l^{-1}$$

*for $1 \leq j \leq m$ and $|n| \leq l^m$.*

We conclude our collection of consequences with Lagrange's four squares theorem.

**Theorem 6.12 (Lagrange)** *Every positive integer is the sum of at most 4 squares.*

**Lemma 6.13** *We can not reduce 4 in the statement of Lagrange's theorem (Theorem 6.12).*

We need an observation of Euler.

**Lemma 6.14** *If $x_0$, $x_1$, $x_2$, $x_3$, $y_0$, $y_1$, $y_2$, $y_3$ are real, then*

$$\begin{aligned}(x_0^2+&x_1^2+x_2^2+x_3^2)(y_0^2+y_1^2+y_2^2+y_3^2)\\ &=(x_0y_0-x_1y_1-x_2y_2-x_3y_3)^2+(x_0y_1+x_1y_0+x_2y_3-x_3y_2)^2+\\ &\quad(x_0y_2-x_1y_3+x_2y_0+x_3y_1)^2+(x_0y_3+x_1y_2-x_2y_1+x_3y_1)^2.\end{aligned}$$

**Exercise 6.15** *In the lectures we will use quaternions to prove Lemma 6.14. Prove the equality by direct verification.*

**Exercise 6.16** *Use complex numbers to prove that, if $x_0$, $x_1$, $y_0$, $y_1$, are real, then*

$$(x_0^2+x_1^2)(y_0^2+y_1^2)=(x_0y_0-x_1y_1)^2+(x_0y_1-x_1y_0)^2.$$

**Lemma 6.17** *Lagrange's four square theorem will follow if we can show that every odd prime is the sum of at most four squares.*

We shall also need the volume of a 4 dimensional sphere. A simple argument gives the volume of a unit sphere in any dimension.

**Lemma 6.18** *Let $V_n$ be the (n-dimensional) volume of an $n$ dimensional unit sphere.*

*(i) If $f:[0,\infty)\rightarrow\mathbb{R}$ is a continuous function with $t^{n+2}f(t)\rightarrow 0$ as $t\rightarrow\infty$, then*

$$\int_{\mathbb{R}^n} f(\|\mathbf{x}\|)\,dV(\mathbf{x})=V_n\int_0^\infty f(t)nt^{n-1}\,dt.$$

*(ii)* $V_{2k}=\dfrac{\pi^k}{k!}$, $V_{2k+1}=\dfrac{k!2^{2k+1}\pi^k}{(2k+1)!}$.

Finally we need the apparently more general version of Minkowski's theorem obtained by applying a linear map.

**Theorem 6.19 (Minkowski for general latices)** *We work in in $\mathbb{R}^m$. Let $\Lambda$ be a lattice with fundamental region of volume $L$ and let $\Gamma$ be an open symmetric convex set with volume $V$. If $V>2^mL$, then $\Gamma\cap\Lambda$ contains at least two points.*

We now turn to the proof of the fundamental lemma.

**Lemma 6.20** *Every odd prime is the sum of at most four squares.*

We begin with a simple lemma.

**Lemma 6.21** *Let $p$ be an odd prime.*

*(i) If we work in $\mathbb{Z}_p$, then the set $\{u^2 : u \in \mathbb{Z}_p\}$ has at least $(p+1)/2$ elements.*

*(ii) We can find integers $u$ and $v$ such that $u^2 + v^2 \equiv -1 \bmod p$.*

We now introduce a lattice.

**Lemma 6.22** *Let $p$, $u$ and $v$ be as in Lemma 6.21. If*

$$\Lambda = \{(n,m,a,b) \in \mathbb{Z}^4 \,:\, nu + mv \equiv a,\ mu - nv \equiv b \bmod p\}$$

*the $\Lambda$ is a lattice with fundamental region of volume $p^2$.*

*If $(n,m,a,b) \in \Lambda$, then $n^2 + m^2 + a^2 + b^2 \equiv 0 \bmod p$.*

We can now prove Lemma 6.20 and with it Theorem 6.12.

**Exercise 6.23** *(i) Recall that if $p$ is a prime, then the multiplicative group $(\mathbb{Z}_p \setminus \{0\}, \times)$ is cyclic. (This is the subject of Exercise 10.15.) Deduce that if $p = 4k+1$, then there is an element $u$ in $(\mathbb{Z}_p \setminus \{0\}, \times)$ of order 4. Show that $u^2 = -1$.*

*(ii) If $p$ and $u$ are as in (i) show that*

$$\Lambda = \{(n,m) \in \mathbb{Z}^2 \,:\, m \equiv n \bmod p\}$$

*is a lattice and deduce that there exist $n$, $m$ with $n^2 + m^2 = p$. (This is a result of Fermat. Every prime congruent to 1 modulo 4 is the sum of two squares.)*

*(iii) Show that any odd number which is the sum of two squares must be congruent to 1 modulo 4.*

# 7 A preliminary look at Fourier transforms

If $f : \mathbb{R} \to \mathbb{C}$ is integrable on each finite interval $[a,b]$ (in the Riemann or Lebesgue sense) and $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$ we[3] shall say that $f$ is *appropriate*. (This is non-standard notation.) If $f$ is appropriate define

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)\exp(-i\lambda t)\,dt.$$

[3] The majority of my auditors who know Lebesgue integration will prefer the formulations '$f \in L^1(\mathbb{R})$' or '$f$ measurable and $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$'.

**Lemma 7.1** *If $f$ is appropriate, $\hat{f} : \mathbb{R} \to \mathbb{C}$ is continuous. and bounded.*

Our first problem is that even when $f$ is appropriate $\hat{f}$ need not be.

