Partial Solutions for Exercises in Naive Decision Making

T. W. Körner

Introduction

Here is a miscellaneous collection of hints, answers, partial answers and remarks on some of the exercises in the book. I have written in haste in the hope that others will help me correct at leisure. I am sure that they are stuffed with errors ranging from the TEXtual through to the arithmetical and not excluding serious mathematical mistakes. I would appreciate the opportunity to correct at least some of these problems. Please tell me of any errors, unbridgeable gaps, misnumberings etc. I welcome suggestions for additions.

ALL COMMENTS GRATEFULLY RECEIVED.

If you can, please use LATEX 2_ε or its relatives for mathematics. If not, please use plain text. My e-mail is **twk@dpmms.cam.ac.uk**. You may safely assume that I am both lazy and stupid so that a message saying 'Presumably you have already realised the mistake in Exercise Z' is less useful than one which says 'I think you have made a mistake in Exercise Z because you have have assumed that the sum is necessarily larger than the integral. One way round this problem is to assume that f is decreasing.'

When I was young, I used to be surprised when the answer in the back of the book was wrong. I could not believe that the wise and gifted people who wrote textbooks could possibly make mistakes. I am no longer surprised.

To avoid disappointment note that Exercise Z^* means that there is no comment. Exercise Z? means that I still need to work on the remarks. Note also that what is given is *at most* a sketch and often very much less.

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Exercise 1.1.1

$$\frac{y(cb)}{ca} = \frac{c(yb)}{ca} = \frac{yb}{a}.$$

Exercise 1.1.2

Under the new arrangement you end up with $y(v+1) - y = yv$ in your pocket if the horse wins and you lose y if the horse loses. This is the same as before.

Exercise 1.1.3

You should get an upside down V shape.

Exercise 1.1.4

If $u_1 s < u_2(Y - s)$, then

$$\min\big(u_1(s), u_2(Y - s)\big) = u_1 s.$$

If t is small and strictly positive, we have

$$u_1(s + t) < u_2(Y - (s + t))$$

and

$$\begin{aligned}\min\big(u_1(s + t), u_2(Y - (s + t))\big) &= u_1(s + t) \\ &> u_1 t \\ &= \min\big(u_1 s, u_2(Y - s)\big).\end{aligned}$$

If $u_1 s > u_2(Y - s)$, then

$$\min\big(u_1(s), u_2(Y - s)\big) = u_2(Y - s).$$

If t is small and strictly positive (and $u_1 s > u_2(Y - s)$), then

$$u_1(s + t) > u_2(Y - (s + t))$$

and

$$\begin{aligned}\min\big(u_1(s + t), u_2(Y - (s + t))\big) &= u_1(Y - (s + t)) \\ &< u_1(Y - t) \\ &= \min\big(u_1 s, u_2(Y - s)\big).\end{aligned}$$

Exercise 1.1.6

(i) If $s \leq s^*$, then

$$\min\big(u_1 s^*, u_2(Y - s^*)\big) = u_1 s^* \geq u_1 s \geq \min\big(u_1 s, u_2(Y - s)\big).$$

(ii) If $s \neq s^*$ then either $s > s^*$ or $s^* > s$.

If $s > s^*$, then $Y - s^* > Y - s$ so

$$\min\big(u_1 s^*, u_2(Y - s^*)\big) = u_2(Y - s^*) > u_2(Y - s) \geq \min\big(u_1 s, u_2(Y - s)\big).$$

If $s < s^*$, then

$$\min\big(u_1 s^*, u_2(Y - s^*)\big) = u_1 s^* > u_1 s \geq \min\big(u_1 s, u_2(Y - s)\big).$$

Exercise 1.1.8

If

$$u_1 y_1^* = u_1 y_2^* = \cdots = u_n y_n^* \text{ and } y_1^* + y_2^* + \cdots + y_n^* = Y,$$

then, writing $L = u_1 y_1^*$, we have

$$y_j^* = L u_j^{-1}$$

so

$$L(u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}) = y_1 + y_2 + \cdots + y_n = Y$$

and

$$L = \frac{Y}{u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}}$$

so

$$y_j^* = \frac{Y u_j^{-1}}{u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}}$$

and the solution, if it exists, is unique.

If we set

$$w_j^* = \frac{Y u_j^{-1}}{u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}},$$

then

$$u_1 w_1^* = u_1 w_2^* = \cdots = u_n w_n^* \text{ and } w_1^* + w_2^* + \cdots + w_n^* = Y$$

so we do indeed have a solution.

Exercise 1.1.9

(i) If $y_j > y_j^*$ for all j, then
$$Y = y_1 + y_2 + \cdots + y_n > y_1^* + y_2^* + \cdots + y_n^* = Y$$
which is impossible. Thus we can find a k with
$$y_k \leq y_k^*$$
so
$$\min_j u_j y_j^* = u_k y_k^* \geq u_k y_k \geq \min_j u_j y_j.$$

(ii) If there exists a k with
$$y_k < y_k^*$$
then
$$\min_j u_j y_j^* = u_k y_k^* > u_k y_k \geq \min_j u_j y_j.$$
Thus, under the conditions stated, $y_j \geq y_j^*$.

If there exists a k with
$$y_k > y_k^*$$
then
$$Y = y_1 + y_2 + \cdots + y_n > y_1^* + y_2^* + \cdots + y_n^* = Y$$
which is impossible. Thus $y_j = y_j^*$ for all j.

Exercise 1.2.1

(1) You gain y if the horse loses, and lose vy if it wins.

(2) You gain y if the horse loses, and lose $uy - y = (v+1)y - y = vy$ if it it wins.

(3) You gain $-yv + (y+1)v = y$ if the horse loses, and lose vy if it wins.

Exercise 1.2.2

(i) Since

$$y_1^* + y_2^* + \cdots + y_n^* = Y \text{ and } y_1 + y_2 + \cdots + y_n = Y,$$

there must be a k such that

$$y_k \geq y_k^*$$

and so

$$\max_j u_j y_j^* = u_k y_k^* \leq u_k y_k \leq \max_j u_j y_j.$$

(ii) If

$$\max_j u_j y_j^* = \max_j u_j y_j$$

then the argument of (i) shows that

$$y_j \geq y_j^*$$

for all j. But

$$y_1^* + y_2^* + \cdots + y_n^* = Y \text{ and } y_1 + y_2 + \cdots + y_n = Y,$$

so, if $y_k > y_k^*$ for some k,

$$y_1^* + y_2^* + \cdots + y_n^* < \ y_1 + y_2 + \cdots + y_n,$$

which is impossible, so $y_j = y_j^*$ for all j.

(iii) We minimise our maximum loss by taking

$$y_j =_j^* = \frac{Y u_j^{-1}}{u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}}$$

and our loss, whatever the outcome, is

$$\begin{aligned} L &= Y - u_j \frac{Y u_j^{-1}}{u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}} \\ &= Y \left(1 - \frac{1}{u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}} \right). \end{aligned}$$

We lose if $L < 0$, that is to say, if

$$1 - \frac{1}{u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}} < 0$$

ie if

$$\frac{1}{u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1}} < 1$$

ie if

$$u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1} > 1.$$

The remaining statements follow similarly.

Exercise 1.2.3

If $a_k \neq b_k$, then there is no loss in generality in supposing $a_k < b_k$ Thus

$$\begin{aligned}
\frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n} &= \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_{k-1}} + \frac{1}{c_k} + \frac{1}{c_{k+1}} + \cdots + \frac{1}{c_n} \\
&= \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_{k-1}} + \frac{1}{b_k} + \frac{1}{c_{k+1}} + \cdots + \frac{1}{c_n} \\
&\leq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{b_k} + \frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n} \\
&< \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{a_k} + \frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n} \\
&= 1
\end{aligned}$$

Thus betting with the first bookmaker when $a_j > b_j$ and with the second when $a_j \leq b_j$ is equivalent to betting with a bookmaker who allows a certain profit (unless $a_j = b_j$ for all j).

Exercise 1.2.4

(i) The effective pay out multiplier is αu_j so you can only make a certain profit if

$$\frac{1}{\alpha u_1}+\frac{1}{\alpha u_2}+\cdots+\frac{1}{\alpha u_n}<1$$

ie

$$\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_n}<\alpha.$$

If you *take* then the effective pay out multiplier is u_j/α so you can only make a certain profit if

$$\frac{1}{\alpha^{-1}u_1}+\frac{1}{\alpha^{-1}u_2}+\cdots+\frac{1}{\alpha^{-1}u_n}>1$$

ie

$$\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_n}>\frac{1}{\alpha}.$$

(ii) If you make bets you will bet on horse j with the bookmaker with higher payout multiplier on that horse. By (i) you can only make a certain profit if

$$\frac{1}{\max(a_1,b_1)}+\frac{1}{\max(a_2,b_2)}+\cdots+\frac{1}{\max(a_n,b_n)}<\alpha.$$

If you take bets you will bet on horse j with the bookmaker with lowest payout multiplier on that horse. By (i) you can only make a certain profit if

$$\frac{1}{\min(a_1,b_1)}+\frac{1}{\min(a_2,b_2)}+\cdots+\frac{1}{\min(a_n,b_n)}>\frac{1}{\alpha}.$$

(iii) If $a\geq b$.

$$\frac{1}{\min(a,b)}-\frac{1}{\max(a,b)}=\frac{1}{b}-\frac{1}{a}=\left|\frac{1}{a}-\frac{1}{b}\right|.$$

If $a<b$.

$$\frac{1}{\min(a,b)}-\frac{1}{\max(a,b)}=\frac{1}{a}-\frac{1}{b}=\left|\frac{1}{a}-\frac{1}{b}\right|.$$

Thus, by (ii), if we can make a certain profit both making a bet and taking it

$$\begin{aligned}
&\left|\frac{1}{a_1}-\frac{1}{b_1}\right|+\left|\frac{1}{a_2}-\frac{1}{b_2}\right|+\cdots+\left|\frac{1}{a_n}-\frac{1}{b_n}\right| \\
&\qquad=\left(\frac{1}{\min(a_1,b_1)}+\frac{1}{\min(a_2,b_2)}+\cdots+\frac{1}{\min(a_n,b_n)}\right) \\
&\qquad\quad-\left(\frac{1}{\max(a_1,b_1)}+\frac{1}{\max(a_2,b_2)}+\cdots+\frac{1}{\max(a_n,b_n)}\right) \\
&\qquad>\frac{1}{\alpha}-\alpha=\frac{(1-\alpha^2)}{\alpha}.
\end{aligned}$$

(iv) Suppose $a_j/b_j \geq 1/\alpha^2$. If we bet 1 with the second bookmaker that horse j will lose and

$$\frac{\alpha b_j^{-1}+1}{\alpha a_j+1}$$

with the first bookmaker that horse j will win, then if horse j wins we win

$$\alpha a_j\frac{\alpha b_j^{-1}+1}{\alpha a_j+1}-1=\frac{\alpha^2 a_j b_j^{-1}-1}{\alpha a_j+1}$$

and if horse j loses we win

$$\alpha b_j^{-1}-\frac{\alpha b_j^{-1}+1}{\alpha a_j+1}=\frac{\alpha^2 a_j b_j^{-1}-1}{\alpha a_j+1}.$$

In either case, since $\alpha^2 a_j b_j^{-1}-1>0$ so we can guarantee to make money.

Exercise 1.4.1

(i) Our expected gain is

$$125\times\frac{1}{216}\times 0+75\times\frac{1}{216}\times\frac{9}{4}+15\times\frac{1}{216}\times 3+1\times\frac{1}{216}\times 4-1=\frac{871}{864}-1=\frac{5}{864}$$

so the game is now favourable to us.

(ii) Our expected gain is

$$125\times\frac{1}{216}\times 0+75\times\frac{1}{216}\times\frac{9}{4}+15\times\frac{1}{216}\times 3+1\times\frac{1}{216}\times 10-1=\frac{205}{216}$$

so the game remains favourable to the banker.

(iii) The players will all play hearts (since the dice are symmetric but the rewards are not) but there expected gain is

$$125\times\frac{1}{216}\times 0+75\times\frac{1}{216}\times\frac{9}{4}+15\times\frac{1}{216}\times 3+1\times\frac{1}{216}\times 20-1=\frac{215}{216}$$

so the game remains favourable to the banker.

Exercise 1.5.1

A straight line sloping up if $a_1 > a_2$ and down if $a_2 > a_1$.

Exercise 1.5.4

Observe that

$$\begin{aligned} a_nY - (a_1y_1 + a_2y_2 + \cdots + a_ny_n) \\ &= (a_n - a_1)y_1 + (a_n - a_2)y_2 + \cdots + (a_n - a_n)y_n \\ &\leq 0 \end{aligned}$$

with equality if and only if $y_j = 0$ whenever $a_j > a_n$.

Exercise 1.5.5

(i) By our earlier discussion, I should bet on the horses for which $p_j u_j = p_j/q_j$ is a maximum.

(ii) I wish to maximise

$$Y - \sum_{j=1}^{n} y_j p_j/q_j$$

subject to

$$Y = \sum_{j=1}^{n} y_j p_j/q_j$$

and this is equivalent to minimising $\sum_{j=1}^{n} y_j/q_j$ subject to the condition $Y = \sum_{j=1}^{n} y_j/q_j$. By much the same arguments as before, I should I should take bets on the horses for which p_j/q_j is a minimum and so q_j/p_j is a maximum.

Exercise 1.5.6

If A make a bet with B on the j th horse and B can take it if and only if

$$\alpha p_j \leq \alpha^{-1} q_j^{-1}$$

ie

$$\frac{p_j}{q_j} \geq \frac{1}{\alpha^2}.$$

Thus A can bet on some horse with B if and only if

$$\max_j \frac{p_j}{q_j} \geq \frac{1}{\alpha^2}.$$

Similarly B can bet on some horse with A if and only if

$$\max_j \frac{q_j}{p_j} \geq \frac{1}{\alpha^2}.$$

Exercise 1.5.7

(i) The expected score with one die is

$$\sum_{j=1}^{6} j\,\Pr(\text{throw } j) = \frac{1}{6}\sum_{j=1}^{6} j = \frac{21}{6} = \frac{7}{2}.$$

(ii) If the die shows 1, 2 or 3 our expected score of 7/2 with a new throw is better than our expected score if we stick so we should throw. If the die shows 4, 5 or 6 our expected score of 7/2 with a new throw is less than our expected score if we stick so we should stick.

(iii) We can consider throwing a pair of dice to get X and Y. Our score Z is given by $Z = X$ if $X \geq 4$ and $Z = Y$ if $X < 4$. Our expected score in *High Dice with Free Turn* is

$$\begin{aligned}
\mathbb{E}Z &= \sum_{j=4}^{6} j\,\Pr(X = j) + \sum_{j=1}^{3}\sum_{i=1}^{6} \Pr(X = j,\, Y = i)\\
&= \frac{15}{6} + \Pr(X \leq 3) \times \text{expected score with one die}\\
&\frac{5}{2} + \frac{1}{2} \times \frac{7}{2} = \frac{17}{4}.
\end{aligned}$$

Exercise 1.5.8

Corrected by Nigel White and Liangpeng Zhang. (Two seperate mistakes so the reader is warned there may be more)

(i) It is obviously stupid to rethrow a die with higher score than the other so we need only look at a die with the lowest score. By Exercise 1.5.7 we should rethrow this dice if and only if it shows three or less. We can consider throwing three of dice to get X, Y and Z. Our score W is given by $Z = X + Y$ if $X, Y \geq 4$ and $W = \max\{X, Y\} + Z$ if $\min\{X, Y\} < 4$. Our expected score in High Dice with two dice is

$$\begin{aligned}
W = &\sum_{i\geq 4, j\geq 4} (i+j)\Pr(X=i, Y=j) + \sum_{i\geq 4, 3\geq j} (i+k)\Pr(X=i, Y=j, Z=k) \\
&\quad + \sum_{3\geq i, j\geq 4} (k+j)\Pr(X=i, Y=j, Z=k) \\
&\quad + \sum_{3\geq i, j\geq 3} (\max\{i,j\}+k)\Pr(X=i, Y=j, Z=k) \\
= &\sum_{i\geq 4} i\Pr(X=i) + \sum_{j\geq 4} j\Pr(y=j) + \Pr(\min\{X,Y\}\leq 3)\mathbb{E}Z \\
&\quad + \sum_{i\leq 3, j\leq 3} \max\{i,j\}\Pr(X=i, Y=j) \\
= &\, 2\times\frac{1}{6}(6+5+4) + \frac{3}{4}\times 72 + \left(\frac{5}{36}\times 3 + \frac{3}{36}\times 2 + \frac{1}{36}\times 1\right) = \frac{593}{72}
\end{aligned}$$

(ii) If we play high dice with two dice and free turn the we throw once and if the total shown exceeds the expected value of high dice with two dice (in this case a total of 9 or greater) we do not throw both dice (otherwise we throw both). If we do not throw both we now look at the lowest die. If its value shown exceeds the value of high dice with one die (in this case 4 or greater) we stop otherwise we throw.

(iii) If we play high dice with three dice we throw once, remove the die with highest score and follow tactics for high dice with two dice and free score. If we play high dice with three dice and free turn the we throw once and if the total shown exceeds the expected value of high dice with three dice we do not throw all three dice but remove the highest die and play high dice with two die. (Otherwise we play high dice with three dice.) Continuing we obtain best tactics for any number of dice.

Exercise 1.5.9

We should fix all dice showing 6 and throw the rest. (Clearly we should fix the sixes.) If a large number of dice are thrown it is very likely that a six will be thrown and (since we are happy to fix the six) we are essentially playing a new game with fewer dice but the maximum possible additional score.

Exercise 1.5.10

If the second player's first throw beats the first player's final score she should stick (since she has won). If the second player's first is beaten by the first player's final score she can do no worse by throwing her lowest die so she should. (If the player has final score 12 and she has, say, $(5, 5)$ the second player could chose to do nothing; we have just given the simplest rule.)

If second player's first score equals the first player's final score then she should throw the lowest dice again if and only if the probability of strictly increasing the number shown (and winning) is greater than the probability of strictly decreasing the number shown (and losing) so she should throw again if and only if her lowest die shows 1, 2 or 3.

Exercise 1.6.1

Unless $p_k = 0$ you should place a very small sum of money on the kth horse since you will then scoop the pool if it wins.

Exercise 1.6.4*

Exercise 1.6.5

(i) We have

$$\begin{aligned} f(y) &= \text{our expected profit}\\ &= -\text{expected profit other bettors}\\ &= \text{cost other bets} - \text{expected value other bets}\\ &= T - \left(\frac{p_1t_1(T+y)}{t_1+y} + p_2(T+y)\right) \end{aligned}$$

(ii) We have

$$\begin{aligned} f(y) &= (t_1+t_2) - \left(\frac{p_1t_1(t_1+t_2+y)}{t_1+y} + p_2(t_1+t_2+y)\right)\\ &= t_1+t_2-p_1t_1-p_2t_1-p_2t_2-p_2y-\frac{p_1t_1t_2}{t_1+y}\\ &= p_1t_2-p_2y-\frac{p_1t_1t_2}{t_1+y}. \end{aligned}$$

Thus

$$\begin{aligned} f'(y) &= -p_2 + \frac{p_1t_1t_2}{(t_1+y)^2}\\ f''(y) &= -\frac{2p_1t_1t_2}{(t_1+y)^3} < 0 \end{aligned}$$

Thus f' is strictly decreasing and $f'(y) > 0$ for $y < Y_0$ $f'(y) < 0$ for $y > Y_0$ where

$$\frac{p_1t_1}{(t_1+Y_0)^2} = \frac{p_2t_2}{t_1^2}.$$

Thus $f(y)$ is increasing as y increases from 0 (but at a decreasing rate) to Y_0 and then decreases.

(iii) Observe that

$$p_1t_2 - \frac{p_1t_1t_2}{t_1+y} \to 0$$

but

$$-p_2y \to -\infty$$

so $f(y) \to -\infty$ as $y \to \infty$.

(iii) As we put more money on the first horse we drive the odds on the first horse down and on the second horse up. Eventually we drive the odds on the first horse below the correct odds and are now placing worse and worse bets. The backers of the second horse watch with fascinated pleasure as we use our money to drive the odds on their favoured horse to astronomical heights.

Exercise 1.6.6

(i) Suppose $p_1/t_1 \geq p_2/t_2$. People will bet on the first horse until the total z_1 placed satisfies

$$\frac{p_1}{t_1 + z_1} = \frac{p_2}{t_2}.$$

Thereafter the sums z_1, z_2 bet on on the two horses will satisfy

$$\frac{p_1}{t_1 + z_1} = \frac{p_2}{t_2 + z_2}$$

so

$$\frac{t_1 + z_1}{t_2 + z_1} = \frac{p_1}{p_2}.$$

and everybody's expected winnings equal the amount they bet.

(ii) Once $z_2 > 0$ the tote ratios are the probabilities, ie

$$\frac{t_j + z_j}{T + Z} = p_j$$

where $Z = z_1 + z_2$, $T = t_1 + t_2$

On the other hand if the syndicate bets y_1, y_2 the tote ratios are

$$\frac{t_j}{T + Y} = \frac{(p_j t_j)^{1/2}}{(p_1 t_1)^{1/2} + (p_2 t_2)^{1/2}}.$$

(iv) In the first case the new bettors are (a) knowledgeable and (b) competing against each other so the move the odds towards the true odds.

In the second case the new bettors are (a) knowledgeable and (b) wish to extract the most money from the early bettors so they move the odds to those which give the early bettors smallest expected winnings.

Exercise 1.6.7

(i) We have

$$f'(q) = -\frac{A}{q^2} + \frac{B}{(1-q)^2}$$
$$= \frac{A(1-q)^2 - Bq^2}{q^2(1-q)^2}$$

so $f'(q) < 0$ for $0 < q < q_0$ and $0 < f'(q)$ for $q_0 < q < 1$ where q_0 is the positive solution of

$$A(1-q)^2 = Bq^2$$

and so

$$A^{1/2}(1-q_0) = B^{1/2}q_0$$

ie

$$q_0 = \frac{A^{1/2}}{A^{1/2} + B^{1/2}}.$$

(taking positive square roots throughout).

Thus f has a unique minimum at q_0 with value

$$f(q_0) = A\frac{A^{1/2} + B^{1/2}}{A^{1/2}} + B\frac{A^{1/2} + B^{1/2}}{B^{1/2}} = (A^{1/2} + B^{1/2})^2.$$

(ii) Let $q = u_1^{-1}$ so

$$q + \frac{1}{u_2} = 1$$

and $u_2 = (1-q)^{-1}$.

The bookmaker wishes to minimise

$$f(q) = u_1p_1t_1 + u_2p_2t_2 = \frac{p_1t_1}{q} + \frac{p_2t_2}{1-q}$$

so, by the first part she should take

$$\frac{1}{u_1} = \frac{(p_1t_1)^{1/2}}{(p_1t_1)^{1/2} + (p_2t_2)^{1/2}} \text{ and } \frac{1}{u_2} = \frac{(p_2t_2)^{1/2}}{(p_1t_1)^{1/2} + (p_2t_2)^{1/2}}.$$

Exercise 1.7.2

Write
$$a = \frac{t_1}{p_1} = \frac{t_2}{p_2}$$
so
$$1 = p_1 + p_2 = \frac{t_1}{a} = \frac{t_2}{a} = \frac{T}{a}$$
whence $a = T$. Set $q = u_1^{-1}$ so that $u_1 = q^{-1}$, $u_2 = (1-q)^{-1}$. Then (using Exercise 1.6.7),
$$\begin{aligned} f(q) &= p_1 t_1 u_1 + p_2 t_2 u_2 \\ &= \frac{ap_1^2}{q} + \frac{ap_2^2}{1-q} \\ &\geq ap_1^2 q_0 + \frac{ap_2^2}{1-q_0} \\ &= a(p_1+p_2)^2 = a = T \end{aligned}$$
with equality if and only if
$$q = q_0 = \frac{p_1}{p_1+p_2} = p_1$$
ie if and only if $u_1 = 1/p_1$ and $u_2 = 1/p_2$.

Exercise 1.7.3

Write $Y = y_1 + y_2$. Using Exercise 1.7.2 with
$$u_1 = \frac{t_1 + y_1}{T + Y},\ u_2 = \frac{t_2 + y_2}{T + Y}$$
we see that our expected profit
$$p_1 t_1 \frac{T + Y}{t_1 + y_1} + p_2 t_2 \frac{T + Y}{t_2 + y_2} \geq T$$
with equality if and only if
$$u_1 = \frac{1}{p_1},\ u_2 = \frac{1}{p_2}$$
ie
$$\frac{t_1 + y_1}{T + Y} = p_1,\ \frac{t_2 + y_2}{T + Y} = p_2.$$
Now
$$\frac{t_j + y_j}{T + Y} = p_j \Leftrightarrow t_j + y_j = Tp_j + Yp_j$$
$$\Leftrightarrow y_j = Yp_j$$
so the expected value of our bet is strictly greater than T unless $y_j = Yp_j$.

Exercise 1.7.5

Observe that if we take the expected value of our bet is

$$u_1p_1t_1 + u_2p_2t_2 + \cdots + u_np_nt_n$$

and if we take $t_j = p_jT$

$$u_1p_1t_1 + u_2p_2t_2 + \cdots + u_np_nt_n \geq T$$

for all choices of u_j.

If we choose particular t_j, then we know that we can find $u_j > 0$ with

$$u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1} = 1.$$

and

$$u_1p_1t_1 + u_2p_2t_2 + \cdots + u_np_nt_n = (p_1t_1)^{1/2} + (p_2t_2)^{1/2} + \cdots + (p_nt_n)^{1/2}.$$

Now

$$\frac{d}{dt}At^{1/2} + B(t_0 - t)^{1/2} = \frac{A(t_0 - t)^{1/2} - Bt^{1/2}}{2t^{1/2}(t_0 - t)^{1/2}}$$

and by looking at the sign of the derivative we see that there is a unique maximum for $t_0 > t > 0$ when

$$A(t_0 - t)^{1/2} = Bt^{1/2}$$

ie

$$At^{-1/2} = B(t_0 - t)^{1/2}$$

so by our usual argument

$$u_1p_1t_1 + u_2p_2t_2 + \cdots + u_np_nt_n$$

is maximised uniquely when

$$p_j = at_j$$

ie $t_j = p_jT$. Thus we should take $t_j = p_jT$.

Now suppose that

$$u_1^{-1} + u_2^{-1} + \cdots + u_n^{-1} = U.$$

If we set $v_j = Uu_j$ then

$$v_1^{-1} + v_2^{-1} + \cdots + v_n^{-1} = 1$$

and the expected value of our bet is

$$U(v_1p_1t_1 + v_2p_2t_2 + \cdots + v_np_nt_n)$$

so the maximisation problem is unchanged.

Exercise 1.7.6

(i) This was shown in Exercise 1.7.5. If $t_k = p_k T$ the expected value of the bet is

$$u_1 p_1 t_1 + u_2 p_2 t_2 + \cdots + u_n p_n t_n = u_1 p_1^2 T + u_2 p_2^2 T + \cdots + p_n^2 T \geq TK$$

with equality only if $u_j = p_j^{-1} K$ when the value of the bet is TK.

(ii) This is essentially the same result as (i). The bookie should choose $u_j = p_j^{-1} K$ and then the bettor $t_k = p_k T$. The value of the bet is TK.

Exercise 1.7.7

(i) We are seeking to maximise

$$(1-\alpha)\sum_{j=1}^{n} t_j \frac{T+Y}{t_j+y_j} p_j$$

which is the same as seeking to maximise

$$\sum_{j=1}^{n} t_j \frac{T+Y}{t_j+y_j} p_j.$$

The recommendations are unaltered.

If we bet a small amount we are essentially offered a payout ratio on the jth horse given by

$$u_j = (1-\alpha)\frac{Y}{y_j}.$$

We can only make a profit if

$$\max u_j p_j > 1.$$

Thus, if

$$(1-\alpha)\frac{Y}{y_j} < 1$$

for all j there is no way we can make a positive expected profit.

If we bet a small amount last we are essentially offered a payout ratio on the jth horse given by

$$u_j = \frac{Y-b}{y_j}.$$

We should bet on the horse which has largest value of $(Y-b)p_j/y_j$ ie the largest value of p_j/y_j as before.

If we bet first we do not know the final payout ratios so as before we should bet $p_j T$ on the jth horse.

Exercise 1.7.8

(i) By Theorem 1.6.3 observing that $T+Y$ is constant,

$$\frac{p_1 y_1}{t_1+y_1}+\frac{p_2 y_2}{t_2+y_2},$$

attains a unique maximum for y_1+y_2+Y and $y_1, y_2 \geq 0$ at $y_1 = y_1^*$, $y_2 = y_2^* = Y - y_1^*$ where

$$y_1^* = Y \text{ for } \frac{p_1 t_1}{(t_1+Y)^2} \geq \frac{p_2}{t_2}$$

and, otherwise

$$\frac{p_1 t_1}{(t_1+y_1^*)^2} = \frac{p_1 t_2}{(t_2+y_2^*)^2}.$$

(ii) As usual, we assume that a minimum is attained. By (i) this maximum must be at $y_j = y_j^*$ obeying the conditions

$$\frac{p_j t_j}{(t_j+y_j^*)^2} \geq \frac{p_k}{t_k} \text{ and } y_k^* = 0,$$

or

$$\frac{p_j t_j}{(t_j+y_j^*+y_k^*)^2} < \frac{p_k}{t_k} \text{ and } \frac{p_j t_j}{(t_j+y_j^*)^2} = \frac{p_k t_k}{(t_k+y_k^*)^2}.$$

(iii) Since

$$\frac{p_1}{t_1} \geq \frac{p_2}{t_2} \geq \cdots \geq \frac{p_n}{t_n}$$

we can find an r with $1 \leq r \leq n$ such that

$$\frac{p_1 t_1}{(t_1+y_1^*)^2} = \frac{p_2 t_2}{(t_2+y_2^*)^2} = \cdots = \frac{p_r t_r}{(t_r+y_r^*)^2}$$

and, if $r \leq n-1$,

$$y_{r+1}^* = y_{r+2}^* = \cdots = y_n^* = 0, \text{ and } \frac{p_r t_r}{(t_r+y_r^*)^2} \geq \frac{p_{r+1}}{t_{r+1}}.$$

(iv) Suppose

$$\frac{p_1}{t_1} \geq \frac{p_2}{t_2} \geq \cdots \geq \frac{p_n}{t_n}$$

Starting from zero slowly increase your bet y_1 on the first horse until

$$\frac{p_1 t_1}{(t_1+y_1)^2} = \frac{p_2}{t_2}$$

Now slowly increase your bets y_1 and y_2 on the first two horses in such a way that

$$\frac{p_1 t_1}{(t_1+y_1)^2} = \frac{p_2 t_2}{(t_2+y_2)^2}$$

until

$$\frac{p_1 t_1}{(t_1+y_1)^2} = \frac{p_2 t_2}{(t_2+y_2)^2} = \frac{p_3}{t_3}.$$

Now slowly increase your bets $y_1,\ y_2,\ y_3$ on the first three horses in such a way that

$$\frac{p_1t_1}{(t_1+y_1)^2} = \frac{p_2t_2}{(t_2+y_2)^2} = \frac{p_3t_3}{(t_3+y_3)^2}.$$

Once we are betting on all the horses we have

$$t_j + y_j = (p_jt_j)^{1/2}$$

so the expected value (to the other bettors) of their bet is

$$E = \sum_{j=1}^{n} \frac{p_jt_j}{t_j+y_j}(T+Y) \qquad = (T+Y)\sum_{j=1}^{n}(p_jt_j)^{1/2}$$

$$\left(\sum_{j=1}^{n}(p_jt_j)^{1/2}\right)^2$$

so our expected winnings are

$$\sum_{j=1}^{n} t_j - \left(\sum_{j=1}^{n}(p_jt_j)^{1/2}\right)^2$$

independent of how much more we bet. There is no reason to bet more.

Exercise 2.1.1

(i) $\Pr(\{\omega\}) = \sum_{\rho\in\omega} p(\rho) = p(\omega)$.

(ii) $\Pr(A) = \sum_{\omega\in A\setminus B} p(\omega) + \sum_{\omega\in B} p(\omega) \geq \sum_{\omega\in B} p(\omega) = \Pr(B)$.

(iii) $\Omega \supseteq A \supseteq \varnothing$ so, by (ii),

$$1 = \Pr(\Omega) \geq \Pr(A) \geq \Pr(\varnothing) = 0.$$

(iv) We have

$$\Pr(A\cup B) = \sum_{\omega\in A\cup B} p(\omega) = \sum_{\omega\in A} p(\omega) + \sum_{\omega\in B} p(\omega) = \Pr(A) + \Pr(B).$$

Exercise 2.1.2

Since $[(r-1)2^{-n}, r2^{-n})$ is a translate of $[0, 2^{-n})$ it should have the same probability for each $1 \leq r \leq 2^{-n}$. Now

$$\begin{aligned} 1 = \Pr([0,1)) &= \Pr\left(\bigcup_{r=1}^{2^n} [(r-1)2^{-n}, r2^{-n})\right) \\ &= \sum_{r=1}^{2^n} \Pr\big([(r-1)2^{-n}, r2^{-n})\big) = 2^n \Pr\big([0, 2^{-n})\big) \end{aligned}$$

so

$$\Pr\big([(r-1)2^{-n}, r2^{-n})\big) = \Pr\big([0, 2^{-n})\big) = 2^{-n}.$$

Since $\{\omega\} \subset [(r-1)2^{-n}, r2^{-n})$ for some r, $0 \leq \Pr(\{\omega\}) \leq 2^{-n}$ for all n and so $\Pr(\{\omega\}) = 0$ for all $\omega \in [0,1)$.

A	A	B	B	C	C
B	C	C	A	A	B
C	B	A	C	B	A
D	C	D	C	B	B

TABLE 1. Possible deals with 3 cards

A	A	A	A	A	A	B	B	C	D	C	D	B	B	C	D	C	D	B	B	C	D	C	D
B	B	C	D	C	D	A	A	A	A	A	A	C	D	B	B	D	C	C	D	B	B	D	C
C	D	B	B	D	C	C	D	B	B	D	C	A	A	A	A	A	A	D	C	D	C	B	B
D	C	D	C	B	B	D	C	D	C	B	B	D	C	D	C	B	B	A	A	A	A	A	A
D	C	D	C	B	B	D	C	D	C	B	B	D	C	D	C	B	B	A	A	A	A	A	A
E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E

TABLE 2. Possible deals with 5 cards, E last

EXERCISE 2.2.1

There are 24 deals with E last so there must be 24 deals with E in rth place with r taking one of the 4 values 1, 2, 3, 4, 5 so $120 = 24 \times 5$ deals in all.

EXERCISE 2.2.3*

Exercise 2.2.4

(i) Let $B(n,r,b,g)$ be the number of ways that we can arrange the cards if cards of the same colour are indistinguishable. Now suppose we number the n cards so that they are all distinguishable. The red cards can now be arranged in $r!$ ways, the blue cards can be arranged in $b!$ ways and the green cards in $g!$. Thus the total number of ways of arranging our cards without exchanging colours is $r!b!g!$ and the total number of ways of arranging our cards in any way we wish is $B(n,r,b,g)r!b!g!$. But we already know that the total number of ways of arranging our cards in any way we wish is $n!$ so

$$B(n,r,b,g) = \frac{n!}{r!b!g!}$$

as required.

(ii) Suppose we have m_j cards of colour j and $\sum_{j=1}^{k} m_j = n$. Let $B(n,\mathbf{m})$ be the number of ways that we can arrange the cards if cards of the same colour are indistinguishable. Now suppose we number the n cards so that they are all distinguishable. The j colour cards can now be arranged in $n_j!$ ways. Thus the total number of ways of arranging our cards without exchanging colours is $m_1!m_2!\dots m_k!$ and the total number of ways of arranging our cards in any way we wish is $B(n,\mathbf{m})m_1!m_2!\dots m_k!$. But we already know that the total number of ways of arranging our cards in any way we wish is $n!$ so

$$B(n,\mathbf{m}) = \frac{n!}{m_1!m_2!\dots m_k!}.$$

Exercise 2.2.5

(i) We have

$$(x+y)^n = (x+y)(x+y)(x+y)\dots(x+y)$$

so the coefficient of $x^{n-r}y^r$ is the number of ways we can select x from r distinct terms $(x+y)$ ie $\binom{n}{r}$.

(ii) If $r+b+g \neq n$ the coefficient is 0.

If $r+b+g = n$ we have

$$(x+y+z)^n = (x+y+z)(x+y+z)(x+y+z)\dots(x+y+z)$$

so the coefficient of $x^r y^b z^g$ is the number of ways we can select x from r distinct terms $(x+y)$, y from b distinct remaining terms and z from the g remaining so the coefficient is

$$\frac{n!}{r!b!g!}.$$

H	H	H	H	H	H	H	H	T	T	T	T	T	T	T	T
H	H	H	H	T	T	T	T	H	H	H	H	T	T	T	T
H	H	T	T	H	H	T	T	H	H	T	T	H	H	T	T
H	T	H	T	H	T	H	T	H	T	H	T	H	T	H	T

TABLE 3. *

Two-sided die thrown four times

A	A	A	A	B	B	B	B	C	C	C	C	D	D	D	D
A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D

TABLE 4. *

four-sided die thrown twice

EXERCISE 2.2.6

To get a four-sided die thrown thrice we need a first row consisting of 16 A's followed by 16 B's and so on with the table just given repeated four times underneath.

EXERCISE 2.2.8*

EXERCISE 2.3.1* (SEE EXERCISE 2.3.10)

Exercise 2.3.3

A bet a times as large as another should be a times as valuable.

The sum of the values of two separate bets should be the same as the value of the bets made together.

If every outcome of one bet is no worse than the outcome of the other bet the value of the first bet cannot be less than the value of the second.

Exercise 2.3.7

$$\mathbb{I}_{A^c}(\omega) = \begin{cases} 0 & \text{if } \omega \in A, \\ 1 & \text{if } \omega \notin A, \end{cases}$$

and

$$1 - \mathbb{I}_A(\omega) = \begin{cases} 1 - 1 = 0 & \text{if } \omega \in A, \\ 1 - 0 = 1 & \text{if } \omega \notin A, \end{cases}$$

so

$$\mathbb{I}_{A^c}(\omega) = 1 - \mathbb{I}_A(\omega)$$

for all ω and so

$$\mathbb{I}_{A^c} = 1 - \mathbb{I}_A.$$

(1) Thus

$$\begin{aligned} \mathbb{I}_{(A\cup B)^c} &= 1 - \mathbb{I}_{A\cup B} = 1 - (\mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_A\mathbb{I}_B) \\ &= 1 - \mathbb{I}_A - \mathbb{I}_B + \mathbb{I}_A\mathbb{I}_B) = (1 - \mathbb{I}_A)(1 - \mathbb{I}_B) \\ &= \mathbb{I}_{A^c}\mathbb{I}_{B^c} = \mathbb{I}_{A^c\cap B^c} \end{aligned}$$

and so $(A \cup B)^c = A^c \cap B^c$.

(2) Similarly

$$\begin{aligned} \mathbb{I}_{(A\cap B)^c} &= 1 - \mathbb{I}_{A\cap B} = 1 - \mathbb{I}_A\mathbb{I}_B \\ &= (1 - \mathbb{I}_A) + (1 - \mathbb{I}_B) - (1 - \mathbb{I}_A)(1 - \mathbb{I}_B) \\ &= \mathbb{I}_{A^c} + \mathbb{I}_{B^c} - \mathbb{I}_{A^c}\mathbb{I}_{B^c} = \mathbb{I}_{A^c\cup B^c} \end{aligned}$$

and so $(A \cap B)^c = A^c \cup B^c$.

(3) Finally

$$\mathbb{I}_{(A^c)^c} = 1 - \mathbb{I}_{A^c} = 1 - (1 - \mathbb{I}_A) = \mathbb{I}_A$$

so $(A^c)^c = A$.

We could have obtained (2) from the (1) and (3) by the argument

$$A \cap B = (A^c)^c \cap (B^c)^c = A^c \cup B^c.$$

We have

$$\begin{aligned} \mathbb{I}_{A\setminus B} &= \mathbb{I}_{A\cap B^c} = \mathbb{I}_A\mathbb{I}_{B^c} \\ &= \mathbb{I}_A(1 - \mathbb{I}_B) = \mathbb{I}_A - \mathbb{I}_A\mathbb{I}_B \end{aligned}$$

and, noting that $\mathbb{I}_C^2 = \mathbb{I}_C$,

$$\begin{aligned} \mathbb{I}_{A\triangle B} &= \mathbb{I}_{(A\setminus B)\cup(A\setminus B)} = \mathbb{I}_{A\setminus B} + \mathbb{I}_{B\setminus A} - \mathbb{I}_{A\setminus B}\mathbb{I}_{B\setminus A} \\ &= \mathbb{I}_A - \mathbb{I}_A\mathbb{I}_B + \mathbb{I}_B - \mathbb{I}_A\mathbb{I}_B - (\mathbb{I}_A - \mathbb{I}_A\mathbb{I}_B)(\mathbb{I}_B - \mathbb{I}_A\mathbb{I}_B) \\ &= \mathbb{I}_A + \mathbb{I}_B - 2\mathbb{I}_A\mathbb{I}_B + 0 = \mathbb{I}_A + \mathbb{I}_B - 2\mathbb{I}_A\mathbb{I}_B \end{aligned}$$

(Or think what the answer should be and then verify.)

By inspection, $\mathbb{I}_\Omega = 1$ and $\mathbb{I}_\varnothing = 0$.

Exercise 2.3.8

(i) $\sum_{\omega\in\Omega}\mathbb{I}_D(\omega)=\sum_{\omega\in D}1=|D|$.

(ii) We have

$$\begin{aligned}|A\cup B\cup C|&=\sum_{\omega\in\Omega}\mathbb{I}_{A\cup B\cup C}(\omega)\\&=\sum_{\omega\in\Omega}\big(\mathbb{I}_A(\omega)+\mathbb{I}_B(\omega)+\mathbb{I}_C(\omega)\\&\qquad-\mathbb{I}_{A\cap B}(\omega)-\mathbb{I}_{B\cap C}(\omega)-\mathbb{I}_{C\cap A}(\omega)+\mathbb{I}_{A\cap B\cap C}(\omega)\big)\\&=|A|+|B|+|C|\\&\qquad-|A\cap B|-|B\cap C|-|C\cap A|+|A\cap B\cap C|.\end{aligned}$$

(iii) The number of students who are mathematicians or wear glasses or are musical (or any combination) is the number of mathematicians plus the number of glasses wearers plus the number of musicians correcting for counting those with two of those properties twice by subtracting the number of musical mathematicians plus the number of mathematical glasses wearers plus the the number of goggled musicians correcting for the fact that those who have all three properties have been counted three times and then subtracted three times by adding them in again.

(iv) If n has a factor r then $n=rs$ with $r\leq n^{1/2}$ and/or $s\leq n^{1/2}$. Thus if n is composite it has a prime factor no greater than $n^{1/2}$. If $p<49$ and p is not divisible by 2, 3 or 5 it must be prime. Let A_q be the set of strictly positive integers less than 49 divisible by q. Then

$$\begin{gathered}|A_2|=24,\ |A_3|=16,\ |A_5|=9,\\|A_2\cap A_3|=|A_6|=8,\ |A_3\cap A_5|=|A_{15}|=3,|A_2\cap A_5|=|A_{10}|=4,\\|A_2\cap A_3\cap A_5|=|A_{30}|=1.\end{gathered}$$

Thus

$$|A_2\cup A_3\cup A_5|=24+16+9-8-3-4+1$$

and there are 35 integers between 1 and 48 divisible by at least one of 2, 3 or 5. Thus $48-35-1=12$ integers between 2 and 48 are not divisible by 2, 3 or 5. Since 2, 3 and 5 are prime it follows that there are 15 primes less than 49.

(v) We have

$$\begin{aligned}
\mathbb{I}_{\bigcup_{i=1}^4 A_i} &= 1 - \mathbb{I}_{(\bigcap_{i=1}^4 A_i)^c}\\
&= 1 - \prod_{i=1}^4 \mathbb{I}_{A_i^c}\\
&= 1 - \prod_{i=1}^4 (1 - \mathbb{I}_{A_i})\\
&= \sum_{i=1}^4 \mathbb{I}_{A_i} - \sum_{i<j} \mathbb{I}_{A_i}\mathbb{I}_{A_j}\\
&\qquad + \sum_{i<j<k} \mathbb{I}_{A_i}\mathbb{I}_{A_j}\mathbb{I}_{A_k} + \mathbb{I}_{A_1}\mathbb{I}_{A_2}\mathbb{I}_{A_3}\mathbb{I}_{A_4}\\
&= \sum_{i=1}^4 \mathbb{I}_{A_i} - \sum_{i<j} \mathbb{I}_{A_i\cap A_j}\\
&\qquad + \sum_{i<j<k} \mathbb{I}_{A_i\cap A_j\cap A_k} - \mathbb{I}_{A_1\cap A_2\cap A_3\cap A_4}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\bigcup_{i=1}^4 A_i\right| &= \mathbb{E}\mathbb{I}_{\bigcup_{i=1}^4 A_i}\\
&= \sum_{i=1}^4 \mathbb{E}\mathbb{I}_{A_i} - \sum_{i<j} \mathbb{E}\mathbb{I}_{A_i\cap A_j}\\
&\qquad + \sum_{i<j<k} \mathbb{E}\mathbb{I}_{A_i\cap A_j\cap A_k} - \mathbb{E}\mathbb{I}_{A_1\cap A_2\cap A_3\cap A_4}\\
&= \sum_{i=1}^4 |A_i| - \sum_{i<j} |A_i\cap A_j|\\
&\qquad + \sum_{i<j<k} |A_i\cap A_j\cap A_k| - |A_2\cap A_2\cap A_3\cap A_4|
\end{aligned}$$

(vi) If n has a factor r then $n = rs$ with $r \leq n^{1/2}$ and/or $s \leq n^{1/2}$. Thus if n is composite it has a prime factor no greater than $n^{1/2}$. If $p \leq 100$ and p is not divisible by 2, 3, 4 or 7 it must be prime. Let A_q be the set of strictly positive no greater than than 100 divisible by q.

Then

$$|A_2| = 50,\ |A_3| = 33,\ |A_5| = 20,\ |A_7| = 14$$
$$|A_2 \cap A_3| = |A_6| = 16,\ |A_2 \cap A_5| = |A_{10}| = 10\ |A_2 \cap A_7| = |A_{14}| = 7,$$
$$|A_3 \cap A_5| = |A_{15}| = 6,\ |A_3 \cap A_7| = |A_{21}| = 4,\ |A_5 \cap A_7| = |A_{35}| = 2,$$
$$|A_2 \cap A_3 \cap A_5| = |A_{30}| = 3,\ |A_2 \cap A_3 \cap A_7| = |A_{42}| = 2,$$
$$|A_2 \cap A_5 \cap A_7| = |A_{70}| = 1,\ |A_3 \cap A_5 \cap A_7| = |A_{105}| = 0$$
$$|A_2 \cap A_3 \cap A_5 \cap A_7| = 0$$

Thus

$$|A_2 \cup A_3 \cup A_5 \cup A_7| = 50+33+20+14-16-10-7-6-4-2+3+2+1 = 78$$

and there are 78 integers between 1 and 100 divisible by at least one of 2, 3, 5 or 7. Thus $100 - 78 - 1 = 21$ integers between 2 and 100 are not divisible by 2, 3, 5 or 7. Since 2, 3, 5 and 7 are prime it follows that there are 25 primes less than 100.

Exercise 2.3.10

(i) Paint the cards numbered i, j and k blue and the rest red.

(ii) $A_1 \cap A_2 \cap A_3$ is the event that the first card dealt is 1, the second 2and the third 3.

$$\Pr(A_1 \cap A_2 \cap A_3) = \frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}.$$

(iii) There are $n \times (n-1) \times (n-2)$ of choosing three distinct integers in some order and

$$\binom{n}{3} = \frac{n \times (n-1) \times (n-2)}{3!}$$

of obtaining three distinct integers in a particular order so

$$\sum_{i<j<k} \Pr(A_i \cap A_j \cap A_k) = \sum_{i<j<k} \frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} = \frac{1}{3!}.$$

(iv) The number of ways of choosing integers $i_1, i_2, \dots, i_r$ with $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ is $\binom{n}{r}$.