**Example 7.2** *If $f$ is the indicator function of $[a, b]$ (that is, $f(x) = 1$ if $x \in [a, b]$, $f(x) = 0$ otherwise), then*

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)| \, d\lambda = \infty.$$

This turns out not to matter very much but should be borne in mind.

When we try to imitate our treatment of Fourier series we find that we need to interchange the order of integration of two infinite integrals. If we use Lebesgue integration we can use a very powerful theorem.

**Theorem 7.3 (Fubini's theorem)** *If $f : \mathbb{R}^2 \to \mathbb{C}$ is measurable and either of the two integrals*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dx \, dy \text{ and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dy \, dx$$

*exists and is finite, then they both do and the integrals*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy \text{ and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx.$$

*exist and are finite and equal.*

If we use Riemann integration, then we have a slogan.

**Pretheorem 7.4** *If $f : \mathbb{R}^2 \to \mathbb{C}$ is well behaved and either of the two integrals*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dx \, dy \text{ and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)| \, dy \, dx$$

*is finite, then they both are and*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy \text{ and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx.$$

In every case that we need the pretheorem can be turned into a theorem but the proofs become more and more tedious as we weaken the conditions on $f$.

**Exercise 7.5** *In this exercise we use Riemann integration and derive a simple Fubini type theorem.*

*(i) If $I$ and $J$ are intervals on $\mathbb{R}$ (so $I$ could have the form $[a,b]$, $[a,b)$, $(a,b)$ or $(a,b)$) and we write $\mathbb{I}_{I\times J}(x,y)=1$ if $(x,y)\in I\times J$, $\mathbb{I}_{I\times J}(x,y)=0$, otherwise show that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{I}_{I\times J}(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbb{I}_{I\times J}(x,y)\,dy\,dx.$$

*(ii) Suppose that $I_r$ and $J_r$ are intervals on $\mathbb{R}$ and that $\lambda_r\in\mathbb{C}$ $[1\leq r\leq n]$. If $f=\sum_{r=1}^{n}\mathbb{I}_{I_r\times J_r}$ show that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx.$$

*(iii) Suppose that $f:\mathbb{R}^2\to\mathbb{C}$ is continuous and that $I$ and $J$ are intervals on $\mathbb{R}$. If $g(x,y)=\mathbb{I}_{I\times J}(x,y)f(x,y)$ show using (ii) that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)\,dy\,dx.$$

*(iv) Suppose that $f:\mathbb{R}^2\to\mathbb{C}$ is continuous and that there exists a real constant $A$ such that*

$$|f(x,y)|\leq A(1+x^2)^{-1}(1+y^2)^{-1}. \qquad \bigstar$$

*Show, using (ii), that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dy\,dx.$$

Conditions like ★ imposing some rate of decrease at infinity play an important role in Fourier analysis.

In the next section we shall adopt a slightly more sophisticated approach to the Fourier transform than that given in the next exercise but the results are sufficient for many purposes and the exercise gives an interesting review of earlier work. We shall need the following definition.

**Definition 7.6** *We say that $f$ is piecewise continuous if, for each $R>0$, $f$ continuous at all but finitely many points of $(-R,R)$ and*

$$f(t)=\lim_{h\to 0+}f(t-h)$$

*for all $t$.*

(Different authors use different definitions. They are all the same 'in spirit' but not 'in logic'.)

**Exercise 7.7** *If $f$ is appropriate and $R > 0$ we define*

$$\sigma_R(f,t) = \frac{1}{2\pi}\int_{-R}^{R}\left(1 - \frac{|\lambda|}{R}\right)\hat{f}(\lambda)\exp(i\lambda t)\,d\lambda.$$

*(It will become clear that this is the analogue of the Fejér sum.)*

*(i) (For users of the Lebesgue integral) By adapting the proof of Theorem 3.6 show that if $f \in L^1$ and $f$ is continuous at $t$ then*

$$\sigma_R(f,t) \to f(t)$$

*as $R \to \infty$. Is the result necessarily true if $f$ is not continuous at $t$? Give reasons.*

*(i') (For users of the Riemann integral) By adapting the proof of Theorem 3.6 show that if $f$ is continuous and there there exists a real constant $A$ such that*

$$|f(x)| \le A(1 + x^2)^{-1}$$

*for all $x$, show that*

$$\sigma_R(f,t) \to f(t)$$

*for all $t$.*

*Without going into detail convince yourself that the result continues to hold if we replace the condition '$f$ continuous' by the condition '$f$ piecewise continuous' and the conclusion by*

$$\sigma_R(f,t) \to f(t)$$

*at all $t$ where $f$ is continuous. (All we need is a slight extension of Exercise 7.5 (iv).)*

*(ii) (For users of the Lebesgue integral) Suppose that $f$ and $g$ are piecewise continuous $L^1$ functions. Show that, if $\hat{f}(\lambda) = \hat{g}(\lambda)$ for all $\lambda$, then $f(t) = g(t)$ for all $t$.*

*(ii') (For users of the Riemann integral) Suppose $f$ and $g$ are piecewise continuous and there there exists a real constant $A$ such that*

$$|f(x)|, |g(x)| \le A(1 + x^2)^{-1}$$

*for all $x$. Show that, if $\hat{f}(\lambda) = \hat{g}(\lambda)$ for all $\lambda$, then $f(t) = g(t)$ for all $t$.*

## 8 The Fourier transform and Heisenberg's inequality

In this section we starting by looking at some of the ideas of the previous section from a slightly different angle. I shall state results using Lebesgue measure but students using Riemann integration will find appropriate modifications as exercises.

Here are a few standard results on Fourier transforms.

**Lemma 8.1** *(i) If $f \in L^1(\mathbb{R})$ and we write $f_a(x) = f(x-a)$, then*

$$\hat{f}_a(\lambda) = e^{i\lambda a}\hat{f}(\lambda).$$

*(ii) If $f$ is differentiable, $f'$ is continuous, $f,\ f' \in L^1(\mathbb{R})$ and $f(t) \to 0$ as $|t| \to \infty$, then*

$$i\lambda\widehat{f'}(\lambda) = \hat{f}(\lambda).$$

*(iii) If $f,\ g \in L^1(\mathbb{R})$ and we write*

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

*then $f * g \in L^1(\mathbb{R})$ and*

$$\widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda).$$

**Exercise 8.2** *Prove similar results for the Riemann integral. (Feel free to use strong hypotheses.)*

We pay particular attention to the Gaussian (or heat, or error) kernel $E(x) = (2\pi)^{-1/2}\exp(-x^2/2)$.

**Lemma 8.3** $\hat{E}(\lambda) = (2\pi)^{1/2}E(\lambda)$

**Exercise 8.4** *If I prove Lemma 8.3, I shall do so by setting up a differential equation. Obtain Lemma 8.3 by complex variable techniques.*

We use the following neat formula.

**Lemma 8.5** *If $f, g \in L^1(\mathbb{R})$ then the products $\hat{f} \times g,\ f \times \hat{g} \in L^1(\mathbb{R})$ and*

$$\int_{\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{\infty}^{\infty} f(\lambda)\hat{g}(\lambda)\,d\lambda.$$

**Exercise 8.6** *(For those using Riemann integration. You will need to refer back to the exercises in Section 7.) Suppose $f$ and $g$ are continuous and there there exists a real constant $A$ such that*

$$|f(x)|, |g(x)| \leq A(1+x^2)^{-1}$$

*for all $x$ and*

$$\hat{f}(\lambda)|, |\hat{g}(\lambda)| \leq A(1+\lambda^2)^{-1}$$

*for all $\lambda$. Show that*

$$\int_{\infty}^{\infty} \hat{f}(x)g(x)\,dx = \int_{\infty}^{\infty} f(\lambda)\hat{g}(\lambda)\,d\lambda.$$

*Without going into detail convince yourself that the hypothesis '$f$ and $g$ are continuous' can be replaced by '$f$ and $g$ are piecewise continuous'.*

By taking $f = E_h$ where $E_h(x) = h^{-1}E(h^{-1}(x))$ $[h > 0]$ in Lemma 8.5 we obtain a nice pointwise inversion result.

**Theorem 8.7** *If $f, \hat{f} \in L^1$ and $f$ is continuous at $t$, then $\hat{\hat{f}}(t) = 2\pi f(-t)$.*

**Exercise 8.8** *(For those using Riemann integration.) Suppose that $f$ is piecewise continuous and there exists a real constant $A$ such that*

$$|f(x)| \leq A(1+x^2)^{-1}$$

*for all $x$ and*

$$|\hat{f}(\lambda)| \leq A(1+\lambda^2)^{-1}$$

*for all $\lambda$. Show that if $f$ is continuous at $t$, then $\hat{\hat{f}}(t) = 2\pi f(-t)$.*

**Exercise 8.9** *Suppose that $f$ satisfies the conditions of Theorem 8.7 (if you use Lebesgue integration) or Exercise 8.8 (if you use Riemann integration). If $t$ is a point where $f(t+) = \lim_{h\to 0+} f(t+h)$ and $f(t-) = \lim_{h\to 0+} f(t-h)$ both exist show that*

$$\hat{\hat{f}}(-t) = \pi(f(t+) + f(t-)).$$

**Exercise 8.10** *Suppose that $G$, $\hat{G} \in L^1(\mathbb{R})$, $G(0) \neq 0$ and $\int_{-\infty}^{\infty} G(t)\,dt \neq 0$. Show that if we replace $E$ by $G$ in our arguments we still obtain*

$$\int_{-\infty}^{\infty} \hat{f}(\lambda)\,dt = Af(0)$$

*for all $f$ satisfying appropriate conditions and some constant $A$.*

We can now prove a version of Parseval's formula for Fourier transforms.

**Theorem 8.11** *If $f$, $g$, $\hat{f}$, $\hat{g} \in L^1(\mathbb{R})$ and $f$ and $g$ are continuous, then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)\hat{g}(\lambda)^* \, d\lambda = \int_{-\infty}^{\infty} f(t)g(t)^* \, dt$$

*and*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \, d\lambda = \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

**Exercise 8.12** *Prove similar results for the Riemann integral. (Feel free to use strong hypotheses.)*

The next exercise contains a more sophisticated version of Parseval's theorem.