If $1 \leq i_1 < i_2 < \cdots < i_r \leq n$

$$\Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = \frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-r+1}.$$

Thus

$$\sum_{i_1<i_2<\cdots<i_r} \Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = \binom{n}{r} \frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-r+1} = \frac{1}{r!}$$

Using the inclusion-exclusion formula

$$\begin{aligned}\Pr\left(\bigcup_i A_i\right) &\\ &= \sum_i \Pr(A_i) - \sum_{i<j} \Pr(A_i \cap A_j) + \sum_{i<j<k} \Pr(A_i \cap A_j \cap A_k) - \dots \\ &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1} \frac{1}{n!}.\end{aligned}$$

(v) $\bigcup_i A_i$ is the event that at least one card is dealt in the same place as the number it bears. Thus the probability that no card is dealt in the same place as the number it bears is

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}.$$

(vi) Let p_n be the probability that no card from a pack of n is dealt in the same place as the number it bears. Then (to the number of places shown)

$$\begin{aligned} p_1 &= 0 \\ p_2 &= .50000 \\ p_3 &= .33333 \\ p_4 &= .37500 \\ p_5 &= .36667 \\ p_6 &= .36806 \end{aligned}$$

Setting $x = -1$ in Exercise A.10 (iv) we obtain

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \approx e^{-1}$$

for large n. We observe that

$$e^{-1} = 0.36788$$

to the number of places shown.

(vii) To a very good level of approximation the probability that no pair will consist of identical cards is e^{-1} and the probability that at least one pair will consist of identical cards is

$$1 - e^{-1} \approx .63212$$

(viii) The probability that the first k cards will be dealt in their correct place and the remainder will all be misplaced is

$$\begin{aligned} &\frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-k-1} \\ &\qquad\qquad \times \Pr(\text{a pack of } n-k \text{ cards will be dealt out of order}) \\ &\approx \frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-k-1} e^{-1}. \end{aligned}$$

By symmetry the probability that a particular k cards will be dealt in their correct place and the remainder will all be misplaced is approximately

$$\frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-k-1} e^{-1}.$$

There are $\binom{n}{k}$ ways of choosing k cards so the probability that some set of k cards will be dealt in their correct place and the remainder will all be misplaced is approximately

$$\binom{n}{k} \times \frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-k-1} e^{-1} = \frac{e^{-1}}{k!}.$$

Exercise 2.3.11

(i) Let A_i be the event that the committee can meet on the ith date.

$$\Pr(\text{committee can meet}) = \Pr\left(\bigcup_{i=1}^{n} A_i\right)$$

$$\sum_{1\leq i\leq n} \Pr(A_i) - \sum_{1\leq i<j\leq n} \Pr(A_i\cap A_j) + \sum_{1\leq i<j<k\leq n} \Pr(A_i\cap A_j\cap A_k) - \dots$$

$$= \binom{n}{1}\Pr(A_1) - \binom{n}{2}\Pr(A_1\cap A_2) + \binom{n}{2}\Pr(A_1\cap A_2\cap A_3) - \dots$$

$$+ \binom{n}{1}p^m - \binom{n}{2}p^{2m} + \dots + (-1)^{n-1}\binom{n}{n}p^{nm}$$

We have

$$P(1/2,8,30) = \sum_{r=1}^{30}(-1)^{r-1}a_r$$

with

$$a_r = \binom{30}{r}2^{8r} = \binom{30}{r}(256)^{-1}$$

so $a_r > 0$ and $a_{r+1}/a_r \leq 30/256 < 1$. Thus

$$P(1/2,8,30) \approx \sum_{r=1}^{k}(-1)^{r-1}a_r$$

with error less in magnitude less than a_{r+1}. (Truncating an alternating decreasing sum produces an error less than the first term neglected.

Now $a_1 \approx 0.117$ and $a_2 \approx 0.006$ so $P(1/2,8,30) \approx .12$.

(ii) As before

$$\Pr(\text{committee can meet}) = \Pr\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= \binom{n}{1}\Pr(A_1) - \binom{n}{2}\Pr(A_1\cap A_2) + \binom{n}{2}\Pr(A_1\cap A_2\cap A_3) - \dots$$

$$= \binom{n}{1}\left(\frac{k}{n}\right)^m - \binom{n}{2}\left(\frac{k(k-1)}{n(n-1)}\right)^m + \binom{n}{3}\left(\frac{k(k-1)(k-2)}{n(n-1)(n-2)}\right)^m$$

$$- \dots + (-1)^{k-1}\binom{n}{k}\left(\frac{k(k-1)(k-2)\dots 1}{n(n-1)(n-2)\dots(n-k+1)}\right)^m.$$

Observe that if $m(n-k) < n$ then there must be a possible meeting so $Q(k,m,n) = 1$, but if $p < 1$ then there is always a strictly positive probability that the y cannot meet so $P(p,m,n) < 1$. Thus $P(k/n,m,n) \neq Q(k,m,n)$ whenever $m(n-k) < n$. (Of course even if $m(n-k) > n$ the inequality will usually hold but this is harder to prove.) [Thanks to Nigel White for a correction.]

If $k = 15$, $m = 8$, $n = 30$ then it is still true that the terms are decreasing in size and alternating in sign so the first term (which is the same) dominates. Essentially the terms which are affected by non-independence are very small.

Exercise 2.4.3

$$\begin{aligned}\Pr(A) = \Pr(B) &= \Pr(\text{two heads or two tails in succession})\\ &= 1/4 + 1/4 = 1/2.\end{aligned}$$

We have

$$\begin{aligned}\Pr(A \cap B) &= \Pr(\text{three heads or three tails})\\ &= 1/8 + 1/8 = 1/4 = \Pr(A)\Pr(B)\end{aligned}$$

so A and B are independent.

Exercise 2.4.4

(i) Since $(A^c \cap B) \cap (A \cap B) = \varnothing$ and $(A^c \cap B) \cup (A \cap B) = B$,

$$\Pr(A^c \cap B) + \Pr(A \cap B) = \Pr(B)$$

Thus if A and B are independent

$$\begin{aligned}\Pr(A^c \cap B) &= \Pr(B) - \Pr(A \cap B) = \Pr(A) - \Pr(A)\Pr(B)\\ &= (1 - \Pr(A))\Pr(B) = \Pr(A^c)\Pr(B)\end{aligned}$$

so A^c and B are independent.

Since B and A^c are independent so are B^c and A^c.

(ii) If A and B are independent then, by (i), so are A^c and B^c. If further $A \cup B = \Omega$ then $A^c \cap B^c = \varnothing$ so

$$0 = \Pr(A^c \cap B^c) = \Pr(A^c)\Pr(B^c)$$

and at least one of A^c and B^c has probability 0 and so at least one of A and B has probability 1.

(iii) Let $p_j = \Pr(\{\omega_j\})$. If A and B have the properties stated

$$p_3 = \Pr(A \cap B) = pq$$

so

$$p_4 = Pr(B) - p_3 = (1-p)q$$

and, by independence of complements

$$p_5 = \Pr(A^c \cap B^c) = \Pr(A^c)\Pr(B^c) = (1-p)(1-q).$$

Reversing our arguments we see that if we set $p_1 = p_2 = p(1-q)/2$, $p_3 = pq$, $p_4 = (1-p)q$ and $p_5 = (1-p)(1-q)$ they satisfy the stated conditions. (And indeed we could have guessed the solution if we had been awake.)

In Lemma 2.4.2, writing $|X|$ for the number of elements in X,

$$|\Omega| = |U| \times |V| \geq |A| \times |B|.$$

and this is not true here.

Exercise 2.4.6

(i) Just as in Exercise 2.4.3,

$$\Pr(A) = \Pr(B) = \Pr(C) = 1/2$$
$$\Pr(A \cap B) = 1/4 = \Pr(A)\Pr(B), \quad \Pr(B \cap C) = 1/4 = \Pr(B)\Pr(C),$$
$$\Pr(C \cap A) = 1/4 \Pr(C)\Pr(A).$$

However, if A and B occur, then all the throws are heads or all the throws are tails so C occurs. Thus

$$A \cap B \cap C = A \cap B$$

and

$$\Pr(A \cap B \cap C) = 1/4 \neq 1/8 = \Pr(A)\Pr(B)\Pr(C).$$

(ii) Let $p_r = \Pr(\{\omega_r\})$. Then

$$p_4 = \Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C) = 1/27$$

and

$$p_2 + p_4 = \Pr(A \cap B) = \Pr(A)\Pr(B) = 1/9,$$
$$p_3 + p_4 = \Pr(A \cap C) = \Pr(A)\Pr(C) = 1/9$$

so

$$p_2 = p_3 = 2/27.$$

Now

$$p_1 + p_2 + p_3 + p_4 = \Pr(A) = 1/3$$
$$p_2 + p_4 + p_5 = \Pr(B) = 1/3$$
$$p_3 + p_4 + p_6 = \Pr(C) = 1/3$$

so

$$p_1 = 5/27, \ p_5 = p_6 = 2/9$$

whilst since

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1$$

we have $p_7 = 5/27$.

Since all the p_j are positive and the working is reversible we have found an appropriate set of probability Pr (and it is unique). We observe that

$$\Pr(B \cap C) = p_4 = 2/27 \neq 1/9 = \Pr(B)\Pr(C).$$

Exercise 2.4.7

If A, B and C are independent then we know that they are independent in pairs so (see eg Exercise 2.4.4) A, B and C^c are independent in pairs. Further

$$\begin{aligned}
\Pr(A \cap B \cap C^c) &= \Pr(A \cap B) - \Pr(A \cap B \cap C) \\
&= \Pr(A) \Pr(B) - \Pr(A) \Pr(B) \Pr(C) \\
&= \Pr(A) \Pr(B) \bigl(1 - \Pr(C)\bigr) \\
&= \Pr(A) \Pr(B) \Pr(C^c)
\end{aligned}$$

Exercise 2.4.8

We seek to prove that if U, V and W are probability spaces with associated probabilities $\Pr_U$, $\Pr_V$ and $\Pr_V$ then we can define a probability Pr on $\Omega = U \times V \times W$ in such a way that

$$\Pr(\{(u, v, w)\}) = \Pr_U(\{u\})\Pr_V(\{v\})\Pr_W(\{w\})$$

for all $(u, v, w) \in \Omega$. Further, with this choice of Pr, if $E \subseteq U$, $F \subseteq V$ and $G \subseteq W$ the events

$$E \times V \times W,\ U \times F \times W,\ U \times V \times G$$

are independent.

To this end observe that

$$\begin{aligned}\sum_{(u,v,w)\in\Omega} &\Pr_U(\{u\})\Pr_V(\{v\})\Pr_W(\{w\})\\ &= \sum_{u\in U}\Pr_U(\{u\})\sum_{v\in V}\Pr_V(\{v\})\sum_{w\in W}\Pr_W(\{w\})\end{aligned}$$

and $\Pr_U(\{u\})\Pr_V(\{v\})\Pr_W(\{w\}) \geq 0$ for all $(u, v, w) \in \Omega$ so Pr is indeed a probability.

Further

$$\begin{aligned}\Pr(E \times V \times W \cap U \times F \times W \cap U \times V \times G) &= \Pr(E \times F \times G)\\ = \sum_{(u,v,w)\in E\times F\times G} &\Pr_U(\{u\})\Pr_V(\{v\})\Pr_W(\{w\})\\ = \sum_{u\in E}\Pr_U(\{u\})&\sum_{v\in F}\Pr_V(\{v\})\sum_{w\in G}\Pr_W(\{w\})\\ = \Pr(E \times V \times W)\,&\Pr(U \times F \times W)\,\Pr(U \times V \times G)\end{aligned}$$

and independence in pairs can be proved similarly. Thus the events

$$E \times V \times W,\ U \times F \times W,\ U \times V \times G$$

are indeed independent.

Exercise 2.4.12*

Exercise 2.4.13

The argument is not very different to that of Lemma 2.4.2.

Observe that $\Pr(\{\omega_{(j,k)}\}) = p_j q_k \geq 0$ and

$$\sum_{\omega\in\Omega} \Pr(\{\omega\}) = \sum_{j=1}^{J}\sum_{k=1}^{K} p_j q_k = \sum_{j=1}^{J} p_j \sum_{k=1}^{K} q_k = 1^2 = 1$$

$$\begin{aligned}\Pr(X = c,\, Y = d) &= \sum_{c_j=c,\, d_k=d} \Pr(\{\omega_{(j,k)}\})\\ &= \sum_{c_j=c,\, d_k=d} p_j q_k = \sum_{c_j=c} p_j \sum_{d_k=d} q_k\\ &= \Pr(X = c)\Pr(Y = d)\end{aligned}$$

so X and Y are independent and $\mathbb{E}X\mathbb{E}Y = \mathbb{E}XY$.

$\mathbb{E}X$ is the value the bet on the first race if we place 1 unit. $\mathbb{E}Y$ is the value the bet on the second race if we place 1 unit. $\mathbb{E}XY$ is the value of our bet if we place 1 unit on it and the bet our winnings on the second race.

Exercise 2.4.14

$$\mathbb{E}X = \mathbb{E}Y = \tfrac{1}{2} \times 1 + \tfrac{1}{2} \times 0$$

Now

$$X(\omega_1)Y(\omega_1) = X(\omega_2)Y(\omega_2) = 0$$

so $XY = 0$ and

$$\mathbb{E}XY = 0 \neq \tfrac{1}{4} = \mathbb{E}X\mathbb{E}Y$$

On the other hand $X^2 = X$ so

$$\mathbb{E}X^2 = \tfrac{1}{2} \neq \tfrac{1}{4} = (\mathbb{E}X)^2.$$

Exercise 2.4.15

$$\begin{aligned}\Pr(X=0,\ Y=0) &= \Pr(\{\omega_5\}) = \frac{1}{5}\\ &\neq \frac{1}{25} = \Pr(\{\omega_5\})^2 \Pr(X=0)\Pr(Y=0)\end{aligned}$$

so X and Y are not independent.

$$\Pr(XY=1) = \Pr(\{\omega_1,\,\omega_4\}) = \frac{2}{5}$$

$$\Pr(XY=-1) = \Pr(\{\omega_2,\,\omega_3\}) = \frac{2}{5}$$

$$\Pr(XY=0) = \Pr(\{\omega_5\}) = \frac{1}{5}$$

$$\mathbb{E}X = \mathbb{E}Y = \mathbb{E}XY\frac{2}{5}\times 1 + \frac{2}{5}\times(-1) = 0$$

so $\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y$.

Exercise 2.4.16

(iii) Paint the high cards red and the low cards blue. The probability asked for in (i) is the probability that if I deal 3 cards from a pack of 4 red and 4 blue cards I get all blue or all red ie

$$2 \times \Pr(\text{all blue}) = 2 \times \frac{4}{8} \times \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}.$$

The probability asked for in (i) is the probability that if I deal 6 cards from a pack of 4 red and 4 blue cards I get 3 blue and 3 red. The probability of any particular hand of this type (say $RRRBBB$) is

$$\frac{4}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{14} \times \frac{2}{5} = \frac{1}{35}$$

There are

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

such hands so the probability that the hand will contain three of the four smallest cards and three of the four largest cards is $20/35 = 4/7$.

EXERCISE 2.4.17

(i) Write

$$A = \{\omega \in \Omega : X(\omega) \in U\} \text{ and } B = \{\omega \in \Omega : Y(\omega) \in V\}.$$

Then

$$\begin{aligned}
\Pr(A \cap B) &= \sum_{X(\omega)\in U,\, Y(\omega)\in V} \Pr(\{\omega\}) \\
&= \sum_{u\in U,\, v\in V} \Pr(X(\omega) = u,\ Y(\omega) = v) \\
&= \sum_{u\in U,\, v\in V} \Pr(X(\omega) = u) \Pr(Y(\omega) = v) \\
&= \sum_{u\in U} \Pr(X(\omega) = u) \sum v \in V \Pr(Y(\omega) = v) \\
&= \Pr(A) \Pr(B)
\end{aligned}$$

(ii) Suppose s, $t \in \mathbb{R}$. Then, using (i) and writing

$$f^{-1}(s) = \{x : f(x) = s\},\ g^{-1}(t) = \{y : g(y) = t\},$$

we have

$$\begin{aligned}
\Pr(f(X)(\omega) = s,\ g(Y)(\omega) = t) &= \Pr\big(X(\omega) \in f^{-1}(s),\ Y(\omega) \in g^{-1}(t)\big) \\
&= \Pr\big(X(\omega) \in f^{-1}(s)\big) \Pr\big(Y(\omega) \in g^{-1}(t)\big) \\
&= \Pr(f(X)(\omega) = s) \Pr(g(Y)(\omega) = t)
\end{aligned}$$

(iii) If X_1, X_2, $\dots$, X_n are independent random variables and U_1, U_2, $\dots$, U_n are subsets of $\mathbb{R}$ then the events $\{\omega \in \Omega : X_j(\omega) \in U_j\}$ are independent.

We can prove this by induction on n. It is certainly true for $n = 1$. If it is true for all $n \leq m - 1$ then the result for $n = m$ follows from the observation that

$$\begin{aligned}
\Pr\left(\bigcap_{j=1}^{m} X_j^{-1}(U_j)\right) &= \sum_{X_j(\omega)\in U_j \text{ for all } j} \Pr(\{\omega\}) \\
&= \sum_{u_j\in U_j \text{ for all } j} \Pr(X_1(\omega)=u_1,\ X_2(\omega_2)=u_2,\ \ldots,\ X_m(\omega)=u_m) \\
&= \sum_{u_j\in U_j \text{ for all } j} \prod_{j=1}^{m} \Pr(X_j(\omega)=u_j) \\
&= \sum_{u_1\in U_1}\sum_{u_2\in U_2}\cdots\sum_{u_m\in U_m}\prod_{j=1}^{m} \Pr(X_j(\omega)=u_j) \\
&= \prod_{j=1}^{m}\sum_{u_j\in U_j} \Pr(X_j(\omega)=u_j) \\
&= \prod_{j=1}^{m} \Pr\left(X_j^{-1}(U_j)\right).
\end{aligned}$$

If $X_1, X_2, \ldots, X_n$ are independent random variables and $f_j : \mathbb{R} \to \mathbb{R}$ then the sets

$$\{\omega \in \Omega \,:\, f_j(X_j)(\omega) = t_j\} = \{\omega \in \Omega \,:\, X_j(\omega) \in f_j^{-1}(t_j)\}$$

are independent and so the random variables $f_j(X_j)$ are independent.

Exercise 2.4.21

$$\begin{aligned}
\Pr(X = 1) &= \Pr(\text{initial head}) = 1/2\\
\Pr(X = -1) &= \Pr(\text{initial tail}) = 1/2\\
\Pr(Y = 2) &= \Pr(\text{initial head}) = 1/2\\
\Pr(Y = -2) &= \Pr(\text{initial tail}) = 1/2\\
\Pr(Z = 1) &= \Pr(\text{second head}) = 1/2\\
\Pr(Z = -1) &= \Pr(\text{second tail}) = 1/2\\
\Pr(W = 1) &= \Pr(\text{HH}) + \Pr(\text{TT}) = 1/4 + 1/4 = 1/2\\
\Pr(W = -1) &= \Pr(\text{TH}) + \Pr(\text{HT}) = 1/4 + 1/4 = 1/2\\
\Pr(V = 1) &= \Pr(\text{HHH}) + \Pr(\text{TTT}) = 1/8 + 1/8 = 1/4\\
\Pr(V = -1) &= 1 - \Pr(W = 1) = 3/4
\end{aligned}$$

Thus

$$\begin{gathered}
\Pr(X = 1) = \Pr(Z = 1) = \Pr(W = 1) = 1/2;\\
\Pr(X = -1) = \Pr(Z = -1) = \Pr(W = -1) = 1/2;\\
\Pr(X = 1) = 1/2 \neq 0 = \Pr(Y = 1);\\
\Pr(X = 1) = 1/2 \neq 1/4 = \Pr(V = 1);\\
\Pr(V = 1) = 1/4 \neq 0 = \Pr(Y = 1)
\end{gathered}$$

and the results follow.

Exercise 2.4.22

(i) If $2 \leq r \leq 7$

$$\Pr(\text{throw } r) = \sum_{s=1}^{r-1} \Pr(\text{first die } s, \text{ second } r - s) \frac{r-1}{36}.$$

By symmetry, if $7 \leq r \leq 12$,

$$\Pr(\text{throw } r) = \frac{13 - r}{36}.$$

(ii) We have

$$\begin{aligned}
&(x + x^2 + x^3 + x^4 + x^5 + x^6) \times (x + x^2 + x^3 + x^4 + x^5 + x^6) = \\
&x^2 + x^3 + x^4 + x^5 + x^6 + x^7 \\
&\quad +x^3 + x^4 + x^5 + x^6 + x^7 + x^8 \\
&\qquad +x^4 + x^5 + x^6 + x^7 + x^8 + x^9 \\
&\qquad\quad +x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} \\
&\qquad\qquad +x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} \\
&\qquad\qquad\quad +x^7 + x^8 + x^9 + x^{10} + x^{11} + x12 = \\
&x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x12
\end{aligned}$$

(iii) Let $X = Y + Z$ with Y the throw on the first die, Z on the second.

$$\begin{aligned}
\Pr(X = 3) &= \Pr(Y = 2, Z = 1) = \Pr(Y = 2)\Pr(Z = 1) = 6/36 \\
\Pr(X = 4) &= \Pr(Y = 3, Z = 1) = \Pr(Y = 3)\Pr(Z = 1) = 2/36 \\
\Pr(X = 6) &= \Pr(Y = 5, Z = 1) = \Pr(Y = 5)\Pr(Z = 1) = 2/36 \\
\Pr(X = 7) &= \Pr(Y = 2, Z = 5) = \Pr(Y = 2)\Pr(Z = 5) = 3/36 \\
\Pr(X = 14) &= \Pr(Y = 7, Z = 7) = \Pr(Y = 7)\Pr(Z = 7) = 1/36 \\
\Pr(X = 15) &= \Pr(Y = 7, Z = 8) = \Pr(Y = 7)\Pr(Z = 8) = 1/36
\end{aligned}$$

and

$$\begin{aligned}
\Pr(X=8) &= \Pr(Y=2, Z=6) + \Pr(Y=3, Z=5) + \Pr(Y=7, Z=1)\\
&= \Pr(Y=2)\Pr(Z=6) + \Pr(Y=3)\Pr(Z=5)\\
&\qquad + \Pr(Y=7)\Pr(Z=1) = 6/36\\
\Pr(X=9) &= \Pr(Y=2, Z=7) + \Pr(Y=3, Z=6)\\
&= \Pr(Y=2)\Pr(Z=7) + \Pr(Y=3)\Pr(Z=7) = 4/36\\
\Pr(X=10) &= \Pr(Y=2, Z=8) + \Pr(Y=5, Z=5) + \Pr(Y=7, Z=3)\\
&= \Pr(Y=2)\Pr(Z=8) + \Pr(Y=5)\Pr(Z=5)\\
&\qquad + \Pr(Y=7)\Pr(Z=3) = 5/36\\
\Pr(X=11) &= \Pr(Y=3, Z=8) + \Pr(Y=5, Z=6)\\
&= \Pr(Y=3)\Pr(Z=8) + \Pr(Y=5)\Pr(Z=6) = 2/36\\
\Pr(X=12) &= \Pr(Y=5, Z=7) + \Pr(Y=7, Z=5)\\
&= \Pr(Y=5)\Pr(Z=7) + \Pr(Y=7)\Pr(Z=5) = 2/36\\
\Pr(X=13) &= \Pr(Y=7, Z=6) + \Pr(Y=5, Z=8)\\
&= \Pr(Y=7)\Pr(Z=6) + \Pr(Y=5)\Pr(Z=8) = 2/36
\end{aligned}$$

(iv) Exactly the same calculations show that

$$\begin{aligned}
&(3x^2 + x^3 + x^5 + x^7)(2x + x^5 + x^6 + x^7 + x^8)\\
&\quad = 6x^3 + 2x^4 + 2x^6 + 3x^7 + 6x^8 + 4x^9 + 5x^{10}\\
&\qquad\qquad + 2x^{11} + 2x^{12} + 2x^{13} + x^{14} + x^{15}
\end{aligned}$$

(v) We have

$$\begin{aligned}
&(x + x^2 + x^3 + x^4 + x^5 + x^6)^2\\
&\qquad = x(1+x)(1+x^2+x^4)x(1+x+x^2)(1+x^3)\\
&\qquad = x(1+x^3)(1+x^2+x^4)x(1+x)(1+x+x^2)\\
&\qquad = (x + x^3 + x^4 + x^5 + x^6 + x^8)(x + 2x^2 + 2x^3 + x^4)
\end{aligned}$$

so using the parallelism between coefficient and probability calculations (or just rechecking) we find that $(1, 3, 4, 5, 6, 8)$, $(1, 2, 2, 3, 3, 4)$ are non-standard dice with the same probabilities for totals as standard dice.

(vi) A non-standard pair of dice correspond to a pair of polynomials P and Q each of degree 5 with positive integer coefficients such that

$$P(x)Q(x) = (1 + x + x^2 + x^3 + x^4 + x^5)^2$$

Now

$$(x-1)^2 P(x)Q(x) = (x^6 - 1)^2 = \left(\prod_{k=1}^{6}(x - \omega^k)\right)^2$$

so

$$P(x)Q(x) = \left(\prod_{k=1}^{5}(x - \omega^k)\right)^2$$

with $\omega = \exp(\pi i/3)$.

The non-real roots of a real polynomial occur in conjugate pairs so $P(x)$ and $Q(x)$ must be products of $x+1$, $(x-\omega)(x-\bar{\omega}) = (x^2+x+1)$ and $(x-\omega^2)(x-\bar{\omega}^2) = (x^2-x+1)$. By working through the possibilities, we see that the only choices for P and Q are those that have already been considered in (v).

Exercise 2.4.23

Suppose such a die exists.

We must have

$$11^{-1} = \Pr(\text{total } 2) = \Pr(\text{throw } (1,1)) = p_1^2$$

Thus $p_1 = 11^{-1/2}$. Similarly $p_6 = 11^{-1/2}$. Thus

$$\begin{aligned}\Pr(\text{total } 7) &\geq \Pr(\text{throw } (1,6) \text{ or } (6,1))\\ &= p_1 p_6 + p_6 p_1 = 2/11\end{aligned}$$

Thus no such die exists.

Exercise 2.4.24

My proposed solution was wrong (and I think I underestimated the work involved). Here with my thanks is a solution from Matthew Towers.

The player, P, and the banker, B, draw cards uniformly at random with replacement from the set $\{2, 3, 4\}$. Each gets one card face up. P may continue to draw more cards, up to a total of 3. When they stop, B must draw if they have 2 or 3 and must not draw if they have a 4. The scoring is then

- If P has a total of at least 7 they get -1, regardless of B's score.
- If P has < 7 and B has at least 7, P gets 1.
- If both P and B have less then 7 then if the scores are equal P gets 0, otherwise P gets 1 if they have the higher score and -1 if they have the lower score.

We then have the following payoffs and expected values:

	B's 1st card									
	2			3			4	P's EV if B's 1st card is		
	B's next card									
P stops at	2	3	4	2	3	4		2	3	4
3	-1	-1	-1	-1	-1	1	-1	-1	$-1/3$	-1
4	0	-1	-1	-1	-1	1	0	$-2/3$	$-1/3$	0
5	1	0	-1	0	-1	1	1	0	0	1
6	1	1	0	1	0	1	1	$2/3$	$2/3$	1
$\geqslant 7$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

P's first card could be 2, 3, or 4.

- P's first card is 2. They should draw again, getting 4, 5, or 6. Suppose they have 4.
  - If B has 4 then stopping gets 0 and drawing gets an expected value of $(1/3) \cdot 1 + (2/3) \cdot (-1) = -1/3$ so P should stop.
  - If B has 3, stopping has expected value $-1/3$ (from the table) and drawing has expected value $(1/3) \cdot$(expected score if B has 3 and P stops at 6)$+ (2/3) \cdot (-1) = -4/9$ so P shold hold.
  - If B has 2, stopping has an expected value of $-2/3$ from the table and drawing has expected value $(1/3)$(expected score if B has 2 and P stops at 6)$+ (2/3) \cdot (-1) = -4/9$ so P should draw.

Now suppose P has 5 after the second draw. It can't make sense to draw again, so their expectation is $1/3$ (conditioning on the three possibilities for B's card) Finally if P has 6 after the 2nd draw, they should not draw again. The expected score is $7/9 = (1/3)(2/3) + (1/3)(2/3) + (1/3)(3/3)$ from the table.

Following this strategy, with first card 2 P has an EV of $(1/3)((1/3)\cdot 0+(1/3)(-4/9)+(1/3)(-1))+(1/3)(1/3)+(1/3)(7/0) = 23/81$.

- P's first card is 3. If B has 2 or 4 then clearly P must draw. If B has 3 then stopping has expectation $(1/3)(-1)+(1/3)(-1)+(1/3)\cdot 1 = -1/3$ and drawing expects $(1/3)\cdot 0 + (1/3)(2/3) + (1/3)(-1) = -1/9$ from the table, so P should draw in this case too. Their expectation is $(1/3)$(expectation if they stop at 5)$+$ $(1/3)$(expectation if they stop at 6)$+(1/3)$(expectation if they stop at 7) which from the table is $(1/3)(1/3) + (1/3)(7/9) + (1/3)(-1) = 1/27$.
- P's first card is 4. Stopping has an expected value of $-2/3$ if B has 2, $-1/3$ if B has 3, and 0 if B has 4. If P draws, the expectations are as in the first bullet point when P had 4 after their second draw. Thus P should stop if B has 3 or 4, expecting $-1/3$ and 0), and draw if B has 2, expecting $-4/9$. Their expected score is $(1/3)(-4/9)+(1/3)(-1/3)+(1/3)\cdot 0 = -7/27$.

Overall the expected score for P is $(1/3)(23/81) + (1/3)(1/27) + (1/3)(-7/27) = 5/243$.

(ii) If the player follows the rule 'draw if your hand is 4 or less', she will usually end up with a 6 and the banker will usually end up with a 6, so the player will usually win.

Exercise 2.4.25

(i) You should attack all the questions with $p_j = 1$ first (since your probability of losing is 0) and all the questions with $p_j = 0$ last.

(ii) Let X_r be your expected winnings from the r th question in oder. Your probability of actually answering the rth question correctly is

$$p_{j(1)}p_{j(2)}\dots p_{j(r)}$$

so

$$\mathbb{E}X_r = p_{j(1)}p_{j(2)}\dots p_{j(r)}a_{j(r)}$$

and your total expected winnings is

$$\begin{aligned}\mathbb{E}\sum_{r=1}^{n} X_r &= \sum_{r=1}^{n}\mathbb{E}X_r\\ &= \sum_{r=1}^{n} p_{j(1)}p_{j(2)}\dots p_{j(r)}a_{j(r)}.\end{aligned}$$

(iii) From (ii)

$$\begin{aligned}e_A - e_B &= qp_ia_i + qp_ip_ja_j - qp_ja_j + qp_jp_ia_i\\ &= q\big(p_i(1-p_j)a_i - p_j(1-p_i)a_j\big)\end{aligned}$$

We prefer plan A to plan B if $e_A - e_B > 0$ so if

$$q\big(p_i(1-p_j)a_i - p_j(1-p_i)a_j\big) > 0$$

so if

$$\frac{a_ip_i}{1-p_i} > \frac{a_jp_j}{1-p_j}.$$

The remaining cases follow the same pattern.

(iv) If we do not follow this strategy, part (iii) shows there is a better strategy.

(v) If $p_1 = p_2 = \dots = p_n$ then p_j we should choose the questions in decresing order of the a_j. This is reasonable since the probability of sucessfully answering exactly r questions does not depend on order so we wish to get the largest rewards first.

If $a_1 = a_2 = \dots = a_n$ we should choose the questions in decreasing order of p_j. This is reasonable since we are now trying to make the game last as long as possible.

If the p_j are very small then we should choose the question for which a_jp_j is largest. This is reasonable since we are very unlikely to get the chance to answer more than one question.

Exercise 2.5.1

The probability of a hand with three aces in specified places and the remainder non aces (for example $AAANNNNNNNNNN$ where A is an ace and N a non-ace) is

$$\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{48}{49} \times \frac{47}{48} \times \cdots \times \frac{38}{39} = \frac{24 \times 38}{52 \times 51 \times 50 \times 49}$$

and there are

$$\binom{13}{3} = \frac{13 \times 12 \times 11}{3 \times 2} = 13 \times 22$$

such hands. The probability of exactly three aces is thus

$$\frac{13 \times 22 \times 24 \times 37}{52 \times 51 \times 50 \times 48} \approx 0.0412.$$

Exercise 2.5.2

If some one stakes 1 probability of winning is 1/32 and if they win they get 28 so the expected gain is

$$28 \times \frac{1}{32} - 1 = \frac{7}{8} - 1 = -\frac{1}{8}$$

(ie 'they lost one eighth part of all the money they played for').

$$\begin{aligned}
&\Pr(\text{point comes up in 22 games}) \\
&\qquad = 1 - \Pr(\text{point does not come up up in 22 games}) \\
&\qquad = 1 - \big(\Pr(\text{point does not come up up in one game}\big)^{22} \\
&\qquad = 1 - \left(\frac{31}{32}\right)^{22} \\
&\qquad \approx 0.5027
\end{aligned}$$

Thus the Master of the Balls is betting at favourable odds. (But not very favourable, he must much prefer his standard game.)

Exercise 2.5.4

Since f is increasing $f(a) \geq f(x)$ for $x \geq a$. Since $f(x) > 0$ and $f(a)$, it follows that $1 \leq f(x)/f(a)$ for $x \geq a$ and $0 \leq f(x)/f(a)$ for all $x < a$. Thus
$$\mathbb{I}_{(a,\infty)}(x) \leq \frac{f(x)}{f(a)}$$
for all x.

It follows that
$$\begin{aligned}
\Pr(X \geq a) &= \mathbb{E}\mathbb{I}_{(a,\infty)}(X) \\
&\leq \mathbb{E}\left(\frac{f(X)}{f(a)}\right) \\
&= \frac{\mathbb{E}f(X)}{f(a)}.
\end{aligned}$$

EXERCISE 2.5.8

Observe that $\Pr(\omega_j) \geq 0$ and
$$\Pr(\omega_1) + \Pr(\omega_2) + \Pr(\omega_3) = 1.$$

$$\mathbb{E}X = 0(1-p) + ap/2 - ap/2 = 0$$
and
$$\operatorname{var} X = \mathbb{E}X^2 = 0(1-p) + a^2p/2 - a^2p/2 = a^2.$$
Thus
$$\Pr(|X - \mathbb{E}X|) = \Pr(\omega_2) + \Pr(\omega_3) = p = \frac{\operatorname{var} X}{a^2}.$$

Exercise 2.5.10

(i) Observe that

$$\begin{aligned}\mathbb{E}(X-a)^2 &= \mathbb{E}\big((X-\mathbb{E}X)+(\mathbb{E}X-a)\big)^2\\ &= \mathbb{E}\big((X-\mathbb{E}X)^2+2(\mathbb{E}X-a)(X-\mathbb{E}X)(\mathbb{E}X-a)^2\big)^2\\ &= \mathbb{E}\big(X-\mathbb{E}X\big)^2+2(\mathbb{E}X-a)\mathbb{E}(X-\mathbb{E}X)+\mathbb{E}(\mathbb{E}X-a)^2\\ &= \mathbb{E}\big(X-\mathbb{E}X\big)^2+2(\mathbb{E}X-a)0+\mathbb{E}(\mathbb{E}X-a)^2\\ &== \operatorname{var}X+(\mathbb{E}X-a)^2.\end{aligned}$$

Thus $\mathbb{E}(X-a)^2$ is minimised by taking $a=\mathbb{E}X$.

(ii) We have

$$\operatorname{var}X+\operatorname{var}Y=\operatorname{var}X+\operatorname{var}(-1)X=\operatorname{var}X+(-1)^2\operatorname{var}X=2\operatorname{var}X.$$

and

$$\operatorname{var}(X+Y)=\operatorname{var}0=0.$$

Since X and Y are not independent Lemma 2.5.9 (iv) is not relevant.

(iii) We have

$$\begin{aligned}\operatorname{var}(X+Y) &= \mathbb{E}\big(X+Y-\mathbb{E}(X+Y)\big)^2=\mathbb{E}(X+Y-\mathbb{E}X-\mathbb{E}Y)^2\\ &= \mathbb{E}\big(X^2+Y^2+(\mathbb{E}X)^2+(\mathbb{E}Y)^2+2XY-2(\mathbb{E}X)X-2(\mathbb{E}Y)X\\ &\qquad -2(\mathbb{E}X)Y-2(\mathbb{E}Y)Y+2(\mathbb{E}X)(\mathbb{E}Y)\big)\\ &= \mathbb{E}X^2+\mathbb{E}Y^2+(\mathbb{E}X)^2+(\mathbb{E}Y)^2+2(\mathbb{E}X)(\mathbb{E}Y)\\ &\qquad -2(\mathbb{E}X)^2-2(\mathbb{E}X)(\mathbb{E}Y)\\ &\qquad -2(\mathbb{E}X)(\mathbb{E}Y)-2(\mathbb{E}Y)^2+2(\mathbb{E}X)(\mathbb{E}Y)\\ &= \mathbb{E}X^2-\mathbb{E}X^2+\mathbb{E}Y^2-\mathbb{E}Y^2\\ &= \operatorname{var}X+\operatorname{var}Y.\end{aligned}$$

Exercise 2.5.11

Write $\mathbb{A}X = \mathbb{E}|X - \mathbb{E}X|$ and $\mathbb{B}X = \mathbb{E}(X - \mathbb{E}X)^4$

(i) Both carry over since

$$(X + a) - \mathbb{E}(X + a) = X - \mathbb{E}X,$$

so $\mathbb{A}(X + a) = \mathbb{A}X$ and $\mathbb{B}(X + a) = \mathbb{B}X$.

(ii) Similar formulae.

$$\mathbb{A}aX = \mathbb{E}|aX - a\mathbb{E}X| = \mathbb{E}|a||X - \mathbb{E}X| = |a|\mathbb{A}X.$$
$$\mathbb{B}aX = \mathbb{E}(aX - a\mathbb{E}X)^4 = \mathbb{E}a^4(X - \mathbb{E}X)^4 = |a|\mathbb{B}X.$$

(iii) There do not seem to be appropriate formulae.

(iv) No direct carry over. If $\Pr(X = 2) = 1/3$, $\Pr(X = -1) = 2/3$, $\Pr(Y = 1) = \Pr(Y = -1) = 1/2$ and X and Y are independent then

$$\Pr(X + Y = 3) = 1/6,\ \Pr(X + Y = 1) = 1/6,\ \Pr(X + Y = 0) = 1/3$$
$$\Pr(X + Y = -2) = 1/3$$

We have $\mathbb{E}X = \mathbb{E}Y = 0$, so $\mathbb{E}(X + Y) = 0$, and

$$\mathbb{A}X = \mathbb{E}|X| = 2 \times 1/3 + 1 \times 2/3 = 4/3$$
$$\mathbb{A}Y = \mathbb{E}|Y| = 2 \times 1/2 = 1$$

and

$$\begin{aligned}\mathbb{A}(X + Y) &= \mathbb{E}|X + Y| = 3 \times 1/6 + 1 \times 1/6 + 0 \times 1/3 + 2 \times 1/3\\ &= 4/3 \neq \mathbb{A}X + \mathbb{A}Y\end{aligned}$$

$$\mathbb{B}X = \mathbb{E}X^4 = 16 \times 1/3 + 1 \times 2/3 = 6$$
$$\mathbb{B}Y = \mathbb{E}Y^4 = 2 \times 1/2 = 1$$

and

$$\begin{aligned}\mathbb{B}(X + Y) &= \mathbb{E}(X + Y)^4\\ &= 81 \times 1/6 + 1 \times 1/6 + 0 \times 1/3 + 4 \times 1/3\\ &= 89/6 \neq \mathbb{B}X + \mathbb{B}Y.\end{aligned}$$

Exercise 2.5.12

Use induction or direct calculation. Let $a_j = \mathbb{E}X_j$ and set $Y_j = X_j - a_j$. The Y_j are independent and $\mathbb{E}Y_j = 0$. Thus

$$
\begin{aligned}
\operatorname{var}(\sum_{j=1}^{n} X_j) &= \operatorname{var}\left(\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} a_j\right)) \\
&= \operatorname{var}\left(\sum_{j=1}^{n} Y_j\right) \\
&= \mathbb{E}\left(\sum_{j=1}^{n} Y_j\right)^2 \\
&= \mathbb{E}\left(\sum_{j=1}^{n} Y_j^2 + 2\sum_{1\le j<i\le n} Y_iY_j\right) \\
&= \sum_{j=1}^{n} \mathbb{E}Y_j^2 + 2\sum_{1\le j<i\le n} \mathbb{E}(Y_iY_j) \\
&= \sum_{j=1}^{n} \mathbb{E}Y_j^2 + 2\sum_{1\le j<i\le n} \mathbb{E}Y_i\mathbb{E}Y_j \\
&= \sum_{j=1}^{n} \mathbb{E}\operatorname{var} Y_j + 2\sum_{1\le j<i\le n} 0 = \sum_{j=1}^{n} \mathbb{E}\operatorname{var} X_j
\end{aligned}
$$

Exercise 2.5.14

Set

$$Z = (Y_1 + Y_2 + \cdots + Y_n) - (\mu_1 + \mu_2 + \cdots + \mu_n)$$

Then

$$\mathbb{E}Z = (\mathbb{E}Y_1 - \mu_1) + (\mathbb{E}Y_2 - \mu_2) + \cdots + (\mathbb{E}Y_n - \mu_n) = 0$$

and

$$\operatorname{var} Z = \operatorname{var}(Y_1 + Y_2 + \cdots + Y_n) = \operatorname{var} Y_1 + \operatorname{var} Y_2 + \cdots + \operatorname{var} Y_n \leq n\sigma^2.$$

Thus by Tchebychev's inequality

$$\Pr(|Z| \geq n^{1/2}a) \leq \frac{\operatorname{var} Z}{(n^{1/2}a)^2} \leq \frac{\sigma^2}{a^2}$$

and this is the required inequality.

Exercise 2.5.15

Set $Z_j = Y_j - \mathbb{E}Y_j$. Then $\mathbb{E}Z_j = 0$, $\operatorname{var} Z_j = \operatorname{var} Y_j \leq \sigma^2$ and $Z_j \leq Y_j - \mu$. Thus

$$\begin{aligned}
&\Pr\big(Y_1 + Y_2 + \cdots + Y_n > (1-c)n\mu\big)\\
&\qquad= \Pr\big((Y_1-\mu) + (Y_2-\mu) + \cdots + (Y_n-\mu) > -cn\mu\big)\\
&\qquad\geq \Pr\big(Z_1 + Z_2 + \cdots + Z_n > -cn\mu\big)\\
&\qquad= 1 - \Pr\big(Z_1 + Z_2 + \cdots + Z_n) \leq -cn\mu\big)\\
&\qquad\geq 1 - \Pr\big(|Z_1 + Z_2 + \cdots + Z_n| \geq cn\mu\big)\\
&\qquad\geq 1 - \frac{n\sigma^2}{(cn\mu)^2} = 1 - \frac{c^2\sigma^2}{n\mu^2} \geq 1 - b
\end{aligned}$$

whenever

$$n \geq \frac{bc^2\sigma^2}{\mu^2}.$$

So take

$$N = 1 + \text{integer part of } \frac{bc^2\sigma^2}{\mu^2}$$

If we take a long series of similar bets with return greater than a certain proportion of our stake then with high probability the average return over a long series of bets will not be much smaller than that proportion.

Exercise 2.5.17

Let $\Omega = \{0,1\}^n$ with

$$\Pr(\boldsymbol{a}) = \prod_{j=1}^{n} (1-2^{-j})^{1-a_j} 2^{-ja_j}.$$

and

$$X_j(\boldsymbol{a}) = a_j 2^j.$$

Exercise 2.5.18

(i) $\mathbb{E}X_j = p \times 1 + (1-p) \times 0 = p$ and $\mathbb{E}X_j^2 = \mathbb{E}X_j = p$ so

$$\operatorname{var} X_j = (\mathbb{E}X_j)^2 - \mathbb{E}X_j^2 = p - p^2 = p(1-p).$$

Since the X_j are independent

$$\operatorname{var}(X_1 + X_2 + \cdots + X_n) = np(1-p).$$

(ii) Completing the square

$$p(1-p) = p - p^2 = \tfrac{1}{4} - (p - \tfrac{1}{2})^2 \leq \tfrac{1}{4}$$

with equality if and only if $p = \frac{1}{2}$.

(iii) We thus have, using Tchebychev's inequality,

$$\begin{aligned}
&\Pr(|\bar{X} - p| \geq a) \\
&\quad = \Pr\big(|X_1 + X_2 + \cdots + X_n \\
&\qquad\qquad - \mathbb{E}X_1 - \mathbb{E}X_2 + \cdots + \mathbb{E}X_n| \geq na\big) \\
&\quad \leq \frac{np(1-p)}{(na)^2} = \frac{p(1-p)}{na^2} \leq \frac{1}{4a^2}
\end{aligned}$$

(iv) Set $a = 1/10$, $n = 1000$ in (iii).

(v) We must take $n = 4000$.

EXERCISE 2.5.19

(i) Observe that

$$\begin{aligned}0 \le \sum_{r=1}^{m}(p_r - m^{-1})^2 &= \sum_{r=1}^{m}(p_r^2 - 2m^{-1}p_r + m^{-2})\\ &= \sum_{r=1}^{m} p_r^2 - 2\sum_{r=1}^{m} p_r + m^{-1} = \sum_{r=1}^{m} p_r^2 - m^{-1}\end{aligned}$$

with equality only if $p_r = m^{-1}$ for each r.

(ii) If you choose the jth grotto

$$\begin{aligned}Z &= \sum_{k=1}^{n} X_{jk}\\ &= \text{number others at } j\text{th grotto}\\ &= \text{number others at your grotto}\end{aligned}$$

as required.

Since Y and X_{jk} are independent

$$\mathbb{E}Y_j X_{jk} = \mathbb{E}Y\mathbb{E}X_{jk} = p_j^2$$

and so

$$\begin{aligned}\mathbb{E}Z &= \mathbb{E}\sum_{j=1}^{m}\sum_{k=1}^{n} Y_j X_{jk}\\ &= \sum_{j=1}^{m}\sum_{k=1}^{n} \mathbb{E}Y_j X_{jk} = \sum_{j=1}^{m} np_j^2\\ &= n\sum_{j=1}^{m} p_j^2.\end{aligned}$$

By part (i)

$$\mathbb{E}Z = n\sum_{j=1}^{m} p_j^2 \ge \frac{n}{m}$$

with equality only if $p_r = m^{-1}$ for each r, which is the desired result.

If $m > 1$ the expected number of hermits at your grotto including yourself is

$$1 + \mathbb{E}Z \ge 1 + \frac{n}{m} > \frac{n+1}{m}$$

the average number of hermits per grotto. You are more likely to choose a popular grotto and popular grottos are more likely to have crowds of hermits.

Exercise 2.5.20

(i) Observe that $X_j = 1$ if and only if exactly one child at the jth desk has the sniffles at the beginning of the day. Thus

$$\Pr(X_j = 1) = 2p(1-p),\ \Pr(X_j = 0) = 1 - 2p(1-p).$$

Thus

$$\mathbb{E}X_j = \mathbb{E}X_j^2 = 2p(1-p)$$

and

$$\begin{aligned}\operatorname{var} X_j &= \mathbb{E}X_j^2 - (\mathbb{E}X_j)^2 \\ &= 2p(1-p)\big(1 - 2p(1-p)\big) = 2p(1-p)(1-2p+2p^2).\end{aligned}$$

It follows that

$$\mathbb{E}X = \sum_{j=1}^{n} \mathbb{E}X_j = 2np(1-p)$$

and, since the X_j are independent,

$$\operatorname{var} X = \sum_{j=1}^{n} \operatorname{var} X_j = np(1-p)(1-2p+2p^2).$$

Since

$$p(1-p) = p - p^2 = \tfrac{1}{4} - (p - \tfrac{1}{2})^2$$

it follows that $\mu_{p,n}$ takes its largest value when $p = 1/2$.