**Exercise 8.13** *(This requires Lebesgue measure.) The present course is rather old fashioned, not least in the way it thinks of Fourier transforms $\hat{f}$ in terms of its values $\hat{f}(\lambda)$ at points $\lambda$, rather than an object in its own right. Here is one of several ways in which a more general view gives a more elegant theory.*

*(i) Let $S$ be the set of infinitely differentiable functions $f$ with*

$$x^n f^{(m)}(x) \to 0$$

*as $|x| \to \infty$ for all integers $n, m \geq 0$. Show that if $f \in S$, then $\hat{f} \in S$.*

*(ii) Let $\mathbb{I}_{[a,b]}(x) = 1$ for $x \in [a,b]$, $\mathbb{I}_{[a,b]}(x) = 0$ otherwise. Show that if $E_h$ is defined as above, then*

$$\|\mathbb{I}_{[a,b]} - E_h * \mathbb{I}_{[a,b]}\|_2 \to 0$$

*as $h \to 0+$. Deduce or prove otherwise that $S$ is $L^2$ norm dense in $L^2$.*

*(iii) By taking $g = \hat{f}$ in Lemma 8.5 show that*

$$\|\hat{f}\|_2^2 = 2\pi\|f\|_2^2$$

*for all $f \in S$.*

*(iv) Deduce that there is a unique continuous mapping $\mathcal{F} : L^2 \to L^2$ with $\mathcal{F}(f) = \hat{f}$ for all $f \in S$. (Uniqueness is easy but you should take care proving existence.)*

*(v) Show that $\mathcal{F} : L^2 \to L^2$ is linear and that*

$$\|\mathcal{F}(f)\|_2^2 = 2\pi\|f\|_2^2$$

*for all $f \in L^2$.*

*If we define $\mathcal{J} : L^2 \to L^2$ by $(\mathcal{J}f)(t) = f(-t)$ show that $\mathcal{F}^2 = 2\pi\mathcal{J}$.*

*(vi) If we wish to work in $L^2$, it makes sense to use a different normalising factor and call $\mathcal{G} = (2\pi)^{-1/2}\mathcal{F}$ the Fourier transform. Show that $\mathcal{G}^4 = I$ and that $\mathcal{G} : L^2 \to L^2$ is a bijective linear isometry.*

We now come to one of the key facts about the Fourier transform (some would say one of the key facts about the world we live in).

**Theorem 8.14 (Heisenberg's inequality)** *If $f$ is reasonably well behaved, then*

$$\frac{\int_{-\infty}^{\infty} \lambda^2 |\hat{f}(\lambda)|^2 \, d\lambda}{\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \, d\lambda} \times \frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx}{\int_{-\infty}^{\infty} |f(x)|^2 \, dx} \geq \frac{1}{4}.$$

*If equality holds, then $f(x) = A\exp(-bx^2)$ for some $b > 0$.*

**Exercise 8.15** *Write down explicit conditions for Theorem 8.14.*

The extension of Heisenberg's inequality to all $f \in L^2$ is given in Section 2.8 of the beautiful book [1] of Dym and McKean.

# 9 The Poisson formula

The following remarkable observation is called Poisson's formula.

**Theorem 9.1** *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous function such that $\sum_{m=-\infty}^{\infty} |\hat{f}(m)|$ converges and $\sum_{n=-\infty}^{\infty} |f(2\pi n + x)|$ converges uniformly on $[-\pi, \pi]$. Then*

$$\sum_{m=-\infty}^{\infty} \hat{f}(m) = 2\pi \sum_{n=-\infty}^{\infty} f(2\pi n).$$

It is possible to adjust the hypotheses on $f$ in Poisson's formula in various ways though some hypotheses there must be. We shall simply think of $f$ as 'well behaved'. The following rather simple lemma will suffice for our needs.

**Lemma 9.2** *If $f : \mathbb{R} \to \mathbb{C}$ is a twice continuously differentiable function such that $\int_{-\infty}^{\infty} |f(x)| \, dx$, $\int_{-\infty}^{\infty} |f'(x)| \, dx$ and $\int_{-\infty}^{\infty} |f''(x)| \, dx$ converge whilst $f'(x) \to 0$ and $x^{-2}f(x) \to 0$ as $|x| \to \infty$, then $f$ satisfies the conditions of Theorem 9.1.*

**Exercise 9.3** *(i) By applying Poisson's formula to the function $f$ defined by $f(x) = \exp(-t|x|/2\pi)$ show that*

$$2(1-e^{-t})^{-1} = 1 + \sum_{n=-\infty}^{\infty} 2t(t^2+4\pi^2n^2)^{-1}.$$

*(ii) By expanding $(t^2+4\pi n^2)^{-1}$ and (carefully) interchanging sums deduce that*

$$2(1-e^{-t})^{-1} = 1 + 2t^{-1} + \sum_{m=0}^{\infty} c_m t^m$$

*where $c_{2m} = 0$ and*

$$c_{2m+1} = a_{2m+1} \sum_{n=1}^{\infty} n^{-2m}$$

*for some value of $a_{2m+1}$ to be given explicitly.*

*(iii) Hence obtain Euler's formula*

$$\sum_{n=1}^{\infty} n^{-2m} = (-1)^{m-1}2^{2m-1}b_{2m-1}\pi^{2m}/(2m-1)!$$

*for $m \geq 1$, where the $b_m$ are defined by the formula*

$$(e^y-1)^{-1} = y^{-1} - 2^{-1} + \sum_{n=1}^{\infty} b_n y^n/n!$$

*(The $b_n$ are called Bernoulli numbers.)*

**Exercise 9.4** *Suppose $f$ satisfies the conditions of Lemma 9.2. Show that*

$$K \sum_{m=-\infty}^{\infty} \hat{f}(Km) = 2\pi \sum_{n=-\infty}^{\infty} f(2\pi K^{-1}n)$$

*for all $K > 0$. What is the corresponding result when $K < 0$.*

*By letting $K \to 0+$ deduce that*

$$\hat{\hat{f}}(0) = 2\pi f(0).$$

*(There is some interest in just seeing that this is so, but it is more profitable to give a rigorous proof.) Deduce in the usual way that*

$$\hat{\hat{f}}(t) = 2\pi f(-t)$$

*for all $t$.*

Poisson's formula has a particularly interesting consequence.