(ii) Observe that so

$$\begin{aligned}\Pr(Y_1 = 1) &= \Pr(\text{exactly one of the two children starts with sniffles}) \\ &= 2\frac{k}{2n} \times \frac{2n-k}{2n-1} = \frac{k(2n-k)}{n(2n-1)}\end{aligned}$$

and

$$\mathbb{E}Y_1 = \mathbb{E}Y_1^2 = \Pr(Y_1 = 1) = \frac{k(2n-k)}{n(2n-1)}$$

Thus using symmetry

$$\mathbb{E}Y = \sum_{j=1}^{n} \mathbb{E}Y_j = n\mathbb{E}Y_1 = \frac{k(2n-k)}{2n-1}$$

Next observe that Y_1Y_2 takes the values 0 and 1 and that $Y_1Y_2 = 1$ if and only if exactly one child at each of the two desks starts with the sniffles. Thus

$$\begin{aligned}\mathbb{E}Y_1Y_2 &= \Pr(Y_1Y_2 = 1)\\ &= 4 \times \frac{k(2n-k)}{2n(2n-1)} \times \frac{(k-1)(2n-k-1)}{(2n-2)(2n-3)}\\ &= \frac{k(k-1)(2n-k)(2n-k-1)}{n(n-1)(2n-1)(2n-3)}\end{aligned}$$

Now, using symmetry,

$$\begin{aligned}\mathbb{E}Y^2 &= \mathbb{E}\left(\sum_{j=1}^{n} Y_j\right)^2\\ &= \mathbb{E}\left(\sum_j Y_j^2 + \sum_{i\neq j} Y_iY_j\right)\\ &= \sum_j \mathbb{E}Y_j^2 + \sum_{i\neq j} \mathbb{E}Y_iY_j\\ &= n\mathbb{E}Y_1^2 + n(n-1)\mathbb{E}Y_iY_j\\ &= \frac{2k(2n-k)}{2n-1} + \frac{k(k-1)(2n-k)(2n-k-1)}{(2n-1)(2n-3)}\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{var} Y &= \mathbb{E}Y^2 - (\mathbb{E}Y)^2\\ &= \frac{2k(2n-k)}{2n-1} + \frac{k(k-1)(2n-k)(2n-k-1)}{(2n-1)(2n-3)} - \left(\frac{k(2n-k)}{2n-1}\right)^2\\ &= \frac{k(2n-k)}{2n-1}\left(4 + \frac{(k-1)(2n-k-1)}{2n-3} - \frac{k(2n-k)}{2n-1}\right)\end{aligned}$$

(iii) Writing $p = k/2n$ we have

$$\mathbb{E}Y = n\frac{2p(1-p)}{1-(2n)^{-1}} \neq \mathbb{E}X$$

so that $\tilde{\mu}_{k,n} \neq \mu_{k/2n,n}$ if $2n-1 \geq k \geq 1$.

On the other hand, setting $p_n = k_n/2n$, we have

$$\frac{\tilde{\mu}_{k_n,n}}{n} = \frac{2p_n(1-p_n)}{1-(2n)^{-1}} \to 2p(1-p)$$

and

$$\frac{\mu_{p,n}}{n} = 2p(1-p)$$

so

$$\frac{\tilde{\mu}_{k_n,n}}{\mu_{p,n}} \to 1.$$

Exercise 2.6.1

$n =$	5	10	20	30
$2^{-n} \approx$	3.1×10^{-2}	9.8×10^{-4}	9.5×10^{-7}	9.3×10^{-9}
$(11/5)^n \approx$	5.1×10	2.7×10^{2}	7.1×10^{6}	1.9×10^{10}
$(11/10)^n \approx$	1.6	2.5	6.7	1.7×10

Exercise 2.6.3

(i) Observe that Y_j depends only on the jth throw and the tosses are independent.

(ii) If Z_1, Z_2, $\ldots Z_n$ are independent so are $f(Z_1)$, $f(Z_2)$, $\ldots f(Z_n)$.

(iii) Tchebychev's inequality.

(iv) $\log X_n = \sum_{j=1}^n \log Y_j$.

(v) Take $a = \delta$, $N > \tilde{\sigma}^2 \delta^{-2} \epsilon^{-1}$.

(vi) If we take $\delta = \log k$, then

$$\begin{aligned}\left|\frac{\log X_n}{n} - \tilde{\mu}\right| < \delta &\Leftrightarrow \tilde{\mu} - \delta < \frac{\log X_n}{n} < \tilde{\mu} + \delta \\ &\Leftrightarrow \log(k^{-1}L) < \frac{\log X_n}{n} < \log(kL) \\ &\Leftrightarrow (k^{-1}L)^n < X_n < (kL)^n\end{aligned}$$

and the result follows.

Exercise 2.6.5

(i) Since t_B is a strict maximum for the function

$$f(t) = p\log\big(1+(u-1)t)\big) + (1-p)\log(1-t)$$

Choose $\delta > 0$ with

$$f(t_B) - 3\delta > f(t_A),\ f(t_B).$$

By Theorem 2.6.2, there exists an N such that

$$\Pr\left(\left|\frac{\log X_n}{n} - f(t_A)\right| \geq \delta\right) < \epsilon/3$$
$$\Pr\left(\left|\frac{\log X_n}{n} - f(t_B))\right| \geq \delta\right) < \epsilon/3$$
$$\Pr\left(\left|\frac{\log X_n}{n} - f(t_C))\right| \geq \delta\right) < \epsilon/3$$

for all $n \geq N$. Thus

$$\begin{aligned}
\Pr(X_n, Z_n < Y_n) &= \Pr(\log X_n,\, \log Z_n < \log Y_n)\\
&\geq \Pr\left(\left|\frac{\log X_n}{n} - f(t_A)\right|,\ \left|\frac{\log Y_n}{n} - f(t_B))\right|,\right.\\
&\qquad\qquad \left.\left|\frac{\log Z_n}{n} - f(t_C)\right| \leq \delta\right)\\
&\leq 1 - \Pr\left(\left|\frac{\log X_n}{n} - f(t_A)\right| \geq \delta\right)\\
&\qquad - \Pr\left(\left|\frac{\log Y_n}{n} - f(t_B)\right| \geq \delta\right)\\
&\qquad - \Pr\left(\left|\frac{\log Z_n}{n} - f(t_C)\right| \leq \delta\right)\\
&> 1 - \epsilon.
\end{aligned}$$

[Thanks to Nigel White for corrections.)

(ii) Since f is strictly increasing on $[0, t_B]$ we can find a $\delta > 0$ such that $f(t_A) > 2\delta$. We can find an N such that, if $n \geq N$

$$\Pr\left(\left|\frac{\log X_n}{n} - f(t_B)\right| \geq \delta\right) < \epsilon$$

and so

$$\Pr(X_n > 1) \geq 1 - \Pr\left(\left|\frac{\log X_n}{n} - f(t_B)\right| \geq \delta\right) < \epsilon.$$

(iii) Observe that f is strictly decreasing (and continuous) on $[t_B, 1)$ with $f(t) \to -\infty$ as $t \to 1-$. Since $t_B > 0$ there is a t_D with $1 > t_D >$

t_B such that

$$f(t)\begin{cases} > 0 & \text{for } t_B \le t < t_D, \\ = 0 & \text{if } t = t_D, \\ < 0 & \text{for } t_D \le t < 1. \end{cases}$$

The stated result follows using the arguments of (ii).

Exercise 2.6.6

A Kelly bettor never bets with $u \leq 1$ so $u - 1 > 0$ and we have

$$\frac{pu - 1}{u - 1} \leq \frac{pu - p}{u - 1} = p.$$

Exercise 2.6.7

We have

$$t = \frac{(11/10) - 1}{(11/5) - 1} = \frac{1}{12}.$$

With this choice

$$\Pr(Y_j = .9) = \Pr(Y_j = .1.12) = 1/2$$
$$\tilde{\mu} = \mathbb{E} \log Y_j \approx 0.00419$$
$$\tilde{\sigma}^2 = \operatorname{var}(\log Y_j) = \mathbb{E}(\log Y_j)^2 (\mathbb{E} \log Y_j)^2 \approx 0.00831$$

Thus

$$\mathbb{E} \sum_{j=1}^{175} \log Y_j \approx \log 2.$$

Rest of calculation needs redoing so continue at your own risk.

If we take $n = 500$ and try our Tchebychev estimate we get

$$\Pr\left(|\log Y_1 + \log Y_2 + \cdots + \log Y_{500} - 1.99| \geq 1.3\right) \leq \frac{500 \times \tilde{\sigma}^2}{1.3^2} \approx 3.5$$

which tells us nothing.

If we take $n = 5000$ and try our Tchebychev estimate we get

$$\Pr\left(|\log Y_1 + \log Y_2 + \cdots + \log Y_{5000} - 19.9| \geq 19\right) \leq \frac{5000 \times \tilde{\sigma}^2}{19^2} \approx 0.16.$$

Thus

$$\log Y_1 + \log Y_2 + \cdots + \log Y_{5000} - 19.9 > 19$$

and so

$$\log Y_1 + \log Y_2 + \cdots + \log Y_{5000} > \log 2$$

with probability at least .74. In fact the situation is rather better than this but Exercises 2.6.1 and 2.6.7 show that it is hard for individuals to make a living by gambling at only slightly favourable odds.

(ii) We have

$$\begin{aligned}
\tilde{\mu} &= \mathbb{E}\log Y_1\\
&= \frac{1}{2}\log\big(tu+(1-t)\big)+\frac{1}{2}\log(1-t)\\
&= \frac{1}{2}\big(\log\big(1+(u-1)t\big)+\log(1-t)\big)\\
&= \frac{1}{2}\big(\log\big(1+(u-1)t\big)+\log(1-t)\big)\\
&= \frac{1}{2}\left((u-1)t-\frac{(u-1)^2t^2}{2}-t-\frac{t^2}{2}+\dots\right)\\
&= \frac{1}{2}\left((u-2)t-\frac{(u-1)^2t^2}{2}-\frac{t^2}{2}+\dots\right)\\
&= \frac{1}{2}\left(2(u-1)t^2-\frac{(u-1)^2t^2}{2}-\frac{t^2}{2}\right)\\
&= \frac{1}{2}(u-\tfrac{1}{2}(u-2)^2)t^2+\dots
\end{aligned}$$

so $\tilde{\mu}$ behaves like t^2 for small t.

Also

$$\begin{aligned}
\mathbb{E}Y_1^2 &= \frac{1}{2}\Bigg(\log\big(tu+(1-t)\big)\Bigg)+\frac{1}{2}\Bigg(\log\big(tu+(1-t)\big)\Bigg)\\
&= \frac{1}{2}\Bigg(\big((u-1)t\big)^2+t^2+\dots\Bigg)\\
&= \frac{u^2-2u+2}{2}t^2
\end{aligned}$$

so $\mathbb{E}Y_1^2$ and thus $\tilde{\sigma}^2$ behave like t^2 and when t is small the the standard deviation is very much bigger than the difference of the mean from zero. It is going to take a very large number of throws for the law of large numbers in our form (or indeed in any form) to give us any certainty.

Exercise 2.6.8

Kelly suggests gambling 4/5 of your fortune at each go.

$n =$	20	19	18	17	16	15	14
probability	.122	.270	.285	.095	.045	.015	.004
(i) gives	3325.3	1108.4	369.5	123.2	41.1	13.7	4.6
(ii) gives	127482.3	14164.7	1573.9	174.9	19.4	2.2	.23
(iii) gives	1048576.0	0	0	0	0	0	0

Exercise 2.6.9

Can write out more or less generally. The following is on the less general side.

Let $T < 1$. Suppose that we are presented with a series of independent bets with probability p_j of success and our jth bet has payout ratio $u_j < U$ with
$$\frac{p_j u_j - 1}{u_j - 1} < T$$
] Suppose we start with ! we bet a proportion t_j of our fortune on each go with $t_j < T$. Then our fortune satisfies $X_0 = 1$ and
$$X_{j+1} = \begin{cases} X_j(t_j u + (1 - t_j)) & \text{if the } j\text{th throw is heads,} \\ X_j(1 - t) & \text{if the } j\text{th throw is tails.} \end{cases}$$
We set
$$Y_{j+1} = \frac{X_{j+1}}{X_j} = \begin{cases} t_j u_j + (1 - t_j) & \text{if the } j\text{th throw is heads,} \\ 1 - t_j & \text{if the } j\text{th throw is tails.} \end{cases}$$

Observe that
$$\operatorname{var} \log Y_j \leq (\log(1 - T))^2 + (\log T + \log U)^2$$
so we may apply Tchebychev's theorem to tell us that when n is very large the probability that
$$\frac{\log X_n}{n} = \frac{\log Y_1 + \log Y_2 + \cdots + \log Y_n}{n}$$
differs greatly from
$$\frac{\mathbb{E} \log Y_1 + \mathbb{E} \log Y_2 + \cdots + \mathbb{E} \log Y_n}{n}$$
is small.

Since maximising $n^{-1} \log X_n$ is equivalent to maximising X_n, I wish to maximise $\mathbb{E} \log Y_j$ and so follow the Kelly criterion.

More generally (with much the same proof) offered the a sequence of bets which multiply my fortune by Y_j or Z_j on the jth go, then provided
$$\operatorname{var} \log Y_j, \operatorname{var} \log Z_j < K$$
and
$$\mathbb{E} \log Y_j \geq k \mathbb{E} \log Z_j$$
for some fixed K and some fixed $k > 1$, I should choose Y_j to maximise my fortune (in the long run with high probability).

Exercise 2.6.10

Write $Z_j = 1$ if the jth throw is heads, $Z_j = 0$ if tails. Observe that

$$\mathbb{E}Z_j = p, \ \operatorname{var} Z_j = p(1-p) = \tfrac{1}{4} - (p - \tfrac{1}{2})^2 \leq \tfrac{1}{4}.$$

(i) If $pu < 1$, then using Tchebychev's inequality,

$$\begin{aligned}
\Pr(X_n(t) < X_n(0)) &= \Pr\left(\sum_{j=1}^{n} tuZ_j - tn < 0\right) \\
&= \Pr\left(\sum_{j=1}^{n} Z_j < 0\right) \\
&\geq \Pr\left(\left|\sum_{j=1}^{n}(Z_j - \mathbb{E}Z_j)\right) < np\right) \\
&\geq 1 - \frac{1}{n^2p^2}\operatorname{var}\left(\sum_{j=1}^{n} Z_j\right) \\
&\geq 1 - \frac{1}{4np^2} \geq 1 - \epsilon
\end{aligned}$$

for $n > 4^{-1}\epsilon^{-1}p^{-2}$.

(ii) If $pu > 1$, then using Tchebychev's inequality,

$$\begin{aligned}
\Pr(X_n(t) < X_n(1)) &= \Pr\left(\sum_{j=1}^{n} u(1-t)Z_j - (1-t)n > 0\right) \\
&= \Pr\left(\sum_{j=1}^{n} uZ_j - n > 0\right) \\
&\geq \Pr\left(\left|\sum_{j=1}^{n}(Z_j - \mathbb{E}Z_j)\right) \leq n(p - u^{-1})\right) \\
&\geq 1 - \frac{1}{n^2(p-u^{-1})^2}\operatorname{var}\left(\sum_{j=1}^{n} Z_j\right) \\
&\geq 1 - \frac{1}{np^2} \geq 1 - \epsilon
\end{aligned}$$

for $n > 4^{-1}\epsilon^{-1}(p - u^{-1})^{-2}$.

Exercise 2.7.1

(i) If the first horse wins, she gets back

$$\frac{u_2}{u_1+u_2} \times u_1 = \frac{u_1 u_2}{u_1+u_2} = \frac{1}{u_1^{-1}+u_2^{-1}} = K.$$

A similar calculation works for the second horse.

(ii) If the jth horse wins $X = p_j u_j$ for the Kelly bettor. Thus

$$\begin{aligned}\mathbb{E}\log X &= p_1 \log p_1 u_1 + p_2 \log p_2 u_2\\ &= p_1 \log p_2 + p_2 \log p_2 + p_1 \log u_1 + p_2 \log u_2.\end{aligned}$$

If we set $t = K^{-1}u_1^{-1}$, we have $u_2^{-1} = K^{-1}(1-t)$ and

$$F(u_1, u_2) = f(t)$$

with

$$f(t) = \log K + p_1 \log p_2 + p_2 \log p_2 - p_1 \log t - p_2 \log(1-t).$$

Since

$$f'(t) = -\frac{p_1}{t} + \frac{p_2}{1-t} = \frac{t-p_1}{t(1-t)}$$

$f(t)$ attains a strict maximum for $0 < t < 1$ when $t = p_1$.

Thus

$$\mathbb{E}\log X = F(u_1, u_2) \geq F(Kp_1, Kp_2) = \log K$$

with equality only when $u_j = Kp_j^{-1}$.

(iii) Without loss of generality, suppose $u_1 p_1 \geq u_2 p_2$. For the maximum expectation bettor,

$$\begin{aligned}\mathbb{E}X &= u_1 p_1\\ &= \frac{u_1}{u_1+u_2} \times u_1 p_1 + \frac{u_2}{u_1+u_2} \times u_1 p_1\\ &\geq \frac{u_1}{u_1+u_2} \times u_2 p_2 + \frac{u_2}{u_1+u_2} \times u_1 p_1\\ &= \frac{u_1 u_2}{u_1+u_2} = K\end{aligned}$$

with equality only when $u_j = Kp_j^{-1}$.

Exercise 2.7.2

(i) We have

$$g'(x) = \frac{d}{dx}\left(p_i \log\left((u_i s_i + s_{n+1}) + u_i x\right) - p_j \log\left((u_j s_i + s_{n+1}) + u_j x\right)\right)$$
$$\frac{p_i u_i}{(u_i s_i + s_{n+1}) + u_i x} - \frac{p_j u_j}{(u_j s_j + s_{n+1}) + u_j x}$$

If $s_i, s_j \neq 0$ we must have $g'(0) = 0$ so

$$\frac{p_i u_i}{u_i s_i + s_{n+1}} = \frac{p_j u_j}{u_j s_j + s_{n+1}},$$

If $s_j = 0$ we must have $s_{n+1} \neq 0$ (otherwise we lose our entire fortune) and $g'(0) \geq 0$ so

$$\frac{p_i u_i}{u_i s_i + s_{n+1}} > \frac{p_j u_j}{s_{n+1}}.$$

If $s_i = 0$ and $s_j > 0$ we must have $g'(0) < 0$ which is impossible. Thus if $s_i = 0$ we have $s_j = 0$.

(ii) Let

$$h(x) = f(s_1 + x, s_2, \ldots, s_{n+1} - x)$$

Then

$$\begin{aligned}
&h'(x)\\
&= \frac{d}{dx}\left(p_1 \log\left((u_1 - 1)x + u_1 s_1 + s_{n+1}\right) + \sum_{j=2}^{n} p_j \log(u_j s_j + s_{n+1} - x)\right)\\
&= \frac{p_1(u_1 - 1)}{(u_1 - 1)x + u_1 s_1 + s_{n+1}} - \sum_{j=2}^{n} \frac{p_j}{u_j s_j + s_{n+1} - x}
\end{aligned}$$

so

$$h'(0) = \frac{p_1(u_1 - 1)}{u_1 s_1 + s_{n+1}} - \sum_{j=2}^{n} \frac{p_j}{u_j s_j + s_{n+1}}$$

If $s_1 \neq 0$ and $s_{n+1} \neq 0$ then $h'(0) = 0$ and we have

$$\text{(C)} \quad \frac{p_1}{u_1 s_1 + s_{n+1}} + \frac{p_2}{u_2 s_2 + s_{n+1}} + \cdots + \frac{p_n}{u_n s_n + s_{n+1}} = \frac{p_1 u_1}{u_1 s_1 + s_{n+1}}.$$

If $s_{n+1} = 0$ we must have $h'(0) \geq 0$ and so

$$\text{(B)} \qquad \frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n} \leq 1 \text{ and } s_{n+1} = 0.$$

If $s_1 = 0$ (so $s_j = 0$ for $1 \leq j \leq n$ and $s_{n+1} = 1$) by our earlier results, we must have $h'(0) \leq 0$ so

$$\sum_{j=1}^{n} p_j \geq p_1 u_1$$

ie $p_1 u_1 \leq 1$ (case A).

(iii) In case (A) we have $s_j = 0$ for $1 \leq j \leq n$ so we do not bet. In case (B) s_{n+1} so we bet everything and by (i)

$$\frac{p_j}{s_j} = \frac{p_i}{s_i}$$

for all $1 \leq i, j \leq n$ so $s_i = \alpha p_i$ for some α. Summing, $\alpha = 1$ and so $s_i = \alpha p_i$

(iv) In case (C) parts (i) and (ii) tells us that that there is an m with $1 \leq m < n$ and a $k > 0$ such that

$$\frac{p_j u_j}{u_j s_j + s_{n+1}} = k \text{ for } 1 \leq j \leq m,$$

$$s_j = 0,\ \frac{p_j u_j}{s_{n+1}} \leq k \text{ for } m+1 \leq j \leq n$$

and

$$\frac{p_1}{u_1 s_1 + s_{n+1}} + \frac{p_2}{u_2 s_2 + s_{n+1}} + \cdots + \frac{p_n}{u_n s_n + s_{n+1}} = k.$$

Thus

$$p_j u_j = k u_j s_j + k s_{n+1}$$

and

$$s_j = k^{-1} p_j - s_{n+1} u_j^{-1}$$

for $1 \leq j \leq m$, $s_j = 0$.

The equation

$$\frac{p_1}{u_1 s_1 + s_{n+1}} + \frac{p_2}{u_2 s_2 + s_{n+1}} + \cdots + \frac{p_n}{u_n s_n + s_{n+1}} = k$$

yields

$$\frac{k}{u_1} + \frac{k}{u_2} + \cdots + \frac{k}{u_m} + \frac{p_{m+1}}{s_{n+1}} + \cdots + \frac{p_n}{s_{n+1}} = k$$

so

$$s_{n+1} = k^{-1} \frac{1-p}{1-T}$$

where $p = p_1 + p_2 + \cdots + p_m$ and

$$T = \frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_m}.$$

But

$$s_1 + s_2 + \cdots + s_{n+1} = 1$$

so

$$k^{-1}p - Ts_{n+1} + s_{n+1} = 1$$

and $k = 1$. Hence

$$s_j = \begin{cases} p_j - \frac{s_{n+1}}{u_j} & \text{for } 1 \leq j \leq m, \\ 0 & \text{for } m+1 \leq j \leq n \end{cases}$$

and

$$s_{n+1} = \frac{1-p}{1-T}.$$

Exercise 3.1.1

(i) We have

$$\mathbb{E}X = \tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{2} \times 2 = \tfrac{5}{4}$$

and

$$\mathbb{E} \log Y = -\tfrac{1}{2} \log 2 + \tfrac{1}{2} \log 2 = 0.$$

(ii) We have

$$\mathbb{E}X = \tfrac{1}{2} \times 0 + \tfrac{1}{2} \times \tfrac{11}{4} = \tfrac{11}{8} > \tfrac{5}{4}$$

and

$$\mathbb{E} \log Y = -\tfrac{1}{2} \log 1 + \tfrac{1}{2} \log \tfrac{5}{4} = 0 = \tfrac{1}{2} \log \tfrac{5}{4} > 0.$$

Exercise 3.2.1

$$\begin{aligned}
&(1-x)(1+x+x^2+\cdots+x^m)\\
&\qquad = (1+x+x^2+\cdots+x^m) - (x+x^2+x^3+\cdots+x^{m+1})\\
&\qquad = 1-x^{m+1}
\end{aligned}$$

so provided $x \neq 1$ we can divide by $x-1$ to get

$$1+x+x^2+\cdots+x^m = \frac{1-x^{m+1}}{1-x}.$$

Exercise 3.2.2

If I start with x, I will have $k(x-1)$ just before the second withdrawal so I need

$$k(x-1) = x$$

ie

$$x = \frac{k}{k-1}.$$

The working is reversible.

Exercise 3.2.3

$$l = \frac{k}{k-p}$$

so $kl - pl = k$ and

$$p = k\frac{l-1}{l}$$

The figures given suggest

$$p \approx 1.04 \times \frac{11}{12} \approx .95.$$

Exercise 3.2.4

If you want to draw a fixed income for the rest of your life and leave your present fortune to your children you can only enjoy an income $a/(a+b)$ times as much as if you bought an annuity and left nothing.

(In C.S. Forster's *The African Queen* the heroine's father leaves her nothing because he has put his little money into an annuity.)

If b is large, then there is a high chance of your dying soon so the annuity company can be generous. If b is small, the annuity company expects to be paying out for many years and will pay close to bank interest.

Perhaps $a = b = 1/25$. These figures are plucked out of the air but it is easy to find bank interest rates and not hard to find annuity rates.

Exercise 3.2.5

$$\begin{aligned}
\mathbb{E}\,\mathrm{cost} &= \sum_{j=0}^{N-1} (j+1)\,\Pr(\text{die in } j\text{-th year}) \\
&= \sum_{j=0}^{N-1} (j+1)/N = N^{-1} \sum_{j=1}^{N} j \\
&= (N+1)/2
\end{aligned}$$

Exercise 3.2.6

(i) We have, by differentiation

$$1 + 2x + \cdots + mx^{m-1} = \frac{1 - x^{m+1}}{(1-x)^2} - \frac{(m+1)x^m}{1-x}$$

so differentiating again

$$\begin{aligned}(1 \times 2)+&(2 \times 3)x + (3 \times 4)x^2 \cdots + m(m-1)x^{m-2}\\ &= \frac{d}{dx}\left(\frac{1 - x^{m+1}}{(1-x)^2} - \frac{(m+1)x^m}{1-x}\right)\\ &= 2\frac{1 - x^{m+1}}{(1-x)^3} - 2\frac{(m+1)x^m}{(1-x)^2} - \frac{m(m+1)x^m}{1-x}.\end{aligned}$$

(ii) (Corrected by Nigel White.) We have

$$\begin{aligned}1^2 &+ 2^2x + 3^2x^2 \cdots + (m-1)^2x^{m-2}\\ &= (1 \times 2) + (2 \times 3)x + (3 \times 4)x^2 \cdots + m(m-1)x^{m-2}\\ &\quad - 1 - 2x - \cdots - (m-1)x^{m-2}\\ &= 2\frac{1 - x^{m+1}}{(1-x)^3} - 2\frac{(m+1)x^m}{(1-x)^2} - \frac{m(m+1)x^m}{1-x} + \frac{1 - x^m}{(1-x)^2} - \frac{mx^{m-1}}{1-x}\\ &= 2\frac{1 - x^{m+1}}{(1-x)^3} - \frac{(2m+1)x^m + 1}{(1-x)^2} - \frac{m(mx - x + 1))x^{m-1} - 1}{1-x}.\end{aligned}$$

(iii) The probability that there is a payout at the beginning of year j is

$$\begin{aligned}1 - \Pr(\text{both dead at beginning of } j) &= 1 - j^2/N^2\\ &= N^{-2}(N^2 - j(j-1) - j)\end{aligned}$$

so the expected value of the annuity is

$$\begin{aligned}\sum_{j=0}^{N-1} &N^{-2}(N^2 - j(j-1) - j)\\ &= N + N^{-2}\sum_{j=0}^{N-1}\big((j+1)j(j-1) - j(j-1)(j-2)\big)/3\\ &\qquad - \big((j+1)j - j(j-1)\big)/2\\ &= N - N^{-1}(N+1)(N-1)/3 - N^{-1}(N+1)/2\\ &= N - \frac{2N^2 - 3N + 1}{6N}.\end{aligned}$$

If $k \neq 1$, the expected value of the annuity is

$$\sum_{j=0}^{N-1} N^{-2}(N^2 - j^2 k^{-1})$$

which can be computed by (ii).

Exercise 3.2.7

(i) We have

$$\begin{aligned} A_r &= 1 + k^{-1}q_{r,r+1} + k^{-2}q_{r,r+2} + \cdots + k^{-N}q_{r,N} \\ &= 1 + k^{-1}p_r + k^{-2}p_r q_{r+1,r+2} + \cdots + k^{r-N}p_r q_{r+1,N} \\ &= 1 + k^{-1}p_r(1 + k^{-1}q_{r+1,r+2} + \cdots + k^{r+1-N}q_{r+1,N}) \\ &= 1 + p_r k^{-1}A_{r+1} \end{aligned}$$

If I want to pay an annuity of 1 unit a year to someone of age r, I can pay them 1 now and bank $k^{-1}p_r k^{-1}A_{r+1}$. At the end of the year with probability p_r they will still be alive and I will have $p_r A_{r+1}$ to buy an annuity then.

(ii) Since

$$q_{r,s} = p_r p_{r+1} \dots p_{r+s-1}, \text{ and } q_{r,s+1} = p_{r+1} \dots p_{r+s}$$

we have

$$p_{r+s} = q_{r,s+1}/q_{r,s}.$$

for all $0 \leq s \leq N-1$. Thus

$$p_s = q_{0,s+1}/q_{0,s}.$$

for $0 \leq s \leq N-1$.

$q_{0,s}$ is the probability that a newborn will live at least s years. No one lives longer than N.

Exercise 3.2.8

The probability that at least one of the pair will be alive at the beginning of the uth year is

$$\begin{aligned}1 - \Pr(&\text{both dead at beginning of } j\text{th year})\\ &= 1-(1-q_{r,r+u})(1-q_{t,t+u})\\ &= q_{r,r+u}+q_{t,t+u}-q_{r,r+u}q_{t,t+u}\end{aligned}$$

so the expected present value of the joint annuity is

$$\sum_{u=)}^{N} k^{-j}(q_{r,r+u}+q_{t,t+u}-q_{r,r+u}q_{t,t+u})$$

where $q_{i,j}=0$ if $j \geq N$.

Exercise 3.3.1

We have
$$f'(t) = \frac{r'(t)}{s(t)} - \frac{r(t)s'(t)}{s(t)^2}$$
so
$$\frac{f'(t)}{f(t)} = \frac{r'(t)}{r(t)} - \frac{s'(t)}{s(t)}.$$
Since
$$\frac{r'(t)}{r(t)} - \frac{s'(t)}{s(t)} = p - pq\frac{s(t)}{r(t)},$$
we get
$$\frac{f'(t)}{f(t)} = p - \frac{pq}{f(t)}$$
and
$$f'(t) = pf(t) - pq$$
whence
$$\frac{f'(t)}{f(t) - q} = p.$$
It follows that
$$\frac{d}{dt}\big(\log(f(t) - q) - pt\big) = \frac{f'(t)}{f(t) - q} - p = 0$$
and so
$$\log(f(t) - q) - pt = A$$
for some constant A.

Sine $r(0) = s(0)$, $f(0) = 1$ and, setting $t = 0$ in the last equation of the previous paragraph, $A = \log(1 - q)$. Thus
$$\log(f(t) - q) - pt = pt + \log(1 - q)$$
and applying exp to both sides
$$f(t) - q = (1 - q)e^{pt}$$
that is to say
$$\frac{r(t)}{s(t)} = f(t) = q + (1 - q)e^{pt}$$
whence
$$s(t) = \frac{r(t)}{q + (1 - q)e^{pt}}.$$

Exercise 3.3.2

One half of the population dies from A and $1/4$ from B so the probability of dying from A is $1/2$ and of dying from B is $1/4$. However, as stated the probability of dying from an attack of A is $1/2$ and the probability of dying from an attack of B is $1/2$.

The probability of dying from an attack of smallpox, is $1/8$ and that the probability of dying from smallpox is $1/13$ because, as may be seen from Bernoulli's table, many people die young and so may have escaped attacks of smallpox. The probability of dying from smallpox cannot be smaller than the probability of dying an attack of smallpox because in order to die from smallpox you must first have an attack of smallpox.

Exercise 3.3.3*

Exercise 3.3.4*

Exercise 3.3.5

As the age increases the probability that you will have had an attack of smallpox in your life up to then increases. However, if you are inoculated you will eventually die of something else so the number 'saved from smallpox' must start to decrease.

Exercise 3.3.6

If there is no smallpox $p = 0$ and $R(t) = r(t)$ so

$$R'(t) = -R(t)u(t)$$

so

$$\frac{R'(t)}{R(t)} = -u(t).$$

Earlier we showed that

$$\frac{r'(t)}{r(t)} = -u(t) - pq\frac{s(t)}{r(t)}$$

so that

$$\frac{R'(t)}{R(t)} - \frac{r'(t)}{r(t)} = pq\frac{s(t)}{r(t)}.$$

But in Exercise 3.3.1 we showed that

$$\frac{s(t)}{r(t)} = \frac{1}{q + (1-q)e^{pt}}$$

so

$$\frac{R'(t)}{R(t)} - \frac{r'(t)}{r(t)} = \frac{pq}{q + (1-q)e^{pt}}.$$

Now

$$\frac{pq}{q + (1-q)e^{pt}} = p - \frac{p(1-q)e^{pt}}{q + (1-q)e^{pt}}$$

so

$$\begin{aligned}
&\frac{d}{dt}\big(\log R(t) - \log r(t) - pt + \log(q + (1-q)e^{pt})\big)\\
&\qquad = \frac{R'(t)}{R(t)} - \frac{r'(t)}{r(t)} - p + p - \frac{p(1-q)e^{pt}}{q + (1-q)e^{pt}} = 0
\end{aligned}$$

and

$$\log R(t) - \log r(t) - pt + \log(q + (1-q)e^{pt}) = C$$

where C is a constant.

Now $R(0) = r(0)$ so setting $t = 0$ in the last formula of the previous paragraph we get $C = 0$. Thus

$$\log R(t) - \log r(t) = pt - \log(q + (1-q)e^{pt})$$

and applying exp to both sides we get

$$\frac{R(t)}{r(t)} = \frac{e^{pt}}{q + (1-q)e^{pt}}.$$

As $t \to \infty$

$$\frac{R(t)}{r(t)} = \frac{1}{qe^{-pt} + (1-q)} \to \frac{1}{1-q}.$$

Suppose smallpox does not kill but with probability q colours your hair green. When t is large almost everybody will have had smallpox and only a proportion $1-q$ will not have green hair. The result is what we expect.

Exercise 3.4.1

If we expect that the 'unlucky' children would if uninoculated suffer the same diseases (including smallpox) and die in the same proportions as other children then we expect

$$D(n) \approx D(n-1) \times \frac{B(n)}{B(n-1)}.$$

(If we think that the 'unlucky' children are more likely to die of ordinary smallpox then this improves Bernoulli's figures in the sense of increasing the expected lifetime with universal inoculation.)

The number of children who would be alive at n if not inoculated at birth but would die from inoculation must be subtracted from the size of the population of age n calculated by assuming that inoculation is riskless.

Exercise 3.4.2

If someone in immune with expected lifetime u then with probability $(1-a)$ they will die one year hence and with probability a they will survive one year and their expected lifetime at the beginning of that year will be u. Thus

$$u = (1-a) + a(1+u) = 1 = au$$

and $u = (1-a)^{-1}$.

If someone who is not immune and will not be inoculated has expected lifetime v then

(1) With probability $(1-b-c)$ they will die one year hence.

(2) With probability b they will be alive but not immune in a year's time and their expected lifetime at the beginning of that year will be v.

(3) With probability c they will be alive and immune in a year's time and their expected lifetime at the beginning of that year will be u.

Thus

$$v = (1-b-c) + b(1+v) + c(1+u) = 1 + bv + cu.$$

Thus

$$(1-b)v = 1 + cu$$

and

$$v = \frac{1}{1-b}\left(1 + \frac{c}{1-a}\right).$$

(The same results can be obtained by summing appropriate geometric series.)

If the individual is inoculated they have a probability of dying of r otherwise their expected life time is $(1-a)^{-1}$. Thus there expected lifetime is

$$\frac{1-r}{1-a}$$

so the 'gain in expected lifetime' is

$$\frac{1-r}{1-a} - \frac{1}{1-b}\left(1 + \frac{c}{1-a}\right).$$

They will have strictly positive gain if

$$\frac{1-r}{1-a} > \frac{1}{1-b}\left(1 + \frac{c}{1-a}\right).$$

If a child is inoculate late it runs the additional risk of dying from smallpox until it is inoculated.

Exercise 3.4.3

We have already estimated the probability of dying from other causes between the ages of $n-1$ and n as

$$p_{n-1} \approx \frac{C(n)}{B(n-1)}$$

and we have estimates for p the probability of catching smallpox in a year and q of dying from an attack.

Thus if u_n is the probability that a uninoculated child of 4 will survive to age n without catching smallpox and v_n is the probability that a uninoculated child of 4 will survive to age n having caught smallpox at sometime

$$u_n \approx (1-p)(1-p_{n-1})u_{n-1}$$
$$v_n \approx (1-p_{n-1})v_{n-1} + (1-p_{n-1})pqu_{n-1}$$

with $u_4 = 1$, $v_4 = 0$. The probability that a uninoculated child of 4 will survive to age n is $u_n + v_n$ and its expected lifetime is approximately $\sum_{n=5}^{80} u_n + v_n$.

If w_n is the probability that a inoculated child of 4 will survive to age n then

$$w_n = (1-p_{n-1})w_{n-1}$$

and its expected lifetime is approximately $\sum_{n=5}^{80} w_n$.

Exercise 3.4.4*

Exercise 3.4.5

This must be very much a back of an envelope calculation. Suppose we have a country with a population of 70 million an a lifetime expectancy of 70 years. This suggests a million births a year. If the birth figure remained unaltered but smallpox became endemic and the risk of dying from other causes remained the same then, since the death rate for young people is presently negligible and essentially everyone would expect to get smallpox, there would be a million cases of smallpox a year. If the death rate was one in 14 this would give 70 thousand deaths a year. If we took the overoptimistic assumption that modern medicine could get this down to 1 in a hundred we would still have 10 thousand deaths a year.

Exercise 3.5.4

(Corrected by Nigel White.) We have $f'(t) = 1/t$ and $f''(t) = -1/t^2 < 0$ so f is a smooth concave function.

With probability 1/2 my fortune will be

$$2a - av = a(2-v)$$

and with probability 1/2 my fortune will be

$$\tfrac{1}{2}a + kav - av = a(\tfrac{1}{2} + k - v).$$

Thus

$$\mathbb{E}f(Y) = g(v) = \frac{1}{2}\log\big(a(2-v)\big) + \frac{1}{2}\log\big(a(\tfrac{1}{2} + (k-1)v)\big).$$

Now

$$g'(v) = \frac{1}{2}\left(-\frac{1}{2-v} + \frac{k-1}{\frac{1}{2} + (k-1)v}\right) = \frac{1}{2} \times \frac{(2k - \frac{5}{2}) - 2(k-1)v}{(2-v)(\frac{1}{2} + (k-1)v)}.$$

Thus, if we take

$$v_0 = \frac{2k - \frac{5}{2}}{2(k-1)}$$

g is strictly increasing for $v < v_0$ and strictly decreasing for $v > v_0$. Thus I will buy insurance only if

$$v_0 > 0$$

and I will then set $v = v_0$.

The insurance companies expected gain is

$$\begin{aligned} u(k) &= av_0 - \tfrac{1}{2}kav_0 \\ &= \frac{a}{8}\frac{(-4k^2 + 13k - 10)}{k-1} \end{aligned}$$

so

$$u'(k) = -\frac{a}{8}\frac{2k^2 - 8k + 3}{(k-1)^2}$$

and the insurance company should take

$$k = 3/2.$$

Exercise 3.5.6

(i) Let
$$\Pr(X = x) = t,\ \Pr(X = y) = 1 - t.$$
Then
$$tf(x) + (1-t)f(y) = \mathbb{E}f(X) \leq f(\mathbb{E}X) = f\big(tx + (1-t)y\big)$$
for all $0 \leq t \leq 1$.

(ii) If $x < v < y$ and we set $t = (y-v)/(y-x)$ then $0 < t < 1$ and
$$\begin{aligned} tx + (1-t)y &= \frac{(y-v)x + \big((y-x)-(y-v)\big)y}{y-x} \\ &= \frac{(y-v)x + (v-x)\big)y}{y-x} = v. \end{aligned}$$
Thus
$$tf(x) + (1-t)f(y) \leq f(v) = tf(v) + (1-t)f(v)$$
so
$$t\big(f(x) - f(v)\big) \leq (1-t)\big(f(v) - f(y)\big)$$
whence
$$(y-v)\big(f(x) - f(v)\big) \leq (v-x)\big(f(v) - f(y)\big).$$
It follows that
$$(y-v)\big(f(v) - f(x)\big) \geq (v-x)\big(f(y) - f(v)\big)$$
and
$$\frac{f(v) - f(x)}{v-x} \geq \frac{f(y) - f(v)}{y-v}.$$

(iii) By (ii)
$$\frac{f(b) - f(a)}{b-a} \geq \frac{f(c) - f(b)}{c-b} \geq \frac{f(d) - f(c)}{d-c}.$$
Note that it follows that
$$\frac{f(a) - f(b)}{a-b} \geq \frac{f(c) - f(d)}{c-d}.$$

(iii) If $|k| < (y-x)/2$ we may apply (ii) to get
$$\frac{f(x+k) - f(x)}{k} \leq \frac{f(y+k) - f(y)}{k}.$$
Allowing $k \to 0$ we get
$$f'(x) \leq f'(y)$$
(remember that we suppose f well behaved). Since f' is decreasing $f''(t) \leq 0$ for all t.

Exercise 3.5.7

$$\mathbb{E}X_1 = 74, \ \operatorname{var} X_1 = 1$$
$$\mathbb{E}X_2 = 75, \ \operatorname{var} X_2 = 25$$
$$\mathbb{E}X_3 = 80, \ \operatorname{var} X_3 = \frac{2}{3}10^2 = \frac{200}{3}$$

The first player never wins, the second player wins with probability $1/2$ and the third with probability $1/3$. I should choose the second player. (Moral, the mean and variance do not tell you everything.)

Exercise 3.5.8

If I am in a shop and suddenly the price of everything is doubled but the amount of money in my purse is doubled I will make the same choices.

Exercise 3.5.9

(i) We use induction on n. The result is trivially true when $n = 1$. Suppose it is true for $n = m$. Then Using Theorem 3.5.3, we have (if $t_{m+1} \neq 1$)

$$\begin{aligned} f(t_1x_1{+}t_2x_2 + \cdots + t_{m+1}x_{m+1}) \\ &= f\left((1-t_{m+1})\sum_{j=1}^{m}\frac{t_j}{1-t_{m+1}}x_j + t_{m+1}x_{m+1}\right) \\ &\leq (1-t_{m+1})f\left(\sum_{j=1}^{m}\frac{t_j}{1-t_{m+1}}\right) + t_{m+1}f(x_{m+1}). \end{aligned}$$

But

$$\frac{t_j}{1-t_{m+1}} \geq 0 \text{ and } \sum_{j=1}^{m}\frac{t_j}{1-t_{m+1}}$$

so our inductive hypothesis shows that

$$\sum_{j=1}^{m}\frac{t_j}{1-t_{m+1}}f(x_j) \leq f\left(\sum_{j=1}^{m}\frac{t_j}{1-t_{m+1}}x_j\right)$$

and so

$$f(t_1x_1 + t_2x_2 + \cdots + t_{m+1}x_{m+1}) \leq f(t_1x_1 + t_2x_2 + \cdots + t_nx_{m+1}).$$

The full result now follows by induction.

(ii) Since $f''(t) = -1/t^2$ we know that f is concave so

$$\frac{1}{n}\log x_1 + \frac{1}{n}\log x_2 + \cdots + \frac{1}{n}\log x_n \leq \log\left(\frac{1}{n}x_1 + \frac{1}{n}x_2 + \cdots + \frac{1}{n}x_n\right)$$

that is to say

$$\log(x_1x_2\dots x_n)^{1/n} \leq \log\frac{x_1 + x_2 + \cdots + x_n}{n}$$

so applying exp to both sides

$$(x_1x_2\dots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

(iii) We have

$$g'(x) = x^{p^{-1}-1}(1 + x^{p^{-1}})^{(p-1)}$$

so

$$\begin{aligned} g''(x) &= (p^{-1}-1)x^{-2+p^{-1}}(1 + x^{p^{-1}})^{(p-1)} \\ &\qquad + (p-1)(p^{-1}-1)x^{-2+2p^{-1}}(1 + x^{p^{-1}})^{p-2} \\ &< 0 \end{aligned}$$

so g is concave.

Taking

$$x_k = \frac{b_k^p}{a_k^p} \text{ and } t_k = a_k^p,$$

Jensen's inequality

$$t_1 g(x_1) + t_2 g(x_2) + \cdots + t_n g(x_n) \leq g(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n)$$

gives

$$\sum_{k=1}^{n} a_k^p \left(1 + \frac{b_k}{a_k}\right)^p \leq \left(1 + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\right)^p$$

which simplifies to

$$\sum_{k=1}^{n} a_k^p (a_k + b_k)^p)^p \leq \left(1 + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\right)^p$$

and taking pth roots we obtain the required inequality.

(iv) Take $T = \left(\sum_{j=1}^{n} t_j^p\right)^{1/p}$ and set $a_j = t_j/T$, $b_j = s_j/T$. The inequality of (iii) the holds and, multiplying through by T, we obtain

$$\begin{aligned}\big((t_1 + s_1)^p + &(t_2 + s_2)^p + \cdots + (t_n + s_n)^p\big)^{1/p} \\ &\leq \left(t_1^p + t_2^p + \cdots + t_n^p\right)^{1/p} + \left(s_1^p + s_2^p + \cdots + s_n^p\right)^{1/p}.\end{aligned}$$

Thus

$$\begin{aligned}&\left(|u_1 + v_1|^p + |u_2 + v_2|^p + \cdots + |u_n + v_n|^p\right)^{1/p} \\ &\quad\leq \left((|u_1| + |v_1|)^p + (|u_2| + |v_2|)^p + \cdots + (|u_n| + |v_n|^p)\right)^{1/p} \\ &\quad\leq \left(|u_1|^p + |u_2|^p + \cdots + |u_n|^p\right)^{1/p} + \left(|v_1|^p + |v_2|^p + \cdots + |v_n^p|\right)^{1/p}.\end{aligned}$$

$$\begin{aligned}|OU| + |OV| &= (u_1^2 + u_2^2)^{1/2} + (v_1^2 + v_2^2)^{1/2} \\ &= (u_1^2 + u_2^2)^{1/2} + \left((-v_1)^2 + (-v_2)^2\right)^{1/2} \\ &\geq \left((u_1 - v_1)^2 + (u_2 - v_2)^2\right)^{1/2} \\ &= |UV|\end{aligned}$$

In three dimensions

$$\begin{aligned}|OU| + |OV| &= (u_1^2 + u_2^2 + u_3^2)^{1/2} + (v_1^2 + v_2^2 + v_3^2)^{1/2} \\ &= (u_1^2 + u_2^2 + u_3^2)^{1/2} + \left((-v_1)^2 + (-v_2)^2 + (-v_3)^2\right)^{1/2} \\ &\geq \left((u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2\right)^{1/2} \\ &= |UV|\end{aligned}$$

(v) $h'(x) = -(\log x + 1)$ so $h''(x) = -1/x$. Thus

$$h(tu + (1-t)v) \geq th(u) + (1-t)h(v)$$

for $1 \geq t \geq 0$, u, $v > 0$. Taking

$$t = \frac{a}{a+b},\ u = \frac{x}{a},\ v = \frac{y}{b}$$

this gives

$$-\frac{x+y}{a+b}\log\frac{x+y}{a+b} \geq -\frac{1}{a+b}x\log\frac{x}{a} - \frac{1}{a+b}y\log\frac{y}{b}$$

so, multiplying by $-(a+b)$ we obtain

$$(x+y)\log\frac{x+y}{a+b} \leq x\log\frac{x}{a} + y\log\frac{y}{b}$$

for all a, b, x, $y > 0$.

Exercise 3.5.10

It is sufficient to deal with the case $a, b \geq 0$. Observe that

$$f(x) = -x^p$$

defines a concave function so, by Jensen's inequality,

$$f(\tfrac{1}{2}a + \tfrac{1}{2}b) \geq \tfrac{1}{2}f(a) + \tfrac{1}{2}f(b)$$

so

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

But, if $a = b = 1$,

$$(a+b)^p = 2^{p-1}(a^p + b^p) \neq 0$$

so $c_p = 2^{p-1}$ is the required constant.

Exercise 3.5.11

Observe that $p_i q_j > 0$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j = \sum_{i=1}^{n} p_i sum_{j=1}^{m} q_j = 1 \times 1 = 1.$$

Thus by Jensen's inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j h(t_{ij}) \geq h\left(\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j t_{ij}\right)$$

for all $t_{ij} > 0$. Setting

$$t_{ij} = \frac{\pi_{ij}}{p_i q_j},$$

we get

$$-\sum_{i=1}^{n}\sum_{j=1}^{m} \pi_{ij} \log \frac{\pi_{ij}}{p_i q_j} \geq h(0) = 0$$

so $I \leq 0$.

We have equality when $\pi_{ij} = p_i q_j$. (And only then, by drawing a diagram we see that we have strict inequality unless all the t_{ij} are equal.)

Suppose X and Y are random variables with X taking distinct values $x_1, x_2, \ldots, x_n$ and Y taking distinct values $y_1, y_2, \ldots, y_m$. Suppose $\Pr(X = x_i, Y = y_j) > 0$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Then

$$I = \sum_{i=1}^{n}\sum_{j=1}^{m} \Pr(X = x_i, Y = y_j) \log \frac{\Pr(X = x_i, Y = y_j)}{\Pr(X = x_i)\Pr(Y = y_j)}$$

is maximised when X and Y are independent.

Exercise 4.1.1*

Exercise 4.1.2

If $k_n = 2^n - 1$, then

$$k_{n+1} = 2k_n + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1.$$

But $k_1 = 1 = 2^1 - 1$ so, by induction,

$$k_n = 2^n - 1$$

for all $n \geq 1$.

Exercise 4.1.3

With 256 disks at one operation per Planck time it will take at least

$$(2^{256} - 1) \times 5 \times 10^{-44} \times \frac{1}{31600000} \approx 1.8 \times 10^{26}$$

years. She should stick to old fashioned ways.

Exercise 4.1.4

Let $P(n)$ be the statement that, if we follow the rules we will move a tower of size clockwise one place if n is odd and anticlockwise one place if n is even taking $2^n - 1$ moves.

By inspection $P(1)$ is true.

Now suppose $P(n)$ is true. By the inductive hypothesis the first $2^{n+1} - 1$ moves will move the top n pieces one place clockwise if n is odd, anticlockwise if n is even. The largest piece is now exposed and moved to the empty peg anticlockwise if n is odd, clockwise if n is even by the 2^{n+1}th move. The remaining 2^n moves take the top n pieces one place clockwise if n is odd, anticlockwise if n is even so they form a tower over the largest piece. Thus $P(n+1)$ is true.

The desired result follows by induction.

Exercise 4.2.2

(i) If d divides a and b then $a = k \times d$, $b = l \times d$ so $a = k \times d$, $-b = (-l) \times d$. Thus d is no greater than the s greatest common divisor d' of a and $-b$. Similarly $d' \leq d$ so $d = d'$.

(ii) $182 = 2 \times 7 \times 13$ and $140 = 2 \times 5 \times 7$ so the greatest common divisor is $2 \times 7 = 14$.

(iii) $815\,055 = 3 \times 5 \times 67 \times 811$, and $208\,427 = 257 \times 811$ all factors being prime. (It is not hard for me to multiply primes together. It is much harder for you to find them.) The greatest common divisor is 811.

Exercise 4.2.4

If c divides a and b then $a = kc$, $b = lc$ and using the notation of Lemma 4.2.3,

$$d = ma + nb = mkc + nlc = (mk + nl)c$$

so c divides d.

Exercise 4.2.5

(i) If $a_n \neq 0$

$$\sum_{j=0}^{n} a_j x^j = a_n \left(x^n + \sum_{j=0}^{n-1} \frac{a_j}{a_n} x^j \right).$$

(ii) Long division.

The result is true when P has degree 0. (If $n = 0$ then $K = P/Q$, $R = 0$; if $n \geq 1$ then $K = 0$, $R = P$.)

Suppose it is true whenever P has degree m or less. If P has degree $m+1$ either $m + 1 < n$ and $K = 0$, $R = P$ or

$$P(t) = b_{m+1-n} t^{m+1-n} Q(t) + S(t)$$

where b_{m+1-n} is the value of the leading coefficient of P divided by the value of the leading coefficient of Q and S has degree at most m. By the inductive hypothesis,

$$S(t) = L(t)Q(t) + R(t)$$

where R has degree strictly less than n and so

$$P(t) = \big(b_{m+1-n} t^{m+1-n} + L(t)\big) Q(t) + R(t).$$

The required result is thus true when P has degree $m + 1$ or less.

The conclusion follows by induction.

(iii) Since 1 divides P and Q and no polynomial of degree strictly higher than the degree of P divides P there must be a monic polynomial S of highest degree dividing P and Q.

(iv) Consider the collection of polynomials $U(x)P(x) + V(x)Q(x)$ with U and V not both zero. This must contain a polynomial of lowest degree and so a monic polynomial T of the same degree. Since S is a factor of P and Q, S divides T.

Now $P = KT + R$ where R has degree less than T. Thus (for appropriate U and V)

$$R = P - KT = P - K(UP + VQ) = (1 - KU)P + (-KV)Q$$

and so $R = 0$. Thus T divides P and similarly T divides Q. Thus The degree of T is less than or equal to the degree of S and $T = S$. (Thus T and S must be unique.)

(v) We call S the highest common factor of P and Q. Since

$$S(x) = U(x)P(x) + V(x)Q(x)$$

it follows that if $W = AS$ we have $W = (US)P + (UV)Q$.

If $W = \tilde{U}P + \tilde{V}Q$ then since S divides P and Q, S divides W.

Exercise 4.2.6

(i) We want

$$1 = 13u + 10v$$

and a little experimentation (or thought about last digits) suggests $u = 7$, $v = -9$.

(ii) Rather you than me.

Exercise 4.2.9

$$|a|,\ |b| \leq 10^{300} = 1000^{100} < 2^{1000}.$$

Now apply Lemma 4.2.8.

Exercise 4.2.11*

Exercise 4.2.12

Suppose they start with n coconuts and the captain's initial share is n_1, the First Mate's share n_2 and so on down to the cabin boy who gets n_5. Let the the number of remaining coconuts be $5m$. We have

$$\begin{aligned}
n &= 5n_1 + 1\\
4n_1 &= 5n_2 + 1\\
4n_2 &= 5n_3 + 1\\
4n_3 &= 5n_4 + 1\\
4n_4 &= 5n_5 + 1\\
4n_5 &= 5m
\end{aligned}$$

Thus

$$\begin{aligned}
4^5 n &= 5 \times 4^5 n_1 + 4^5\\
&= 5 \times 4^4(5n_2 + 1) + 4^5 = 5^2 \times 4^4 n_2 + 5 \times 4^4 + 4^5\\
&\vdots\\
&= 5^6 m + +4 \times 5^4 + \cdots + 4^5)\\
&= 5^6 m + \frac{5^6 - 4^6}{5 - 4} - 5^5 = 5^6 m + 5^6 - 4^6 - 5^5
\end{aligned}$$

Thus

$$\bigstar \qquad 1024n = 15625m + 8404.$$

Let us apply Euclid's algorithm to 15625 and 1024 We have

$$\begin{aligned}
15625 = 15 \times 1024 + 2651024 &= 4 \times 265 - 36\\
265 &= 7 \times 36 = 13\\
36 &= 3 \times 13 - 3\\
13 &= 4 \times 3 + 1.
\end{aligned}$$

Thus

$$\begin{aligned}
1 &= 13 - 4 \times 3 = 13 - 4 \times (3 \times 13 - 36) = 4 \times 36 - 11 \times 13\\
&= 4 \times 36 - 11 \times (265 - 7 \times 36) = 81 \times 36 - 11 \times 265\\
&= 81 \times (4 \times 265 - 1024) - 11 \times 265 = 313 \times 265 - 81 \times 1024\\
&= 313 \times (15625 - 15 \times 1024) - 81 \times 1024\\
&= 313 \times 15625 - 4776 \times 1024.
\end{aligned}$$

Thus one solution of $\bigstar$ is

$$n = -8404 \times 4776 = -40137504, \ m = -8404 \times 313 = -2630452.$$

But since $1024 = 4^5$ and $15625 = 5^6$ are coprime we know that the solutions of ★ are given by

$$n = -40137504 + 15625k$$
$$m = -2630452 + 1024k$$

with k integer. Since $2630452 = 2569 \times 1024 - 204$ we see that $m = 204$ gives the smallest possible positive value of m corresponding to the smallest positive value of n which is 3121.

Exercise 4.2.13

This is really a repetition of known results. If d divides a and b then it must divide the highest common factor. But, by Lemma 4.2.10 (ii) the highest common factor must divide d. Thus $|d|$ is the highest common factor.

Exercise 4.2.14

(i) The set of strictly positive integers divisible by a and b contains $|ab|$ so is non-empty and has a least member.

(ii) Choose the integer v so that

$$0 \leq n + ve < e.$$

Now a and b divide $n + ve$ so, by definition of e, $n + ve = 0$. Thus $n = (-v)e$ and e divides n.

(iii) Observe that

$$\frac{ab}{d} = a\frac{b}{d} = b\frac{a}{d}$$

so a and b divide e. Thus $ab/d \geq e$.

By part (ii), e divides ab so ab/e is an integer with

$$a = \frac{ab}{e}\frac{e}{b},\ b = \frac{ab}{e}\frac{e}{a}$$

Thus $ab/e \leq d$ so $ab \leq de \leq ab$ and $ab = de$.

If a and b are not necessarily positive repeat the argument with $|a|$ and $|b|$.

(iv) We have

$$\begin{aligned}
22015 &= 4 \times 5291 + 851\\
5291 &= 6 \times 851 + 185\\
851 &= 5 \times 185 - 74\\
185 &= 2 \times 74 + 37\\
74 &= 2 \times 37
\end{aligned}$$

Thus 22 015 and 5291 have highest common divisor 37 and lowest common multiple

$$\frac{22015 \times 5291}{37} = 3148145.$$

Exercise 4.2.15

Given polynomials P_1, P_2 with the degree ∂P_1 no larger than ∂P_2 then either

$$P_1 = Q_1 P_2$$

for some polynomial Q and (after multiplication by a constant to make it monic) P_2 is the required polynomial or

$$P_1 = Q_1 P_2 + P_3$$

with $\partial P_3 < \partial P_2$ but $P_3 \neq 0$. We observe that if S divides P_2 and P_3 then S divides P_1 and P_2 and conversely so (if $\gcd(A, B)$ is the monic polynomial of highest degree dividing A and B)

$$\gcd(P_1, P_2) = \gcd(P_2, P_3).$$

Further if

$$A_2 P_2 + A_3 P_3 = B$$

then

$$A_3 P_1 + (A_2 - Q_1) P_2 = B.$$

We can repeat the process but since the degree of at least one of the polynomial pairs decreases at each step the process must terminate giving us $S = \gcd(P_1, P_2)$. Reversing the process as indicated computes U and V such that

$$S(x) = U(x)P(x) + V(x)Q(x).$$

Exercise 4.3.1*

Exercise 4.3.2

(i) In 279 months a certain number of years and 3 months will have passed. Count 3 months forward.

(ii) Every breakfast is 2 hours earlier so the 200th breakfast is 400 hours earlier, that is to say a certain number of days and 16 hours earlier which is to say a certain other number of days and 8 hours later. She will breakfast at 4 am.

(iii) Odd.

Exercise 4.3.5

Observe that
$$10 \equiv 1 \mod 9$$
so
$$10^r \equiv 1^r \equiv 1 \mod 9$$
$$a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_0 \equiv a_n + a_{n-1} + \cdots + a_0$$
and
$$\big(a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_0\big) \times \big(b_n 10^n + b_{n-1}10^{n-1} + \cdots + b_0\big)$$
$$\equiv (a_n + a_{n-1} + \cdots + a_0) \times (b_n + b_{n-1} + \cdots + b_0).$$

(ii) Casting out nines produces an equation involving smaller integers which remains true modulo 9. After a finite number of steps we have integers between 0 and 8 which must be equal.

Reverse obviously false
$$18 \to 9 \to 0$$
and
$$3 \times 9 \to 3 \times 0 = 0$$
but $18 \neq 3$.

Since
$$\big(a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_0\big) \times \big(b_n 10^n + b_{n-1}10^{n-1} + \cdots + b_0\big)$$
$$\equiv (a_n + a_{n-1} + \cdots + a_0) \times (b_n + b_{n-1} + \cdots + b_0).$$
and
$$\big(a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_0\big) \times \big(b_n 10^n + b_{n-1}10^{n-1} + \cdots + b_0\big)$$
$$\equiv (a_n + a_{n-1} + \cdots + a_0) \times (b_n + b_{n-1} + \cdots + b_0).$$
modulo 9 the method will also work for addition and subtraction.

Exercise 4.3.6

(i)Observe that if $n \geq 2$

$$0 \equiv 0^n \equiv (1+1)^n \equiv \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

modulo 2 thus there must be an even number of r with

$$\binom{n}{r} \equiv 1 \mod 2$$

and this is what is claimed.

If $n = 0$ then $(1+1)^n \equiv (1+1)^0 \equiv 1 \not\equiv 0$ modulo 2.

(ii) We have

$$\begin{array}{c} 1 \\ 11 \\ 101 \\ 1111 \\ 10001 \\ 110011 \\ 1010101 \\ 11111111 \\ 100000001 \\ 1100000011 \\ 10100000101 \end{array}$$

Exercise 4.3.7

By observing that $x = -1$ is a solution and that any other solution must have the form $-1 + K$ with K divisible by 6 and 10 whilst everything of this form is a solution we see that the solutions are

$$-1 + 30n$$

Now $1000000 \equiv 10$ modulo 30 so the least number of the required form is 1000019.

Exercise 4.3.8

We wish to solve

$$\begin{aligned} x &\equiv 2 \mod 3, \\ x &\equiv 1 \mod 4. \end{aligned}$$

If x is a solution so is $x + 12k$ and inspection of the table shows that there is precisely one solution (viz $x = 5$) with $0 \leq x \leq 11$.

Thus there are 3 solutions with $0 \leq x \leq 35$

(ii) In the following table $r_1 \equiv r \mod 4$, $0 \leq r_1 \leq 3$ and $r_2 \equiv r \mod 6$, $0 \leq r \leq 6$

$r =$	0	1	2	3	4	5	6	7	8	9	10	11
$r_1 =$	0	1	2	3	0	1	2	3	0	1	2	3
$r_2 =$	0	1	2	3	4	5	0	1	2	3	4	5

We note that

$$\begin{aligned} x &\equiv x + 12n \mod 4, \\ x &\equiv x + 12n \mod 6. \end{aligned}$$

Thus by inspection of the table

$$\begin{aligned} x &\equiv 2 \mod 4, \\ x &\equiv 4 \mod 6. \end{aligned}$$

has 4 solutions with $0 \leq x \leq 47$ but

$$\begin{aligned} x &\equiv 3 \mod 4, \\ x &\equiv 4 \mod 6. \end{aligned}$$

has none.

Exercise 4.3.11

We first seek to solve

$$\begin{aligned} y &\equiv 1 \quad \text{mod } 17, \\ y &\equiv 0 \quad \text{mod } 19. \end{aligned}$$

that is, to find a, b such that

$$19a + 17b = 1$$

Now

$$19 - 17 = 2$$

and

$$17 = 2 \times 8 + 1$$

so

$$1 = (19 - 17) \times (-8) + 17 = (-8) \times 19 + 9 \times 17$$

Thus $y = (-8) \times 19 = -152$ is a possible solution.

If we set $z = 9 \times 17 = 153$ then

$$\begin{aligned} z &\equiv 0 \quad \text{mod } 17, \\ z &\equiv 1 \quad \text{mod } 19. \end{aligned}$$

Thus (since 17 and 19 are coprime) the general solution to our problem is

$$x = 3 \times (-152) - 2 \times (153) + n \times 17 \times 19 = -762 + 323n = 207 + 323m$$

with m an integer.

The smallest positive solution is 207.

Exercise 4.3.12

Since a and b have highest common factor 1, Euclid's algorithm tells us that there exist m and n such that

$$am + bn = 1.$$

Similarly, there exist r and s such that

$$ar + cs = 1.$$

Thus

$$1 = (am + bn)(ar + cs) = a(mar + mcs + bnr) + bc(ns).$$

Setting $N = mar + mcs + bnr$ and $M = bc$, we have

$$1 = Na + Mbc$$

so a and bc have highest common factor 1.

Exercise 4.3.13

By Exercise 4.3.12, a_1 and a_2a_3 are coprime so we can find a z_1 such that

$$\begin{aligned} z_1 &\equiv 1 \mod a_1, \\ z_1 &\equiv 0 \mod a_2a_3. \end{aligned}$$

Thus

$$\begin{aligned} z_1 &\equiv 1 \mod a_1, \\ z_1 &\equiv 0 \mod a_2, \\ z_1 &\equiv 0 \mod a_3 \end{aligned}$$

More generally we can find z_j with $z_j \equiv 1 \mod a_j$ and $z_k \equiv 1 \mod a_k$ for $k \neq j$. Now $x = u_1z_1 + u_2z_2 + u_3z_3$ satisfies ★★.

If x_0 and x_1 are solutions of ★★ then $x_[0 - x_1$ is divisible by a_1, a_2 and a_3 so $x_0 - x_1$ is divisible by the lowest common multiple of a_1, a_2 and a_3 ie by $|a_1a_2a_3|$. Thus

$$x_0 \equiv x_1 \mod a_1a_2a_3.$$

The converse is immediate.

The same arguments show that if we have non zero integers a_k $[1 \leq k \leq n]$ such that each pair (a_i, a_j) with $i \neq j$ has highest common divisor 1 then given integers u_k we can find an x such that

$$x \equiv u_k \mod a_k.$$

If x_0 is a solution then the solutions are given by

$$x_0 \equiv x_1 \mod a_1a_2 \dots a_n.$$

Exercise 4.3.14

Note that 3, 5, 7 are coprime. By rapid search (or we could use Euclid's algorithm).

$$\begin{aligned} -35 &\equiv 1 \mod 3, \\ -35 &\equiv 0 \mod 5, \\ -35 &\equiv 0 \mod 7. \end{aligned}$$

$$\begin{aligned} 21 &\equiv 0 \mod 3, \\ 21 &\equiv 1 \mod 5, \\ 21 &\equiv 0 \mod 7. \end{aligned}$$

$$\begin{aligned} 15 &\equiv 0 \mod 3, \\ 15 &\equiv 0 \mod 5, \\ 15 &\equiv 1 \mod 7. \end{aligned}$$

Sunzi's problem is to solve

$$\begin{aligned} x &\equiv 2 \mod 3, \\ x &\equiv 3 \mod 5, \\ x &\equiv 2 \mod 7. \end{aligned}$$

Thus the general solution is

$$x = -70 + 63 + 30 + 105n = 23 + 105n$$

and the least positive solution is 23.

Exercise 4.3.15

$$A \equiv 0 - 1039470 \equiv 0 \mod 3$$
$$A \equiv (3^2 + 0 - (-1)^7 - 9) \times 1 - 0 \equiv 1 \mod 5$$
$$A \equiv (1^2 + 0 - (2^3)^4 \times 4 - 3) \times (-1) - 5 \equiv -(1 - 4 - 3) - 5 \equiv 1 \mod 7$$

so using the results of Exercise 4.3.14 their age

$$x = 0 \times (-35) + 1 \times 21 + 1 \times 15 + 105n = 36 + 105n$$

so they are 36.

Exercise 4.3.16

If n, $m.\,k$, r are strictly positive integers with

$$n = mk + r$$

and $m > r$ then

$$\begin{aligned} 2^n - 1 &= 2^n - 2^r + 2^r - 1 \\ &= 2^r(2^m - 1)(2^{(k-1)m} + 2^{(k-2)m} + \cdots + 1) + 2^r - 1 \end{aligned}$$

so

$$2^n - 1 = M(2^m - 1) + (2^r - 1)$$

and Euclid's algorithm works in parallel with the indices.

Thus $\gcd(2^n - 1, 2^m - 1) = 2^{\gcd(n,m)} - 1$.

Exercise 4.3.17

(i) If d divides a_i and a_j and

★★
$$\begin{aligned} x &\equiv u_i \mod a_i, \\ x &\equiv u_j \mod a_j \end{aligned}$$

then since a_i divides $x-u_i$ and a_j divides $x-u_j$ it follows that d divides $x-u_i$ and $x-u_j$ so d divides

$$u_i - u_j = (x-u_j) - (x-u_i)$$

and

$$u_i \equiv u_j \mod d.$$

Taking $d = \gcd(a_i, a_j)$ gives the desired result.

Suppose

$$\begin{aligned} x &\equiv u_1 \mod a_1, \\ x &\equiv u_2 \mod a_2, \\ x &\equiv u_3 \mod a_3. \end{aligned}$$

Then

$$\begin{aligned} x' &\equiv u_1 \mod a_1, \\ x' &\equiv u_2 \mod a_2, \\ x' &\equiv u_3 \mod a_3. \end{aligned}$$

if and only if

$$\begin{aligned} x - x' &\equiv 0 \mod a_1, \\ x - x' &\equiv 0 \mod a_2, \\ x - x' &\equiv u_3 \mod a_3. \end{aligned}$$

that is to say a_j divides $x - x'$ for each j.

Let us write e for the lowest common multiple of a_1, a_2 and a_3. Suppose a_j divides y for each j, then we can find a k such that

$$e > y - ke \geq 0.$$

Since a_j divides $y - ke$ for each j it follows by minimality that $y = ke$ so e divides y. The converse is trivial.

Thus

$$\begin{aligned} x' &\equiv u_1 \mod a_1, \\ x' &\equiv u_2 \mod a_2, \\ x' &\equiv u_3 \mod a_3. \end{aligned}$$

if and only if e divides $x - x'$, ie if and only if

$$x \equiv x' \mod e$$

(ii) Set $x_1 = u_1$

Now $x_1 - u_2$ and a_1 are both divisible by the highest common divisor of a_1 and a_2 so by Euclid's algorithm we can find k and m such that

$$ka_1 + ma_2 = x_1 - u_2.$$

Thus setting

$$x_2 = x_1 - ka_1$$

we have

$$\begin{aligned} x_2 &\equiv x_1 \equiv u_1 \mod a_1, \\ x_2 &\equiv x_1 - ka_1 \equiv u_2 + ma_2 \equiv u_2 \mod a_2. \end{aligned}$$

Let f be the lowest common multiple of a_1 and a_2. We observe that $x_1 - u_3$ and f are both divisible by the highest common divisor of f and a_3 so by Euclid's algorithm we can find K and M such that

$$kf + Ma_3 = x_2 - u_3.$$

Thus setting

$$x_3 = x_2 - Kf$$

we have

$$\begin{aligned} x_3 &\equiv x_2 - Kf \equiv x_2 \equiv u_1 \mod a_1, \\ x_3 &\equiv x_2 - Kf \equiv x_2 \equiv u_2 \mod a_2, \\ x_3 &\equiv x_2 - Kf \equiv u_3 - Ma_3 \equiv u_3 \mod a_3 \end{aligned}$$

(iii) The system

★★
$$\begin{aligned} x &\equiv u_1 \mod a_1, \\ x &\equiv u_2 \mod a_2, \\ x &\equiv u_3 \mod a_3 \end{aligned}$$

is soluble if and only if

$$u_i \equiv u_j \mod \gcd(a_i, a_j)$$

for all $1 \leq i < j \leq 3$. If x is a solution of ★★, then x' is a solution of ★★ if and only if $x \equiv x' \mod e$ where e is the least common multiple of the a_i

The system

★
$$\begin{aligned} x &\equiv u_1 \mod a_1, \\ x &\equiv u_2 \mod a_2, \\ x &\equiv u_3 \mod a_3, \\ &\vdots \\ x &\equiv u_n \mod a_n \end{aligned}$$

is soluble if and only if

$$u_i \equiv u_j \mod \gcd(a_i, a_j)$$

for all $1 \leq i < j \leq n$. If x is a solution of ★, then x' is a solution of ★ if and only if $x \equiv x' \mod e$ where e is the least common multiple of the a_i.

Exercise 4.4.2

If n is not prime then we can find r and s such that $n = rs$ and $r, s \geq 2$ so $r \not\equiv 0$ and $s \not\equiv 0$ but

$$r \times s \equiv 0 \mod n.$$

Exercise 4.4.4

(ii) Suppose $\mathbf{a}$ is a book number. If $a_j = c_j$ for $j \neq k$ then, if $\mathbf{c}$ satisfies the check,

$$\begin{aligned} 0 &\equiv 10c_1 + 9c_2 + \cdots + 2c_9 + c_{10} \\ &\equiv 10c_1 + 9c_2 + \cdots + 2c_9 + c_{10} - 10a_1 + 9a_2 + \cdots + 2a_9 + a_{10} \\ &\equiv (11-k)(c_k - a_k) \end{aligned}$$

modulo 11 so, since 11 is prime

$$c_k - a_k \equiv 0$$

and so $c_k = a_k$.

(iii) If $a_j = b_j = 0$ for $1 \leq j \leq 8$ and $a_9 = 5$, $a_{10} = 1$, $b_9 = b_{10} = 0$ then $\mathbf{a}$ and $\mathbf{b}$ are both ISBNs but differ in exactly two places.

(iv) and (v) Suppose $\mathbf{a}$ is a book number. If $a_j = c_j$ for $j \neq p, q$ $a_p = c_q$ and $a_q = c_p$ then, if $\mathbf{c}$ satisfies the check,

$$\begin{aligned} 0 &\equiv 10c_1 + 9c_2 + \cdots + 2c_9 + c_{10} \\ &\equiv 10c_1 + 9c_2 + \cdots + 2c_9 + c_{10} - 10a_1 + 9a_2 + \cdots + 2a_9 + a_{10} \\ &\equiv (11-q)(c_q - c_p) + (11-p)(c_p - c_q) \\ &\equiv (q-p)(c_p - c_q) \end{aligned}$$

modulo 11 so, since 11 is prime

$$c_p - c_q \equiv 0$$

and so $c_j = a_j$ for all j. The check works for any single transposition.

Exercise 4.4.6

By Fermat's little theorem

$$10^{p-1} \equiv 1 \mod p$$

so that

$$10^{(p-1)k} \equiv 1 \mod p$$

and $10^{(p-1)k} - 1$ is divisible by p for each $k \geq 1$. If $p \neq 3$ this tells us that $(10^{(p-1)k} - 1)/9$ (which has the form $111\ldots1$) is divisible by p for each $k \geq 1$.

If $p = 3$, then, since $10^k \equiv 1^k \equiv 1 \mod 3$, we have $111\ldots1$ divisible by 3 whenever the number has $3k$ digits.

Exercise 4.4.8

(i) If $r^2 \equiv a^2$, then $r \equiv a \mod 2$.

(ii) The equation $r^2 \equiv u \mod 2$ has exactly one solution for all u

Exercise 4.4.9

If
$$am^2 + bm + c \equiv 0 \mod p$$
then
$$m^2 + a^{-1}b + a^{-1}c \equiv 0$$
so
$$(m + 2^{-1}a^{-1}b)^2 \equiv 4^{-1}(ba^{-1})^2 - a^{-1}c$$
and
$$\big(2a(m + 2^{-1}a^{-1}b)\big)^2 \equiv b^2 - 4ac$$
so $b^2 - 4ac$ is a square.

Conversely if $b^2 - 4ac = u^2$ then arguing as before
$$am^2 + bm + c \equiv 0 \mod p$$
if and only if
$$\big(2a(m + 2^{-1}a^{-1}b)\big)^2 \equiv u^2$$
Thus $\star$ holds if and only if
$$\big(2a(m + 2^{-1}a^{-1}b) - u\big)\big(2a(m + 2^{-1}a^{-1}b) - u\big) \equiv 0$$
so if and only if
$$2a(m + 2^{-1}a^{-1}b) - u \equiv 0 \text{ or } 2a(m + 2^{-1}a^{-1}b) - u \equiv 0$$
Thus the solutions of $\bigstar$ are given by
$$m \equiv 2^{-1}(-b \pm u) \mod p.$$

If we work modulo 2 then $a = 1$ and
$$0^2 + b0 + c = c,\ 1^2 + b1 + c = 1 + b + c.$$
The equation has solution 0 if $c \equiv 0$ and the solution 1 if $b + c \equiv 1$.

Exercise 4.4.11

(i) If $u^2 \equiv -1$ then by Lemma 4.4.10 (ii)

$$(m^2)^{(p+1)/4} \equiv m \text{ or } (um^2)^{(p+1)/4} \equiv m \mod p$$

for all m contradicting Lemma 4.4.7 (iii).

(ii) If $u^2 \equiv a$ and $v^2 \equiv -a$ then $(u^{-1}v)^2 \equiv -1$ which is impossible by (i).

(iii) Thus at most one of a and $-a$ has a square root for each $a \not\equiv 0$. But by Lemma 4.4.7 (iii) exactly half of the non-zero integers modulo p have square roots so exactly one of a and $-a$ has a square root for each $a \not\equiv 0$.

(iv) Thus using (iii) and Lemma 4.4.10, $u^2 \equiv a$ has solution if and only if $u^{(p+1)/2} \equiv u$.

Exercise 4.4.12

We have
$$2^{10} \equiv 16 \times 16 \times 4 \equiv (-3)^2 \times 4 \equiv -2$$
so $2^5 \equiv -6$ is a square root of -2. Thus $u^2 \equiv 2$ has no solution and $u^2 \equiv -2$ has solutions $u \equiv \pm 6$.

We have
$$3^{10} \equiv 27^3 \times 3 \equiv 8^3 \times 3 \equiv 7 \times 8 \times 3 \equiv -3$$
so $3^5 \equiv 8 \times 9 \equiv -4$ is a square root of -3. Thus $u^2 \equiv 3$ has no solution and $u^2 \equiv -3$ has solutions $u \equiv \pm 4$.

Exercise 4.4.13

Observe that $a^{2^{r+1}} = (a^{2^r})^2$ so by induction a^{2^n} can be computed by squaring $n-1$ times.

We may compute $2, 2^2, \dots, 2^{n-1}$ with $n-2$ multiplications and then

$$2^m = \prod_{w_j=1, j\geq 1} 2^j$$

with $n-1$ multiplications so we have used at most $2n$ multiplications.

Modulo 43 we have

$$\begin{gathered} 2^2 \equiv 4,\ 2^4 \equiv 16, \\ 2^8 \equiv 16^2 \equiv 256 \equiv -2, \\ 2^{16} \equiv 4. \end{gathered}$$

Thus

$$2^{22} \equiv 2^{16} \times 2^4 \times 2^2 \equiv 4 \times 16 \times 4 \equiv 16^2 \equiv -2$$

and so $u \equiv 2 \mod p$ has no solutions by Exercise 4.4.11 (iv).

Exercise 4.4.14

(i) We draw up a table of squares modulo 21

$n \equiv$	0	1	2	3	4	5	6	7	8	9	10
$n^2 \equiv$	0	1	4	9	-5	4	-6	7	1	-3	-5

Thus

$n^2 \equiv 0$ has one solution $n \equiv 0$,

$n^2 \equiv 1$ has four solutions $n \equiv \pm 1, \pm 8$,

$n^2 \equiv 4$ has four solutions $n \equiv \pm 2, \pm 5$,

$n^2 \equiv -5$ has four solutions $n \equiv \pm 4, \pm 10$,

$n^2 \equiv -3$ has two solutions $n \equiv \pm 9$,

$n^2 \equiv -6$ has two solutions $n \equiv \pm 6$,

$n^2 \equiv 7$ has two solutions $n \equiv \pm 7$.

$n^2 \equiv 9$ has two solutions $n \equiv \pm 3$.

Otherwise $n^2 \equiv u$ has no root.

(ii) We draw up a table of squares modulo 15

$n \equiv$	0	1	2	3	4	5	6	7
$n^2 \equiv$	0	1	4	-6	1	-5	6	4

Thus

$n^2 \equiv 0$ has one solution $n \equiv 0$,

$n^2 \equiv 1$ has four solutions $n \equiv \pm 1, \pm 4$,

$n^2 \equiv 4$ has four solutions $n \equiv \pm 2 \pm 7$,

$n^2 \equiv -5$ has two solutions $n \equiv \pm 5$,

$n^2 \equiv 6$ has two solutions $n \equiv \pm 6$,

$n^2 \equiv -6$ has two solutions $n \equiv \pm 3$.

Otherwise $n^2 \equiv u$ has no root.

Exercise 4.5.1

Either observe that if $a \not\equiv 0 \mod p$ and $a \not\equiv 0 \mod q$ then a and pq have highest common factor 1 so by Euclid's algorithm we can find u and v such that

$$au + pqv = 1$$

and so $au \equiv 1 \mod pq$.

Or use Euclid to show that there exist r, s with

$$ar \equiv 1 \mod p \text{ and } as \equiv 1 \mod q.$$

Now use the Chinese remainder theorem to give a u with

$$u \equiv r \mod p \text{ and } u \equiv s \mod q$$

so

$$au \equiv 1 \mod p \text{ and } au \equiv 1 \mod q$$

so by the uniqueness part of the Chinese remainder theorem

$$au \equiv 1 \mod pq.$$

Exercise 4.5.2

$u^2 \equiv a \mod pq$ has one solution corresponding to each pair (r, s) of solutions

$$\begin{aligned} r^2 &\equiv 0 \mod p \\ s^2 &\equiv 0 \mod q \end{aligned}$$

(i) If $a \equiv 0 \mod pq$ then the only pair is $(0, 0)$ so exactly one solution.

(ii) If $a \equiv 0 \mod q$, $a \equiv u^2 \mod p$ with $a \not\equiv 0 \mod p$ then two pairs $(\pm u, 0)$ so two solutions.

If $a \equiv 0 \mod q$, $a \not\equiv v^2 \mod p$ for any v then two pairs $(\pm u, 0)$ so two solutions.

Similar with roles of p and q reversed.

If $a \not\equiv 0 \mod p$ and $a \not\equiv 0 \mod q$ then either $a \equiv u^2 \mod p$, $a \equiv v^2 \mod q$ so four pairs $(\pm u, \pm v)$ and four solutions or no solutions.

(iii) There are $(p-1)/2$ non-zero values of u^2 modulo p and $(p-1)/2$ non-zero values of v^2 modulo 2.

We thus have

$$\begin{aligned} |\{u \in A \,:\, u^2 \equiv 0 \mod p\}| &= \frac{p+1}{2} \\ |\{u \in A \,:\, u^2 \equiv 0 \mod q\}| &= \frac{q+1}{2} \\ |\{u \in A \,:\, u^2 \equiv 0 \mod pq\}| &= 1 \end{aligned}$$

$$|\{u \in A \,:\, u^2 \equiv a \mod pq \text{ has exactly two solutions}\}| = \frac{p+q-2}{2}$$

There are $(p-1)/ \times (q-1)/2$ pairs (r, s) with

$$\begin{aligned} r^2 &\equiv a \mod p \\ s^2 &\equiv a \mod q \end{aligned}$$

when $a \not\equiv 0 \mod p$, $a \not\equiv 0 \mod q$ so

$$\begin{aligned} &|\{u \in A \,:\, u^2 \equiv a \mod pq \text{ has exactly four solutions}\}| \\ &\quad = (p-1)(q-1)/4. \end{aligned}$$

Since there are $pq - 1$ non-zero integers modulo pq

$$\begin{aligned} &|\{u \in A \,:\, u^2 \equiv a \mod pq \text{ has no solutions}\}| \\ &\quad = pq - 1 - \frac{p+q-2}{2} - \frac{p-1}{2}\frac{q-1}{2} = \frac{pq-1}{2} + \frac{(p-1)(q-1)}{4} \end{aligned}$$

EXERCISE 4.5.3*

Exercise 4.5.4

(Corrected by Nigel White.)

If $q = 2$ we know that $u^2 = a$ always has exactly one root. Thus

$$\begin{aligned} r^2 &\equiv a \mod 2 \\ s^2 &\equiv a \mod q \end{aligned}$$

has exactly the same number of solution pairs as

$$s^2 \equiv a \mod q$$

Thus

(i) $u^2 \equiv a \mod 2q$ has exactly one solution if $a \equiv 0 \mod q$.

(ii) If $a \not\equiv 0 \mod q$ then $u^2 \equiv a \mod 2q$ either has no solutions or has exactly two solutions.

(iii) Let $A = \{a : 1 \leq a \leq 2q - 1\}$

$$|\{u \in A : u^2 \equiv a \mod 2q \text{ has exactly two solutions}\}| = (q-1),$$

$$|\{u \in A : u^2 \equiv a \mod pq \text{ has no solutions}\}| = (q-1)$$

Exercise 4.5.6

If p and q are odd primes and $a \equiv 0 \mod q$, $a \not\equiv 0 \mod p$ then if v is a solution of $u^2 \equiv a \mod pq$ it follows that $-v$ is the other distinct solution.

To see this observe that

$$\begin{aligned} u^2 \equiv a \mod pq &\Leftrightarrow u^2 \equiv a \mod p \text{ and } u^2 \equiv 0 \mod q \\ &\Leftrightarrow (u-v)(u+v) \equiv 0 \mod p \text{ and } u \equiv 0 \mod q \\ &\Leftrightarrow u \equiv \pm v \mod p \text{ and } u \equiv \pm v \mod q \\ &\Leftrightarrow u \equiv \pm v \mod pq \end{aligned}$$

Exercise 4.5.7

(i) Since $10^{300} \approx 2^{1000}$, we know from Lemma 4.2.8 that we need only a few thousand operations.

(ii) Our proof of Lemma 4.3.10 showed that we need only apply Euclid's algorithm and the method of Lemma4.2.10 twice (actually we need only do it once) and do few more steps.

(iii) Since $10^{300} \approx 2^{1000}$, Lemma 4.4.13 shows that we need only a few thousand operations.

Exercise 4.5.11

Roughly speaking, we will get a satisfactory answer once in a thousand times but if we ask our question 200 000 times the probability that we will get less than 64 is very small (how small can be estimated using Theorem 10.5.3 or less well using Tchebychev) and, if we have 64 satisfactory answers, then Exercise 4.5.10 tells us that with high probability we can determine p and q rapidly.

Exercise 4.5.12

By considering cases we find $1333 = 31 \times 43$. We wish to solve

$$u^2 \equiv -3 \mod 31$$
$$u^2 \equiv 11 \mod 43$$

Now

$$(-3)^2 \equiv 9,\ (-3)^4 \equiv 12,\ (-3)^8 \equiv 11 \mod 31$$

so $u \equiv \pm 11 \mod 31$.

Also,

$$11^2 \equiv -8,\ 11^4 \equiv 21,\ 11^8 \equiv 11 \mod 43$$
$$11^{11} \equiv 11 \times 11^2 \times 11^8 \equiv (-8) \times (-8)^2 \times (-8)^8 \equiv -21 \mod 43$$

so $u \equiv \pm 21 \mod 43$.

We now need to use the Chinese remainder theorem. Applying Euclid's algorithm we have

$$43 = 1 \times 31 + 12$$
$$31 = 3 \times 12 - 5$$
$$12 = 2 \times 5 + 2$$
$$5 = 2 \times 2 + 1$$

Thus

$$\begin{aligned} 1 = 5 - 2 \times 2 = 5 - 2 \times (12 - 2 \times 5) = 5 \times 5 - 2 \times 12 \\ - 2 \times 12 + 5 \times (3 \times 12 - 31) = 13 \times 12 - 5 \times 31 \\ = 13 \times (43 - 31) - 5 \times 31 = 13 \times 43 - 18 \times 31 \end{aligned}$$

and

$$559 \equiv 13 \times 43 \equiv 1 \mod 31$$
$$559 \equiv 0 \mod 43$$

$$-558 \equiv 0 \mod 31$$
$$-558 \equiv 1 \mod 43$$

Thus the possible roots are

$$559 \times 11 + 558 \times 21 \equiv 538 \text{and} - 538 \equiv 795$$
$$559 \times 11 - 558 \times 21 \equiv 1096 \text{and} - 1096 \equiv 237.$$

Exercise 4.5.13

Write u and v for the first and second encoded message. Observe that N and N' are coprime. (If not, I really have been stupid, since Euclid's algorithm will now give SNDO the common factor.) SNDO can use the known N and N' together with the Chinese remainder theorem to compute w with $w \equiv m^2 \pmod{NN'}$ and $0 \leq w \leq NN' - 1$. But $0 \leq m^2 \leq NN' - 1$ so $w = m^2$ and m is the positive square root of m.

[SNDO uses Euclid's algorithm to find a, b with $aN + bN' = 1$. If $w \equiv bN'u + aNv \pmod{NN'}$, then $w \equiv u \pmod{N}$ and $w \equiv v \pmod{N'}$.]

SNDO is no further forward in reading other messages. For we have proved that even if we encode messages known to SNDO and give SNDO the results the code is still unbroken. Effectively SNDO knows m and m^2 (modulo N) in one case and nothing else (since N' and m^2 (modulo N') are irrelevant).

Exercise 4.5.14

My problem is, knowing b and s with

$$r^2 + br - s \equiv 0 \mod pq,$$

to find r. I can do this by using the Chinese remainder theorem and solving

$$r^2 + br - s \equiv 0 \mod p$$
$$r^2 + br - s \equiv 0 \mod q.$$

The solutions of these equations are

$$r \equiv 2^{-1}(b \pm u) \mod p$$
$$r \equiv 2^{-1}(b \pm v) \mod q.$$

where u and v are solutions of

$$u^2 \equiv b^2 - 4s \mod p$$
$$v^2 \equiv b^2 - 4s \mod q.$$

Thus I can decode the messages by finding square roots and using the Chinese remainder theorem.

If I can decode messages, then I can solve

$$r^2 + br - s \equiv 0 \mod pq$$

and so find w with

$$w^2 \equiv b^2 - 4s \mod pq.$$

But given any n I can set $s = 4^{-1} \times (b^2 - n)$ to obtain a square root of n modulo pq. The new system is exactly secure as the old.

Exercise 4.6.2

If $a_j = a_k$ with $r \geq k \geq j \geq 1$ then
$$a_1 = T^{-j+1}(a_j) = T^{-j+1}a_k = a_{k-j+1}$$
and $r > k - j + 1 \geq 1$ which is impossible.

Observe that
$$T^k a_{s+1} = T^{k+1} a_s = \begin{cases} a_{s+2} & \text{if } 1 \leq s \leq r-1 \\ a_1 & \text{if } s = r \end{cases}$$

If
$$\{a_1,\, a_2, \ldots,\, a_r\} \cap \{b_1, b_2, \ldots, b_k\} \neq \varnothing$$
then we can find p and q such that $a_p = b_q$. If we write down the cycle expressions starting with a_p and b_q we will get the same cycles. Thus, by the previous paragraph, $r = k$,
$$\{a_1,\, a_2, \ldots,\, a_r\} = \{b_1, b_2, \ldots, b_r\},$$
and taking $u = q - p$, we have and $b_s = T^u a_s$ for all s.

Exercise 4.6.3

(a) We have

$$(1\ 2\ 4\ 8\ 16\ 15\ 13\ 9)(3\ 6\ 12\ 7\ 14\ 11\ 5\ 10)(17)$$

(b) We have

$$(1\ 6\ 11\ 16\ 4\ 9\ 14\ 2\ 7\ 12\ 17\ 5\ 10\ 15\ 3\ 8\ 13)$$

(c) We have

$$(1\ 3\ 9\ 10\ 13\ 5\ 15\ 11\ 16\ 14\ 8\ 7\ 4\ 12\ 2\ 6)(17)$$

(d) We have $T_4(16) = 1 = T_4(1)$.

(e) $S_1(5) = S_1(15)$ so S_1 is not a shuffle.

(f) S_2 is a shuffle given by

$$(1\ 6\ 11)(2\ 7\ 8)(3\ 8\ 11)(4\ 9\ 12)(5\ 10\ 15)$$

(g) S_3 is a shuffle given by

$$(1)(2\ 8)(3\ 12)(4)(5)(6)(7\ 13)(9)(10)(11)(14)$$

Exercise 4.6.4

(i) Observe that

$$T^k a_j = a_j \text{ for all } j \Leftrightarrow T^k a_1 = a_1.$$

Suppose that $T^k a_1 = a_1$. Choose m such that

$$r > k - mr \geq 0.$$

Then $T^{k-mr} a_1 = T^k (T^r)^{-m} a_1 = a_1$ and so, since r is minimal, $k - mr = 0$. Thus $k = mr$. The converse is immediate.

(ii) Using (i),

$$\begin{aligned} T^k x = x \text{ for all } x &\Leftrightarrow k \text{ is divisible by } d_k \text{ for all } 1 \leq k \leq m \\ &\Leftrightarrow k \text{ is divisible by } \operatorname{lcm}(d_1, d_2, \ldots, d_m) \end{aligned}$$

(iii) T_1 has period 16, T_2 has period 17, T_3 has period 16, S_2 has period 3 and S_3 has period 2.

Exercise 4.6.6

(i) Without loss of generality $u \geq v \geq 2$ and
$$uv \geq 2u \geq u + v.$$

(ii) By Exercise 4.2.14
$$uv = \gcd(u,v) \operatorname{lcm}(u,v) = \operatorname{lcm}(u,v).$$

(iii) Suppose that d_1, d_2, ..., d_k are strictly positive integers and d_1 and d_2 have highest common factor $c \geq 2$. Then setting $\tilde{d}_1 = d_1/c$, $\tilde{d}_j = d_j$ for $k \geq j \geq 2$ we have

$$\operatorname{lcm}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_k) = \operatorname{lcm}(d_1, d_2, \dots, d_k)$$
and
$$\tilde{d}_1 + \tilde{d}_2 + \cdots + \tilde{d}_k \leq d_1 + d_2 + \cdots + d_k.$$

Thus there is no loss of generality in supposing d_1, d_2, ... d_k coprime. Suppose d_k is not a power of a prime. Then we can write $d_k = \tilde{d}_k \times \tilde{d}_{k+1}$ with $\tilde{d}_k$ and $\tilde{d}_{k+1}$ coprime. Setting $\tilde{d}_j = d_j$ for $1 \leq j \leq k-1$ we have (by part (i))
$$\tilde{d}_1 + \tilde{d}_2 + \cdots + \tilde{d}_k + \tilde{d}_{k+1} \leq d_1 + d_2 + \cdots + d_k$$
and
$$\begin{aligned}\operatorname{lcm}(d_1, d_2, \dots, d_k) &= d_1 d_2 \dots d_k\\ &= \tilde{d}_1 \tilde{d}_2 \dots \tilde{d}_k \tilde{d}_{k+1}\\ &= \operatorname{lcm}(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_k, \tilde{d}_{k+1})\end{aligned}$$

Thus $\operatorname{lcm}(d_1, d_2, \dots, d_k)$ is maximised subject to $d_j \geq 1$ and $d_1 + d_2 + \cdots + d_k \leq n$, $m \geq 1$ by taking the d_j powers of distinct primes.

(iv) By Lemma 4.6.5 and part (iii) the longest period of a shuffle with n cards is given by the maximum value of
$$p_1^{m(1)} p^{m(2)} \dots p_l^{m(l)}$$
where l is a strictly positive integer, p_1, p_2, ..., p_l are distinct primes and m_1, m_2, ..., m_l are strictly positive integers with
$$p_1^{m(1)} + p^{m(2)} + \cdots + p_l^{m(l)} \leq n.$$

Exercise 4.6.7

I do not know any better method than exhaustive search using Exercise4.6.6 (iv).

$n = 1$. Longest period 1.

$n = 2$. Longest period 2.

$n = 3$. Longest period 3.

$n = 4$. Since $2^2 > 3$, longest period 4.

$n = 5$. Since $2 \times 3 > 5$, longest period 6.

$n = 6$. Since $2 \times 3 > 5$, longest period 6.

$n = 7$. Since $3 \times 2^2 > 7$, longest period 12.

$n = 8$. Longest period $3 \times 5 = 15$.

$n = 9$. Since $2^2 \times 5 > 2^4 > 3 \times 5$, longest period 20.

$n = 10$. Longest period $2 \times 3 \times 5 = 30$.

$n = 11$. Longest period $2 \times 3 \times 5 = 30$.

$n = 12$. Longest period $2^2 \times 3 \times 5 = 60$.

Exercise 4.6.8

(i) We have

$n = 1$, shuffle is (12), 2 shuffles to return.

$n = 2$, shuffle is (1243), 4 shuffles to return.

$n = 3$ shuffle is (124)(365), 3 shuffles to return.

$n = 4$, shuffle is (124875)(36), 6 shuffles to return.

$n = 5$, shuffle is (12485[10]9736), 10 shuffles to return.

$n = 6$, shuffle is (124836[12][11]95[10]7), 12 shuffles to return.

$n = 7$, shuffle is (1248)(36[12]9)(5[10])(7[14][13][11]), 4 shuffles to return.

When $n = 4$, cards 3 and 6 just swap places each time (so return to original position after any even number of shuffles). When $n = 7$, cards 5 and 10 just swap places each time (so return to original position after any even number of shuffles).

(ii) Mr Jonas prefers the out-shuffle because the first and last cards do not move. (Very useful indeed.)

We have

$n = 1$, shuffle is (1)92), stays in place.

$n = 2$, shuffle is (1)(23)(4), 2 shuffles to return.

$n = 3$ shuffle is (1)(2354)(6), 4 shuffles to return.

$n = 4$, shuffle is (1)(235)(476)(8), 3 shuffles to return.

$n = 5$, shuffle is (1)(235986)(47)([10]), 6 shuffles to return.

$n = 6$, shuffle is (1)(23596[11][10]847)([12], 10 shuffles to return.

$n = 7$, shuffle is (1)(235947[13][12][10]6[11]8)(14), 12 shuffles to return.