**Lemma 9.5** *If $g : \mathbb{R} \to \mathbb{C}$ is twice continuously differentiable and $g(t) = 0$ for $|t| \geq \pi$, then $g$ is completely determined by the values of $\hat{g}(m)$ for integer $m$.*

Taking $g = \hat{f}$ and remembering the inversion formula we obtain the following result.

**Pretheorem 9.6** *If $f : \mathbb{R} \to \mathbb{C}$ is a well behaved function with $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \pi$ then $f$ is determined by its values at integer points.*

We call this a pretheorem because we have not specified what 'well behaved' should mean.

The simplest approach is via the *sinc function*

$$\operatorname{sinc}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(ix\lambda)\, d\lambda.$$

We state the most immediately useful properties of sinc.

**Lemma 9.7** *(i)* $\operatorname{sinc}(0) = 1$,

*(ii)* $\operatorname{sinc}(n) = 0$ *if* $n \in \mathbb{Z}$ *but* $n \neq 0$.

(We note also that although, strictly speaking, $\widehat{\operatorname{sinc}}(\lambda)$ is not defined for us, since $\int_{-\infty}^{\infty} |\operatorname{sinc}(x)|\, dx = \infty$, we are strongly tempted to say that $\widehat{\operatorname{sinc}}(\lambda) = 1$ if $|\lambda| < \pi$ and $\widehat{\operatorname{sinc}}(\lambda) = 0$ if $|\lambda| > \pi$.)

We can, at once, prove that Pretheorem 9.6 is best possible.

**Lemma 9.8** *If $\epsilon > 0$, then we can find an infinitely differentiable non-zero $f$ such that $\hat{f}(\lambda) = 0$ for $|\lambda| > \pi + \epsilon$ but $f(n) = 0$ for all $n \in \mathbb{Z}$.*

**Exercise 9.9** *In Lemma 9.8 show that we can take $f \in S$ where $S$ is the class discussed in Exercise 8.13.*

We can also show how to recover the function of Pretheorem 9.6 from its values at integer points.

**Theorem 9.10** *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous function with $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$. If $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \pi$ then*

$$\sum_{n=-N}^{N} f(n) \operatorname{sinc}(t - n) \to f(t)$$

*as uniformly as $N \to \infty$.*

Thus Pretheorem 9.6 holds under very general conditions. We state it in a lightly generalised form.

**Theorem 9.11 (Shannon's Theorem)** *Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous function with $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$ and that $K > 0$. If $\hat{f}(\lambda) = 0$ for $|\lambda| \geq K$, then then $f$ is determined by its values at points of the form $n\pi K^{-1}$ with $n \in \mathbb{Z}$.*

Theorem 9.11 belongs to the same circle of ideas as Heisenberg's inequality. It is the key to such devices as the CD.

# 10 Fourier analysis on finite Abelian groups

The methods of Fourier analysis can be extended to the much wider system of locally compact Abelian groups. We shall hint at how this is done by looking at the special case of finite Abelian groups.

Recall that classical Fourier analysis deals with formulae like

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n(t)$$

where $e_n(t) = \exp(int)$. The clue to further extension lies in the following observation.

**Lemma 10.1** *Consider the Abelian group $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and the subgroup $S = \{z : |z| = 1\}$ of $(\mathbb{C} \setminus \{0\}, \times)$. The continuous homomorphisms $\theta : \mathbb{T} \to S$ are precisely the functions $e_n : \mathbb{T} \to S$ given by $e_n(t) = \exp(int)$ with $n \in \mathbb{Z}$.*

**Exercise 10.2** *(i) Find (with proof) all the continuous homomorphisms $\theta : (\mathbb{R}, +) \to (S, \times)$. What is the connection with Fourier transforms?*

*(ii) (Only for those who know Zorn's lemma[4].) Assuming Zorn's lemma, show that any linearly independent set in a vector space can be extended to a basis. If we consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ show that there exists a linear map $T : \mathbb{R} \to \mathbb{R}$ such that $T(1) = 1$, $T(\sqrt{2}) = 0$. Deduce the existence of a function $T : \mathbb{R} \to \mathbb{R}$ such that $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}$ which is not continuous (with respect to the usual metric). Show that, if we accept Zorn's lemma, there exist discontinuous homomorphisms $\theta : (\mathbb{R}, +) \to (S, \times)$.*

This suggests the following definition.

[4] And, particularly, those who only know Zorn's lemma.

**Definition 10.3** *If $G$ is a finite Abelian group, we say that a homomorphism $\chi : G \to S$ is a* character. *We write $\hat{G}$ for the collection of such characters.*

In this section we shall accumulate a substantial amount of information about $\hat{G}$ by a succession of small steps.

**Lemma 10.4** *Let $G$ be a finite Abelian group.*
*(i) If $x \in G$ has order $m$ and $\chi \in \hat{G}$, then $\chi(x)$ is an $m$th root of unity.*
*(ii) $\hat{G}$ is a finite Abelian group under pointwise multiplication.*

To go further we consider, for each finite Abelian group $G$, the collection $C(G)$ of functions $f : G \to \mathbb{C}$. If $G$ has order $|G|$, then $C(G)$ is a vector space of dimension $N$ which can be made into a complex inner product space by means of the inner product

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x) g(x)^*.$$

**Exercise 10.5** *Verify the statements just made.*

**Lemma 10.6** *Let $G$ be a finite Abelian group. The elements of $\hat{G}$ form an orthonormal system in $C(G)$.*

Does $\hat{G}$ form an orthonormal basis of $C(G)$? The next lemma tells us how we may hope to resolve this question.