Exercise 4.6.9

This is a simple generalisation of Lemma 4.4.3 (i). Since $\gcd(a, n) = 1$ Euclid's algorithm tells us that we can find u and v such that

$$au + nv = 1$$

and so

$$au \equiv 1 \mod n.$$

Exercise 4.6.11

(i) Observe that

$$A_{j(1)} \cap A_{j(2)} \cap \dots A_{j(u)}$$
$$= \{rnp_{j(1)}^{-k_{j(1)}} p_{j(2)}^{-k_{j(2)}} \dots p_{j(s)}^{-k_{j(s)}} \ : \ 1 \leq r \leq p_{j(1)}^{k_{j(1)}} p_{j(2)}^{k_{j(2)}} \dots p_{j(s)}^{k_{j(s)}}\}$$

and so

$$|A_{j(1)} \cap A_{j(2)} \cap \dots A_{j(u)}| = \frac{n}{p_{j(1)}^{k_{j(1)}} p_{j(2)}^{k_{j(2)}} \dots p_{j(s)}^{k_{j(s)}}}.$$

(ii) By the inclusion-exclusion formula

$$\left|\bigcup_{j=1}^{u} A_j\right| = \sum_{u=1}^{s} (-1)^{s+1} \sum_{1 \leq j(1) < j(2) < \dots < j(s) \leq u} |A_{j(1)} \cap A_{j(2)} \cap \dots A_{j(u)}|$$
$$= \sum_{u=1}^{s} (-1)^{s+1} \sum_{1 \leq j(1) < j(2) < \dots < j(s) \leq u} \frac{n}{p_{j(1)}^{k_{j(1)}} p_{j(2)}^{k_{j(2)}} \dots p_{j(s)}^{k_{j(s)}}}$$

so

$$\phi(n) = \left|X \setminus \bigcup_{j=1}^{u} A_j\right| = |X| - \left|\bigcup_{j=1}^{u} A_j\right|$$
$$= n\left(1 - \sum_{u=1}^{s} (-1)^{s} \sum_{1 \leq j(1) < j(2) < \dots < j(s) \leq u} \frac{1}{p_{j(1)}^{k_{j(1)}} p_{j(2)}^{k_{j(2)}} \dots p_{j(s)}^{k_{j(s)}}}\right)$$
$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_u}\right)$$
$$= p_1^{k_1}\left(1 - \frac{1}{p_1}\right) p_2^{k_2}\left(1 - \frac{1}{p_2}\right) \dots p_u^{k_u}\left(1 - \frac{1}{p_u}\right).$$

(iii) We have $\phi(p) = p(1 - p^{-1}) = p - 1$ when p is a prime so

$$\phi(2) = 1,\ \phi(3) = 2,\ \phi(4) = 2^2 \times \frac{1}{2} = 2,\ \phi(5) = 4,$$
$$\phi(6) = 2 \times \frac{1}{2} \times 3 \times \frac{2}{3} = 2,\ \phi(7) = 6,\ \phi(8) = 8 \times \frac{1}{2} = 4,$$
$$\phi(9) = 9 \times \frac{2}{3} = 6,\ \phi(10) = 2 \times \frac{1}{2} \times 5 \times \frac{4}{5} = 4,$$
$$\phi(11) = 10,\ \phi(12) = 2^2 \times \frac{1}{2} \times 3 \times \frac{2}{3} = 4.$$

Exercise 4.6.13

If p is prime, then $\phi(p) = p(1 - p^{-1}) = p - 1$ and

$$a^{p-1} \equiv a^{\phi(p)} \equiv 1 \mod p$$

whenever $\gcd(a, p) = 1$ ie whenever $a \not\equiv 0 \mod p$.

Exercise 4.6.14

(i) Observe that
$$u^2 \equiv 1 \Leftrightarrow (u-1)(u+1) \equiv 0 \Leftrightarrow u \equiv \pm 1 \mod p.$$
Thus if $rs \equiv 1$ and $r \not\equiv \pm 1$ we have $r \not\equiv s$. By choosing exactly one member of each pair r, s with $rs \equiv 1$ and $r, s \not\equiv \pm 1$ we obtain a B with the required properties. Now
$$\begin{aligned}\prod_{r\in A} r = 1 \times -1 \times \prod_{u\in B} u \prod_{u\notin B} u \qquad &\equiv 1 \times -1 \times \prod_{u\in B} u \prod_{u\in B} u^{-1}\\ \equiv 1 \times -1 \times \prod_{u\in B} uu^{-1} \equiv -1 &\end{aligned}$$
modulo p.

(ii) If n is not a prime and $n > 4$ we know that either $n = rs$ with $n - 1 \geq r > s > 1$ so rs divides $(n-1)!$ and
$$(n-1)! \equiv 0 \mod n$$
or $n = r^2$ and $n - 1 \geq 2r > r$ so $(2r) \times r$ and thus r^2 divides $(n-1)!$ and
$$(n-1)! \equiv 0 \mod n.$$

Since $(2-1)! \equiv 1! \equiv 1 \mod 2$ and
$$3! \equiv 6 \equiv 2 \not\equiv -1 \mod 4$$
the required result now follows fro (i).

(iii) We have $p = 4n + 1$ so
$$-1 \equiv (p-1)! \equiv \prod_{r=1}^{2n} r \prod_{r=1}^{2n}(-r) \quad (-1)^{2n}(2n)! \equiv \big((p-1)/2\big)!^2 \mod p$$

If $p \equiv -1 \mod 4$ so $p = 4n - 1$ the same argument gives
$$-1 \equiv (p-1)! \equiv \prod_{r=1}^{2n-1} r \prod_{r=1}^{2n-1}(-r) \quad (-1)^{2n-1}(2n)! \equiv -\big((p-1)/2\big)!^2$$
so
$$\Big(\big((p-1)/2\big)!\Big)^2 \equiv 1 \mod p$$

(iv) We know that $u^2 \equiv 1 \mod pq$ has four distinct roots a, $-a$, b, $-b$. Call the set of four roots X. Thus if $rs \equiv 1$ and $r \notin X$ we have $r \not\equiv s$. By choosing exactly one member of each pair r, s with $rs \equiv 1$ and $r, s \not\equiv \pm 1$, we obtain a B such that $B \cap X = \varnothing$ and if $u \notin X$ $u^{-1} \in B$ if and only if $u \notin B$.

Thus

$$\prod_{r\in A} r = \prod_{u\in X} u \times \prod_{u\in B} u \prod_{u\notin B} u \qquad \equiv a^2b^2 \times \prod_{u\in B} u \prod_{u\in B} u^{-1}$$

$$\equiv -1 \times -1 \times \prod_{u\in B} uu^{-1} \equiv 1$$

modulo pq.

Exercise 4.6.15

(i) Observe that, if $1 \le r \le n$, then
$$2r \equiv 2r \mod 2n+1$$
and, if $n+1 \le r \le 2n$, then
$$2(r-n)-1 \equiv 2r-(2n+1) \equiv 2r \mod 2n+1.$$

If we start in position j, then, after m shuffles, we are at $2^m j$ modulo $2n+1$. By the Euler–Fermat theorem,
$$2^{\phi(2n+1)} j \equiv 1 \times j \equiv j \mod 2n+1,$$
so the pack returns to its original order after $\phi(2n+1)$ shuffles.

Since 53 is prime, $\phi(53) = 52$ and a standard pack returns to its original order after 52 in-shuffles.

(ii) Observe that, if $2 \le r \le n$, then
$$2(r-1) \equiv 2(r-1) \mod 2n-1$$
and, if $n+1 \le r \le 2n-1$, then
$$2((r-1)-n)-1 \equiv 2(r-1)-(2n-1) \equiv 2r-1 \mod 2n-1.$$

We ignore the fixed first and last card and use the renumbering. If we start in position j, then, after m shuffles, we are at $2^m j$ modulo $2n-1$. By the Euler–Fermat theorem,
$$2^{\phi(2n-1)} j \equiv 1 \times j \equiv j \mod 2n-1,$$
so the pack returns to its original order after $\phi(2n-1)$ out-shuffles.

Since $51 = 3 \times 17$, we have $\phi(51) = \phi(3)\phi(17) = 2 \times 16$ and a standard pack returns to its original order after 16 out-shuffles.

(iii) Working modulo 51, we have $2^2 \equiv 4$, $2^4 \equiv 16$ and
$$2^{16} \equiv 16^4 \equiv 64^2 \times 16 \equiv 13^2 \times 16 \equiv 16^2 \equiv 64 \times 4 \equiv 13 \times 4 \equiv 1.$$
Thus
$$2^8 j \equiv 1 \times j \equiv j \mod 51,$$
and a standard pack returns to its original order after 8 out-shuffles.

The statement that a pack returns after m shuffles does not exclude the statement that it returns after k shuffles. [This is the minimum answer. A more expansive discussion would remark that if a pack returns after m shuffles and after after k shuffles, then it will return after the highest common factor of m and k shuffles.]

(iv) There is really no problem in just calculating powers of 2 up to the 26th. Since the minimum return period must divide any return period and since 2 must return to 2 after a return period this does it.

Shorter ways are possible, since the only possible periods are 1 (obviously not a period), 2, 4, 13 and 26. But $2^2 \equiv 4$ modulo 53, $2^4 \equiv 16$ and

$$2^{13} \equiv 2 \times 64^2 \equiv 2 \times 11^2 \equiv 30$$

whilst

$$2^{26} \equiv 900 \equiv -1.$$

No doubt even quicker ways are possible but only at the expense of thought.

Exercise 5.1.1

No changes are required if the reals are distinct.

If the reals are not distinct then we can (for example) instruct our assistant not to swap cards if the numbers they bear are equal. We will find one of the largest cards. (So, if there is only one largest card, we will find it.)

Exercise 5.1.2*

Exercise 5.1.3

The pack contains n pairs. Place the larger of each pair in pile A and the smaller of each pair in pile B. The requires n comparisons. Pile A must contain the largest card in the pack so find the largest card in pile A using $n-1$ operations. Pile B must contain the smallest card in the pack so find the smallest card in pile A using $n-1$ operations. We have used $3n-2$ operations.

Exercise 5.2.1*

Exercise 5.2.2

Let $k \in A$ if and only if the kth card is a record.

$$X_1 + X_2 + \cdots + X_n = \sum_{j \in A} 1 = |A|$$

where $|A|$ is the total number of records.

$$\mathbb{E}X_j = 1 \times \Pr(X_j = 1) + 0 \times \Pr(X_j = 0) = \frac{1}{j}$$

so

$$\mathbb{E}Y = \mathbb{E}X_1 + \mathbb{E}X_2 + \cdots + \mathbb{E}X_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Exercise 5.2.3

(i) If $n \leq x \leq n+1$ then since f is increasing $f(n) \leq f(x)$ and $f(x) \leq f(n+1)$.

Since
$$f(n) \leq f(x) \leq f(n+1) \text{ for } n \leq x \leq n+1$$
we have
$$\int_n^{n+1} f(n)\,dx \leq \int_n^{n+1} f(x)\,dx \leq [\int_n^{n+1} f(n+1)\,dx$$
so
$$f(n) \leq \int_n^{n+1} f(x)\,dx \leq f(n+1).$$
Thus
$$\sum_{n=1}^{N-1} f(n) \leq \sum_{n=1}^{N-1} \int_n^{n+1} f(x)\,dx \leq \sum_{n=1}^{N-1} f(n+1).$$
ie
$$\sum_{n=1}^{N-1} f(n) \leq \int_1^N f(x)\,dx \leq \sum_{n=2}^{N} f(n).$$

If $f(x) > 0$ for all x then $f(1) > 0$ so
$$\sum_{n=1}^{N} f(n) \geq \sum_{n=2}^{N} f(n) \geq \int_1^N f(x)\,dx.$$
Replacing N by $N+1$ in the inequalities of the previous paragraph
$$\sum_{n=1}^{N} f(n) \leq \sum_{n=1}^{N+1} \int_1^N f(x)\,dx$$

(iii) The area under the big rectangles includes the area under the curve so
$$\sum_{n=1}^{N-1} \int_n^{n+1} f(x)\,dx \leq \sum_{n=1}^{N-1} f(n+1).$$

The area under the small rectangles is included includes the area under the curve so
$$\sum_{n=1}^{N-1} f(n) \leq \sum_{n=1}^{N} \int_1^N f(x)\,dx.$$

(iii) Suppose that g is a well behaved decreasing function. Then
$$g(n) \geq g(x) \geq g(n+1) \text{ for } n \leq x \leq n+1$$

so

$$\int_n^{n+1} g(n)\,dx \geq \int_n^{n+1} g(x)\,dx \leq \int_n^{n+1} g(n+1)\,dx$$

and summing

$$\sum_{n=1}^{N-1} g(n) \geq \sum_{n=1}^{N-1} \int_n^{n+1} g(x)\,dx \geq \sum_{n=1}^{N-1} g(n+1),$$

ie

$$\sum_{n=1}^{N-1} g(n) \geq \int_1^N g(x)\,dx \geq \sum_{n=2}^{N} g(n).$$

(Or we could simply set $f = -g$ and apply (i).)

(iv) We have $g(x) = 1/x$ decreasing so

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n} &= \sum_{n=1}^{N} g(n)\\ &\geq \int_1^{N-1} g(x)\,dx\\ &= \int_1^{N-1} \frac{1}{x}\,dx\\ &\geq \int_1^{N} \frac{1}{x}\,dx = \log N\end{aligned}$$

and

$$\begin{aligned}\sum_{n=2}^{N} \frac{1}{n} &= \sum_{n=2}^{N} g(n)\\ &\leq \int_1^{N-1} g(x)\,dx\\ &\leq \int_1^{N} g(x)\,dx = \log N\end{aligned}$$

so that

$$\sum_{n=1}^{N} \frac{1}{n} \leq 1 + \log N.$$

Exercise 5.2.4

Set $X_k = 1$, if the kth card is a record, and $X_k = 0$, if not. Then

$$Y = X_n + X_2 + \cdots + X_{[bn]}$$

is the total number of records. It follows that

$$\mathbb{E}Y = \mathbb{E}X_n + \mathbb{E}X_2 + \cdots + \mathbb{E}X_{[bn]} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{[bn]}.$$

Now

$$\begin{aligned}\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{[bn]} &\le \sum_{j=n-1}^{[bn]-1} \int_j^{j+1} \frac{1}{x}\,dx \\ &= \int_n^{[bn]} \frac{1}{x}\,dx = \log([bn]/n)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{[bn]} &\ge \sum_{j=n}^{[bn]} \int_j^{j+1} \frac{1}{x}\,dx \\ &\ge \sum_{j=n}^{[bn]-1} \int_j^{j+1} \frac{1}{x}\,dx \\ &= \int_{n-1}^{[bn]} \frac{1}{x}\,dx = \log([bn]/(n-1))\end{aligned}$$

Since $[bn]/n \to b$ and $[bn]/(n-1) = ([bn]/n)\big(n/(n-1)\big) \to 0$ as $n \to \infty$ the expected number of records between the nth and the last card is approximately $\log b$.

1	1	1	1	1	1	2	2	3	4	3	4	2	2	3	4	3	4	2	2	3	4	3	4
2	**2**	**3**	**4**	**3**	**4**	1	1	1	1	1	1	**3**	**4**	2	2	**4**	3	**3**	**4**	2	2	**4**	3
3	4	2	2	4	3	**3**	**4**	2	2	**4**	3	1	1	1	1	1	1	4	3	**4**	3	2	2
4	3	4	3	2	2	4	3	**4**	**3**	2	**2**	4	3	**4**	**3**	2	**2**	1	1	1	**1**	1	**1**

Exercise 5.3.1

(i) If $m = 3$ we win if either the third card is 4 which happens with probability $1/4$ or the fourth card is 4 and the third is not 3 (since the third card should not be the largest of the three first cards) which happens with probability $(1/4) \times (2/3) = 1/6$. The total probability of winning is $5/12$.

(ii) If $m = 1$ we win if the first card is 4 ie with probability $1/4$. If $m = 4$ we win if the first card is 4 ie with probability $1/4$.

If $m = 2$ we look at the table marking our choice with boldface. The probability of winning is $11/24$.

(iii) With three cards if $m = 1$ or 3 we must pick the card we are on so we have probability $1/3$ of winning. With $m = 2$ we win if the second card is 3 or if our second card is 1 and the last card 3 so we have probability $1/3 + 1/6 = 1/2$ of winning.

EXERCISE 5.3.4

If $m = 1$ the probability of success is $1/5$.

If $m = 2$ the probability of success is
$$\frac{4}{5}\times\frac{1}{4}+\frac{1}{2}\times\frac{4}{5}\times\frac{3}{4}\times\frac{1}{3}+\frac{1}{3}\times\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2}+\frac{1}{4}\times\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}\times\frac{1}{2}=\frac{5}{12}.$$

If $m = 3$ the probability of success is
$$\frac{3}{5}\times\frac{2}{4}+\frac{1}{2}\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}+\frac{1}{3}\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}=\frac{13}{30}.$$

If $m \geq 4$ the probability of the largest number being among the mth or larger is $2/5 < 5/12$. Since $13/30 > 5/12$ we should take $m = 3$.

Exercise 5.3.6

The probability of dealing u given cards before the tnth card and v after is

$$\frac{[tn]}{n}\times\frac{[tn]-1}{n-1}\times\dots\frac{[tn]-u+1}{n-u+1}$$
$$\times\frac{n-[tn]}{n-u}\times\frac{n-[tn]-1}{n-u-1}\times\dots\frac{[tn]-u-v+1}{n-u-v+1}$$
$$=(n^{-1}[tn])\times\frac{n^{-1}[tn]-n^{-1}}{1-n^{-1}}\times\dots\frac{n^{-1}[tn]-n^{-1}(u-1)}{1-n^{-1}(u-1)}$$
$$\times\frac{1-n^{-1}[tn]}{1-n^{-1}u}\times\frac{1-n^{-1}[tn]-n^{-1}}{1-n^{-1}(u-1)}\times\dots\frac{n^{1}[tn]-n^{-1}(v-1)}{1-n^{-1}(u+v-1)}$$
$$\to t^u(1-t)^v.$$

as $n\to\infty$. Thus

$$\Pr(\text{largest } k \text{ cards after } [tn],\ k+1\text{st card before})\approx t(1-t)^k$$

and so

$$\begin{aligned}&\Pr(\text{largest } k \text{ cards after } [tn],\ k+1\text{st card before}\\&\qquad\text{and largest card before other } k-1 \text{ largest})\\&=\frac{t(1-t)^k}{k}\end{aligned}$$

so as before

$$\Pr(\text{stop at largest card})\approx t(1-t)+\frac{t(1-t)^2}{2}+\frac{t(1-t)^3}{3}+\dots.$$

Exercise 5.3.7

If $f(t) = -t \log t$, then

$$f'(t) = -\log t - 1$$

so $f'(t) > 0$ and $f(t)$ is increasing for $t < e^{-1}$, while $f'(t) < 0$ and $f(t)$ is decreasing for $t > e^{-1}$. Thus f attains its maximum at e^{-1}.

Exercise 5.3.9

(i) If the largest card is dealt before m then no new card will be a record and we will stop at the last card and it will not be a record. If we stop at the last card and it is not a record no card on or after the mth can be a record so none of them can be the largest in the pack. Thus the largest in the pack is in the first $m-1$.

If n is large

$$\begin{aligned}\Pr(\text{stop at last card and it is not largest}) &= \Pr(\text{largest in first } m-1)\\ &= \frac{m-1}{n} \approx e^{-1}\end{aligned}$$

$$\begin{aligned}\Pr(\text{stop at last card and it is largest}) &\leq \Pr(\text{last card is largest})\\ &= \frac{1}{n} \approx 0\end{aligned}$$

Thus

$$\Pr(\text{stop at last card} \approx e^{-1}$$

Also

$$\begin{aligned}\Pr(\text{choose largest card or stop at last}) &= \Pr(\text{choose largest})\\ &\quad + \Pr(\text{stop at last card and it is not largest})\\ &\approx 2e^{-1}\end{aligned}$$

Thus the probability that we neither choose the last card nor the largest is approximately $1 - 2e^{-1}$.

(ii) If n is large

$$\begin{aligned}&\Pr(\text{stop at last card and it is not largest})\\ &= \Pr(\text{largest in first } m-1)\\ &= \frac{tn-1}{n}\end{aligned}$$

$$\begin{aligned}\Pr(\text{stop at last card and it is largest}) &\leq \Pr(\text{last card is largest})\\ &= \frac{1}{n}\end{aligned}$$

so

$$\Pr(\text{stop at last card}) \leq t$$

Also

$$\begin{aligned}&\Pr(\text{none of the } k \text{ largest cards is turned over before the } m\text{th card})\\ &= \frac{n-m}{n} \times \frac{n-m-1}{n-1} \times \cdots \times \frac{n-m-k+1}{n-k+1}\\ &\leq (1-t)^k\end{aligned}$$

But, if we do not stop at one of the highest k cards, then either none of the k largest cards is turned over before the mth or at least one of the k largest cards is turned over before the mth in which case we must have stopped at the last card.

$$\begin{aligned}
&\Pr(\text{stop at one of the } k \text{ th largest cards}) \\
&\geq 1 - \Pr(\text{non of the } k \text{ largest cards is turned over before the } m\text{th}) \\
&\quad - \Pr(\text{stop at last card}) \\
&\geq 1 - t - (1-t)^k
\end{aligned}$$

If we choose $\epsilon/2 > t > \epsilon/4$ and k such that $(1-\epsilon/4)^k < \epsilon/2$

$$\Pr(\text{stop at one of the } k \text{ largest cards}) \geq 1 - \epsilon$$

however large n is.

EXERCISE 5.4.1

(i) Is the first digit 0? Is the second digit 0? Is the third digit 0?

Label the objects by sequences of n zeros and ones (there are 2^n such sequences). Now ask 'Is the jth digit 0?' for $1 \leq j \leq n$.

(ii) If N is the number of head words then we are told that $N \leq 2^{20}$ so the result follows from (i).

We could label the kth word by the sequence

$$\mathbf{x}_k = x_{1,k}x_{2,k}\ldots x_{20,k}$$

with

$$k = \sum_{j=1}^{20} 2^{j-1}x_{j,k}.$$

Exercise 5.4.4

(i) We have for all values of n

$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \\ &\leq n \times n \times n \times \cdots \times n \times n = n^n \end{aligned}$$

and

$$\begin{aligned} n! &= \prod_{r=1}^{n} r \geq \prod_{r=n/2}^{n} r \\ &\geq \prod_{r=n/2}^{n} \frac{n}{2} = \left(\frac{n}{2}\right)^{n/2}. \end{aligned}$$

Thus

$$\begin{aligned} n \log n = \log n^n \geq \log n! &\geq \log \left(\frac{n}{2}\right)^{n/2} \\ &= \frac{n}{2}(\log n - \log 2) \geq \frac{n}{4} \log n \end{aligned}$$

if $n \geq 4$.

(iii) There are lots of different variations. For example, if n is odd and $n \geq 9$,

$$\begin{aligned} n! &= \prod_{r=1}^{n} r \geq \prod_{r=(n-1)/2}^{n} r \\ &\geq \prod_{r=(n-1)/2}^{n} \frac{n}{3} = \left(\frac{n+1}{2}\right)^{n/3} \\ &\geq \left(\frac{n}{2}\right)^{n/3}. \end{aligned}$$

Thus

$$\begin{aligned} n \log n = \log n^n \geq \log n! &\geq \log \left(\frac{n}{2}\right)^{n/3} \\ &= \frac{n}{2}(\log n - \log 3) \geq \frac{n}{4} \log n. \end{aligned}$$

Exercise 5.4.6

$$\begin{aligned} a_2 &= 2^2 + 2 \times 2 = 2 \times 4 = 8 \\ a_3 &= 2^3 + 2 \times 8 = 3 \times 8 = 24 \\ a_4 &= 2^4 + 2 \times 3 \times 8 = 4 \times 2^4 = 64 \\ a_5 &= 2^5 + 2 \times 4 \times 2^4 = 5 \times 2^5 = 160. \end{aligned}$$

If $a_m = m2^m$ then

$$a_{m+1} = 2^{m+1} + 2a_m = 2^{m+1} + m2^m = (m+1)2^m.$$

Since $a_1 = 2 = 1 \times 2^1$ it follows by induction that

$$a_m = m2^m$$

for all $m \geq 1$.

Exercise 5.4.7

We could add $2^m - n$ dummy cards.

If $2^{m-1} < n \leq 2^m$, we can sort a pack of n cards in $m2^m$ operations. But

$$(m-1)\log 2 \leq \log n$$

so

$$m \leq \frac{1+\log n}{\log 2}$$

so

$$m2^m \leq \frac{1+\log n}{\log 2} 2n.$$

Exercise 5.4.9

Suppose that we know that $e_r \leq 4r \log r$ for all $2 \leq r \leq n-1$. By ★ and Lemma 5.4.8

$$\begin{aligned}
e_n &= \frac{n^2}{n-1} + \frac{2}{n-1}(e_2 + \cdots + e_{n-1}) \\
&\leq \frac{n^2}{n-1} + \frac{4}{n-1}\sum_{j=2}^{n-1} j \log j \\
&\leq \frac{n^2}{n-1} + \frac{4}{n-1}\left(\frac{n^2 \log n}{2} - \frac{n^2}{4}\right) \\
&= \frac{2n^2 \log n}{n-1} \\
&\leq 4n \log n.
\end{aligned}$$

Since

$$e_2 = 1 \leq 4 \times 2 \times \log 2$$

the result follows by induction.

Exercise 5.4.10

Let $\mathbb{I}_{[a,\infty)}(t) = 1$ if $t \geq a$ and $\mathbb{I}_{[a,\infty)}(t) = 0$ if $t < a$.

$$a\mathbb{I}_{[a,\infty)}(t) \leq X$$

for all $t \geq 0$, so

$$a\mathbb{I}_{[a,\infty)}(X) \leq X$$

and

$$a \Pr(X \geq a) = \mathbb{E}\big(a\mathbb{I}_{[a,\infty)}(X)\big) \leq \mathbb{E}X$$

as stated.

Thus the probability that the number of operation required is greater than a times the expected number is less than $1/a$.

(Of course if we work harder we can get better bounds but this is already satisfactory for many purposes.)

Exercise 5.4.11

If the pack is well shuffled then the two packs we construct in the first round will be well shuffled and so, by induction, will all the further packs we examine. But if a pack is well shuffled the bottom card will be equally likely to be any card in the pack so choosing it will be the same as choosing a random card.

If the pack is already sorted, then at each stage we divide into a pack of one card and a pack of the rest taking

$$(n-1)+(n-2)+\cdots+1=\frac{n(n-1)}{2}\approx\frac{n^2}{2}$$

(or thereabouts depending on the fine details of our procedure) operations.

Exercise 5.4.12

$$\frac{(8 \times 10^9)^2}{2} = 4 \times 10^{18}.$$

The long way requires 4×10^9 seconds or about 12000 years (see Exercise 4.1.3).

On the other hand

$$4 \times (8 \times 10^9) \times \log(8 \times 10^9) \approx 10^9 \times 730$$

so the short way takes about 12 minutes.

Exercise 5.5.1

Observe, for example, that

$$\min_{1\leq j\leq n} a_j = -\max_{1\leq j\leq n} -a_j.$$

Exercise 5.5.2

Starting from the first airport we may choose the first port of call in $n-2$ ways (we cannot choose the starting or end point), the second in $n-3$ ways, $\ldots$, the $n-2$nd in one way and the fly to the destination airport. Thus there are

$$(n-2) \times (n-3) \times \ldots 1 = (n-2)!$$

routes.

Exercise 5.5.3*

Exercise 5.5.4

(i) We have

$$\frac{n^2}{4} - r(n-r) = \frac{n^2 - 4nr + 4r^2}{4} = \frac{(r - \frac{1}{2}n)^2}{4} \geq 0.$$

Exercise 5.5.5

Let us write

$$a = \{P, W, G, C\},\ b = \{P, W, G\}, c = \{P, W, C\}$$
$$f = \{P, G, C\},\ g = \{P, C\},\ h = \varnothing$$

Then

$$d_{ab} = d_{ba} = d_{bc} = d_{cb} = d_{bf} = d_{fb} = d_{cg} = d_{gc} = d_{fg} = d_{gf} = 2$$

$d_{gh} = d_{hg} = 1$ and all other distances are ∞. We get shortest paths $abcgh$ and $abfgh$ both of length 7.

(ii) An obvious notation is to write (n, m, k) for the state when there are n pints in the 8 pint jug, m in the 5 pint jug and k in the 3 pint jug. We observe that the distance from any (n, m, k) to $(8, 0, 0)$ is 1 (just pour everything in to the big jug). On the other hand if $8 > n > 0$, $5 > m > 0$ and $3 > k > 0$ then the distance from any other point is ∞ (how do you know when to stop pouring?).

In fact there are very few paths and a little thought gives

$$(8, 0, 0) \to (3, 5, 0) \to (3, 2, 3) \to (6, 2, 0) \to (6, 0, 2)$$
$$\to (1, 5, 2) \to (1, 4, 3) \to (0, 4, 4).$$

Exercise 5.5.6

(i) At the first stage A_1 is a 'new settled town'. All the signposts point upwards.

At the nth stage we simply make the 'new settled town' an 'old settled town'.

(ii) We claim that at the beginning of the rth stage the signposts in the settled towns show the shortest route to A_1. Assuming this observe that the shortest route to A_1 from a given unsettled town through a settled town is *either* via an old settled town in which case the distance shown on the signpost will be no larger than the distance to the new settled town plus the shortest distance from the new settled town and the town council will do nothing *or* the distance shown will be greater in which case the town council will make the appropriate alteration. In each case the signpost will now show the shortest distance from the unsettled town to A_1 via a settled town. Just as in the old algorithm an unsettled town which now has a sign showing a distance no larger than any other unsettled town has a signpost showing the next town on the shortest route to A_1 and the total distance required. If we choose one of these as our new settled town the inductive hypothesis holds with r replaced by $r+1$. The case $r=1$ is trivial so algorithm works by induction.

(iii) Each town council makes at most one comparison and there are n councils. We then have to find the smallest of the distances shown on at most n signposts and this requires less than n comparisons. Thus we need less than $2n$ comparisons at each stage. Since there are n steps we need at most $2n^2$ comparisons in all.

Exercise 5.5.7*

Exercise 5.5.9

(i) There are n^2 ordered pairs (i,k) and to compute r_{ik} we need to find the smallest of the n numbers $p_{ij}+q_{jk}$. This requires $n-1$ comparisons so we use $n^2\times(n-1)\leq n^3$ comparisons.

(ii) Let $A=U\bullet(V\bullet W)$ and $B=(U\bullet V)\bullet W$.

We can find a S such that

$$a_{ij}=\min_s\big(u_{rs}+\min_t(v_{st}+w_{tj})\big)=u_{iS}+\min_t(v_{St}+w_{tj})$$

and a T such that

$$\min_t(v_{St}+w_{tj})=v_{ST}+w_{Tj}.$$

Now

$$\min_s(u_{is}+v_{sT})\leq u_{iS}+v_{ST}$$

so

$$\begin{aligned}b_{ij}&\leq\min_t\big(\min_s(u_{is}+v_{st})+w_{tj}\big)\\&\leq\min_s(u_{is}+v_{sT})+w_{Tj}\\&\leq u_{iS}+v_{ST}+w_{Tj}=a_{ij}.\end{aligned}$$

Similarly $a_{ij}\leq b_{ij}$ so $a_{ij}=b_{ij}$ as required.

Exercise 5.5.10

(i) and (ii) Sometimes false. Set

$$U = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 4 \\ 0 & 4 & 4 \end{pmatrix}, \ V = \begin{pmatrix} 0 & 4 & 4 \\ 4 & 0 & 1 \\ 4 & 1 & 0 \end{pmatrix}.$$

Then

$$\min_j (u_{1j} + v_{j3}) = \min(0 + 4, 1 + 1, 4 + 0) = 2$$

and

$$\min_j (v_{1j} + u_{j3}) = \min(0 + 4, 4 + 4, 4 + 0) = 4$$

so $U \bullet V \neq V \bullet U$ although U and V are symmetric.

(iii) Always true. Write $A = V^T$, $B = U^T$, $C = (U \bullet V)^T$ $D = V^T \bullet U^T$. Then

$$\begin{aligned} d_{ik} &= \min_j (a_{ij} + b_{jk}) = \min_j (a_{ji} + b_{jk}) \\ &= \min_j (b_{kj} + a_{ji}) = c_{ij} \end{aligned}$$

as required.

(iv) Sometimes false. Take

$$U = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}, \ V = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \ W = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

Then

$$U \bullet W = U, \ V \bullet W = V$$

but

$$U + V = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix}$$

so

$$(U + V) \bullet W = W \neq (U + V) = U \bullet W + V \bullet W.$$

Exercise 5.5.11

Clearly

$$d_{ij}^{[m]} = d_{ij} = \text{shortest distance from } A_i \text{ to } A_j \text{ through at most 1 town.}$$

Suppose that $d_{ij}^{[r]}$ is the shortest distance from A_i to A_j by a route passing through at most r towns. If we want to travel from A_i to A_j we must go to a town A_k (possibly A_i or A_j) and then take the shortest rout from A_k to A_j passing through at most $r-1$ towns. Thus

$$\begin{aligned}
&\text{shortest distance from } A_i \text{ to } A_j \text{ through at most } r \text{ towns}\\
&\qquad = \min_k(d_{ik} + \text{shortest distance from } A_k \text{ to } A_j \text{ through}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{at most } r-1 \text{ towns})\\
&\qquad = \min_k(d_{ik} + d_{kj}^{[r-1]}) = d_{ij}^{[r]}.
\end{aligned}$$

The stated result follows by induction.

(ii) Since the distance between any two towns is positive there can be no advantage in visiting the same town twice so the shortest route (if one exists) must pass through at most $n-1$ towns. (Similarly, but more simply, if any route exists, then one exists passing through $n-1$ towns or fewer.) Thus $D^{[m]} = D^{[n-1]}$ for all $m \geq n-1$. and, if $d_{ij}^{[n-1]} = \infty$, there is no route of any kind from A_i to A_j. Since $D^{[m]} = D^{[n-1]}$ for all $m \geq n-1$ $d_{ij}^{[n]}$ is the length of the shortest path from i to j.

Exercise 5.5.12

(i) Since $2^{N-1} \leq n$ we have, on taking logarithms,

$$(N-1)\log 2 \leq \log n$$

so

$$N \leq 1 + \frac{\log n}{\log 2}.$$

(ii) Choose N so that $2^{N-1} \leq n \leq 2^N$. Then we can compute

$$D^{[n]} = D^{[2^N]}$$

in less than

$$N \leq 1 + \frac{\log n}{\log 2} \leq K \log n$$

operations. (We can take $K = 1$ if n is not too small. For the purposes of the question we could take $K = 10$.) so we can compute $D^{[n]}$ with less than $Kn^3 \log n$ comparisons.

Exercise 5.5.13

$$
\begin{aligned}
d_{i(1)i(2)} + d_{i(2)i(3)} + \cdots + d_{i(r-1)i(r)} &= 1 - 2 + 1 - 2 + \ldots \\
&= \begin{cases} 2 - r/2 & \text{if } r \text{ is even} \\ -(r-1)/2 & \text{if } r \text{ is odd} \end{cases}
\end{aligned}
$$

Exercise 5.5.16

(i) If
$$d_{j(1)j(2)} + d_{j(2)j(3)} + \cdots + d_{j(s-1)j(s)} = -k < 0$$
then defining $j(rs+t) = j(t)$ for $1 \leq t \leq s$, $r \geq 1$ we have
$$\sum_{u=1}^{Ns-1} d_{j(u)j(u+1)} = -Nk \to -\infty$$
as $n \to \infty$.

(ii) Given k and l, take a path from A_k to $A_{j(1)}$, go round the length decreasing loop many times, then take a path from $A_{j(1)}$ to A_l.

Exercise 5.5.17

The closest town to A_1 is A_2, the closest town to A_1 via A_2 or directly is A_3 reached via A_2 so our algorithm gives the path $A_1A_2A_3$. There is only one other possible path from A_1 to A_3. It is $A_1A_4A_3$ of length 0 and is thus the shortest path.

Exercise 5.5.18

(i) Since there are only n towns any cycle without repetition contains at most n towns. Given a length decreasing cycle follow it until the first return. Either we have performed a negative cycle with at most n towns or we may throw away the first cycle and obtain a smaller negative cycle. Since the process terminates (we have only a finite number of cycle steps) we must eventually find a negative cycle with at most n towns ie we can find $r \leq n$ and i with $d_{kk}^{[n]} < 0$. Automatically $d_{kk}^{[n]} < 0$.

(ii) If $d_{kk}^{[n]} \geq 0$, then there is no shorter path from k to k than staying at k. Thus

$$d_{kk}^{[n]} = \text{length of shortest path from } k \text{ to } k \text{ in } n \text{ or less steps} = 0.$$

Since there are only n towns, it follows that, if there is any path from A_i to A_j, then by removing cycles there is a path P involving at most $n-1$ steps so having

$$d_{ij}^{[n]} \leq \text{length of } P < \infty.$$

Since there are no negative cycles, removing cycles decreases the length of paths from A_i to A_j. Thus the shortest paths from A_i to A_j have at most $n-1$ steps and so have length equal to $d_{ij}^{[n]}$.

Exercise 5.5.19

Before 1835 melt down US gold dollars. Exchange in France for French gold coins at a slight loss. Exchange French gold coins for French silver coins. Melt down French silver coins and use to buy US silver coins. Exchange US silver coins for US gold coins and start again. You have increased your holdings in gold by a little less than $15\frac{1}{2}/15 = 31/30$ but decreased the amount of gold coin in US circulation and increased the amount of silver.

After 1835 run the process backwards.

Exercise 5.5.20

(i) There must be a route from A_1 to A_n. There must be no cycles (if there are then we can go round and round to produce ever longer paths).

We can use Floyd's algorithm to solve the problem with d_{ij} replaced by $-d_{ij}$ (for $d_{ij} \neq \infty$).

(ii) If task A_i must be completed before A_j and take time α, set $d_{ij} = \alpha$. Otherwise take $d_{ij} = 0$. The longest path from A_1 (start building) to A_n (stop building) gives the time to completion.

In general those tasks on the longest path will produce delay if they take longer. (If a task is not on a longest path then increasing the time taken on that task by a small amount will not increase the total time required.)

Exercise 5.5.21

(i) Roads (12), (23), (34), (45), (46) join all towns. The cost is minimal because we need at least 5 roads so a 5 unit cost is cheapest.

(ii) The method works for $n = 2$ since there is only one route.

Suppose it works for $n = m$. Suppose we have $m + 1$ towns with towns A_1 and A_2 the closest. If we have any collection of linking roads for the $m+1$ towns then there is a set of roads linking A_1 to A_2. If we remove one of those roads and build a road direct from A_1 to A_2 then we still have collection of linking roads for the $m + 1$ towns (after the deletion each town must still be linked to at least one of A_1 and A_2, and after the rebuild each town must will be linked to A_1) which will be at least as cheap. Thus a cheapest set of roads will contain the road from A_1 to A_2 so we may start by linking A_1 to A_2. This reduces the problem to linking the 'city' $\{A_1, A_2\}$ and the towns $A_3, \dots, A_{m+1}$. By our inductive hypothesis our method will produce a cheapest solution to this problem and so to the full problem.

The required result follows by induction.

(iii) We first give an upper bound. We need to find the least element of at most n^2 numbers at most n times so we need at most n^3 comparisons.

Next we give a lower bound. If we are on the mth step and $3n/4 \geq m \geq n/4$ then there are at least $[n/4]^2$ possible roads from towns to the city. Thus we need to make to find the least element of at least $[n/4]^2$ numbers at least $[n/2]$ times. Thus (making underestimates) we need at least $(n - 8)^3/32$ comparisons.

Combining these we see that the number of comparisons

(iv) The inductive proof is a repeat of that for (ii).

The method works for $n = 2$ since there is only one route.

Suppose it works for $n = m$. Suppose we have $m + 1$ towns with towns A_1 and A_2 the closest. If we have any collection of linking roads for the $m+1$ towns then there is a set of roads linking A_1 to A_2. If we remove one of those roads and build a road direct from A_1 to A_2 then we still have collection of linking roads for the $m + 1$ towns (after the deletion each town must still be linked to at least one of A_1 and A_2, and after the rebuild each town must will be linked to A_1) which will be at least as cheap. Thus a cheapest set of roads will contain the road from A_1 to A_2 so we may start by linking A_1 to A_2. This reduces the problem to linking the 'city' $\{A_1, A_2\}$ and the towns $A_3, \dots, A_{m+1}$. By our inductive hypothesis our method will produce a cheapest solution to this problem and so to the full problem.

The required result follows by induction.

(v) Initially each town is a conurbation. Read the list of distances in order of increasing size. If you come to a route joining two distinct conurbations build the road and assign the towns in the two conurbations to a single conurbation removing the two original conurbations from the list. When all towns are in a single conurbation stop.

(vi) If we use quicksort we need at most

$$4\frac{n(n-1)}{2}\log\frac{n(n-1)}{2} \leq 8n\log n$$

comparisons.

(vii) If we enter A_1, A_5 or A_6 by a 1 unit route we must leave by a 10 unit route. We must enter and leave all three so either we enter and leave one by a 10 unit route or we use three 10 unit routes. Thus we must use at least two 10 unit routes and our cost is at least 24 units.

A route which is this cheap is given by is given by

$$A_1A_2A_3A_5A_4A_6A_1.$$

Exercise 6.1.1*

Exercise 6.1.2*

Exercise 6.1.3

Consider the table of preferences.

A's preferences	$B > C > D$
B's preferences	$A > C > D$
C's preferences	$D > A > B$
D's preferences	$C > A > B$

The if we group as (A, B) and (C, D) everyone has their first preferences.

Exercise 6.1.4

First stage (A, j)

Second stage (B, j), A

Third stage (B, j), (A, k)

Fourth stage (B, j), (A, k), (C, m)

Fifth stage (D, j), B, (A, k), (C, m)

Sixth (D, j), (A, k), (B, m), C

Seventh (D, j), (A, k), (B, m), (C, l)

Exercise 6.1.5*

Exercise 6.1.6

Those without partners have taken no part in the events. We have thus witnessed an application of the algorithm to the r gentlemen and ladies who have been paired. The pairing is thus stable for those involved.

Exercise 6.1.7

Not terminating Six participants A, B, C, D, E and F.

A preferences $B > C$ and prefers C to any of D, E or F

B preferences $C > A$ and prefers A to any of D, E or F

C preferences $A > B$ and prefers B to any of D, E or F

D, E and F's preferences irrelevant.

A, E and F in room, B, C, D to enter in stated order.

Room sees

$$\begin{aligned} A,E,F &\to (A,B),E,F \to (B,C),A,E,F \\ &\to (A,C),B,E,F \to (B,C),A,E,F \to \dots \end{aligned}$$

Terminating but at unstable point Apply the algorithm to the participants in Exercise 6.1.2 with A and B inside the room, C coming in first and D having preferences $A > B > C$. C matches with A, D is turned down by A and B and proposes to C who must accept. However the system of Exercise 6.1.2 is unstable.

Exercise 6.1.8

(i) The system gives:-

First stage (A, j).

Second and final stage (A, j), (B, k).

The reversed system gives

First stage (j, B).

Second and final stage (j, B), (k, A).

Gentlemen prefer our procedure (they get their first choice). Ladies prefer reversed procedure (they get their first choice).

(ii) Each gentleman enters in turn, proposes to his preferred lady and is accepted.

(iii) The following are stable by inspection.

(A, j) (B, k), (C, l), (D, m)

(A, k) (B, j), (C, l), (D, m)

(A, j) (B, k), (C, m), (D, l)

(A, k) (B, l), (C, m), (D, l).

(vi) Let the gentlemen be A_m the ladies a_m $[1 \leq m \leq 2N]$

If A_{2r-1} has first preference a_{2r-1}, second a_{2r},

A_{2r-1} has first preference a_{2r}, second a_{2r-1},

a_{2r-1} has first preference A_{2r}, second A_{2r-1},

a_{2r} has first preference A_{2r-1}, second A_{2r},

then any pairing with

$$(A_{2r-1}, a_{2r-1}),\ (A_{2r}, a_{2r}) \text{ or } (A_{2r-1}, a_{2r}),\ (A_{2r}, a_{2r-1})$$

for each $1 \leq r \leq N$ is stable and there are 2^N such pairs.

If we add A_{2N+1} with first choice a_{2N+1} and a_{2N+1} with first choice A_{2N+1} then the pairings already given remain stable if we add the pair (A_{2N+1}, a_{2N+1}).

(v) If we have n gentlemen A_j and n ladies a_j such that A_j has first preference a_j and a_j has first preference A_j then the only stable pairing has (A_j, a_j).

Exercise 6.1.12

Suppose that at the first rejection Agatha rejects Albert. Then Agatha must have rejected Albert in favour of someone else, let us say Bertram. (Either Albert proposed and she preferred to stick with Bertram or Bertram proposed and she let go of Albert to take up Bertram.) Thus Agatha prefers Bertram to Albert and (since there were no previous rejections and Bertram starts by proposing to his favourite) Agatha is Bertram's first choice.

There can be no stable solution in which Agatha and Albert are married since Bertram can always leave his wife for Agatha (his first choice) and Agatha will drop Albert for him.

Exercise 6.1.13

Suppose that two applications of our algorithm give two different results. Then there must exist one man Albert, say who is married to two different ladies Bertha and Caroline in the two applications. If he prefers Bertha to Caroline the second application gives a stable arrangement in which his married to someone he likes less well than in the first contrary to Lemma 6.1.9. A similar objection applies if he prefers Caroline to Bertha so we get a contradiction. All applications of our algorithm must give the same result.

Exercise 6.1.15

Since the solution does not depend on the order in which the suitors (copies of universities) enter we can leave the fictitious university copies to last. Just before the first fictitious copy enters the students can be divided into the happy (who have a fiancé) and the unhappy. Each fictitious university copy will be rejected by each happy student and end up with an unhappy student so the same students will fail to find a place and the same students will be assigned to each university regardless of the preferences of the fictitious university copies.

Exercise 6.1.16

[There are many possibilities and these are not claimed to be the most efficient.] (i) Suppose there are N students. We introduce N extra fictitious universities each offering one place and such that the jth fictitious university U_j prefers the j th student to all others. The jth student places the universities she wants to go to in order then university U_j then the remaining fictitious and undesired universities in some order.

Since the solution does not depend on the order in which the suitors (copies of universities) enter we can leave the U_j to last. U_j will propose to student j who will accept (finishing the process) if she has not got an acceptable place or reject in favour of the acceptable place she has already got.

If the universities have a total of M places they introduce M shadow applicants. Each university lists the k applicants it wants in order then $M-k$ shadows, then the M remaining shadows and unwanted applicants. If a university copy proposes to a shadow that university copy will already have been turned down by all its wanted applicants (who will have been accepted by universities they prefer).

(ii) The university of Pismo splits in two 'Scholarship Pismo' and 'Place Pismo' and students may order the list of Scholarship U's and Place U's in any way they wish (eg A with S, B with S, A without S, C with S, …). [This used to be done with College choice in Oxbridge.)

Exercise 6.1.17

Since any lady will leave her present partner if the universal favourite offers himself, the only stable solution will marry him to his first choice. With these two removed the problem reduces with n pairs reduces to the problem with $n-1$ pairs so the only stable choice has the second favourite choosing his favourite amongst the remaining $n-1$ and so on.

In the context of university entrance the top candidate chooses her university, the second chooses from all the universities which are not already full, the third chooses from all the universities which are not already full, and so on.

Exercise 6.1.18

We can replace the non-strict old preferences by new preferences such that

(1) If A (a lady or gentleman) strictly prefers a to b in the old system then A strictly prefers a to b in the new system.

(2) If A gives equal preference a to b in the old system then either A prefers a to b in the new system or A prefers b to a in the new system

We then solve the new problem using Gale-Shapely. Since the result is stable with the new preferences it it is stable under the old.

On the other hand if gentleman A and B like ladies a and b equally and vice versa then neither $(A, a); (B, b)$ nor $(A, b); (B, a)$ satisfy the condition that there do not exist one lady and one gentleman in different pairs by like each other at least as much as their present partner.

Exercise 6.2.1*

Exercise 6.2.2

Since there are only two issues ($2'$) and ($3'$) which deal with 3 issues are vacuously satisfied.

If everybody prefers A to B then at least half the voters prefer A to B and the society prefers A to B.

If everybody prefers B to A then it is false that at least half the voters prefer A to B so the society prefers B to A.

Thus ($1'$) is satisfied in every case.

Exercise 6.2.4*

Exercise 6.3.1

Suppose you choose B and I choose C. The following table shows the winner for the various combinations of throws.

	3	5	7
1	B	B	B
6	C	C	B
8	C	C	C

We see that the probability that C beats B is $5/9$.

Suppose you choose C and I choose A. The following table shows the winner for the various combinations of throws.

	1	6	8
2	A	C	C
4	A	C	C
9	A	A	A

We see that the probability that A beats C is $5/9$.

Exercise 6.3.2

We have

$$\begin{aligned}\Pr(X > Y) &= \Pr(X = 4) = 1 - p\\ \Pr(Y > Z) &= \Pr(Z = 0) = 1 - p\\ \Pr(Z > X) &= \Pr(Z = 3,\ X = 1) = p^2\end{aligned}$$

Since $1-p$ is a decreasing and p^2 an increasing function of p for $p \geq 0$ it follows that $\min(1-p, p^2)$ attains a maximum for $p \geq 0$ when $1-p = p^2$ ie $p^2 + p - 1 = 0$. Thus

$$p = -1 \pm \sqrt{5}2,\ \ p \geq 0$$

so

$$p = \tau = \sqrt{5} - 12$$

(note that $1 > \tau > 0$) maximises

$$\min\big(\Pr(X > Y),\ \Pr(Y > Z),\ \Pr(Z > X)\big)$$

and with this choice

$$\Pr(X > Y) = \Pr(Y > Z) = \Pr(Z > X) = \tau.$$

You will be able to and will choose a more advantageous lane than the red car. He will then been able to and will choose a more advantageous lane than the you. At every stage one of the drivers will be unhappy and choose a new lane.

Exercise 6.3.3

(i) We have

$$\begin{aligned}\Pr(A \text{ beats } B) &= \Pr(A \text{ throws } 4) = \frac{2}{3}\\ \Pr(B \text{ beats } C) &= \Pr(C \text{ throws } 2) = \frac{2}{3}\end{aligned}$$

and

$$\begin{aligned}\Pr(C \text{ beats } D) &= \Pr(C \text{ throws } 6) + \Pr(C \text{ throws } 2 \text{ and } D \text{ throws } 1)\\ &= \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{2}{3}\\ \Pr(D \text{ beats } A &= \Pr(D \text{ throws } 5) + \Pr(D \text{ throws } 1 \text{ and } A \text{ throws } 0)\\ &= \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

A wins against B if B throws 2, 3 or 4 or if B throws 15, 16 or 17 and A throws 18. Thus the probability of A winning against B is

$$\frac{1}{2} + \frac{1}{2} \times 16 = \frac{21}{36}.$$

B wins against C if B throws 15, 16 or 17 or if B throws 2, 3 or 4 and C throws 1 Thus the provability of A winning against B is

$$\frac{1}{2} + \frac{1}{2} \times 16 = \frac{21}{36}.$$

C wins against A unless A throws 18 or A does not throw 18 but C throws 1, Thus the provability of C winning against A is

$$1 - \frac{1}{6} - \frac{5}{6} \times \frac{1}{6} = \frac{25}{36}.$$

Exercise 6.3.4

C and A prefer a to b. A and B prefer b to c. B and C prefer c to a.

By symmetry we need only consider the case when c is left out of the first vote. Then a will win the first vote and then lose the second vote to c.

Exercise 7.1.1

If your opponent chooses A and B then one of them will beat the other. Suppose A beats B. If you always play A then you will draw when your opponent plays A and win when they play B.

Exercise 7.1.3*

Exercise 7.1.5*

Exercise 7.1.7

(i) Whether the second pass is defended or not, it will always be quicker than the first path, whether defended or not. Since Colonel Schröder will always take the second pass, Lieutenant Lukáš must defend the second pass.

(ii) Suppose Lieutenant Lukáš decides to use a random strategy and defend the first pass with probability q and the second with probability $1-q$. If Colonel Schröder uses the first pass his expected time to destination is

$$f_1(q) = 4(1-q) + 6q = 4 + 2q$$

and if he uses the second pass his expected time to destination is

$$f_2(q) = 3(1-q) + 2q = 3 - q.$$

Assuming that Colonel Schröder can guess the chosen q and makes the correct decision, the expected time for his troops to reach their destination will be

$$f(q) = \min(f_1(q), f_2(q)) = f_2(q) = 3 - q$$

Thus f is a decreasing function of q and Lukáš should take $q = 0$ and always defend the second pass.

Exercise 7.2.1*

Exercise 7.2.3

If Calum take $q = \hat{q}$ and Rowena takes $p = \hat{p}$ then Rowena's expected winnings are $e(\hat{p})$, and Calum's expected losses are $f(\hat{q})$. Since Calum's losses are Rowena's gains, $e(\hat{p}) = f(\hat{q})$.

Exercise 7.2.4

(i) He gets a_{11} regardless of what happens.

(ii) Calum will have a non-unique choice if

$$f(q) = \max(a_{12} + (a_{11} - a_{12})q, a_{22} + (a_{21} - a_{22})q)$$

has a horizontal section on which it takes its minimum value ie if

$$a_{11} - a_{12} = 0 \text{ and } a_{12} < \max(a_{21}, a_{22})$$

or

$$a_{21} - a_{22} = 0 \text{ and } a_{22} < \max(a_{11}, a_{12})$$

or

$$a_{11} = a_{12} = a_{21} = a_{22}.$$

(iii) Consider the following matrix of Rowena's winnings

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

Calum can play any strategy which takes the second column with probability $q \geq 1/2$. Rowena must pick the first row (ie has the unique choice $p = 1$).

Exercise 7.3.1

We have an $n \times m$ matrix $A = (a_{ij})$. If Rowena has chosen row i and Calum has chose column j, then Calum pays the amount a_{ij} to Rowena. The object of Rowena is to maximise the expected sum paid to her and the object of Calum is to minimise that sum.

We shall say that Rowena adopts strategy $\mathbf{p}$ if she chooses row i with probability p_i and that Calum adopts strategy $\mathbf{q}$ if he chooses row j with probability q_j. If Rowena adopts strategy $\mathbf{p}$ and Calum adopts strategy $\mathbf{q}$, then the expected gain for Rowena is

$$\begin{aligned} e(\mathbf{p},\mathbf{q}) =& a_{11}p_1q_1 + a_{12}p_1q_2 + a_{13}p_1q_3 + \dots \\ &+ a_{21}p_2q_1 + a_{22}p_2q_2 + a_{23}p_2q_3 + \dots \\ &+ a_{31}p_3q_1 + a_{22}p_3q_2 + a_{23}p_3q_3 + \dots \\ &+ \dots \end{aligned}$$

which may be written more briefly as

$$e(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{n}\sum_{j=m}^{3} a_{ij}p_iq_j = \sum_{i=1}^{n}\sum_{j=m}^{3} p_ia_{ij}q_j$$

or, still more briefly, in matrix notation

$$e(\mathbf{p},\mathbf{q}) = \mathbf{p}^T A\mathbf{q}.$$

Exercise 7.3.5

Take

$$H = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\max_i \min_j h_{ij} = \max\{2, 0\} = 2 \neq 1 = \min\{3, 1\} = \min_i \max_j h_{ij}.$$

Exercise 7.4.1

	[11]	[12]	[21]	[22]
[11]	0	1	-1	0
[12]	-1	0	0	1
[21]	1	0	0	-1
[22]	0	-1	1	0

The game is symmetric so the expected winnings with best play is 0. If row chooses each play with probability $1/4$ the expected value of the outcome for column is 0 whichever column is chosen. Thus row's play is optimal.

The same argument shows that for the general game each player should choose each outcome $[r, s]$ with equal probability.

Exercise 7.4.2*

Exercise 7.4.4

If row plays each row with probability 1/9 the expected value to row of column playing column $[rs]$ is

$$e_{[rs]} = \frac{1}{9}\sum_{[ij]} a_{[rs][ij]}.$$

Thus

$$\begin{aligned} e_{[11]} &= 3/9 = 1/3,\ e_{[12]} = 0,\ e_{[13]} = -3/9 = -1/3,\\ e_{[21]} &= 3/9 = 1/3,\ e_{[22]} = 0,\ e_{[23]} = -3/9 = -1/3,\\ e_{[31]} &= 3/9 = 1/3,\ e_{[32]} = 0,\ e_{[33]} = -3/9 = -1/3. \end{aligned}$$

Since $e_{[11]} > 0$ row's strategy cannot be right. Column can gain an advantage by always guessing 3.

I would always guess 3. (Does not matter how many fingers I hold out.)

Exercise 7.4.5

Since the game is symmetric the outcome of best play must have expected value 0. If row plays $\hat{\mathbf{p}}$. then expected value of the game to row if column plays row $[rs]$ is $e_{[rs]}$ with

$$\begin{aligned}
e_{[11]} &= \tfrac{5}{12} \times (-2) + \tfrac{1}{3} \times 0 + \tfrac{1}{4} \times 4 = \tfrac{1}{6} > 0\\
e_{[12]} &= \tfrac{5}{12} \times 0 + \tfrac{1}{3} \times (-3) + \tfrac{1}{4} \times 4 = 0\\
e_{[13]} &= \tfrac{5}{12} \times 0 + \tfrac{1}{3} \times 0 + \tfrac{1}{4} \times 0 = 0\\
e_{[21]} &= \tfrac{5}{12} \times (-3) + \tfrac{1}{3} \times 4 + \tfrac{1}{4} \times 0 = \tfrac{1}{12}\\
e_{[22]} &= \tfrac{5}{12} \times 0 + \tfrac{1}{3} \times 0 + \tfrac{1}{4} \times 0 = 0\\
e_{[23]} &= \tfrac{5}{12} \times 0 + \tfrac{1}{3} \times 4 + \tfrac{1}{4} \times (-5) = \tfrac{1}{16}\\
e_{[31]} &= \tfrac{5}{12} \times 0 + \tfrac{1}{3} \times 0 + \tfrac{1}{4} \times 0 = 0\\
e_{[32]} &= \tfrac{5}{12} \times 4 + \tfrac{1}{3} \times (-5) + \tfrac{1}{4} \times 0 = 0\\
e_{[33]} &= \tfrac{5}{12} \times 4 + \tfrac{1}{3} \times 0 + \tfrac{1}{4} \times (-6) = \tfrac{1}{6} > 0
\end{aligned}$$

(ii) Follows from the table.

(iii) No. For example, if we know they are always going to play $[2, 2]$ we can always play $[3, 2]$.

Exercise 7.4.6

Rowena will always do better by choosing R_1 rather than R_4. Our system thus reduces to

	C_1	C_2	C_3
R_1	4	−2	−5
R_2	−2	4	3
R_3	−3	6	2

Now Calum can always do better by choosing C_3 rather than C_2. Our system reduces to

	C_1	C_3
R_1	4	−5
R_2	−2	3
R_3	−3	2

Now Rowena can always do better by choosing R_2 rather than R_3. Our system reduces to

	C_1	C_3
R_1	4	−5
R_2	−2	3

Suppose Rowena chooses R_1 with probability p and R_2 with probability $1-p$. If Calum chooses C_1, Rowena's expected winnings are

$$e_1(p) = 4p - 2(1-p) = 6p - 2.$$

If Calum chooses C_3, Rowena's expected winnings are

$$e_2(p) = -5p + 3(1-p) = 3 - 8p.$$

Rowena chooses p to maximise

$$\min(6p-2, 3-8p) = \begin{cases} 6p-2 & \text{if } p \leq 5/14 \\ 3-8p & \text{if } p \geq 5/14. \end{cases}$$

Thus Rowena chooses R_1 with probability $5/14$, R_2 with probability $9/14$ and never chooses R_3 or R_4.

Suppose Calum chooses C_1 with probability q and C_3 with probability $1-q$. If Rowena chooses R_1, Rowena's expected winnings are

$$f_1(q) = 4q - 5(1-q) = 9q - 5.$$

If Rowena chooses R_2, Rowena's expected winnings are

$$f_2(q) = -2q + 3(1-q) = 3 - 5q.$$

Calum chooses q to minimise

$$\max(9q-5, 3-5q) = \begin{cases} 3-5q & \text{if } q \leq 4/7 \\ 9q-5 & \text{if } q \geq 4/7. \end{cases}$$

Thus Calum chooses C_1 with probability $4/7$, C_3 with probability $3/7$ and never chooses R_2.

Observe that Rowena's expected winnings E_j if she plays as advised and Calum chooses C_j are

$$\begin{aligned} E_1 &= 4 \times \tfrac{5}{14} - 2 \times \tfrac{9}{14} = \tfrac{1}{7} \\ E_2 &= (-2) \times \tfrac{5}{14} + 4 \times \tfrac{9}{14} = \tfrac{13}{7} > \tfrac{1}{7} \\ E_1 &= (-5) \times \tfrac{5}{14} + 3 \times \tfrac{9}{14} = \tfrac{1}{7} \end{aligned}$$

and Rowena's expected winnings F_k if Calum plays as advised and she chooses R_k are

$$\begin{aligned} F_1 &= 4 \times \tfrac{4}{7} - 5 \times \tfrac{3}{7} = \tfrac{1}{7} \\ F_2 &= (-2) \times \tfrac{4}{7} + 3 \times \tfrac{3}{7} = \tfrac{1}{7} \\ F_3 &= (-3) \times \tfrac{4}{7} + 2 \times \tfrac{3}{7} = -\tfrac{6}{7} < \tfrac{1}{7} \\ F_4 &= 3 \times \tfrac{4}{7} - 6 \times \tfrac{3}{7} = -\tfrac{6}{7} < \tfrac{1}{7}. \end{aligned}$$

Thus our strategies are indeed optimal.

Exercise 7.4.7

Bingham should always bet if the card is black since he is always better off betting whatever you do. Thus $b = 1$.

If you always fold, then Bingham's expected winnings are

$$e_F = \tfrac{1}{2} + \tfrac{1}{2}\big(p - (1 \times (1-p))\big) = p.$$

If you always call, then Bingham's expected winnings are

$$e_C = 2 \times \tfrac{1}{2} - 2p \times \tfrac{1}{2} - (1-p) \times \tfrac{1}{2} = \tfrac{1}{2}(1-p).$$

Thus if you know what Bingham is going to do, his expected winnings are

$$\min\{p, \tfrac{1}{2}(1-p)\} = \begin{cases} p & \text{if } p \leq \frac{1}{3} \\ \frac{1}{2}(1-p) & \text{if } p \geq \frac{1}{5} \end{cases}$$

so Bingham should take $b = 1$, $p = \frac{1}{3}$ and his expected winnings are $\frac{1}{3}$.

Now consider your play. If Bingham always passes on red (but always bets on black) his expected winnings are

$$f_P = -\tfrac{1}{2} + \tfrac{1}{2}(2q + (1-q)) = \frac{q}{2}.$$

If Bingham always bets, his expected winnings are

$$f_B = 2q \times (\tfrac{1}{2} - \tfrac{1}{2}) + (1-q) = 1-q.$$

Thus if Bingham knows what you are going to do his expected winnings are

$$\max\{\tfrac{1}{2}q, 1-q\} = \begin{cases} 1-q & \text{if } q \leq \frac{2}{3} \\ \frac{1}{2}q & \text{if } q \geq \frac{2}{3} \end{cases}$$

so you should take $q = \frac{2}{3}$ and (as already found earlier) his expected winnings are $\frac{1}{3}$.

Exercise 7.4.8

(i) If $n = 1$, then $a_1 = b_1$. If $n = 2$ $a_1 = b_2$, $a_2 = b_1$. We always have a draw.

(ii) Suppose A always takes $a_j = 1$ or $a_j = 0$. Let X be the set of j where $a_j = 1$, Y the set of j where $b_j = 1$ and Z the set of j where $b_j \geq 2$. Then A wins on battlefields with $j \in X \setminus (Y \cup Z)$ and B wins on battlefields with $j \in Z \cup (Y \setminus X)$. Otherwise we have draws. Writing $|W|$ for the number of elements of W, we have

$$\begin{aligned}
&|X \setminus (Y \cup Z)| - |Z \cup (Y \setminus X)| \\
&\qquad = \big(|X| - |X \cap Y| - |X \cap Z|\big) - \big(|Z| + |Y| - |X \cap Y|\big) \\
&\qquad = \big(|X| - |Z| - |Y|\big) - |X \cap Z| \geq |X| - 2|Z| - |Y| \\
&\qquad = n - 2|Z| - |Y| \geq 0
\end{aligned}$$

so A has a draw or better.

(iii) Without loss of generality we suppose that A has chosen $a_1 \geq a_2 \geq \cdots \geq a_n$. Since $m > n$ we have $a_1 \geq 2$. If B chooses $b_1 = a_1 - 2$, $b_2 = a_2 + 1$, $b_3 = a_3 + 1$, $b_j = a_j$ for $j \geq 4$, she will win.

Exercise 7.6.1

(i) If $a > b$ then

$$h(b) = f(b) + g(b) > f(a) + g(b) > f(a) + g(a) = h(a).$$

Thus h is strictly increasing. Since the sum of continuous functions is continuous, h is continuous.

(ii) $h(0) = f(0) + g(0) = 1 + 1 = 2$ and $h(1) = 0 + 0 = 0$.

(iii) The intermediate value theorem tells us that (i) and (ii) imply the existence of an x_0 with $0 < x_0 < 1$ with $h(x_0) = 1$. Since h is strictly increasing the solution is unique.

Exercise 7.6.2

(i) Suppose $y > x_0$. If $z \geq y$

$$\Pr(\text{George paintballed}) = f(z) < f(x_0).$$

If $z < y$

$$\Pr(\text{George paintballed}) = 1 - g(y) > 1 - g(x_0) = f(x_0).$$

Thus Fred should choose $z < y$ and, with this choice,

$$\Pr(\text{George paintballed}) = f(y) > f(x_0).$$

(ii) Suppose $y < x_0$ so $f(y) + g(y) > 1$. If $z > y$

$$\Pr(\text{George paintballed}) = f(z) < f(y).$$

If $z < y$

$$\Pr(\text{George paintballed}) = 1 - g(y) < f(y).$$

If $z = y$

$$\Pr(\text{George paintballed}) = f(y).$$

Thus Fred should choose $z = y$ and, with this choice,

$$\Pr(\text{George paintballed} = f(y) > f(x_0).$$

(iii) Suppose $y = x_0$. If If $z > y$

$$\Pr(\text{George paintballed}) = f(z) < f(x_0).$$

If $z \leq y$

$$\Pr(\text{George paintballed}) = 1 - g(x_0) = f(x_0).$$

Thus Fred should choose $z \leq x_0$ and, with this choice,

$$\Pr(\text{George paintballed} = f(x_0).$$

(iv) Thus Fred should fire at x_0 and, by the same argument with roles reversed George should fire at x_0.

EXERCISE 7.6.3

If they do not fire simultaneously, either the one who did not fire is paintballed and the firer is not paintballed or the one who fires misses and will certainly be paintballed. The number of paintballed players is 1.

The probability both paintballed is pq, probability neither paintballed is $(1-p)(1-q)$. The expected number of paintballed participants is $p + q = 1$.

If they fire at $x > x_0$ the expected number of paintballed participants is $f(x) + g(x) < 1$.

If they fire at $x < x_0$ the expected number of paintballed participants is $f(x) + g(x) > 1$.

Exercise 7.6.4

It is what one would expected if customers go to the nearest ice-cream seller and customers are evenly distributed.

If the first seller is at $x < 1/2$ then the if the second goes to $y = \frac{1}{2}(x + \frac{1}{2})$ then, since $y > x$, the first seller will have

$$f\bigl(x, \tfrac{1}{2}(x + \tfrac{1}{2})\bigr) = \tfrac{3}{4}x + \tfrac{1}{4} < 1/2.$$

By symmetry, or by repeating the argument, if the first seller is at $x > 1/2$ then there is a y with

$$f(x, y) < 1/2$$

However if $x = 1/2$ then, if $y > 1/2$,

$$f(x, y) = (\tfrac{1}{2} + y)/2 > 1/2$$

and by symmetry, or by repeating the argument, if $y < 1/2$,

$$f(x, y) = (\tfrac{1}{2} + y)/2 > 1/2.$$

Finally

$$f(\tfrac{1}{2}, \tfrac{1}{2}) = 1/2.$$

Thus the first ice-cream seller will choose $x = 1/2$ and by symmetry the second will also chose $y = 1/2$.

Presumably (but this is not really a mathematical question) they will move towards each other and when they have met they will jostle for the more central position slowly moving to the centre.

Exercise 7.6.5

Suppose that the other bus company chooses to start at a minutes past the hour. Then the expected proportion of passengers carried by the first company is

$$\begin{aligned}\frac{1}{60}\sum_{r=0}^{59} f(r,a) &= \frac{1}{60}\sum_{j=0}^{59} f(j,0)\\ &= \frac{1}{60}\left(\frac{1}{2}+\sum_{j=1}^{59}\frac{j}{60}\right)\\ &= \frac{1}{60}\left(\frac{1}{2}+\frac{59}{2}\right)=\frac{1}{2}.\end{aligned}$$

By symmetry, if each company employs its best tactics they must get equal expected return. Since the total expected return (as a proportion of passengers) is 1, if each company employs its best tactics they must get return $1/2$.

If the first company knows the departure time of the second and decides to leave a minutes before, then it will get $(60-a)/60$ of the total carried for $1 \leq a \leq 59$ and $1/2$ if $a=0$. Thus it will leave one minute earlier than the other company. Thus each week the company can move will go 1 minute before the other. (So after the second week the changing company will move its departure time 2 minutes earlier.) The author believes that has actually seen this happen with the dates of college research fellowship interviews.

Exercise 7.6.6

The travellers should finish together (otherwise the one who arrives later could spend a little longer on the horse and arrive sooner yet still not beat the first traveller). Thus the travellers must spend the same time on the horse and the horse must be left at the halfway point. It takes the first rider $x/(2v)$ to get to the halfway point riding and $x/(2u)$ to finish walking. The time taken is

$$\left(\frac{1}{2u}+\frac{1}{2v}\right)x.$$

If they do something more complicated then they must still finish together. If they met earlier then they meet for the first time and the argument above shows that their best average speed up to this point is $\big((2u)^{-1}+(2v)^{-1}\big)^{-1}$. The same holds for the average speed between the kth and $k+1$st meeting so they can not improve their average speed over the course so they can not arrive earlier.

The same arguments show that with the clever horse they can not do better than the first rider riding a certain distance the leaving the horse to walk back toward the second who then mounts and proceeds to the finishing line crossing at the same time. Each must spend the same time in the saddle so the first leaves the horse at $\frac{1}{2}(1+\alpha)x$. The noble and sagacious steed will then walk back meeting the walker at $\frac{1}{2}(1-\alpha)x$.

The time taken by the first traveller is

$$\tfrac{1}{2}(1+\alpha)\frac{x}{v}+\tfrac{1}{2}(1-\alpha)\frac{x}{u}$$

The time taken by the horse is

$$2\times(1+\alpha)\frac{x}{v}+\alpha xw.$$

These two times must be equal so

$$\frac{1+\alpha}{v}+\frac{1-\alpha}{u}=2\frac{1+\alpha}{v}+2\times\alpha w.$$

whence

$$-\frac{1+\alpha}{v}+\frac{1-\alpha}{u}=2\times\alpha w$$

and

$$\alpha=\frac{v^{-1}-u^{-1}}{u^{-1}+v^{-1}+2w^{-1}}.$$

Thus the total time taken is

$$
\begin{aligned}
&\frac{x}{2}\left((v^{-1}+u^{-1})+\frac{u^{-1}-v^{-1}}{u^{-1}+v^{-1}+2w^{-1}}(v^{-1}-u^{-1})\right)\\
&\quad=\left(v^{-1}+u^{-1}-\frac{(u^{-1}-v^{-1})^2}{u^{-1}+v^{-1}+2w^{-1}}\right)\\
&\quad=\frac{x}{1+\frac{u}{v}\frac{v+w}{u+w}}.
\end{aligned}
$$

Exercise 8.1.2

Since t, $1-t$, p_j, $q_j \geq 0$ we have $tp_j \geq 0$, $(1-t)q_j \geq 0$ and

$$tp_j + (1-t)q_j \geq 0.$$

Further

$$\sum_{j=0}^{n}(tp_j + (1-t)q_j) = t\sum_{j=0}^{n} p_j + (1-t)\sum_{j=0}^{n} q_j = t + (1-t) = 1.$$

Thus

$$\mathbf{u}, \mathbf{v} \in \tilde{K},\ 1 \geq t \geq 0 \Rightarrow t\mathbf{u} + (1-t)\mathbf{v} \in \tilde{K}$$

and $\tilde{K}$ is convex.

Exercise 8.1.3

Write

$$K = \{(x, y) \in \mathbb{R}^2 \,:\, f(x) \geq y\}.$$

Then

$$\begin{aligned}
(x_1, y_1),&(x_2, y_2) \in K,\ 1 \geq t \geq 0 \\
&\Rightarrow f(x_1) \geq y_1,\ f(x_2) \geq y_2,\ 1 \geq t \geq 0 \\
&\Rightarrow f\big(tx_1 + (1-t)x_2\big) \geq tf(x_1) + (1-t)f(x_2) \\
&\qquad\qquad \geq ty_1 + (1-t)y_2 \\
&\Rightarrow t(x_1, y_1) + (1-t)(x_2, y_2) \\
&\qquad\qquad = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in K
\end{aligned}$$

so K is convex.

Exercise 8.1.8*

Exercise 8.1.10

(i) If (ax^*, by^*) is a best choice for $K_{a,b}$ when we use the status quo point (ax_0, by_0), then $(x^*, y^*) = (a^{-1}ax^*, b^{-1}by^*)$ is a best choice for $K = (K_{a,b})_{a^{-1},b^{-1}}$ with status quo point $(x_0, y_0) = (a^{-1}ax_0, b^{-1}by_0)$.

(ii) We have

$$
\begin{aligned}
&(u_1, v_1),\ (u_2, v_2) \in L_{a,b},\ 1 \geq t \geq 0\\
&\qquad \Rightarrow (a^{-1}u_1, b^{-1}v_1),\ (a^{-1}u_2, b^{-1}v_2) \in L,\ 1 \geq t \geq 0\\
&\qquad \Rightarrow (ta^{-1}u_1 + (1-t)a^{-1}u_2, tb^{-1}v_1 + (1-t)b^{-1}v_2) \in L\\
&\qquad \Rightarrow \big(a(ta^{-1}u_1 + (1-t)a^{-1}u_2, b(tb^{-1}v_1 + (1-t)b^{-1}v_2) \in L_{a,b}\\
&\qquad \Rightarrow t(u_1, v_1) + (1-t)(u_2, v_2) \in L_{a,b}.
\end{aligned}
$$

Exercise 8.1.12

Observe that

$$t(u + x_1, v + y_1) + (1 - t)(u + x_2, v + y_2) \\ = \big(u + (tx_1 + (1 - t)x_2), v + (ty_1 + (1 - t)y_2)\big).$$

Exercise 8.1.14

We carry out the argument with n people.

Rule 1 (Pareto optimality) Let K be a set of options in $\mathbb{R}^n$. If $\mathbf{x}$ and $\mathbf{y}$are distinct points of K, $x_j \geq y_j$ for all j and $x_k > y_k$ for some k, then $\mathbf{x}$ is preferred to $\mathbf{y}$.

Lemma A Consider a set of options

$$K = \left\{ \mathbf{x} : \sum_{j=1}^{n} x_j \leq 1 \right\}.$$

The set of best choices under Rule 1 is

$$E = \left\{ \mathbf{x} : \sum_{j=1}^{n} x_j \leq 1 \right\}.$$

Proof. If $\sum_{j=1}^{n} x_j < 1$, set $y_1 = x_1 + 1 - \sum_{j=1}^{n} x_j$ and $y_j = x_j$ for $j \geq 2$. Then $\mathbf{y}$ is preferred to $\mathbf{x}$. ■

Rule 2 (Fairness) Let K be a set of options which is symmetric between Alice and Bob in the sense that $\mathbf{x} \in K$, $y_j = x_{\sigma j}$ for all j and some bijection

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

If the status quo point $\mathbf{z}$ has the form

$$\mathbf{z} = (z, z, \ldots z)$$

any best choice must be of the form

$$\mathbf{z} = (w, w, \ldots w).$$

Lemma B Consider a set of options

$$K = \left\{ \mathbf{x} : \sum_{j=1}^{n} \leq 1 \right\}$$

with status quo point $\mathbf{0}$. Under Rule 1 and Rule 2 there is a unique best choice $(1/n, 1/n, \ldots, 1/n)$.

Proof. Since K and $\mathbf{0}$ are symmetric we can apply Rule 2. Observe that the only symmetric point in the set E of Lemma A is

$$\mathbf{x}^* = (1/n, 1/n, \ldots, 1/n).$$

■

Rule 3 (Indifference to rejected alternatives) Let K and K' be a sets of options with $\mathbf{z} \in K' \subseteq K$. Suppose that $\mathbf{x}^* \in K$ is a best choice for K with status quo point $\mathbf{z}$. Then, if $\mathbf{x}^* \in K'$, it will be a best choice for K' with status quo point $\mathbf{z}$.

Lemma C Consider a set of options

$$K \subseteq \left\{ \mathbf{x} : \sum_{j=1}^{n} \leq 1 \right\}$$

with status quo point $\mathbf{0}$. If $(1/n, 1/n, \ldots, 1/n) \in K$, then under Rules 1 to 3 it is the unique best choice.

Lemma D Suppose that K is a convex set such that

$$\mathbf{u} = (1/n, 1/n, \ldots, 1/n) \in K$$

and $\prod_{j=1}^{n} x_j \leq n^{-n}$ for all $\mathbf{x} \in K$ with $x_j \geq 0$ for all j. Then

$$K \subseteq \left\{ \mathbf{x} : \sum_{j=1}^{n} \leq 1 \right\}.$$

Proof. Suppose $\mathbf{x} \in K$ and $x_j \geq 0$ for all j. Then

$$\mathbf{u} + t(\mathbf{x} - \mathbf{u}) = ((1-t)\mathbf{u} + \mathbf{x} \in K$$

for all $1 \geq t \geq 0$ and so

$$n^{-n} \geq \prod_{j=1}^{n} \left(n^{-1} + t(x_j - n^{-1}) \right)$$

and so

$$t \sum_{j=1}^{n} (x_j - n^{-1}) + t^2 P_{\mathbf{x}}(t) \leq 0$$

for some polynomial $P_{\mathbf{x}}(t)$. Thus, dividing by t,

$$\sum_{j=1}^{n} (x_j - n^{-1}) + t P_{\mathbf{x}}(t) \leq 0$$

for all $1 \geq t > 0$. Allowing $t \to 0+$, we obtain

$$\sum_{j=1}^{n} (x_j - n^{-1}) \leq 0$$

ie

$$\sum_{j=1}^{n} x_j \leq 1$$

as stated. ∎

Lemmas C and D combine to give us Lemma E.

Lemma E Suppose that K is a convex set such that

$$\mathbf{u} = (1/n, 1/n, \ldots, 1/n) \in K$$

and $\prod_{j=1}^{n} x_j \leq n^{-n}$ for all $\mathbf{x} \in K$ with $x_j \geq 0$ for all j. If the status quo point is $\mathbf{0}$, then, under Rules 1 to 4, $\mathbf{u}$ is the unique best choice.

Rules 4 and 5 (Scale and translation invariance) If $a_j > 0$ and $\mathbf{b} \in \mathbb{R}^n$ define $T_{\mathbf{a},\mathbf{b}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_{\mathbf{a},\mathbf{b}}(\mathbf{x}) = \mathbf{y}$$

with

$$y_j = a_j x_j + b_j.$$

We demand that, if $\mathbf{x}^*$ is a best choice for some set K with status quo point $\mathbf{u}$, then $T_{\mathbf{a},\mathbf{b}}\mathbf{x}^*$ is a best choice for $T_{\mathbf{a},\mathbf{b}}K$ with status quo point $T_{\mathbf{a},\mathbf{b}}\mathbf{u}$.

Theorem Consider a convex set of options K with status quo point $\mathbf{u}$. If $\mathbf{x}^* \in K$, $x_j^* > u_j$ for all j. and

$$\prod_{j=1}^{n}(x_j - u_j) \leq \prod_{j=1}^{n}(x_j^* - u_j)$$

for all $\mathbf{x} \in K$ with $x_j > u_j$ for all j, then, under Rules 1 to 5, (x^*, y^*) is the unique best choice.

Proof. Use Lemma E together with rules 4 and 5 applied with $\mathbf{b} = -\mathbf{u}$, $a_j = n^{-1}(x_j - u_j)^{-1}$. ■

Exercise 8.2.1*

Exercise 8.2.2

(i) Since the status point is $(0,0)$ the only points (x,y) to consider must have $y \geq 0$ so $y = 0$ so be $(x,0)$. If $0 \leq x < 1$ then $(1,0)$ is preferred to $(x,0)$. There is no point preferred to $(1,0)$ which is therefore a unique best choice.

(ii) Suppose (a) and (b) false. Then we can find $(x,0) \in K$ with $x > 0$ and $(0,y) \in K$ with $y > 0$. By convexity

$$(x/2, y/2) = \tfrac{1}{2}(x,0) + \tfrac{1}{2}(0,y) \in K$$

yet $x/2, y/2 > 0$ contradicting the condition on K.

Part (i) shows that (b) may be false but (a) true. Interchanging x and y this shows that (a) may be false but (b) true.

If we take

$$(a_0, b_0) = (0,0),\ (a_1, b_1) = (0,-1),$$
$$(a_2, b_2) = (-1,0) \text{ and } (a_3, b_3) = (-1,-1)$$

and status quo point $(0,0)$, then the only point (x,y) in $\tilde{K}$ with $x \geq 0$, $y \geq 0$ is $(0,0)$ so both (a) and (b) are true.

(iii) By translation we may suppose $(x_0, y_0) = (0,0)$ Take $K = \tilde{K}$ in part (ii). If (a) and (b) are true then the status point is the unique best point. If (b) is false then by the argument of (i) the unique best point is $(x^*, 0)$ with x^* the largest x with $(0,x) \in K$.

Exercise 8.2.3

(i) Observe that

$$a^2 + b^2 - 2ab = (a - b)^2 \geq 0.$$

(ii) Write

$$\|\mathbf{x}\| = (x^2 + y^2)^{1/2}.$$

Using the triangle inequality (see Exercise 3.5.9 (iv)), we see that, if $1 \geq t \geq 0$,

$$\begin{aligned}
&\mathbf{u}, \mathbf{v} \in D \Rightarrow \|\mathbf{u}\|, \|\mathbf{v}\| < 1\\
&\Rightarrow \|t\mathbf{u} + (1-t)\mathbf{v}\| \leq \|t\mathbf{u}\| + \|(1-t)\mathbf{v}\| = t\|\mathbf{u}\| + (1-t)\|\mathbf{v}\| < 1\\
&\Rightarrow t\mathbf{u} + (1-t)\mathbf{v} \in D
\end{aligned}$$

Thus D is convex.

In a similar manner

$$\begin{aligned}
&\mathbf{u}, \mathbf{v} \in \bar{D} \Rightarrow \|\mathbf{u}\|, \|\mathbf{v}\| \leq 1\\
&\Rightarrow \|t\mathbf{u} + (1-t)\mathbf{v}\| \leq \|t\mathbf{u}\| + \|(1-t)\mathbf{v}\| = t\|\mathbf{u}\| + (1-t)\|\mathbf{v}\| \leq 1\\
&\Rightarrow t\mathbf{u} + (1-t)\mathbf{v} \in \bar{D}
\end{aligned}$$

Thus $\bar{D}$ is convex.

(iii) If $0 < k < 1$ then $(k^{1/2}, k^{1/2}) \in H_k \cap D$ so $H_k \cap D \neq \varnothing$.

However if $k \geq 1$

$$(x, y) \in H_k \Rightarrow xy \geq 1 \Rightarrow x^2 = y^2 \geq 2 \Rightarrow (x, y) \notin D.$$

so $H_k \cap D = \varnothing$.

(iv) Set $x^* = y^* = 1$. Then $(x^*, y^*) \in \bar{D}$ and

$$x^* y^* = 1 \geq \frac{x^2 + y^2}{2} = xy$$

for all $(x, y) \in \bar{D}$ with $x, y \geq 0$.

Exercise 8.2.5

(i) We may take $(x_0, y_0) = (0, 0)$ since
$$\tilde{K} = \{(x - x_0, y - y_0) \,:\, (x, y) \in K\}$$
is also a closed convex set such that

(a) Whenever $(x, y) \in \tilde{K}$ we have $x, y \leq M + |x_0| + |y_0|$,

(b) $(x_1 - x_0, y_1 - y_0) \in \tilde{K}$ and $x_1 - x_0 \geq 0$, $y_1 - y_0 \geq 0$.

(ii) Suppose $(x^*, y^*), (x^{**}, y^{**}) \in K$ distinct and $x^*y^* = x^{**}y^{**} = k$. Then
$$\big(\tfrac{1}{2}(x^* + x^{**}), \tfrac{1}{2}(y^* + y^{**})\big) = \tfrac{1}{2}(x^*, y^*) + \tfrac{1}{2}(x^{**}, y^{**}) \in K$$
but
$$\begin{aligned}\tfrac{1}{2}(x^* + x^{**}) \times \tfrac{1}{2}(y^* + y^{**}) &= \tfrac{1}{4}(x^*y^* + x^{**}y^{**} + x^*y^{**} + x^{**}y^*)\\ &= \frac{k}{4}\left(2 + \frac{x^*}{x^{**}} + \frac{x^{**}}{x^*}\right)\\ &= k + k\left(\left(\frac{x^*}{x^{**}}\right)^{1/2} - \left(\frac{x^{**}}{x^*})\right)^{1/2}\right)^2\\ &> 0\end{aligned}$$
unless
$$\left(\frac{x^*}{x^{**}}\right)^{1/2} - \left(\frac{x^{**}}{x^*}\right)^{1/2} = 0$$
that is to say $x^* = x^{**}$ and so $y^* = y^{**}$. Thus the point (x^*, y^*) is unique if it exists.

(iii) Since $(0, 0) \in K$, $0 \in E$, condition (i) tells us that $e \in E$ implies $e \leq M^2$.

(iv) By the definition of the supremum we can find $k_n \in E$ with $k_n \to k$. Choose $(x_n, y_n) \in K$ with $x_n, y_n \geq 0$ such that $x_n y_n = k_n$.

(v) We know that $0 \leq x_n \leq M$ so we by the theorem of Bolzano–Weierstrass we may find a subsequence $m(j) \to \infty$ and an $x^* \in \mathbb{R}$ such that $x_{m(j)} \to x^*$. We can now find a subsequence $n(j) = m(j(k))$ of the $m(j)$ with $n(j) \to \infty$ and a $y^* \in \mathbb{R}$ such that $y_{n(j)} \to y^*$.

(vi) Now $x_{n(j)} \to x^*$. Since $x_{n(j)} \geq 0$ $x^* \geq 0$. Similarly $y^* \geq 0$. Since K is closed $(x^*, y^*) \in K$. By continuity $x^*y^* = k$ so
$$xy \leq k = x^*y^*$$
whenever $(x, y) \in K$ and x, $y \geq 0$.

Exercise 8.3.1

If we set $X = x + 3$, $Y = y + 1$, we see that we must solve the bargaining problem for $X + Y \leq 13$. By symmetry and Peano we know that $X = Y = 13/2$ so $x = 7/2$ and $y = 11/2$ represent the Nash solution. Since B makes 1 from the new arrangement, A must pay B an extra $9/2$.

Exercise 8.3.2

(i) The expected outcome for each is

$$\begin{aligned} f(q) &= -100q^2 - 5q(1-q) + 5(1-q)q - (1-q)^2 \\ &= -1 + 2q - 101q^2 = -101(q - 101^{-1})^2 + 101^{-2} - 1 \end{aligned}$$

which is maximised by taking $q = 1/101$.

(ii) Since the situation is symmetric between Jules and Jim, the Nash solution will be, so $p_3 = p_4$. If they decide on a choice with $p_j = q_j$ and $q_2 > 0$. Then $p_1 = q_2 + q_4$, $p_2 = q_2$, $p_2 = 0$ will make them happier. Thus for the Nash solution $p_2 = 0$, $p_3 = p_4$.

For both suggested status quo points the only point in the quadrant $\{(x, y) : x, y \geq 0\}$ is $p_3 = p_4 = 1/2$ so this is the Nash solution (with $p_1 = p_2 = 0$).

(iii) Exactly as in (ii) we are looking for a solution $p_3 = p_4 = x$, $p_2 = 0$ $p_1 = 1 - 2x$ with $0 \leq x \leq 1/2$ which maximises

$$x(5 + a) - (1 - 2x) = x(7 + a) - 1.$$

If $a > -7$ we want $x = 1/2$ and $p_3 = p_4 = 1/2$ (with $p_1 = p_2 = 0$). If $a < -7$ we want $x = 0$ and $p_2 = 1$, $p_1 = p_3 = p_4 = 0$ so both always swerve.

Exercise 8.3.3

Since the situation is symmetric between the two prisoners, the Nash solution will be symmetric ie both confess with probability p, one or other confess with probability q, and neither confess with probability $1 - 2q - p$.

The expected sentence for each is then

$$-2p - 3q + 0q - (1 - p - 2q) = -1 - p - q$$

which is minimised by taking $p = q = 0$ and both keeping stum.

Exercise 8.4.5

We need the following result.

Theorem Let

$$\bar{B} = \{(x, y, z \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 \leq 1\}$$
$$\partial B = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1\}.$$

We cannot find a continuous function $F : \bar{B} \to \partial B$ with $F(x, y, z) = (x, y, z)$ for all $(x, y, z) \in \partial B$.

We now prove Theorem 8.4.4.

Suppose that $f : \bar{B} \to \bar{B}$ is a continuous function with no fixed point, that is to say that $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in \bar{B}$.

We can now define a function $F : \bar{B} \to \partial D$ by the recipe 'starting at $f(\mathbf{x})$, draw a line through $\mathbf{x}$ and continue it until it cuts the boundary ∂D at $F(\mathbf{x})$'.

Observe that, if we make a small change in $\mathbf{x}$, there will only be a small change in $f(\mathbf{x})$ (since f is continuous) and so only a small change in $F(\mathbf{x})$. Thus F is continuous. By construction, $F(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial B$ and we have a contradiction with the result of with our first theorem.

Thus no f of the type described can exist and the result follows.

The generalisations are immediate.

Theorem Let

$$\bar{B} = \{(\mathbf{x} \in \mathbb{R}^n \,:\, \|\mathbf{x}\| \leq 1\}$$
$$\partial B = \{(\mathbf{x} \in \mathbb{R}^n \,:\, \|\mathbf{x}\| = 1\}.$$

We cannot find a continuous function $F : \bar{B} \to \partial B$ with $F(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial B$.

Theorem We the notation as before if $f : \bar{B} \to \bar{B}$ is continuous, then there is an $(\mathbf{x}) \in \bar{B}$ such that $f(\mathbf{x}) = (\mathbf{x})$.

Exercise 8.4.7

We go directly to general, the result for 3 being obtained by setting $n = 3$.

Lemma Let
$$\bar{B} = \{(\mathbf{x} \in \mathbb{R}^n \,:\, \|\mathbf{x}\| \leq 1\}.$$
Suppose that E is a set in $\mathbf{R}^n$ such that there exist continuous functions $h_1 : \bar{B} \to E$ and $h_2 : E \to \bar{B}$ with the properties that
$$h_2\big(h_1(\mathbf{x})\big) = \mathbf{x} \text{ for all } \mathbf{x} \in \bar{B}$$
$$h_1\big(h_2(\mathbf{u})\big) = \mathbf{u} \text{ for all } \mathbf{u} \in E.$$
Then, if $h : E \to E$ is continuous, there is an $\mathbf{u}_0 \in E$ such that $h(\mathbf{u}_0) = \mathbf{u}_0$.

The proof runs unaltered.

If we set
$$f(\mathbf{x}) = h_2\Big(f\big(h_1(\mathbf{x})\big)\Big),$$
then f is a continuous function from $\bar{B}$ to $\bar{B}$ and so has a fixed point $\mathbf{x}_0$. Set $\mathbf{u}_0 = h_2(\mathbf{x}_0)$. Then
$$g(\mathbf{u}_0) = g\big(h_2(\mathbf{x}_0)\big) = h_2\left(h_1\left(h_2\Big(g\big(h_1(\mathbf{x}_0)\big)\Big)\right)\right) = h_2\big(f(\mathbf{x}_0)\big) = \mathbf{u}_0.$$

Exercise 8.5.2*

Exercise 8.5.3

Write
$$\mathbb{R}_+ = \{x \in \mathbb{R} \,:\, x \geq 0\}$$
Consider the set
$$E = \left\{(\mathbf{p}, \mathbf{q}, \mathbf{r}) \,:\, \mathbf{p} \in \mathbb{R}^n_+, \mathbf{q} \in \mathbb{R}^m_+, \mathbf{r} \in \mathbb{R}^l_+, \sum_{i=1}^n p_i = \sum_{j=1}^m q_j = \sum_{k=1}^l q_k\right\}.$$
We observe that E is a bounded closed convex subset of $\mathbb{R}^{n+m+l}$.

Let $p_j^{[i]} = 1$ for $i = j$, $p_j^{[i]} = 0$ otherwise. Define
$$u_i(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \max(0, \alpha(\mathbf{p}^{[i]}\mathbf{q}, \mathbf{r}) - \alpha(\mathbf{p}, \mathbf{q}, \mathbf{r})$$
and
$$v_j = (\mathbf{p}, \mathbf{q}, \mathbf{r}) = \max(0, \alpha(\mathbf{p}, \mathbf{q}^{[j]}, \mathbf{r}) - \alpha(\mathbf{p}, \mathbf{q}, \mathbf{r})$$
$$w_k = (\mathbf{p}, \mathbf{q}, \mathbf{r}) = \max(0, \alpha(\mathbf{p}, \mathbf{q}, \mathbf{r}) - \alpha(\mathbf{p}, \mathbf{q}, \mathbf{r}^{[k]})$$
similarly.

We now set $h(\mathbf{p}, \mathbf{q}, \mathbf{r}) = (\mathbf{p}', \mathbf{q}', \mathbf{r}')$ where
$$p_i' = \frac{p_i + u_i(\mathbf{p}, \mathbf{q}, \mathbf{r})}{1 + \sum_{s=1}^n u_s(\mathbf{p}, \mathbf{q}, \mathbf{r})}$$
$$q_j' = \frac{q_j + v_j(\mathbf{p}, \mathbf{q}, \mathbf{r})}{1 + \sum_{t=1}^m v_t(\mathbf{p}, \mathbf{q}, \mathbf{r})}$$
$$r_k' = \frac{r_k + w_k(\mathbf{p}, \mathbf{q}, \mathbf{r})}{1 + \sum_{d=1}^m w_d(\mathbf{p}, \mathbf{q}, \mathbf{r})}.$$

We observe that h maps E to E and h is continuous so h has a fixed point $(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)$.

Suppose, if possible, that
$$\alpha(\mathbf{p}^{[i]}, \mathbf{q}^*, \mathbf{r}^*) > \alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) \text{ whenever } p_i^* > 0.$$
Then
$$p_i^*\alpha(\mathbf{p}^{[i]}, \mathbf{q}^*, \mathbf{r}^*) \geq p_i^*\alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) \text{ for all } i$$
and
$$p_{i_0}^*\alpha(\mathbf{p}^{[i_0]}, \mathbf{q}^*, \mathbf{r}^*) > p_{i_0}^*\alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) \text{ for some } i_0$$
so
$$\alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) = \sum_{i=1}^n p_i^*\alpha(\mathbf{p}^{[i]}, \mathbf{q}^*, \mathbf{r}^*) > \sum_{i=1}^n p_i^*\alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) = \alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)$$
which is absurd. Thus our original assumption must be wrong and there must be some i_1 with
$$\alpha(\mathbf{p}^{[i_1]}, \mathbf{q}^*, \mathbf{r}^*) \leq \alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) \text{ and } p_{i_1}^* > 0.$$

Without loss of generality and to fix ideas, we suppose $i_1 = 1$.

We now know that

$$\alpha(\mathbf{p}^{[1]}, \mathbf{q}^*, \mathbf{r}^*) \leq \alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) \text{ and } p_1^* > 0$$

and so

$$u_1(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) = \max\big(0, \alpha(\mathbf{p}^{[1]}, \mathbf{q}^*, \mathbf{r}^* - \alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)\big) = 0.$$

Thus, by the definition of h,

$$p_1^* = \frac{p_1^* + u_1(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)}{1 + \sum_{i=1}^n u_i(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)} = \frac{p_1^*}{1 + \sum_{i=1}^n u_i(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)}.$$

Since $p_1^* > 0$, it follows that

$$\sum_{i=1}^n u_i(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) = 0$$

and so $u_i(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) = 0$ for all i.