**Lemma 10.7** *Let $G$ be a finite Abelian group. The elements of $\hat{G}$ form an orthonormal basis if and only if given an element $x \in G$ which is not the identity we can find a character $\chi$ with $\chi(x) \neq 1$.*

The way forward is now clear.

**Lemma 10.8** *Suppose that $H$ is a subgroup of a finite Abelian group $G$ and that $\chi \in \hat{H}$. If $K$ is a subgroup of $G$ generated by $H$ and an element $a \in G$, then we can find a $\tilde{\chi} \in \hat{K}$ such that $\tilde{\chi}|H = \chi$.*

**Lemma 10.9** *Let $G$ be a finite Abelian group and $x$ an element of $G$ of order $m$. Then we can find a $\chi \in \hat{G}$ with $\chi(x) = \exp 2\pi i/m$.*

**Theorem 10.10** *If $G$ is a finite Abelian group then $\hat{G}$ has the same number of elements as $G$ and they form an orthonormal basis for $C(G)$.*

**Lemma 10.11** *If $G$ is a finite Abelian group and $f \in C(G)$, then*

$$f = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$$

*where $\hat{f}(\chi) = \langle f, \chi \rangle$.*

**Exercise 10.12** *Suppose that $G$ is a finite Abelian group. Show that, if we define $\theta_x : \hat{G} \to \mathbb{C}$ by $\theta_x(\chi) = \chi(x)$ for $\chi \in \hat{G}$, $x \in G$, then the map $\Theta : G \to \hat{\hat{g}}$ given by $\Theta(x) = \theta_x$ is an isomorphism.*

*If we now identify $x$ with $\theta_x$ (and, so, $G$ with $\hat{\hat{G}}$) show that*

$$|G|\hat{\hat{f}}(x) = f(x^{-1})$$

*for all $f \in C(G)$ and $x \in G$.*

We put 'computational' bones on this section by exhibiting the structure of $G$ and $\hat{G}$.

**Lemma 10.13** *Let $(G, \times)$ be an Abelian group.*

*(i) Suppose that $x, y \in G$ have order $r$ and $s$ with $r$ and $s$ coprime. Then $xy$ has order $rs$.*

*(ii) If $G$ contains elements of order $n$ and $m$, then $G$ contains an element of order the least common multiple of $n$ and $m$.*

**Lemma 10.14** *Let $(G, \times)$ be a finite Abelian group. Then there exists an integer $N$ and an element $k$ such that $k$ has order $N$ and, whenever $x \in G$, we have $x^N = e$.*

**Exercise 10.15** *Let $p$ be a prime. Use Lemma 10.14 together with the fact that a polynomial of degree $k$ can have at most $k$ roots to show that the multiplicative group $(\mathbb{Z}_p \setminus \{0\}, \times)$ is cyclic.*

**Lemma 10.16** *With the hypotheses and notation of Lemma 10.14 we can write $G = K \times H$ where $K$ is the cyclic group generated by $x$ and $H$ is another subgroup of $K$.*

As usual we write $C_n$ for the cyclic group of order $n$.

**Theorem 10.17** *If $G$ is a finite Abelian group we can find $n(1)$, $n(2)$, $\dots n(m)$ with $n(j+1)|n(j)$ such that $G$ is isomorphic to*

$$C_{n(1)} \times C_{n(2)} \times \dots C_{n(m)}.$$

**Lemma 10.18** *If we have two sequences $n(1)$, $n(2)$, $\ldots n(m)$ with $n(j+1)|n(j)$ and $n'(1)$, $n'(2)$, $\ldots n'(m')$ with $n'(j+1)|n'(j)$ then*

$$C_{n(1)} \times C_{n(2)} \times \ldots C_{n(m)} \text{ is isomorphic to } C_{n'(1)} \times C_{n'(2)} \times \ldots C_{n'(m')}$$

*if and only if $m = m'$ and $n(j) = n'(j)$ for each $1 \leq j \leq m$.*

It is easy to identify $\hat{G}$.

**Lemma 10.19** *Suppose that*

$$G = C_{n(1)} \times C_{n(2)} \times \ldots C_{n(m)}$$

*with $C_{n(j)}$ a cyclic group of order $n(j)$ generated by $x_j$. Then the elements of $\hat{G}$ have the form $\chi_{\omega_{n(1)}^{r(1)}, \omega_{n(2)}^{r(2)}, \ldots \omega_{n(m)}^{r(m)}}$ with $\omega_{n(j)} = \exp(2\pi i/n(j)$ and*

$$\chi_{\omega_{n(1)}^{r(1)}, \omega_{n(2)}^{r(2)}, \ldots \omega_{n(m)}^{r(m)}}(x_1^{s(1)} x_2^{s(2)} \ldots x_m^{s(m)}) = \omega_{n(1)}^{r(1)s(1)} \omega_{n(2)}^{r(2)s(2)} \ldots \omega_{n(m)}^{r(m)s(m)}.$$

My readers will see that $\hat{G}$ is isomorphic to $G$ but the more sophisticated algebraists will also see that this is *not a natural isomorphism* (whereas $G$ and $\hat{\hat{G}}$ are *naturally isomorphic*).