Thus

$$\alpha(\mathbf{p}^{[i]}, \mathbf{q}^*, \mathbf{r}^*) \leq \alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)$$

for all i and so

$$\alpha(\mathbf{p}, \mathbf{q}^*, \mathbf{r}^*) = \sum_{i=1}^n p_i \alpha(\mathbf{p}^{[i]}, \mathbf{q}^*, \mathbf{r}^*) \leq \sum_{i=1}^n p_i \alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) = \alpha(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*)$$

for all possible choices of $\mathbf{p}$.

The same argument shows that

$$\beta(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) \geq \beta(\mathbf{p}^*, \mathbf{q}, \mathbf{r}^*) \text{ for all } \mathbf{q},$$
$$\gamma(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*) \geq \gamma(\mathbf{p}^*, \mathbf{q}^*, \mathbf{r}) \text{ for all } \mathbf{r}.$$

Exercise 8.5.4*

Exercise 8.6.1

Suppose that Albert is driving one car and Bertha the other. If Albert chooses left with probability p and right with probability $1-p$ whilst Bertha chooses left with probability q and right with probability $1-q$, then the expected value of the game to Albert is

$$\alpha(p,q) = apq - p(1-q) - q(1-p) = (a+2)pq - p - q = \big((a+2)q-1\big)p - q.$$

If $a > -1$ then writing $q_0 = 1/(a+2)$

$$\text{if } q > q_0, \text{ then } \alpha(1,q) > \alpha(p,q) \text{ for all } 1 > p \geq 0$$
$$\text{if } q < q_0, \text{ then } \alpha(0,q) > \alpha(p,q) \text{ for all } 1 \geq p > 0.$$

In addition, we observe that $\alpha(q_0,q_0) = \alpha(p,q_0)$ for all p. Since similar results apply for Bertha, we see that there are exactly three Nash equilibrium points $(p,q) = (0,0)$ with expected value to each the players of 0, $(p,q) = (1,1)$ with expected value to each the players of a and $(p,q) = (q_0,q_0)$ with expected value to the two players of $-q_0$.

If $a = -1$ then

$$\alpha(p,q) = (q-1)p - q,$$

and

$$\text{if } q < 1, \text{ then } \alpha(0,q) > \alpha(p,q) \text{ for all } 1 \geq p > 0.$$

and $\alpha(p,1) = -1$ for all p. Since similar results apply for Bertha, we see that there are exactly two Nash equilibrium points $(p,q) = (0,0)$ with expected value to each the players of 0 and $(p,q) = (1,1)$ with expected value -1.

If $a < -1$ then, since

$$\alpha(p,q) = apq - p(1-q) - q(1-p) = (a+2)pq - p - q = \big((a+2)q-1\big)p - q,$$

we have

$$\alpha(0,q) > \alpha(p,q) \text{ for all } 1 \geq p > 0 \text{ and all } q.$$

Since similar results apply for Bertha, we see that there is only one Nash equilibrium point $(p,q) = (0,0)$ with expected value to each the players of 0.

Exercise 8.6.2

Suppose that Albert is driving the row car. He chooses left with probability p and Bertha (driving the column car) chooses left with probability q. The expected value of the game to Bertha is

$$\alpha(p,q) = p(aq-(1-q)) + (1-p)(-q+0) = (q(a+2)-1)p - q.$$

If $a \geq -1$, then

$$\text{if } q > (2+a)^{-1} \text{ then } \beta(1,q) > \beta(p,q) \text{ for all } 1 > p \geq 0$$
$$\text{if } q < (2+a)^{-1} \text{ then } \beta(0,q) > \beta(p,q) \text{ for all } 1 \geq p > 0$$
$$\text{if } q = (2+a)^{-1} \text{ then } \beta(0,q) = \beta(p,q) \text{ for all } 1 \geq p \geq 0$$

Since similar results hold for Bertha, there are three Nash equilibrium points $(p,q) = (0,0)$ with expected value to Albert of a and to Bertha of 0, $(p,q) = (1,1)$ with expected value to Albert of 0 and to Bertha of a and $(p,q) = (1-(2+a))^{-1}, (2+a)^{-1})$ with expected value to both players of $-(2+a)^{-1})$.

If $a < -1$ then

$$\beta(0,q) > \beta(p,q) \text{ for all } 1 \geq p > 0$$

and, since similar results hold for Bertha, there is one Nash equilibrium point $(p,q) = (0,1)$ with expected value to Albert of -1 and to Bertha of -1

Exercise 8.6.3

If the value to Row of the various possibilities is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then if Row plays first row with probability p and Column plays first column with probability q the expected value of the game to Row is

$$\begin{aligned} \rho(p,q) &= apq + bp(1-q) + c(1-p)q + d(1-p)(1-q) \\ &= Apq + Bp + Cq + D = (Aq+B)p + Cq + D. \end{aligned}$$

Similarly the expected value of the game to Column is

$$\kappa(p,q) = (\alpha p + \beta)q + \gamma p + \delta$$

If $A+B, B > 0$ then

$$\rho(1,q) > \rho(p,q) \text{ for all } 1 > p \geq 0$$

and we set $p^* = 1$. If $\alpha + \beta \geq 0$ we set $q^* = 1$, if $\alpha + \beta < 0$ we set $q^* = 0$ By inspection we have Nash equilibrium at (p^*, q^*).

If $A+B, B < 0$ then

$$\rho(1,q) > \rho(p,q) \text{ for all } 1 > p \geq 0$$

and we set $p^* = 0$. If $\beta \geq 0$ we set $q^* = 1$, if $\alpha < 0$ we set $q^* = 0$. By inspection we have Nash equilibrium at (p^*, q^*).

If $A+B$ and B do not have the same sign but $\alpha + \beta$ and α do we interchange to role of row and column.

In the remaining cases $A+B$ and B do not have the same sign and $\alpha + \beta$ and β do not have the same sign then, taking $p^* = -\beta/\alpha$, we see that $\kappa(p^*, q)$ does not depend on q and, taking $q^* = -B/A$, we see that $\rho(p, q^*)$ does not depend on q. Thus (p^*, q^*) is a Nash equilibrium point.

Exercise 8.6.4

(ii) As D becomes large (so the costs of fighting become large) p and q become small so the players become more and more dovelike (but it is still worth an occasional hawk play) and the expected payoff

$$\frac{V(D-V)}{2D} = \frac{V}{2}\left(1 - \frac{V}{D}\right) \to \frac{V}{2}$$

the dove pair payoff.

As $D \to V+$ the rewards of being a hawk against a dove remain high but the cost of being a hawk against a hawk diminishes towards zero. $p, q \to 1$ so the players play dove less and less often and the expected payoff

$$\frac{V(D-V)}{2D} = \frac{V}{2}\left(1 - \frac{V}{D}\right) \to 0$$

the payoff for two hawks when the cost of fighting equals the reward.

(iii) If $V = D$ then, with the notation of the previous paragraphs

$$\alpha(p,q) = \frac{V - Dq}{2}p + \frac{V}{2}(1-q)$$

so if $q \neq 1$

$$\alpha(1,q) > \alpha(p,q) \text{ for } 1 > p \geq 0.$$

The only Nash equilibrium point is $(p,q) = (1,1)$ and the payoff for both players is then 0.

Exercise 8.6.5

(i) Since there are slightly more lefters than righters a righter is more likely to collide (with a lefter) than a lefter (with a righter). The average lefter will collect fewer damage points and outbreed the average righter by a bit. Thus the next generation will contain a higher proportion of lefters and the selective pressures against righters will increase. Eventually the flock will consist of lefters (who will then write books on the obvious merits of lefters as demonstrated by their success against righters).

If here are slightly more righters than lefters the reverse process will take place.

(ii) If the proportion of lefters in the population is p then, roughly speaking, the average breeding points a lefter will collect will be

$$L(p) = A\big(pa - (1-p)\big)$$

and the average number that a righter will collect will be

$$R(p) = -Ap.$$

Now

$$L(p) - R(p) = A(p(a+2) - 1).$$

If $a \leq -1$ so $(a+2) \leq 1$ then $L(p) < R(p)$ for all $0 < p \leq 1$ so lefters will die out.

If $a > -1$ then taking $p^* = (a+2)^{-1}$ (and observing that $L(p) - R(p)$ increases as p increases and decreases as p decreases) we see that if a some stage p is at least a bit larger than p^* the lefters will eventually take over, but if p is at least a bit smaller than p^* the righters will.

Exercise 8.6.6

If we release a few bourgeois into a flock of doves then since bourgeois on average, do better than doves in an encounter with a dove and better than doves, in an encounter with other bourgeois the bourgeois are likely to replace the doves.

If we release a few bourgeois into a flock of hawks then since bourgeois on average do better than hawks in an encounter with a hawks and better than hawks in an encounter with other bourgeois, the bourgeois are likely to replace the hawks.

Exercise 9.1.1

Suppose that the distance to the cemetery is x yards and B drives at speed u yards an hour. If A sets of at time 0, then A arrives after a time $x/(4u)$ and B after a time $(x-100)/u$. Thus

$$\frac{x}{4u} < \frac{x-100}{u}.$$

Simplifying, we get

$$x < 4x - 400$$

so $3x > 400$ and $x > 133\frac{1}{3}$.

Exercise 9.1.2

Let us take the time when the race starts to be 0 The hare overtakes the tortoise half a kilometre from the starting post at time $1/(2v)$ so the hare has travelled a distance

$$\left(\frac{1}{2v} - \frac{1}{2}\right).$$

Thus

$$\left(\frac{1}{2v} - \frac{1}{2}\right) V = \frac{1}{2}$$

and

(1) $$\frac{1}{v} - \frac{1}{V} = 1.$$

At the second meeting the tortoise has travelled $X - \frac{5}{4}$ in a time $T = (X - \frac{5}{4})/v$ and the hare has travelled $X + \frac{5}{4}$ running for $T - 1$ hours. Thus

$$\left(\left(X - \frac{5}{4}\right)\frac{1}{v} - 1\right) V = X + \frac{5}{4}$$

and so

(2) $$\left(X - \frac{5}{4}\right)\frac{1}{v} - \left(X + \frac{5}{4}\right)\frac{1}{V} = 1.$$

Substituting in (2) from (1) we get

$$\left(X - \frac{5}{4}\right)\left(1 - \frac{1}{V}\right) - \left(X + \frac{5}{4}\right)\frac{1}{V} = 1.$$

Simplifying gives

$$\left(X - \frac{5}{4}\right) - \frac{5}{2V} = 1$$

so

$$V = \frac{10}{4X - 9}.$$

We have

$$\frac{1}{v} = 1 + \frac{1}{V} = 1 + \frac{4X - 9}{10} = \frac{4X + 1}{10}$$

and so

$$v = \frac{1}{4X - 9}.$$

The hare runs $1/2$ hours to cover a distance $2X$ so

$$\frac{5}{4X - 9} = \frac{V}{2} = 2X$$

and

$$8X^2 - 18X - 5 = 0.$$

Thus

$$X = \frac{9 \pm \sqrt{9^2 + 40}}{8} = \frac{9 \pm 11}{8}.$$

Since $X > 0$ we have $X = 20/8 = 5/2$.

The tortoise travels at 1 kilometre per hour and will take $5/2$ hours to reach the cabbage patch.

Exercise 9.2.3

(i) [Note this part of the execise has been changed in the corrections] If $b \leq 1/4$, then $c \leq 1/4$ so $3b + c(1-b)^2 < 1$. If $b \geq 1/3$, then $3b + c(1-b)^2 > 1$.

(ii) If C misses on his first shot, then either B hits A and C has first shot in a two sided duel against A or B misses A and C has first shot in a two sided duel against B. In either case C has a probability c of hitting his remaining opponent with his first shot and becoming outright winner.

(iii) If b and c are very small then, with high probability, C misses A (by accident or design), B misses A, A hits B, C misses A and A hits C so winning outright.

If b is very close to 1 and c is very small then, with high probability, C misses A (by accident or design), B hits A, C misses B and B hits C so winning outright.

If b and c are very close to 1 then, with high probability, C misses A (by design), B hits A, and C hits B so winning outright.

The verbal arguments can be verified algebraically but there is no need to do so.

Exercise 9.2.4

(i) Since pq is very small compared with $p+q$ the probability that P wins if she shoots first
$$\frac{p}{p+q-pq} \approx \frac{p}{p+q}.$$
Since q is small compared with 1, the probability that P wins if she shoots second
$$\frac{(1-q)p}{p+q-pq} \approx \frac{p}{p+q}.$$

Since P will probably miss with her first shot it makes it makes little difference whether she shoots first or second.

(ii) Since the order of firing in the two-sided duel now makes little difference the participants simply want to make sure that they are faced with the weakest possible opponent in that duel. Thus everyone will fire at the strongest player who is not themselves (thus A fires at B while B and C fire at A).

Since a, b and c are small A is in effect engaging in a two-sided duel in which he has probability b of hitting and his opponent has probability $b+c$ of hitting him. The probability of A surviving this duel is approximately $a/(a+b+c)$. If he survives, he must now duel C with a probability approximately $a/(a+c)$ of surviving to win. His chances of winning outright is thus about
$$P_A = \frac{a}{a+b+c} \times \frac{a}{a+c} = \frac{a^2}{(a+b+c)(a+c)}.$$

The probability of A not surviving the first duel is $(b+c)/(a+b+c)$. If A does not survive, then B and C duel with B having a probability $b/(b+c)$ of winning. Thus B has approximate probability
$$P_B = \frac{b+c}{a+b+c} \times \frac{b}{b+c} = \frac{b}{a+b+c}.$$
of winning outright.

The probability of C winning outright is approximately
$$\begin{aligned} P_C &= 1 - \Pr(A \text{ wins}) - \Pr(B \text{ wins}) \\ &= 1 - \frac{a^2}{(a+b+c)(a+c)} - \frac{b}{a+b+c} \\ &= \frac{2ac+c^2}{(a+b+c)(a+c)} \end{aligned}$$

(iii) We have

$$1 = \frac{a^2}{(a+0+0)(a+0)} > \frac{a^2}{(a+b+c)(a+c)}$$
$$> \frac{a^2}{(a+a+a)(a+a)} = \frac{1}{6}.$$

If a is fixed, $P_A \to 1$ as $b,\ c \to 0$. If A is much better than B and C then he is likely to win. If A, B and C have approximately equal skill then A must engage in a duel in which he has one shot for every two of his more or less equally opponents so two out of every three paintballs are aimed at him and he has probability about $1/3$ of surviving. If he survives then he faces a duel on more or less equal terms for which his survival probability is about $1/2$ thus his worse case chances are about $1/3 \times 1/2 = 1/6$.

We have

$$\frac{1}{2} > \frac{b}{a+b} > \frac{b}{a+b+c} > 0.$$

If a is fixed, $P_B \to 0$ as $b,\ c \to 0$, $P_B \to 1/2$ as $b \to a,\ c \to 0$. Thus P_B can take all values between $1/2$ and 0. B's best chance is if he his almost as good as A so has chance about $1/2$ of surviving to duel with C. If C is incompetent B's chance of triumph remains $1/2$. If P_A is close to 1, P_B must be close to 0.

We have

$$\frac{1}{2} = \frac{2a^2 + a^2}{(a+2a)(a+a)} > \frac{2ac + c^2}{(a+2c)(a+c)} > \frac{2ac + c^2}{(a+b+c)(a+c)} > 0$$

C will always survive to the second duel where his best chance is to be almost as skilled as his opponent in which his chance of triumph is almost $1/2$ so C's best case is when A, B and C are almost equally skilled. If P_A is close to 1, P_C must be close to 0.

Exercise 9.2.6

(i) The fly is approaching B at speed $c+b$ so it takes time
$$t_A(x)=\frac{x}{c+b}.$$
to reach B. The cyclists are approaching each other at speed $a+b$ so the distance apart of the cyclists when the fly reaches B is
$$D_A=x-\frac{x}{c+b}\times a+b=\frac{c-a}{c+b}x.$$

(ii) We have
$$S_A(x)=d_A(x)+S_B(D_A(x)),\ S_B(x)=d_B(x)+S_A(D_B(x))$$
so
$$S_A(x)=ct_A+S_B\left(\frac{c-a}{c+b}x\right)=\frac{cx}{c+b}+S_B\left(\frac{c-a}{c+b}x\right)$$
and
$$S_B(x)=\frac{cx}{c+a}+S_B\left(\frac{c-b}{c+a}x\right).$$
Thus
$$S_A(x)=\frac{cx}{c+b}+\frac{c(c-a)x}{(c+a)(c+b)}+S_A\left(\frac{(c-a)(c-b)}{(c+b)(c+a)}x\right)$$
Now $S_A(x)=xS_A(1)$ so writing $SA(1)=A_1$
$$\left(1-\frac{(c-a)(c-b)}{(c+b)(c+a)}\right)S_1=\frac{c}{c+b}+\frac{c(c-a)}{(c+a)(c+b)}$$
so
$$\big((c+a)(c+b)-(c-a)(c-b)\big)S_1=c(c+a)+c(c-a)$$
and
$$2(a+b)cS_1=2c^2$$
whence
$$S_1=\frac{c}{a+b},S_A(x)=\frac{cx}{a+b}.$$

(iii) We have
$$\begin{aligned}S_A(x)&=\left(\frac{cx}{c+b}+\frac{c(c-a)x}{(c+a)(c+b)}\right)\sum_{j=0}^{\infty}\left(\frac{(c-a)(c-b)}{(c+b)(c+a)}\right)^j\\&=\frac{2c^2x}{(c+a)(c+b)}\left(1-\frac{(c-a)(c-b)}{(c+b)(c+a)}\right)^{-1}\\&=\frac{cx}{a+b}.\end{aligned}$$

(iv) The cyclists meet after a time $x/(a+b)$ and the fly has flown $cx/(a+b)$.

Exercise 9.2.7

(Corrected by Matthew Towers) If on each throw there is a probability a of you winning and a probability b of the banker winning and $1-a-b$ of no decision your probability p of ultimately winning is given by

$$p = a + (1-a-b)p$$

ie $p = a/(a+b)$ the probability of throwing a 7 or 11 is $(6+2)/36 = 8/36$.

If you roll $k \neq 2, 3, 7, 11, 12$ your chances of winning a particular throw are $a = r(k)/36$ and of the banker winning is $b = 1/6$ are as follows

k	r	win probability
4	3	1/3
5	4	2/5
6	5	5/11
8	5	5/11
9	4	2/5
10	3	1/3

so your win probability for the full game is

$$\begin{aligned}
P(\text{win}) = {} & P(\text{win}|\text{initial roll 7, 11})(8/36) \\
& + P(\text{win}|\text{initial roll 2, 3, 12})(4/36) \\
& + P(\text{win}|\text{initial roll not 2, 3, 7, 11, 12})(24/36) \\
= {} & 8/36 + 0 + (24/36)(2 \cdot (3/24) \cdot (1/3) + 2 \cdot (4/24) \cdot (2/5) + 2 \cdot (5/24) \cdot (5/11)) \\
= {} & 244/495 \approx 0.493
\end{aligned}$$

where $2(3/24)(1/3)$ is for 4 or 10, $2(4/24)(2/5)$ is 5 or 9, and $2(5/24)(5/11)$ is 6 or 8. Matthew adds ‘Wikipedia agrees with me, so I must be right’.

Exercise 9.3.4

$$\begin{aligned} u_n &= \sum_{r=1}^{n} 2^r \Pr(r-1 \text{ tails then heads}) \\ &= \sum_{r=1}^{n} 2r2^{-r} = \sum_{r=1}^{n} 1 = n \end{aligned}$$

Exercise 9.3.5

(i) We have

$$\begin{aligned} v_n &= -\sum_{r=1}^{n} \Pr(r-1 \text{ tails then heads}) - (2^n - 1)\Pr(n \text{ tails}) \\ &= 1 - 2^{-n} - \sum_{r=1}^{n} 2^{-r} = 0 \end{aligned}$$

(ii) We have

$$\begin{aligned} v_n &= -\sum_{r=1}^{n} \Pr(r-1 \text{ tails then heads}) \\ &= 1 - 2^{-n} \end{aligned}$$

Exercise 9.3.6

We take $> p > 0$, $p \neq 1/2$.

First St Petersburg.

$$\begin{aligned}
u_n(p) &= \sum_{r=1}^{n} 2^r \Pr(r-1 \text{ tails then heads}) \\
&= (1-p)\sum_{r=1}^{n} p^{r-1}2^r \\
&= 2(1-p)\frac{1-(2p)^n}{1-2p} \\
&= \frac{2(1-p)}{1-2p}(1-(p/2)^n)
\end{aligned}$$

If $p > 1/2$, $u_n(p) \to \infty$ as $n \to \infty$.

If $p < 1/2$
$$u_n(p) \to 2\frac{1-p}{1-2p}$$
and the game is trouble free.

Next Double or Quits with penalty.

$$\begin{aligned}
v_n &= -\sum_{r=1}^{n} \Pr(r-1 \text{ tails then heads}) \\
&\qquad - (2^n-1)\Pr(n \text{ tails}) \\
&= -\sum_{r=1}^{n}(1-p)p^{r-1} + (2^n-1)p^n \\
&= (p^n-1) + (2^n-1)p^n = (2p)^n - 1
\end{aligned}$$

If $p < 1/2$, $u_n(p) \to -1$ as $n \to \infty$.

If $p > 1/2$, $u_n(p) \to \infty$ as $n \to \infty$.

Finally Double or Quits without penalty.

$$\begin{aligned}
v_n &= -\sum_{r=1}^{n} \Pr(r-1 \text{ tails then heads}) \\
&= -\sum_{r=1}^{n}(1-p)p^{r-1} \\
&= (p^n-1) \to -1
\end{aligned}$$

as $n \to \infty$.

Exercise 9.3.7

Without replacement

$$\Pr(\text{suceed on } r\text{th go}) = \begin{cases} 1/n & \text{for } 1 \le r \le n \\ 0 & \text{otherwise.} \end{cases}$$

so

$$a_n = \sum_{r=1}^{n} r/n = (n+1)/2.$$

Lemma 9.3.3 tells us that $b_n = n$ (since with replacement we have probability $1/n$ of getting in on any go independent of what has gone before).

Finally

$$\begin{aligned}
c_n &= \sum_{j=1}^{\infty} j \Pr(\text{in on } j\text{th go}) \\
&= \frac{1}{n} + \sum_{j=2}^{\infty} j \frac{n-1}{n} \left(\frac{n-2}{n-1}\right)^{j-2} \frac{1}{n-2} \\
&= \frac{1}{n} + \frac{(n-1)^2}{n(n-2)} \sum_{j=2}^{\infty} j \left(\frac{1}{n-1}\right)^{j} \\
&= \frac{1}{n} + \frac{(n-1)^2}{n(n-2)} \left(-\frac{1}{n-1} + \sum_{j=1}^{\infty} j \left(\frac{1}{n-1}\right)^{j}\right) \\
&= \frac{1}{n} + \frac{(n-1)^2}{n(n-2)} \left(-\frac{1}{n-1} + (n-1)\right) \\
&= (n-1) + \frac{1}{n}
\end{aligned}$$

Thus

$$\frac{a_n}{b_n} = \frac{1+n^{-1}}{2} \to \frac{1}{2}$$

and

$$\frac{b_n}{c_n} = \frac{1}{1-n^{-1}+n^{-2}} \to 0$$

as $n \to \infty$.

Exercise 9.3.8

(i) All tails has probability $1/4$. One head and one tail (in some order) has probability $1/2$. At least one head has probability $3/4$.

(ii) Each particular outcome has probability $1/8$, so events can have probability $r/8$ and only those values $[0 \leq r \leq 8]$.

(iii) Chose r particular outcomes. If you get one of those shout heads. If not, tails. You have probability $r/2^n$ of shouting heads. $[0 \leq r \leq 2^n]$. If $r = 0$ or $r = 2^n$ you can dispense with the coins and always shout tails or always shout heads.

Exercise 9.3.10

It is probably easiest to separate existence and uniqueness.

Let $P(n)$ be the statement that if $0 \leq r \leq 2^n - 1$, then we can find $e_j \in \{0, 1\}$ such that

$$r = e_1 2^{n-1} + e_2 2^{n-2} + e_3 2^{n-3} + \cdots + e_{n-1} 2 + e_n.$$

$P(1)$ is true (set $e_1 = r$). Suppose $P(n)$ is true and $0 \leq r \leq 2^{n+1} - 1$. If $0 \leq r \leq 2^n - 1$ then by the inductive hypothesis we can find $d_j \in \{0, 1\}$ such that

$$r = d_1 2^{n-1} + d_2 2^{n-2} + d_3 2^{n-3} + \cdots + d_{n-1} 2 + d_n$$

and so, setting $e_j = d_{j-1}$ for $2 \leq j \leq n+1$, $e_1 = 0$ we have $e_j \in \{0, 1\}$ such that

$$r = e_1 2^n + e_2 2^{n-1} + e_3 2^{n-2} + \cdots + e_n 2 + e_{n+1}.$$

If not, then $2^n \leq r \leq 2^{n+1} - 1$ and setting $s = r - 2^n$ we have

$$0 \leq s \leq 2^n - 1$$

and by the inductive hypothesis we can find $d_j \in \{0, 1\}$ such that

$$r = d_1 2^{n-1} + d_2 2^{n-2} + d_3 2^{n-3} + \cdots + d_{n-1} 2 + d_n.$$

Setting $e_j = d_{j-1}$ for $2 \leq j \leq n+1$, $e_1 = 1$ we have $e_j \in \{0, 1\}$ such that

$$r = e_1 2^n + e_2 2^{n-1} + e_3 2^{n-2} + \cdots + e_n 2 + e_{n+1}.$$

By induction $P(n)$ holds for all n.

Let $Q(n)$ be the statement that if $0 \leq r \leq 2^n - 1$, the expansion

$$r = e_1 2^{n-1} + e_2 2^{n-2} + e_3 2^{n-3} + \cdots + e_{n-1} 2 + e_n$$

is unique. $Q(1)$ is true by inspection of cases.

Suppose $Q(n)$ is true and

$$\begin{aligned} e_1 2^n + e_2 2^{n-1} + e_3 2^{n-3} + \cdots + e_n 2 + e_{n+1} \\ = d_1 2^n + d_2 2^{n-1} + d_3 2^{n-3} + \cdots + d_n 2 + d_{n+1} \end{aligned}$$

with $e_j, d_j \in \{0, 1\}$. If $e_1 = 1$, $d_1 = 0$, then

$$e_1 2^n + e_2 2^{n-1} + e_3 2^{n-2} + \cdots + e_n 2 + e_{n+1} \geq 2^n$$

and

$$\begin{aligned} d_1 2^n + d_2 2^{n-1} + d_3 2^{n-3} + \cdots + d_n 2 + d_{n+1} \\ \leq 2^{n-1} + 2^{n-3} + \cdots + 2 + 1 = 2^n - 1 < 2^n \end{aligned}$$

which is impossible. Similarly we can not have $e_1 = 0$, $d_1 = 1$ so $d_1 = e_1$ and

$$e_2 2^{n-1} + e_3 2^{n-2} + \cdots + e_n 2 + e_{n+1} = d_2 2^{n-1} + d_3 2^{n-2} + \cdots + d_n 2 + d_{n+1}$$

whence (since $\sum_{j=2}^{n+1} e_j 2^{n+1-j} \leq 2^n - 1$) by the inductive hypothesis $e_j = d_j$ for all $2 \leq j \leq n+1$. Thus $Q(n+1)$ is true.

By induction $Q(n)$ holds for all n.

Exercise 9.3.14*

Exercise 9.3.15

(i) If
$$p < X_1 2^{-1} + X_2 2^{-2} \cdots + X_n 2^{-n}$$
then the process was halted in round n or earlier with left being recorded.

If
$$p \geq X_1 2^{-1} + X_2 2^{-2} \cdots + X_n 2^{-n} + 2^{-n}$$
then the process was halted in round n or earlier with left being recorded.

(ii) If
$$X_1 2^{-1} + X_2 2^{-2} \cdots + X_n 2^{-n} \leq p < X_1 2^{-1} + X_2 2^{-2} + \cdots + X_n 2^{-n} + 2^{-n-1}$$
then B is true and if
$$X_1 2^{-1} + X_2 2^{-2} \cdots + X_n 2^{-n} + 2^{-n-1} \leq p < X_1 2^{-1} + X_2 2^{-2} + \cdots + X_n 2^{-n} + 2^{-n}$$
then (A) is true. Clearly (A) and (B) cannot both be true.

(iii) If (A) is true the probability of a decision in the $n+1$st round is the probability that $X_{n+1} = 0$ ie $1/2$. If (B) is true the probability of a decision in the $n+1$st round is the probability that $X_{n+1} = 1$ ie $1/2$. Since exactly one of (A) and (B) is true the probability of a decision in the $n+1$st round is $1/2$.

Exercise 9.3.18

Let $X_k = 1$ if the kth purchase contains the jth poet and $X_k = 0$ otherwise. Set $M = (1+\epsilon)Nn$ and

$$S_M = \sum_{k=1}^{M} X_k.$$

Then

$$\mathbb{E}S_M = \sum_{k=1}^{M} \mathbb{E}X_k = Mn^{-1} = (1+\epsilon)N$$

and since the X_j are independent

$$\operatorname{var} S_M = \sum_{k=1}^{M} \operatorname{var} X_k = \sum_{k=1}^{M} \frac{n-1}{n^2} \leq \sum_{k=1}^{M} \frac{1}{n} = (1+\epsilon)N.$$

Thus

$$\begin{aligned}
&\Pr(\text{they obtain fewer than } N \text{ busts of the } j\text{th poet})\\
&\qquad = \Pr(S_M < N)\\
&\qquad \leq \Pr(|S_M - \mathbb{E}S_M| \geq \epsilon N)\\
&\qquad \leq \frac{\operatorname{var} S_M}{(\epsilon N)^2}\\
&\qquad \leq \frac{(1+\epsilon)N}{(\epsilon N)^2} = (1+\epsilon)\epsilon^{-1}N^{-1}
\end{aligned}$$

if $N \geq \epsilon^{-3}$.

Thus, if $N \geq 2n\epsilon^{-3}$,

$$\begin{aligned}
&\Pr(\text{they obtain fewer than } N \text{ busts some poet})\\
&\qquad \leq \sum_{j=1}^{n} \Pr(\text{they obtain fewer than } N \text{ busts of the } j\text{th poet})\\
&\qquad \leq \epsilon
\end{aligned}$$

and they will have at least N full sets of poets with probability at least $1-\epsilon$.

Exercise 9.3.20

(i) No gentleman proposes to the same lady twice. There are n ladies so no gentleman makes more than n proposals. Thus there are no more than n^2 proposals.

(ii) If we have n gentlemen with the same list then each proposes to the favoured lady on entry (giving n proposals). One of them is accepted permanently (though some others may be accepted temporarily). The others then perform our algorithm as if the favoured lady did not exist and we had $n-1$ gentlemen with the same preferences. Thus the number of proposals is

$$n + (n-1) + (n-2) + \cdots + 1 = n(n+1)/2$$

Exercise 9.4.2

We write P_{XY} for the probability that we reach HHT before HTH if the last two throws have been XY and the game is not settled.

If we are at HH then with probability $1/2$ we go to HHT and I win and with probability $1/2$ we go to HHH and the probability I now win is p_{HH}. Thus
$$p_{HH} = \frac{1}{2}(p_{HH} + 1).$$
If we are at HT then with probability $1/2$ we go to HTH and I lose and with probability $1/2$ we go to HTT and the probability I now win is p_{TT}. Thus
$$p_{HT} = \frac{1}{2}(0 + p_{TT}).$$
If we are at TH then with probability $1/2$ we go to THH and the probability I now win is p_{HH} and with probability $1/2$ we go to THT and the probability I now win is p_{HT}. Thus
$$p_{TH} = \frac{1}{2}(p_{HH} + p_{HT}).$$
If we are at TT then with probability $1/2$ we go to TTH and the probability I now win is p_{TH} and with probability $1/2$ we go to TTT and the probability I now win is p_{TT}. Thus
$$p_{TH} = \frac{1}{2}(p_{TH} + p_{HH}).$$

Finally we note that after the first two throws we have probability $1/4$ of being at each of HH, HT, TH, TT so
$$\Pr(\text{I win}) = \frac{1}{4}(p_{HH} + p_{HT} + p_{TH} + p_{TT}).$$

Exercise 9.4.3

(i) We write P_{XY} for the probability that we reach HHT before HTT if the last two throws have been XY and the game is not settled.

$$p_{HH} = \frac{1}{2}(p_{HH} + 1), \quad p_{HT} = \frac{1}{2}(0 + p_{TT})$$

$$P_{TH} = \frac{1}{2}(p_{HH} + p_{HT}), \quad p_{TT} = \frac{1}{2}(p_{TH} + p_{TT}).$$

Thus

$$p_{TT} = \frac{2}{3},\ p_{HH} = 1,\ p_{HT} = \frac{1}{3},\ p_{HT} = \frac{2}{3}$$

and

$$\Pr(\text{I win}) = \frac{1}{4}(p_{HH} + p_{HT} + p_{TH} + p_{TT}) = \frac{2}{3}.$$

(ii) We write P_{XY} for the probability that we reach THH before HHT if the last two throws have been XY and the game is not settled.

$$p_{HH} = \frac{1}{2}(p_{HH} + 0), \quad p_{HT} = \frac{1}{2}(p_{TH} + p_{TT})$$

$$P_{TH} = \frac{1}{2}(1 + p_{HT}), \quad p_{TT} = \frac{1}{2}(p_{TH} + p_{TT}).$$

Thus

$$p_{HH} = 0,\ p_{TT} = p_{TH},\ p_{HT} = p_{TH},\ p_{TH} = 1$$

and so

$$p_{HH} = 0,\ p_{TT} = p_{TH} = p_{HT} = 1$$

and

$$\Pr(\text{I win}) = \frac{1}{4}(p_{HH} + p_{HT} + p_{TH} + p_{TT}) = \frac{3}{4}.$$

The result is also obvious from the diagram. You get to HHT from HHH and HHT but all other paths are blockaded.

(iii) Interchanging T and H, we see that:-

HTT has probability 7/8 of beating TTT,

TTH has probability 2/3 of beating THT,

TTH has probability 2/3 of beating THH,

HTT has probability 3/4 of beating TTH.

Exercise 9.4.4

If the first player chooses HH and the second TH then unless the first two throws are HH the second player must win. Thus the second player has probability 3/4 of winning. Similarly the first player should not choose TT.

If the first player chooses HT then the second player should not choose TT. Suppose the second player chooses HH. Let q_{XY} be the probability that the second player wins starting from XY. We have

$$\begin{aligned}
q_{HH} &= 1\\
q_{HT} &= 0\\
q_{TT} &= \tfrac{1}{2}q_{TT} + \tfrac{1}{2}q_{TH}\\
q_{TH} &= \tfrac{1}{2}q_{HT} + \tfrac{1}{2}q_{HH} = \tfrac{1}{2}
\end{aligned}$$

so $q_{HH} = 0$, $q_{HT} = 1$, $q_{TH} = \frac{1}{2}$, $q_{TT} = q_{TH} = \frac{1}{2}$ and the probability of the second player winning is

$$\frac{q_{HH} + q_{HT} + q_{TH} + q_{TT}}{4} = \frac{1}{2}.$$

By symmetry if the second player chooses TH her probability of winning is also 1/2.

Thus HT and TH are sensible choices for the first player.

If one player chooses TH and the other HT then if the first throw is T the first player must win and if the first throw is H the second player must win.

Exercise 9.4.5

(i) This is just the statement
$$\Pr(A^c) = 1 - \Pr(A).$$

(ii) We have perfect symmetry between heads and tails.

(iii) In order to obtain XYH we must first throw XY but once we have done so we have equal probability of throwing heads or tails.

There is no particular benefit in performing the calculations but but here they are.

Probability HHH beats HTH We write P_{XY} for the probability that we reach HHH before HTH if the last two throws have been XY and the game is not settled.
$$p_{HH} = \frac{1}{2}(1 + p_{HT}), \quad p_{HT} = \frac{1}{2}(0 + p_{TT})$$
$$P_{TH} = \frac{1}{2}(p_{HT} + p_{HH}), \quad p_{TT} = \frac{1}{2}(p_{TH} + p_{TT}).$$
Thus
$$p_{TH} = P_{TT},\ p_{HT} = \frac{1}{2}p_{TT},\ p_{TT} = \frac{1}{4}\left(p_{TT} + \left(1 + \frac{1}{2}p_{TT}\right)\right)$$
and so
$$p_{TH} = p_{TT} = \frac{2}{5},\ p_{HT} = \frac{1}{5},\ P_{HH} = \frac{3}{5}$$
and
$$\Pr(\text{I win}) = \frac{1}{4}(p_{HH} + p_{HT} + p_{TH} + p_{TT}) = \frac{2}{5}.$$

Probability HHH beats HTT We write P_{XY} for the probability that we reach HHH before HTT if the last two throws have been XY and the game is not settled.
$$p_{HH} = \frac{1}{2}(1 + p_{HT}), \quad p_{HT} = \frac{1}{2}(0 + p_{TT})$$
$$P_{TH} = \frac{1}{2}(p_{HT} + p_{HH}), \quad p_{TT} = \frac{1}{2}(p_{TH} + p_{TT}).$$
These are the equations of the previous paragraph.
$$\Pr(\text{I win}) = \frac{2}{5}.$$

Probability HHH beats THT We write P_{XY} for the probability that we reach HHH before THT if the last two throws have been XY and the game is not settled.

$$p_{HH} = \frac{1}{2}(1 + p_{HT}), \quad p_{HT} = \frac{1}{2}(p_{TH} + p_{TT})$$
$$P_{TH} = \frac{1}{2}(1 + p_{HH}), \quad p_{TT} = \frac{1}{2}(p_{TH} + p_{TT}).$$

Thus

$$p_{HT} = P_{TT} = p_{TH} = \frac{1}{2}p_{HH}, \ p_{HH} = \frac{1}{2}\left(1 + \frac{1}{2}p_{HH}\right)$$

and so

$$p_{HH} = \frac{2}{3}, \ p_{HT} = P_{TT} = p_{TH} = \frac{1}{3}$$

and

$$\Pr(\text{I win}) = \frac{1}{4}(p_{HH} + p_{HT} + p_{TH} + p_{TT}) = \frac{5}{12}.$$

Probability HHH beats TTH We write P_{XY} for the probability that we reach HHH before TTH if the last two throws have been XY and the game is not settled.

$$p_{HH} = \frac{1}{2}(1 + p_{HT}), \quad p_{HT} = \frac{1}{2}(p_{TH} + p_{TT})$$
$$p_{TH} = \frac{1}{2}(p_{HT} + p_{HH}), \quad p_{TT} = \frac{1}{2}(0 + p_{TT}).$$

Thus

$$p_{TT} = 0, \ p_{HT} = \frac{1}{3}p_{HH}, \ p_{TH} = \frac{2}{3}p_{HH}, \ p_{HH} = \frac{3}{5}$$

and so

$$p_{TT} = 0, \ p_{HT} = \frac{1}{5}, \ p_{TH} = \frac{2}{5}p_{HH}, \ p_{HH} = \frac{3}{5}$$

and

$$\Pr(\text{I win}) = \frac{1}{4}(p_{HH} + p_{HT} + p_{TH} + p_{TT}) = \frac{3}{10}.$$

Probability HHT beats HTT We write P_{XY} for the probability that we reach HHT before HTT if the last two throws have been XY and the game is not settled.

$$p_{HH} = \frac{1}{2}(1 + p_{HH}), \quad p_{HT} = \frac{1}{2}(0 + p_{TH})$$
$$p_{TH} = \frac{1}{2}(p_{HT} + p_{HH}), \quad p_{TT} = \frac{1}{2}(p_{TH} + p_{TT}).$$

Thus

$$p_{HT} = \frac{1}{2}p_{TH}, \ p_{TT} = p_{TH}, \ p_{HH} = 1, \ p_{TH} = \frac{2}{3}$$

and so

$$p_{TT} = p_{TH} = \frac{2}{3}, \ p_{HT} = \frac{1}{3}, \ p_{HH} = 1$$

and

$$\Pr(\text{I win}) = \frac{1}{4}(p_{HH} + p_{HT} + p_{TH} + p_{TT}) = \frac{2}{3}.$$

Probability HHT *beats* THT We write P_{XY} for the probability that we reach HHT before THT if the last two throws have been XY and the game is not settled.

$$p_{HH} = \frac{1}{2}(1 + p_{HH}), \quad p_{HT} = \frac{1}{2}(p_{TH} + p_{HT})$$
$$p_{TH} = \frac{1}{2}(p_{HH} + 0), \quad p_{TT} = \frac{1}{2}(p_{TH} + p_{TT}).$$

Thus

$$p_{HH} = 1,\ p_{TT} = p_{TH} = p_{TH} = \frac{1}{2}p_{HH} = \frac{1}{2}$$

and so

$$\Pr(\text{I win}) = \frac{1}{4}(p_{HH} + p_{HT} + p_{TH} + p_{TT}) = \frac{5}{8}.$$

Probability HTH *beats* TTH The match will be decided if at least throws have been made and the last two throws were TH. It is equally likely that the throw before the last two was heads or tails so the probability that HTH beats TTH is 1/2.

The remaining entries have already been calculated or may be found by applying rules (i), (ii) and (iii).

Exercise 9.4.6

We do not need to do as many checks as these but it is comforting to observe that

$$\text{Expected value our strategy against } HHH = \frac{1}{4} > 0$$
$$\text{Expected value our strategy against } HHT = \frac{1}{12} > 0$$
$$\text{Expected value our strategy against } HTH = 0$$
$$\text{Expected value our strategy against } HTT = 0$$
$$\text{Expected value our strategy against } THH = 0$$
$$\text{Expected value our strategy against } THT = 0$$
$$\text{Expected value our strategy against } TTH = \frac{1}{12} > 0$$
$$\text{Expected value our strategy against } TTT = \frac{1}{4} > 0$$

By symmetry our expected value against best play is 0 so this is an optimum strategy.

If my opponent just plays one triple XYZ, I will have strictly positive expected winnings against HHH, TTT, HHT and THH.

If my opponent promises to choose either HTH or THT, then inspection of Table 9.2 shows that I should play some combination of HHT and THH and a little thought (or explicit calculation) shows that if he uses best play (subject to his original foolish decision), I should play each option with probability 1/2 and my expected winnings are 7/24.

Exercise 9.5.1

The new game is obtained from the old by adding 3 units to the value of each outcome.

Whatever Little Bonaparte chooses, Spats can do better for himself by pressing. Thus Spats should press. By exactly the same argument Little Bonaparte will press. They get 1 each.

If Bonaparte and Spats could trust each other they could agree not to press and get 2 each.

Exercise 9.5.2

The travellers may not be the most agreeable conversationalists but they are acute reasoners. If there is exactly 1 bore then he knows immediately who he is (since he knows no bores) and resigns on the first evening. If there is more than 1 no one can know for certain that he is a bore. If there are exactly 2 bores then on the first morning each knows that there must be at least 2 bores but is acquainted with only 1 bore. They therefore know that they are bores and resign. If there is more than 2 no one can know for certain that he is a bore. The argument proceeds inductively and shows that if there are exactly k bores they will resign on the kth evening and their resignation will be announced on the kth morning after the announcement. Since there are no resignations on the first 49 mornings everybody must be a bore and they resign en masse and a smirking secretary posts their names on the 50th morning.

Exercise 9.5.5

We have

$$\begin{aligned}
B(I,P) &= 3 + pB(A,P)\\
B(I,N) &= 2 + pB(I,N)\\
B(I,S) &= 2 + pB(I,S)\\
B(I,0) &= 3 + pB(A,O)\\
B(A,P) &= 1 + pB(A,P)\\
B(A,N) &= 0 + pB(I,N)\\
B(A,S) &= 0 + pB(A,S)\\
B(A,0) &= 0 + pB(I,O)
\end{aligned}$$

Thus

$$B(I,N) = \frac{2}{1-p},\ B(I,S) = \frac{2}{1-p},\ B(A,P) = \frac{1}{1-p},\ B(A,S) = 0$$

so

$$B(I,P) = 3 + \frac{p}{1-p},\ B(A,N) = \frac{2p}{1-p}.$$

Also

$$B(I,0) = 3 + pB(A,O) = 3 + p^2B(I,O)$$

so

$$B(I,0) = \frac{3}{1-p^2},\ B(A,0) = \frac{3p}{1-p^2}.$$

Exercise 9.5.6

We have

$$B(I,P) = 3 + p + p^2 + p^3 + \cdots = 3 + \frac{p}{1-p}$$

$$B(I,N) = 2 + 2p + 2p^2 + 2p^3 + \cdots = \frac{2}{1-p}$$

$$B(I,S) = 2 + 2p + 2p^2 + 2p^3 + \cdots = \frac{2}{1-p}$$

$$B(I,O) = 3 + 0p + 3p^2 + 0p^3 + 3p^4 + \cdots = \frac{3}{1-p^2}$$

$$B(A,P) = 1 + p + p^2 + p^3 + \cdots = \frac{1}{1-p}$$

$$B(A,N) = 0 + 2p + 2p^2 + 2p^3 + \cdots = \frac{2p}{1-p}$$

$$B(A,S) = 1 + p + p^2 + p^3 + \cdots = \frac{1}{1-p}$$

$$B(A,O) = 0 + 3p + 0p^2 + 3p^3 + 0p^4 \cdots = \frac{3p}{1-p^2}$$

Exercise 9.5.7

If it is very likely that there will be another game our best choice is to obtain a system where neither press and we share equally a sum which is greater than the total available by non-cooperation.

If it is less likely that there will be another game we take the maximum possible by pressing. If there is a second game we sacrifice a small sum to restablish good relations which we can exploit if there is another round.

If the probability of another game is even smaller then at each round we take the maximum available on that round by pressing.

Exercise 9.5.8

We have $B(A,P) = B(A,S)$ and

$$B(A,P) - B(A,N) = \frac{1-2p}{1-p}$$
$$B(A,P) - B(A,O) = \frac{1-2p}{1-p^2}$$
$$B(A,N) - B(A,O) = \frac{p(2p-1)}{1-p^2}.$$

Thus

$$B(A,P) > B(A,N), B(A,O)$$

for $0 \leq p < 1/2$ and

$$B(A,N) > B(A,O) > B(A,P)$$

for $1/2 < p \leq 1$ whilst

$$B(A,N) = B(A,O) = B(A,P)$$

if $p = 1/2$.

Sonia should play never press if $p > 1/2$ and always press if $p < 1/2$. If $p = 1/2$ all strategies have the same expected pay-off.

Exercise 9.5.9

Using the same notation as before we have

$$\begin{aligned}
B(I,P) &= 6 + p + p^2 + p^3 + \cdots = 6 + \frac{p}{1-p}\\
B(I,N) &= 2 + 2p + 2p^2 + 2p^3 + \cdots = \frac{2}{1-p}\\
B(I,S) &= 2 + 2p + 2p^2 + 2p^3 + \cdots = \frac{2}{1-p}\\
B(I,O) &= 6 + 0p + 6p^2 + 0p^3 + 6p^4 + \cdots = \frac{6}{1-p^2}.
\end{aligned}$$

Thus $B(I,N) = B(I,S)$ and

$$\begin{aligned}
B(I,N) - B(I,P) &= \frac{2-p}{1-p} - 3 = \frac{2p-1}{1-p},\\
B(I,N) - B(I,O) &= \frac{2(1+p)-6}{1-p^2} = \frac{2p-4}{1-p^2},\\
B(I,P) - B(I,O) &= \frac{p}{1+p} - \frac{6p^2}{1-p^2} = \frac{p(1-7p)}{1-p^2}.
\end{aligned}$$

Thus

$$B(I,N) > B(I,P) \text{ for } p > 1/2,\ B(I,P) > B(I,N) \text{ for } 1/2 > p,$$
$$B(I,O) > B(I,N) \text{ for all } p,$$
$$B(I,O) > B(I,P)) \text{ for } p > 1/7,\ B(I,P) > B(I,O) \text{ for } 1/7 > p.$$

and Sonia should do the opposite of Tania if $p > 1/7$ and always press if $p < 1/7$. She can follow either tactic if $p = 1/7$.

The total prize available if one presses and the other does not, is greater than for any other. If the probability of many further games is high the strategies of Tania and Sonia more or less share this total prize.

Exercise 9.5.10

If p is large tactics by which the players always hit on the maximum total pay out and arrange to share more or less equally are optimal.