# 11 Poisson and finite Abelian groups

The following neat discussion is taken from the remarkable book of Audrey Terras [8].

Let us look for an analogue of Poisson's summation formula for finite Abelian groups. We need the following observation.

**Lemma 11.1** *Let $H$ be a subgroup of the finite Abelian group $G$. Consider*

$$H^{\#} = \{\chi \in \hat{G} \,:\, \chi(h) = 1 \text{ for all } h \in H\}.$$

*Then $H^{\#}$ is a subgroup of $\hat{G}$ and there is a natural identification*

$$H^{\#} \cong (H/G)\hat{}.$$

If we look long enough at Poisson's summation formula

$$\sum_{n \in \mathbb{Z}} f(n+y) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \exp(2\pi i n y),$$

we may see a correct analogue for finite Abelian groups.

**Theorem 11.2** *Let $H$ be a subgroup of the finite Abelian group $G$ and let $f \in C(G)$. Then, with the notation of Lemma 11.1,*

$$\frac{1}{|H|}\sum_{h\in H} f(gh) = \sum_{\chi\in H^{\#}} \hat{f}(\chi)\chi(g).$$

(It is easy to get the normalisation wrong here, so it is a good idea to check it by taking $f = 1$.)

Recall[5] the following definitions from coding theory.

**Definition 11.3** *A* linear code *$E$ is a subspace of the vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$ of two elements.*

*The* dual *$E'$ of $E$ is the set*

$$E' = \{\mathbf{a} \in \mathbb{F}_2^n : \sum_{j=1}^{n} a_j x_j = 0 \text{ for all } \mathbf{x} \in \mathbb{F}_2^n\}.$$

*If $\mathbf{x} \in E$ its* Hamming weight *$W(\mathbf{x})$ is given by*

$$W(\mathbf{x}) = |\{j : x_j = 1\}|.$$

*The* weight enumerator polynomial *of $E$ is given by*

$$w_E(x, y) = \sum_{j=1}^{n} A_j x^{n-j} y^j$$

*where $A_j$ is the number of elements of $E$ of Hamming weight $j$.*

**Exercise 11.4** *Show directly that $E'$ is a linear code.*

**Exercise 11.5** *Under the assumptions and with the notation above, show that the following results are true.*

*(i) $w_C$ is a homogeneous polynomial of degree $n$.*
*(ii) If $C$ has rank $r$, then $w_C(1, 1) = 2^r$.*
*(iii) $w_C(1, 0) = 1$.*
*(iv) $w_C(0, 1)$ takes the value $0$ or $1$.*
*(v) $w_C(s, t) = w_C(s, t)$ for all $s$ and $t$ if and only if $w_C(0, 1) = 1$.*

**Lemma 11.6** *A linear code $E$ can be considered as a subgroup of the Abelian group $(\mathbb{F}_2, +)$. With the notation of Lemma 11.1, its dual is $E^{\#}$.*

[5] 'Recall' is here used in the mathematical sense 'I am about to tell you'.

Applying the Poisson formula to $E'$ we obtain the following result.

**Lemma 11.7** *If $E$ is a linear code with dual $E'$ and $f \in C(E')$, then*

$$\sum_{a \in E'} f(a) = |E|^{-1} \sum_{\chi \in E} \hat{f}(\chi).$$

Applied to the function $f(a) = x^{n-W(u)} y^{W(u)}$ this gives the fundamental MacWilliams identity.

**Theorem 11.8 (MacWilliams)** *If $E$ is a linear code with dual $E'$, then*

$$w_{E'}(x, y) = |E|^{-1} w_E(x - y, x + y).$$

**Exercise 11.9** *The repetition code $C$ for $\mathbb{F}_2^n$ consists of the two elements $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. Find the dual and verify the MacWilliams identity.*

# 12 The fast Fourier transform and some applications

If $f : \mathbb{R} \to \mathbb{R}$ is well behaved the natural way to compute $\hat{f}(\lambda)$ is by means of a sum

$$N^{-1} \sum_{j=0}^{N-1} f(x_j) \exp(-i\lambda x_j)$$

which requires at least $N + 1$ multiplications and $N - 1$ additions. To get a good grasp of $\hat{f}$ we might require its value at $N$ points which in turn would require about $2N^2$ arithmetical operations.

This made Fourier methods very expensive in computer time but the situation was transformed by the following simple set of observations.

**Theorem 12.1** *Let $G_N$ be the multiplicative group of $N$th roots of unity and $G_{2N}$ the group of $2N$th roots of unity. Suppose that, given $\omega_N = \exp 2\pi/N$, we can compute all the Fourier coefficients of any $F \in C(G_N)$ using no more than $M$ multiplications and additions. Then, given $\omega_N = \exp 2\pi/N$, we can compute all the Fourier coefficients of any $F \in C(G_{2N})$ using no more than $2M + 8N$ multiplications and additions.*

**Theorem 12.2 (Cooley and Tuckey)** *Let $N = 2^n$ and let $G_N$ be the multiplicative group of $N$th roots of unity. Then, given $\omega_N = \exp 2\pi/N$, we can compute all the Fourier coefficients of any $f \in C(G_N)$ using no more than $n2^{n+2} = 4N \log_2 N$ additions and multiplications.*

This method gives us a large number of Fourier transforms very cheaply. A little thought shows that although the method restricts us in some ways these restrictions are worth accepting.

The next result is left as a simple exercise to the reader but forms part of the course.

**Theorem 12.3** *Let $N = 2^n$ and let $G_N$ be the multiplicative group of $N$th roots of unity. Then, given $\omega_N = \exp 2\pi/N$ and $\hat{f}$, we can compute all the values of any $f \in C(G_N)$ using no more than $n2^{n+2} = 4N \log_2 N$ additions and multiplications.*

Here is a pretty application.

**Theorem 12.4** *Suppose $2^n = N \geq p + q + 1$. Then, given $\omega_N = \exp 2\pi/N$, we can compute all the coefficients $c_u$ of the product $\sum_{u=1}^{p+q} c_u t^u$ of two polynomials $\sum_{r=1}^{p} a_r t^r$ and $\sum_{s=1}^{q} b_s t^s$ in no more than $2^n + 3n2^{n+2} = N + 12N \log_2 N$ additions and multiplications.*

This idea can be made the basis of fast exact multiplication of integers so large that they must be stored in many registers.

# 13 The Radon transform

In a CAT scanner beams of X-rays are shone through the human skull and we seek to reconstruct the interior of the skull by looking at the shadows cast by the X-rays.

Let us look first at an idealised one-dimensional problem in which a ray is shone along the $x$-axis in the direction $x$ increasing. If the ray has intensity $V(x)$ and the material has 'attenuating' power $f(x)$ at at $x$ it is natural to suppose that, to first order, in $\delta x$

$$V(x + \delta x) = (1 - f(x))V(x).$$

First year calculus now gives (assuming reasonable behaviour)

$$\frac{V(X)}{V(Y)} = \exp\left(-\int_X^Y f(x)\, dx\right).$$

This law has been verified by experiment. Assuming that $f(x) = 0$ for large $x$ this means that we can indeed find $\int_{-\infty}^{\infty} f(x)\, dx$ by measuring the attenuation of a ray of light.

Now consider the two-dimensional problem in which we have a well behaved, but unknown function $f : \mathbb{R}^2 \to \mathbb{R}$. In accordance with the previous paragraph we may suppose that we can find the line integral $\int_{-\infty}^{\infty} f(x)\,dx$ of $f$ along any given straight line $l$. Can we reconstruct $f$ from this information? This problem and its higher dimensional analogues is call the 'tomographic problem'. The reader is warned that the nature of the problem depends on the dimension. The two-dimensional problem corresponds to 'slice tomography'.

Let us establish some notation. Write $\mathbf{u}_\theta$ for the unit vector making an angle $\theta$ with the $x$-axis and $\mathbf{v}_\theta$ for the unit vector making an angle $\theta + \pi/2$ with the $x$-axis. We know the line integrals

$$(\mathcal{R}f)(r,\theta) = \int_{-\infty}^{\infty} f(r\mathbf{u}_\theta + t\mathbf{v}_\theta)\,dt$$

and we wish to recover $f$. The function $\mathcal{R}f : \mathbb{R} \times \mathbb{T} \to \mathcal{R}$ is called the Radon transform of $f$.

**Exercise 13.1** *(i) Show that*

$$(\mathcal{R}f)(-r,\theta) = (\mathcal{R}f)(r,\theta+\pi)$$

*(ii) Find $\mathcal{R}f$ in the two cases*
*(a) $f(x,y) = \exp(-(x^2+y^2)/2)$,*
*(b) $f(x,y) = 1$ if $|x|,\ |y| \leq 1$.*

The Radon transform has a remarkable connection with the Fourier transform. Write

$$\mathcal{F}_1(\mathcal{R}f)(s,\theta) = \int_{-\infty}^{\infty} (\mathcal{R}f)(r,\theta)\exp(-irs)\,ds$$

so that $\mathcal{F}_1(\mathcal{R}f)$ is the one-dimensional Fourier transform of $\mathcal{R}f$ with respect to the first transform.

**Lemma 13.2** *Provided $f$ is well behaved*

$$\hat{f}(s\mathbf{u}_\theta) = \mathcal{F}_1(\mathcal{R}f)(s,\theta)$$

*where $\hat{f}$ is the usual two dimensional Fourier transform.*

Thus $f$ is uniquely determined by $\mathcal{R}f$ and $f$ may be obtained from $\mathcal{R}f$ by using one and two-dimensional Fourier transforms.

**Exercise 13.3** *Recover the $f$ in Exercise 13.1 from their Radon transforms.*

# 14 References and further reading

If the elegance and variety of a subject is to be judged by the elegance and variety of the (best) texts on that subject, Fourier Analysis must surely stand high. On the pure side the books of Helson [2] and Katznelson [4] would be my first choice for introductions and this course draws on both. (There is also a nice book by Krantz [7].) If you wish to think about applications, the obvious text is that of Dym and and McKean [1]. The next two recommendations are irrelevant to Part III but, if you go on to work in any field involving classical analysis, Zygmund's treatise [9] is a must and, if you would like a first glimpse at wavelets, (unmentioned in this course) Babarah Hubbard's popularisation *The World According to Wavelets* [3] is splendid light reading.

In preparing this course I have also used [5] and [6] since I find the author sympathetic.

# References


[1] H. Dym and H. P. McKean *Fourier Series and Integrals* Academic Press, 1972.

[2] H. Helson *Harmonic Analysis* Adison–Wesley, 1983.

[3] B. Hubbard *The World According to Wavelets* A. K. Peters, 1996.

[4] Y. Katznelson *An Introduction to Harmonic Analysis* Wiley, 1963. [Now reprinted by CUP]

[5] T. W. Körner *Fourier Analysis* CUP, 1988.

[6] T. W. Körner *Exercises for Fourier Analysis* CUP, 1993.

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