If $2c > d$ and p is large enough always play the same, if $2c < d$ and p is large enough always play the opposite.

Exercise 9.5.11

There is no mathematical argument available but since tit for tats do well in the other type of contest and draw with each other it seems to me that tit for tat remains a good choice.

Exercise 9.5.12

(i) Any program is much more likely to meet a 'tit for tat' than an 'always press'. Since 'tit for tats' do better than 'always press' against 'tit for tats' the 'tit for tats' will gain more points than the 'always presses' and the population of 'always presses' will decline.

(ii) Any program is much more likely to meet an 'always press' than a 'tit for tats'. Since 'tit for tats' do worse than 'always presses' against 'always presses' the 'tit for tats' will gain fewer points than the 'always presses' and the population of 'tit for tats' will decline.

(iii) Any program is much more likely to meet an 'never press' than an 'always press'. Since 'always presses' do better than 'always presses' against 'never presses' than an 'always press' the proportion of 'always presses' will increase. The question of what will happen when there is a large proportion of 'always preses' depends on how we fix the reward and breeding structure.

If we have a mixed flock of 'never presses' and 'tit for tats' then the behaviour of the two types of program in an encounter is the same. There is no tendency for the proportion of 'tit for tats' to grow or decline. (This is not the same as saying that the proportion will stay the same, simply that there is no preferred direction of change.)

Exercise 9.6.1

Consider the map

$$\theta : \{0,1\}^{\mathbb{N}} \to \{H,T\}^{\mathbb{N}}$$

from sequences of 0's and 1's to sequences of heads and tails. Define

$$\theta(\mathbf{x})_{2k} = \begin{cases} H & \text{if } x_k = 0 \\ T & \text{if } x_k = 1 \end{cases}$$

and

$$\theta(\mathbf{x})_{2k+1} = \begin{cases} T & \text{if } x_k = 0 \\ H & \text{if } x_k = 1 \end{cases}$$

Thus the $2k$ and $2k+1$ th throws are HT if the kth entry of $\mathbf{x}$ is 0 and TH if the kth entry of $\mathbf{x}$ is 1 $[0 \leq k]$.

$\theta(\mathbf{x})$ contains no sequence HHH or TTT and θ is injective. Since $\{0,1\}^{\mathbb{N}}$ is uncountable there uncountably many different sequences for which the game goes on for ever.

Exercise 9.6.3

Observe that if the machine spins all wheels the probability of three letters being the same is $3/27 = 1/9$, of no letters being the same is $6/27 = 2/9$ and two the same is $18/27 = 2/3$.

The game will certainly end if hold fails to light up and the machine the produces three distinct letters. The probability of this is $(1/2) \times (2/9) = 8/9$ so the probability the game ends on a particular round is at least $1/9$. Thus he probability that she plays an n th round is no greater than $(8/9)^{n-1}$.

If she must stop on round N her expected number of rounds

$$r_N \leq \sum_{n=1}^{N} \left(\frac{8}{9}\right)^{n-1} \leq 9$$

Since r_N is increasing bounded above it tends to a limit so the expected number of rounds that she plays is finite. Since her winnings and losses on each round are bounded by 3 her expected winnings are finite. (A more sophisticated approach is to use the comparison test.)

Let e_1 be her expected winnings starting with all windows different, e_2 her expected winnings starting with two windows the same and e_3 her expected winnings starting with three windows the same.

$$\begin{aligned}
e_1 &= -10 + \frac{2}{3}e_2 + \frac{1}{9}(30 + e_3)\\
e_2 &= -10 + \frac{1}{2}\left(\frac{2}{3}e_2 + \frac{1}{9}(30 + e_3)\right) + \frac{1}{2}\left(\frac{2}{3}e_2 + \frac{1}{3}(30 + e_3)\right)\\
e_3 &= -10 + \frac{1}{2}\left(\frac{2}{3}e_2 + \frac{1}{9}(30 + e_3)\right) + \frac{1}{2}(30 + e_3)
\end{aligned}$$

Subtracting the second equation from the third gives

$$e_3 - e_2 = \frac{1}{2}(30 + e_3) - \frac{1}{2}\left(\frac{2}{3}e_2 + \frac{1}{3}(30 + e_3)\right)$$

so $e_3 = e_2 + 15$. Substituting back, $e_3 = 15$, $e_2 = 0$ and $e_1 = -5$.

The bettors expected loss is 5 pence.

Exercise 9.6.4

(i) Observe that

$$\begin{aligned}\mathbb{E}X &= \sum_{j=1}^{N} j \Pr(X=j)\\ &= \sum_{j=1}^{N}\sum_{k=1}^{j} \Pr(X=j)\\ &= \sum_{1\leq k\leq j\leq N} \Pr(X=j)\\ &= \sum_{k=1}^{N}\sum_{j=k}^{N} \Pr(X=j)\\ &= \sum_{k=1}^{N} \Pr(X\geq k)\end{aligned}$$

If $Y_N = \min(X, N)$

$$\mathbb{E}Y_N = \sum_{k=1}^{N} \Pr(Y_N \geq k) = \sum_{j=1}^{N} \Pr(X\geq k) \to \sum_{r=1}^{\infty} \Pr(X \geq r)$$

as $N \to \infty$.

(ii) Let X be the number of shots A fires. We have

$$\Pr(X\geq m) = \Pr(m-1 \text{ misses in a row}) = (1-a)^{m-1}$$

so

$$\mathbb{E}\min(N, X) = \sum_{m=1}^{N}(1-a)^{m-1} = \frac{1-(1-a)^{n+1}}{a} - (1-a)^n$$

and

$$\mathbb{E}X = \sum_{m=1}^{\infty}(1-a)^{m-1} = \frac{1}{a}.$$

(iii) Let $B_{j,r}$ be the probability that the jth spot where an acorn is planted does not have an oak after $r-1$ plantings. Then

$$\Pr(\text{at least } r \text{ plantings required}) = \Pr\left(\bigcup_{i=1}^{n} B_{i,r}\right)$$
$$= \sum_{i} \Pr(B_{i,r}) - \sum_{i<j} \Pr(B_{i,r} \cap B_{j,r}) + \sum_{i<j<k} \Pr(B_{i,r} \cap B_{j,r} \cap B_{k,r}) - \dots$$
$$= \binom{n}{1} \Pr(B_{1,r}) - \binom{n}{2} \Pr(B_{1,r} \cap B_{2,r}) + \binom{n}{3} \Pr(B_{1,r} \cap B_{2,r} \cap B_{3,r}) - \dots$$
$$= \binom{n}{1} q^{r} - \binom{n}{2} q^{2r} + \binom{n}{3} q^{3r} - \dots$$

and so

$$\text{expected number of plantings} = \sum_{r=1}^{\infty} \Pr(\text{at least } r \text{ plantings required})$$
$$= \binom{n}{1} \frac{1}{1-q} - \binom{n}{2} \frac{1}{1-q^2} + \binom{n}{3} \frac{1}{1-q^3} - \dots + (-1)^{n-1} \binom{n}{n} \frac{1}{1-q^n}.$$

Exercise 9.6.5

The game will certainly terminate if the next three throws are HHH or TTT (it may terminate otherwise). The probability of three the same is $1/4$, so the probability that the game terminates during the next three throws is at least $1/4$. Thus

$$\Pr(\text{game lasts } 3n \text{ tosses}) \leq (3/4)^n \to 0$$

as $n \to \infty$.

Exercise 9.6.6

Let A_n be the collection of games terminating in n goes or less. Then A_n is finite so countable. Thus $\bigcup_{n=1}^{\infty} A_n$ is countable (being the countable union of countable sets) and this is the stated result.

Exercise 10.1.2

$$\Pr(\text{happy end}) = \Pr(\text{wins 5 goes in succession}) = p^5$$

If $p = 1/4$, $p^5 \approx 0.00098$.

If $p = 3/4$, $p^5 \approx 0.237$.

Exercise 10.1.3

If $q_n = q_0 t^n$ then
$$q_{n+1} = q_n t = q_0 t^{n+1}$$
since $q_0 = t^0 q_0$ the result follows by induction for $n \geq 0$.

The result is also true for $n \leq 0$. Either use simple induction or observe that
$$q_{-(n+1)} = t^{-1} q_{-n}$$
and use the first part.

Exercise 10.1.4

(i) Observe that, if

$$q_n = A + B\left(\frac{p}{1-p}\right)^n$$

then

$$\begin{aligned} pq_{n+1} - q_n + (1-p)q_{n-1} & A\big(p - 1 + (1-p)\big) \\ & + B\left(\frac{1-p}{p}\right)^{n-1}\left(p\left(\frac{1-p}{p}\right)^2 - \left(\frac{1-p}{p}\right) + (1-p)\right) \\ &= 0 + B\left(\frac{1-p}{p}\right)^{n-1}\frac{(1-p)^2 - (1-p) + p(1-p)}{p} = 0 \end{aligned}$$

as stated.

(ii) If $q_0 = 0$ then $A = -B$ so

$$q_n = A\left(1 - \left(\frac{p}{1-p}\right)^n\right).$$

If further $q_{256} = 1$ then

$$A = \left(1 - \left(\frac{p}{1-p}\right)^{256}\right)^{-1}$$

so

$$q_n = \frac{1 - \left(\frac{1-p}{p}\right)^n}{1 - \left(\frac{1-p}{p}\right)^{256}}.$$

Exercise 10.1.5

(i) With (a) I gamble 1120 dollars and have probability p of gaining my end. If I lose (which I do with probability $1-p$) I have 320 dollars and must win three times in a row (probability p^3) to gain my ends. My probability of success is

$$p + (1-p)p^3.$$

With (b) I must win 3 times in row to gain my ends without gambling my back pocket. The probability of this is p^4. Otherwise, with probability $1-p^4$ I gamble my back pocket and have probability p of winning. My probability of success is

$$p^3 + (1-p^3)p = p + (1-p)p^3.$$

(ii) With bold play I have probability p of reaching my goal immediately. Otherwise with probability $1-p$ I will have 1440 dollars and using (a) or (b) I have probability $p_0 = p + (1-p)p^3$ of winning my ends. Thus my probability of success is

$$p + (1-p)p_0.$$

Alternatively I can put 1280 dollars in my back pocket and then use either strategy (a) or (b) (with one dollar being replaced by 50 cents) to gain gain my ends without gambling my back pocket. This has probability p_0. If I fail (which I will with probability $1-p_0$) I gamble my back pocket and have probability p of gaining my ends. Thus my probability of success is

$$p_0 + (1-p_0)p = p + (1-p)p_0.$$

(See *How To Gamble If You Must* by Dubins and Savage Chapter 5, Section 4. The book is reprinted by Dover)

Exercise 10.1.6

The expected gain on a bet of x is

$$pxp^{-1} + (1 - p)0 - x = 0.$$

If your expected gain on any game is zero then it should be zero on any combination of games.

Thus your expected winnings are 0. If you have probability q of gained your desired prize with a particular strategy then your expected winnings are

$$-k + ql + (1 - q)0 = -k + ql.$$

Thus $ql = k$ and $q = k/l$.

Exercise 10.1.7

(i) If $p < 1/2$ she is gambling in unfavourable circumstances and should use bold play as described in this section.

(ii) If f is her fortune on retirement then the probability she is happy is 1 if $f \geq 1$ and zero otherwise.

We have proved the inductive hypothesis for $r = 0$. Suppose it is true for $r = m$. If we have a fortune

$$f = (k + \delta)2^{-m-1} + \delta$$

with $0 \leq k \leq 2^{m+1}$ an integer then if we gamble $(u + \eta)2^{-m-1}$ with $0 \leq u \leq k$ an integer and $0 \leq \eta \leq \delta$ our new fortune will either be $(k+u+\eta+\delta)2^{-m-1}$ or $(k-u-\eta+\delta)$ and by the inductive hypothesis the probability of ultimate success will either be that if we start from

$$2^{-m}[(k + u + \eta + \delta)/2] = 2^{-m}[(k + u)/2]$$

or if we start from

$$2^{-m}[(k - u + \eta - \delta)/2] = 2^{-m}[(k - u)/2]$$

(here $[a]$ is the integer part of a) at year m independent of η and δ. The result thus follows by induction.

(iii) If k is even she should bet $(2r)2^{-m-1}$ with $0 \leq 2r \leq k$ and the probability of ultimate success if she then plays correctly is

$$e(2r, k, m + 1) = \frac{p\big(((k + 2r)/2, m\big) + p\big(((k - 2r)/2, m\big)}{2}$$

She should choose $r = r^*$ to maximise $e(2r, k, m+1)$ and then $p(k, m+1) = e(2r^*, k, m + 1)$.

If k is odd she should bet $(2r + 1)2^{-m-1}$ with $0 \leq 2r + 1 \leq k$ and the probability of ultimate success if she then plays correctly is

$$e(2r + 1, k, m + 1) = \frac{p\big(((k + 2r + 1)/2, m\big) + p\big(((k - 2r - 1)/2, m\big)}{2}$$

She should choose $r = r^*$ to maximise $e(2r + 1, k, m + 1)$ and then $p(k, m + 1) = e(2r^* + 1, k, m + 1)$.

(iv) If n is large we are almost back to the circumstances described in this section and should gamble slowly. She will stake a small proportion of her fortune.

If n is small, small bets can not produce the required outcome. She will bet a relatively large proportion of her fortune.

Exercise 10.2.1

(a) The expected return is

$$\Pr(\text{win}) \times 32 = \frac{32}{33}.$$

The probability of winning with bold play is $1/33 \approx 0.0303$.

(b) The expected return is

$$\Pr(\text{win}) \times 2 = \frac{16}{33} \times 2 = \frac{32}{33}.$$

To win with bold play you need to win 5 times in a row. The probability of winning is

$$\left(\frac{16}{33}\right)^5 \approx 0.0268.$$

Take my advice– when deep in debt,
Set up a bank and play Roulette!
At once distrust you surely lull,
And rook the pigeon and the gull.
The bird will stake his every franc
In wild attempt to break the bank–
But you may stake your life and limb
The bank will end by breaking him!

(Gilbert and Sullivan *The Grand Duke*)

Exercise 10.2.2

(i) Game A has expected value $2^{-10}2^8 = 2^{-2}$ and variance
$$2^{-10}2^{16} - (2^{-2})^2 = 2^6 - 2^{-4}$$
for an investment of one dollar. The probability of winning your desired sum by bold play in game A is 2^{-10}.

Game B has expected value $2^{-3}2^2 = 2^{-1}$ and variance
$$2^{-3}2^4 - (2^{-1})^2 = 1/4.$$
The probability of winning your desired sum by bold play in game B is the probability of winning four times in succession that is to say $(2^{-3})^4 = 10^{-12}$.

Game C has expected value $2^{80}2^{-120} = 2^{-40}$ and variance
$$2^{-120}2^{160} - (2^{-40}) = 2^{40} - 2^{-80}$$
for an investment of one dollar. Bold play in C will involve betting at least 2^{-71} dollars (actually rather more) each go so you will run out of money after at most 2^{71} throws unless you win a throw. The probability of winning in 2^{71} throws is
$$1 - \Pr(\text{losing } 2^{71} \text{ throws}) = 1 - (1 - 2^{-120})^{2^{71}}$$
which is tiny (either by direct calculation or using the binomial expansion and bounding terms).

(ii) The expected value of game D is
$$2^{-10}2^8 + 2^{80}2^{-120} = 2^{-2} + 2^{-40}$$
and its variance is greater than
$$2^{-120} \times 2^{160} - (2^{-2} + 2^{-40})^2 > 2^{39}.$$
If we intend to try and play a few games then the game looks very much like game B which is substantially worse than game A. If we play many games with many small stakes then the law of large numbers says that in practice we are betting in game C with a bet of $k/2$, returning $2^{40}k$ with probability 2^{-80} and the same kind of estimates as before show that our chances of succeeding are negligible.

Exercise 10.2.3

We have
$$\mathbb{E}X = p \times kp^{-1} + (1-p) \times 0 = k$$
and
$$\begin{aligned}\operatorname{var} X &= \mathbb{E}X^2 - (\mathbb{E}X)^2\\ &= p \times (kp^{-1})^2 + (1-p) \times 0 - k^2\\ &= k^2(p^{-1} - 1)\end{aligned}$$

Thus $\operatorname{var} X \to k^2(1-1) = 0$ as $p \to 1-$ and $\operatorname{var} X \to \infty$ as $p \to 0+$.

Exercise 10.2.4

You are broke if you lose. You have 1 if you win. The probability of success is thus $k\mathcal{F} < k$.

Exercise 10.2.5

(i) You have $(n-r)\mathcal{F}$ fortune remaining so you need to win

$$1-(n-r)n^{-1}\mathcal{F}$$

to leave. Thus you take p_r so that

$$\frac{\mathcal{F}}{n}kp_r^{-1}=1-(n-r)n^{-1}\mathcal{F}$$

or, rearranging,

$$p_r=\frac{k\mathcal{F}}{n(1-\mathcal{F})+r\mathcal{F}}.$$

(ii) You will fail if each of your n bets fail so with probability

$$q_n=\prod_{r=1}^{n}\Pr(r\text{th bet fails})=\prod_{r=1}^{n}(1-p_r)$$

so taking logarithms,

$$\log q_n=\sum_{r=1}^{n}\log(1-p_r).$$

(iii) Observe that

$$p_r=\frac{k\mathcal{F}}{n(1-\mathcal{F})}\leq\mathcal{F}(1-\mathcal{F})^{-1}.$$

Exercise 10.2.6

(i) We have

$$g'(x) = 1 + 2x - \frac{1}{1-x} = \frac{x - 2x^2}{1-x} = \frac{x(1-2x)}{1-x} \geq 0$$

for $0 \leq x \leq 1/2$ so g is increasing and

$$0 = g(0) \leq g(x) = x + x^2 + \log(1-x)$$

ie

$$x + x^2 \geq -\log(1-x)$$

for $0 \leq x \leq 1/2$.

On the other hand, setting $f(x) = x + \log(1-x)$, we have

$$f'(x) = 1 - \frac{1}{1-x} \leq 0$$

for $0 \leq x < 1$ so f is decreasing and

$$0 = f(0) \geq f(x) = x + \log(1-x)$$

ie

$$x \leq -\log(1-x)$$

for $0 \leq x < 1$.

(ii) Set $A = \mathcal{F}(1-\mathcal{F})^{-1}$. Provided that $n > 2A$, part (iii) of Exercise 10.2.5 tells us that $0 < p_n < 1/2$ so

$$p_r \leq -\log(1-p_r) \leq p_r + p_r^2 \leq p_r + \frac{A^2}{n^2}$$

whence

$$\sum_{r=1}^{n} p_r \leq -\log q_n \leq \sum_{r=1}^{n} p_r + \frac{A^2}{n}$$

and

$$\log q_n + \sum_{r=1}^{n} p_r \to 0$$

as $n \to \infty$.

We know that if $G : [0,1] \to \mathbb{R}$ is continuous then

$$\frac{1}{n} \sum_{r=1}^{n} G(r/n) \to \int_0^1 G(t)\, dt.$$

Taking

$$G(t) = \frac{1}{(1-\mathcal{F}) + t\mathcal{F}}$$

we obtain

$$\begin{aligned}
\sum_{r=1}^{n} p_r &= k\mathcal{F} \times \frac{1}{n} \sum_{r=1}^{n} \frac{1}{(1-\mathcal{F}) + \frac{r}{n}\mathcal{F}} \\
&\to k\mathcal{F} \int_0^1 \frac{dt}{(1-\mathcal{F}) + t\mathcal{F}} \\
&== -k[\log\big((1-\mathcal{F}) + t\mathcal{F}\big)]_0^1 \\
&= -k\log(1-\mathcal{F})
\end{aligned}$$

as $n \to \infty$.

Thus

$$\log q_n \to k\log(1-\mathcal{F})$$

and so

$$q_n \to (1-\mathcal{F})^k$$

as $n \to \infty$.

Exercise 10.2.7

(i) We have

$$g(x) = k(1-x)^{k-1} - k = k\big((1-x)^{k-1} - 1\big) > 0$$

for $0 < x < 1$, so g is strictly increasing in this range and

$$g(x) > g(0) = 0$$

ie

$$1 - kx > (1-x)^k$$

for $0 < x < 1$.

The probability of failure with the simple bold strategy is $1-k\mathcal{F}$. The probability of failure with the division strategy can be made as close to $(1-\mathcal{F})^k$ as we wish. Thus the division strategy (with n sufficiently large) is always better than our simple bold strategy.

(ii) By the definition of the derivative

$$\frac{1-(1-x)^k}{kx} = \frac{1}{k} \times \frac{h(x)-h(0)}{x} \to \frac{h'(0)}{k} = 1,$$

so when $\mathcal{F}$ is small the ratio of the probabilities of success

$$\frac{1-(1-\mathcal{F})^k}{k\mathcal{F}}$$

is close to 1 and our division strategy not much better better than the simple bold strategy.

Exercise 10.2.8

You must win 5 times in secession so you want

$$p^5 \geq 1/33$$

so $p \geq .497$ (approximately) If we take $p = .497$ the Casino's expectation is about 0.006 (In the game (a) it is about 0.03.)

Exercise 10.2.9*

Exercise 10.2.10

(i) The expected return is $2\,560\,000$ which is what we should expect from a fair game. We expect to get out (on average) what we put in.

(ii) [Corrected version] If $p = .495$ our expected return is

$$10,000 \times .505^{-8} \approx 2\,364\,117.$$

Exercise 10.2.11

This just a particularly wild modification of unlimited double or quits. If you own a' (with $a' < b$) you you announce a bet of $(b' - a)/u$ dollars. If you win, you walk away with b. If you lose you repeat the process.

Since you walk away at the first win and since no strategy will allow you to walk away before you have had at least one win this is a best strategy.

The expected time to your first win is p^{-1} (see the one sided duel) which is independent of a, b and u.

Useful theorems on probability must exclude wild games like this.

Exercise 10.3.1

$$\begin{aligned} aw'' + bw' + cw &= a(kv+u)'' + b(kv+u)' + c(kv+u) \\ &= a(kv''+u'') + b(kv'+u') + c(kv+u) \\ &= k(av''+bv'+cv) + (au''+bu'+cu) \\ &= k0 + f = f. \end{aligned}$$

If

$$au'(x) + bu(x) = f(x)$$

and

$$av'(x) + bv(x) = 0,$$

then, if k is constant and $w = kv + u$,

$$\begin{aligned} aw' + bw &= a(kv+u)' + b(kv+u) \\ &= a(kv'+u') + b(kv+u) \\ &= k(av'+bv) + (au'+bu) \\ &= k0 + f = f. \end{aligned}$$

Exercise 10.3.2

(i) Observe that

$$\begin{aligned}(D-aI)(D-bI)u &= (D-aI)(u'-bu) = (u'-bu)' - a(u'-bu)\\ &= u'' - bu' - au' + abu = (D^2 - (a+b)D + abI)u.\end{aligned}$$

(ii) If $u' - au = f$ then

$$\begin{aligned}\frac{d}{dx}(e^{-ax}u(x)) &= -ae^{-ax}u(x) + e^{-ax}u'(x)\\ &= e^{-ax}(u'(x) - au(x)) = e^{-ax}f(x)\end{aligned}$$

so integrating

$$\big[e^{-as}u(s)\big]_0^x = \int_0^x e^{-as}f(s)\,ds$$

so

$$e^{-ax}u(x) - u(0) = \int_0^x e^{-as}f(s)\,ds$$

and

$$u(x) = u(0)e^{ax} + e^{ax}\int_0^x e^{-as}f(s)\,ds.$$

(iii) Set $v = (D - bI)u$. Then

$$v'(x) - av(x) = 0$$

and, by (ii) with $f(x) = 0$,

$$v(x) = v(0)e^{ax} + e^{ax}\int_0^x e^{-as}0\,ds = v(0)e^{ax}.$$

Now

$$u'(x) - bu(x) = v(x) = v(0) + e^{ax}$$

so, by (ii),

$$\begin{aligned}u(x) &= u(0)e^{bx} + e^{bx}\int_0^x v(0)e^{(a-b)s}\,ds\\ &= u(0)e^{bx} + \frac{v(0)}{a-b}(e^{ax} - e^{bx}) = Ae^{ax} + Be^{bx}.\end{aligned}$$

Thus the only possible solutions are

$$u(x) = Ae^{ax} + Be^{bx}$$

for some pair constants A and B. By inspection every pair of constants give a solution.

To prove the last statement, either examine he argument above carefully or note that we have a solution if and only if

$$\begin{aligned} u_0 &= A + B \\ u_1 &= aA + bB \end{aligned}$$

and, since $a \neq b$, these equations have a unique solution.

(iv) Set $v = (D - aI)u$. Then, as in (iii),

$$v(x) = v(0)e^{ax}$$

so, by (ii), with $f(x) = e^{ax}$

$$\begin{aligned} u(x) &= u(0)e^{ax} + e^{ax} \int_0^x v(0)\, ds \\ &= u(0)e^{ax} + v(0)xe^{ax} \end{aligned}$$

Thus the only possible solutions are

$$u(x) = (A + Bx)e^{ax}$$

for some pair constants A and B. By inspection every pair of constants give a solution.

he prove the last statement either examine he argument above carefully or note that we have a solution if and only if

$$\begin{aligned} u_0 &= A \\ u_1 &= aA + B \end{aligned}$$

and these equations have a unique solution.

(v) Set $v = (D - aI)u$. Then $v(0) = u_1 - au_0$ and, by (ii),

$$v(x) = v(0)e^{ax} + e^{ax} \int_0^x e^{-as} f(s)\, ds,$$

and, using (ii) again

$$u(x) = u(0)e^{bx} + e^{bx} \int_0^x e^{-bs} v(s)\, ds.$$

Thus our system has exactly one solution.

(vi) Use part (i) and parts (iii) and (iv).

Exercise 10.3.3

(i) We have

$$\frac{d}{dx}e^{-ax}u(x) = Ke^{(c-a)x}$$

so

$$e^{-ax}u(x) = A + K\frac{e^{(c-a)x}}{c-a}$$

and

$$u(x) = Ae^{ax} + \frac{K}{c-a}e^{cx}$$

for some constant A.

(ii) We have

$$\frac{d}{dx}e^{-ax}u(x) = K$$

so

$$e^{-ax}u(x) = A + Kx$$

and

$$u(x) = (A + Kx)e^{ax}$$

for some constant A.

(iii) Set $v(x) = u'(x) - au(x)$. Then

$$v'(x) - bv(x) = Ke^{cx}$$

and

$$v(x) = Ce^{bx} + \frac{K}{c-b}e^{cx}$$

so

$$u(x) - au(x) = Ce^{bx} + \frac{K}{c-b}e^{cx}$$

and using linearity or repeating the argument of (i)

$$u(x) = Ae^{ax} + Be^{bx} + \frac{K}{(c-a)(c-b)}e^{cx}$$

for some constants A and B.

(iv) Set $v(x) = u'(x) - au(x)$. As in (iii)

$$u(x) - au(x) = Ce^{bx} + \frac{K}{a-b}e^{ax}$$

so using linearity or repeating the argument of (ii)

$$u(x) = Ae^{ax} + Be^{bx} + \frac{K}{a-b}xe^{ax}$$

for some constants A and B.

(v) Set $v(x) = u'(x) - au(x)$. Then

$$v'(x) - av(x) = Ke^{ax}$$

and

$$v(x) = Ce^{ax} + Kxe^{ax}$$

so

$$\frac{d}{dx} e^{-ax} u(x) = B + Kx$$

whence

$$e^{-ax} u(x) = A + Bx + \frac{K}{2} x^2$$

and

$$u(x) = \left(A + Bx + \frac{K}{2} x^2 \right) e^{-ax}$$

for some constants A and B.

Exercise 10.3.4

(i) If u_n is defined uniquely for some $n \geq 0$, then $u_{n+1} = y_{n+1} + au_n$ is uniquely defined. Since u_0 is uniquely defined, it follows, by induction that u_n is defined uniquely for all $n \geq 0$.

If u_{-n} is defined uniquely for some $n \geq 0$, then $u_{-n-1} = a^{-1}(u_{-n} - y_{-n-1}$ is uniquely defined. Since u_0 is uniquely defined, it follows, by induction that u_{-n} is defined uniquely for all $n \geq 0$. Thus u_n is defined uniquely for all n.

(ii) If u_n and u_{n+1} are defined uniquely for some $n \geq 0$, then u_{n+1} and $u_{n+2} = z_n - au_{n+1} - bu_n$ are uniquely defined. Since u_0 and u_1 are uniquely defined, it follows, by induction that u_n is defined uniquely for all $n \geq 0$.

If u_{-n} and u_{-n-1} are defined uniquely for some $n \geq 0$, then u_{-n-1} and $u_{-n-2} = b^{-1}(z_{-n-2} - au_{-n-1} - u_{-n})$ are uniquely defined. Since u_0 and $u_{-1} = b^{-1}(z_1 - a\tilde{u}_0 - \tilde{u}_1)$ are uniquely defined, it follows, by induction that u_n is defined uniquely for all $n \leq 0$. Thus u_n is defined uniquely for all n.

Exercise 10.3.5

Observe that

$$\begin{aligned}
&aw_{n+2} + bw_{n+1} + cw_n \\
&\quad = a(kv_{n+2} + u_{n+2}) + b(kv_{n+1} + u_{n+1}) + c(kv_n + u_n) \\
&\quad = k(av_{n+2} + bv_{n+1} + cv_n) + (au_{n+2} + bu_{n+1} + cu_n) \\
&\quad = k0 + y_n = y_n.
\end{aligned}$$

Suppose that

$$au_{n+1} + bu_n = y_n$$

and

$$av_{n+1} + bv_n = 0.$$

If k is constant and $w_n = kv_n + u_n$, then

$$\begin{aligned}
aw_{n+1} + bw_n &= a(kv_{n+1} + u_{n+1}) + b(kv_n + u_n) \\
&= k(av_{n+1} + bv_n) + (au_{n+1} + bu_n) \\
&= k0 + y_n = y_n.
\end{aligned}$$

Exercise 10.3.6

(i) We have

$$\begin{aligned}(E-aI)(E-bI)u_n &= (E-aI)u_{n+1} - bu_n\\ &= u_{n+2} - bu_{n+1} - au_{n+1} + abu_n\\ &= u_{n+2} - (a+b)u_{n+1} + abu_n\\ &= (E^2 - (a+b)E + abI)u_n.\end{aligned}$$

(ii) We have

$$(E-aI)v_n = v_{n+1} - av_n = a^{n+1}u_{n+1} - a^n u_n = a^n(u_{n+1} - u_n) = a^n w_n.$$

(iii) We have

$$u_{j+1} - u_j = y_j$$

so

$$u_{n+1} - u_0 = \sum_{j=0}^{n}(u_{j+1} - u_j) = \sum_{j=0}^{n} y_j.$$

for all $n \geq 0$, ie

$$u_n = u_0 + \sum_{j=0}^{n-1} y_j$$

for $n \geq 1$.

Also

$$u_{-(j+1)} - u_{-j} = -y_{-j-1}$$

so, by what we have just shown,

$$u_{-n} = u_0 - \sum_{j=0}^{n} y_{-j-1}$$

ie

$$u_n = u_0 - \sum_{j=1}^{-n} y_{-j}$$

for $n \leq -1$.

Exercise 10.3.7

(i) By Exercise 10.3.6 (iii),

$$u_n = 1$$

for all n.

(ii) By Exercise 10.3.6 (iii),

$$u_n = \frac{b^n - 1}{b - 1}$$

for all n.

(iii) By Exercise 10.3.6 (iii),

$$u_n = n$$

for all n.

(iv) Since $\sum_{j=1}^{n} j = n(n+1)/2$, it follows from Exercise 10.3.6 (iii) that

$$u_n = n(n+1)/2$$

We now use Exercise 10.3.6 (ii) on each of the previous four parts in turn to get.

(v) $u_n = a^n$.

(vi) $u_n = \dfrac{b^n}{b - a}$

(vii) $u_n = na^n$

(viii) $u_n = n(n+1)a^n/2$.

Finally using linearity and parts (v) and (vi).

The general solution of $(E - aI)u_n = 0$ is $u_n = Aa^n$.

The general solution of $(E - aI)u_n = b^n$ with $b \neq a$ is $u_n = Ab^n$.

Exercise 10.3.8

(i) If
$$v_{n+2} - (a+b)v_{n+1} + abv_n = 0,$$
then, setting $u_n = v_{n+1} - av_n$ we get
$$u_{n+1} - bu_n = 0$$
so $u_n = Cb^n$ and
$$v_{n+1} - av_n = Cb^n$$
and by Exercises 10.3.7 (ix) and (v) and linearity
$$v_n = Aa^n + Bb^n.$$
Thus using Exercises 10.3.4 and 10.3.5 we have
$$u_n = Aa^n + Bb^n + v_n,$$
where A and B are freely chosen constants.

(ii) If $a \neq 0$ and

$$v_{n+2} - 2av_{n+1} + a^2 v_n = 0,$$
then, arguing as before,
$$v_{n+1} - av_n = Aa^n$$
so
$$u_n = (A + Bn)a^n + v_n,$$
where A and B are freely chosen constants.

Exercise 10.3.9

(i) Write $v_n = (E - aI)u_n$. Then
$$v_{n+1} - bv_n = Cx^n$$
so
$$v_n = Kb^n + \frac{C}{x-b}x^n$$
for some constant K. Thus
$$u_{n+1} - u_n = Kb^n + \frac{C}{x-b}x^n$$
and
$$u_n = Aa^n + Bb^n + \frac{C}{(x-a)(x-b)}x^n = Aa^n + Bb^n + \frac{C}{x^2-(a+b)x+ab}x^n$$
for some constants A and B.

(ii) Write $v_n = (E - aI)u_n$. Then
$$v_{n+1} - bv_n = Ca^n$$
so
$$v_n = Kb^n + \frac{C}{a-b}a^n$$
for some constant K. Thus
$$u_{n+1} - u_n = Kb^n + \frac{C}{a-b}a^n$$
and
$$u_n = Aa^n + Bb^n + \frac{C}{b-a}na^n.$$

(iii) We have
$$(E - I)^2 u_n = C$$
so
$$(E - I)u_n = B + Cn$$
and
$$u_n = A + Bn + \tfrac{1}{2}Cn^2$$
for some constants A and B.

Exercise 10.3.10

We have
$$\alpha^2 + b\alpha + a = 0$$
so taking complex conjugates
$$\bar{0} = \overline{\alpha^2 + b\alpha + c} = \bar{\alpha}^2 + b\bar{\alpha} + c.$$
Thus $\bar{\alpha}$ is root. Since $\bar{\alpha} \neq \alpha$ we have $\beta = \bar{\alpha}$.

Since $A + B = A\alpha^0 + B\bar{\alpha}^0$ is real we know that
$$\Im A = -\Im B.$$
Let $\alpha = u + iv$ with u and v real. Then
$$A\alpha + B\bar{\alpha} = (\Re A + i\Im A)(u + iv) + (\Re B + i\Im B)(u - iv)$$
is real so
$$u(\Im A + \Im B) + v(\Re A - \Re B) = v(\Re A - \Re B)$$
is real so, since $v \neq 0$, $\Re A = \Re B$ and $B = \bar{A}$.

Conversely if $B = \bar{A}$ then
$$A\alpha^n + B\bar{\alpha}^n = A\alpha^n + \bar{A}\bar{\alpha}^n = \overline{A\alpha^n + \bar{A}\bar{\alpha}^n}$$
so $A\alpha^n + B\bar{\alpha}^n$ is real.

(i) If u_j and u_{j+1} are real then u_{j+1} $u_{j+2} = -bu_{j+1} - cu_j = 0$ are real. Since u_0 and u_1 are real u_j is real for all $j \geq 0$. To show that u_j is real for $j \leq 0$ consider the equation
$$u_{-(n+2)} + c^{-1}bu_{-(n+1)} + c^{-1}au_{-n} = 0.$$

(ii) We have
$$\begin{aligned} u_0 &= A + B \\ u_1 &= A\alpha + B\bar{\alpha} \end{aligned}$$
so
$$(u_1 - \bar{\alpha}u_0) = A(\alpha - \bar{\alpha})$$
and
$$A = \frac{u_1 - \bar{\alpha}u_0}{\alpha - \bar{\alpha}}$$
Similarly
$$B = \frac{u_1 - \alpha u_0}{\bar{\alpha} - \alpha} = \bar{A}$$
so u_n is real for all n.

EXERCISE 10.3.11

(i) We have

$$\begin{aligned}(E-I)\binom{n}{r} &= \binom{n+1}{r} - \binom{n}{r}\\ &= \frac{1}{r!}n(n-1)\dots(n-r+2)\big((n+1)-(n-r+1)\big)\\ &= \frac{1}{(r-1)!}n(n-1)\dots(n-r+2) = \binom{n}{r-1}.\end{aligned}$$

(ii) Using the particular solution of of (i) and the known complementary solution we have

$$u_n = A + \binom{n}{r}$$

for some constant A.

(iii) We use induction on k. If the result is true for $k = m$ then

$$\begin{aligned}(E-I)^{m+1}u_n = 0 &\Leftrightarrow (E-I)u_n = v_n \text{ with } (E-I)^m v_n = 0\\ &\Leftrightarrow (E-I)u_n = \sum_{j=0}^{m-1} A_{m-1-j}\binom{n}{j}\\ &\Leftrightarrow u_n = \sum_{j=0}^{m} A_{m-j}\binom{n}{j}\end{aligned}$$

with A_j arbitrary. The result holds for $k = 1$ so it holds for all k by induction.

(iv) By inspection or by using Exercise 10.3.6,

$$u_n = \sum_{j=0}^{k-1} A_{k-1-j}\binom{n}{j}a^j.$$

(v) By inspection or using induction on k

$$u_n = \binom{n}{r+k}$$

is a solution so the general solution is

$$u_n = \binom{n}{r+k} + \sum_{j=0}^{k-1} A_{k-1-j}\binom{n}{j}.$$

Exercise 10.3.12

Let
$$t^3 + at^2 + bt + c = 0$$
have roots α, β and γ (repeated roots repeated appropriately). Note that since $c \neq 0$ no root is zero.

(i) If α, β, γ are distinct,
$$\begin{aligned}(E^3 + aE^2 + bE + cI)u_n = 0 &\Leftrightarrow \big(E - \beta I)((E - \gamma I)\big)(E - \alpha I)u_n = 0\\ &\Leftrightarrow (E - \alpha I)u_n = B'\beta^n + C'\gamma^n\\ &\Leftrightarrow u_n = A\alpha^n + B\beta^n + C\gamma^n\end{aligned}$$
for general constants B', C', A, B, C.

(ii) If α and β are distinct but $\gamma = \alpha$,
$$\begin{aligned}(E^3 + aE^2 + bE + cI)u_n = 0 &\Leftrightarrow \big(E - \alpha I)((E - \beta I)\big)(E - \alpha I)u_n = 0\\ &\Leftrightarrow (E - \alpha I)u_n = A'\alpha^n + B'\beta^n\\ &\Leftrightarrow u_n = (A + A'n)\alpha^n + B\beta^n\end{aligned}$$
for general constants B', A', A, B.

(iii) If If $\alpha = \beta = \gamma$,
$$\begin{aligned}(E^3 + aE^2 + bE + cI)u_n = 0 &\Leftrightarrow \big(E - \alpha I)((E - \alpha I)\big)(E - \alpha I)u_n = 0\\ &\Leftrightarrow (E - \alpha I)u_n = A' + A'''\alpha^n\\ &\Leftrightarrow u_n = (A + A'n + A''n^2)\alpha^n\end{aligned}$$
for general constants A, A', A'', A'''.

Exercise 10.3.13

(i) Just before the nth minute the flea will be on the black dog with probability $1 - p_{n-1}$ and jump to the white dog with probability w or it will be on the white dog with probability p_{n-1} and stay on the white dog with probability $1 - b$. Thus

$$p_n = (1 - p_{n-1})w + p_{n-1}(1 - b) = w + (1 - b - w)p_{n-1}$$

for $n \geq 1$ and

$$(E - (1 - b - w)I)p_n = w$$

for $n \geq 0$.

We seek a particular solution $p_n = c$, obtaining

$$(b + w)c = w$$

and the general solution

$$p_n = \frac{w}{b + w} + A(1 - b - w)^n.$$

Since $p_0 = 1$, we have $A = b/(b + w)$ so

$$p_n = \frac{w}{b + w} + \frac{b}{w + b}(1 - b - w)^n.$$

Since $0 < b + w < 2$ we have $|1 - b - w| < 1$ so $(1 - b - w)^n \to 0$ and

$$p_n \to \frac{w}{b + w}$$

as $n \to \infty$.

(ii) As before

$$q_n = \frac{w}{b + w} + B(1 - b - w)^n$$

for some constant B. Since $q_0 = 0$, $B = -w/(b + w)$ and

$$q_n = \frac{w}{b + w} - \frac{w}{w + b}(1 - b - w)^n \to \frac{w}{b + w}$$

as $n \to \infty$.

(The process gradually forgets how it starts.)

(iii) If $b + w = 0$ then $b = w = 0$ and the flea stays on the dog it starts on.

If $b + w = 2$ then $b = w = 1$ and after an even number of seconds it it will be on its starting dog and after an odd number on the the other.

(In neither case does the process forget how it started.)

Exercise 10.4.1

We have

$$u_n = \tfrac{1}{2}u_{n+1} + \tfrac{1}{2}u_{n-1}$$

so

$$(E - I)^2 u_n = 0$$

and

$$u_n = A + Bn$$

We have $u_0 = 0$ so $A = 0$ and $u_N = 1$ so $b = 1/N$ and

$$u_n = u_N(n) = \frac{n}{N}.$$

Exercise 10.4.2

If the casino has a fortune of n, then at the next bet it has a probability of p of gaining 1 so its probability of going bankrupt is now u_{n+1}, a probability of r of neither losing nor gaining so its probability of going bankrupt is now u_n, and probability of q of neither losing 1 so its probability of going bankrupt is now u_{n-1}. Thus

$$u_n = pu_{n-1} + ru_n + qu_{n+1}$$

for $1 \leq n \leq N-1$. If $n=0$ it is bankrupt so $u_0 = 0$. If $n = N$ it stops so $u_N = 1$.

(ii) We have

$$(qE^2 - (p+q)E + pI))u_n = 0$$

so

$$(E-I)(E-(p/q))u_n = 0$$

whence

$$u_n = A + B\left(\frac{p}{q}\right)^n.$$

Since $u_0 = 0$ we have $A = -B$ and since, $u_N = 1$, $A = (1-(p/q)^N)^{-1}$. Thus

$$u_n = \frac{1-\left(\frac{p}{q}\right)^n}{1-\left(\frac{p}{q}\right)^N}.$$

(iii) If $p = q$ we get

$$(pE^2 - 2pE + pI))u_n = 0$$

so $(E^2 - E + I))u_n = 0$ and, exactly as in Exercise 10.4.1,

$$u_n = u_N(n) = \frac{n}{N}.$$

(iv) Allowing $N \to \infty$ in (ii) and (iii) we see that bankruptcy is certain if $p \geq q$. If $p < q$ the probability of bankruptcy starting with a fortune n is

$$1 - \left(\frac{p}{q}\right)^n$$

which may be made as small as we wish by taking n large enough.

Exercise 10.4.3

Suppose that the casino has N units. If it takes a bet, then with probability p it will lose, have a new fortune of $N-1$ and expect to survive a further e_{N-1} bets. With probability $1-p$ it will win, pay its owners 1 so have a new fortune of N and expect to survive a further e_N bets. Thus
$$e_n = 1 + pe_{N-1} + (1-p)e_N.$$

Exercise 10.4.4

Suppose that the casino has n units with $1 \leq n \leq N-1$. If it takes a bet, then with probability p it will lose, have a new fortune of $n-1$ and expect to pay out a further f_{n-1} to its owners. With probability $1-p$ it will win, have a new fortune of $n+1$ and expect to pay out a further f_{n+1} to its owners. Thus

$$f_n = pf_{n-1} + (1-p)f_{n+1}.$$

Suppose that the casino has N units. If it takes a bet, then with probability p it will lose, have a new fortune of $N-1$ and expect to pay out a further f_{N-1} to its owners With probability $1-p$ it will win, pay its owners 1 so have a new fortune of N and expect to pay out a further f_N. Thus

$$\bigstar \qquad f_N = pf_{N-1} + (1-p)(1+f_N).$$

If the casino has fortune 0 it is bankrupt so $f_0 = 0$.

Our standard calculation shows that

$$f_n = A\left(1 - \left(\frac{1-p}{p}\right)^n\right)$$

Equation $\bigstar$ now gives

$$p(f_N - f_{N-1}) = 1-p$$

and

$$A = \frac{1}{1-2p}\left(\frac{1-p}{p}\right)^N$$

whence

$$f_n = \frac{1}{1-2p}\left(\frac{1-p}{p}\right)^N\left(1 - \left(\frac{1-p}{p}\right)^n\right)$$

and, so in particular,

$$f_N = \frac{1}{1-2p}\left(\left(\frac{1-p}{p}\right)^N - 1\right).$$

Exercise 10.4.5

Since the game is fair, the expected return to investors is precisely what they put in. Thus $f_n = n$.

Exercise 10.4.6

The expected time to bankruptcy e_n satisfies

$$e_n = 1 + pe_{n-1} + qe_{n+1} + re_n$$

for $1 \leq n \leq N-1$ while $e_0 = 0$ and

$$e_N = 1 + pe_{N-1} + (q+r)e_N.$$

Thus

$$(qE^2 - (p+q)E + p)e_n = 1$$

or

$$(qE - p)(E-1)e_n = 1$$

We try a particular solution $e_n = Kn$ obtaining $K = (q-p)^{-1}$ and a general solution

$$e_n = A + B\left(\frac{p}{q}\right)^n + \frac{n}{q-p}$$

Since $e_0 = 0$, $B = -A$ and

$$e_n = A\left(1 - \left(\frac{p}{q}\right)^n\right) + \frac{n}{q-p}.$$

The condition $e_N = 1 + pe_{N-1} + (q+r)e_N$ gives

$$e_N - e_{N-1} = p^{-1}$$

so

$$A\left(\left(\frac{p}{q}\right)^N - \left(\frac{p}{q}\right)^{N-1}\right) = \frac{1}{q-p} - \frac{1}{p}$$

whence we obtain A and e_N.

The expected sum withdrawn f_n if the casino starts with n satisfies

$$f_n = pf_{n-1} + rf_n + qf_{n+1}$$

if $1 \leq n \leq N-1$ and $f_0 = 0$,

$$f_N = pf_{N-1} + rf_N + q(1 + f_N) = q + (1-p)f_N) + pf_{N-1}$$

so

$$pf_N = q + pf_{N-1}.$$

We have

$$(qE - p)(E-1)f_n = 1$$

so

$$f_n = A + B\left(\frac{p}{q}\right)^n$$

and since $f_0 = 0$, $B = -A$ and

$$f_n = A\left(1 - \left(\frac{p}{q}\right)^n\right).$$

Applying the other end condition

$$A\left(\frac{p}{q}\right)^{N-1}\left(1-\frac{p}{q}\right)=\frac{q}{p}$$

whence we obtain A and f_N.

Exercise 10.4.7

$Y_n(r)$ is my fortune if I play fair heads and tails for n goes starting with a fortune of r and I am not stopped by bankruptcy. The first value of n, if any, with $Y_n(r) = 0$ corresponds to bankruptcy.

If I start with n and avoid bankruptcy by reaching N then my probability of escaping bankruptcy is n/N. Allowing $N \to \infty$ we see that if I play indefinitely the probability of bankruptcy is 1.

If I have a fortune of $N - 1 \geq n \geq 1$ then at the next bet I have a probability of $1/2$ of gaining 1 so I will have lasted one further turn and my expected survival time will be e_{r+1} and a probability of $1/2$ of gaining 1 so I will have lasted one further turn and my expected survival time will be e_{r-1}. Thus

$$\bigstar \qquad e_n = 1 + \tfrac{1}{2}(e_{n+1} + e_{n-1})$$

for $N - 1 \geq n \geq 1$. Since I stop at 0 and at N we have $e_0 = e_N = 0$.

$\bigstar$ gives $e_{n+1} - 2e_n + e_{n-1} = -2$. We seek a solution of the form Cn^2 obtaining

$$C(2n^2 + 2 - 2n^2) = -2$$

when $C = 1$. Since $u_{n+1} - 2u_n + u_{n-1} = 0$ has the general solution $A + bN$ we have

$$e_n = -n^2 + Bn + A$$

Since $e_0 = e_N = 0$, $a = 0$ and $B = -N$ whence

$$e_n = e_n(N) = n(N - n).$$

If n is fixed

$$e_n(N) = n(N - n) = nN - n^2 \to \infty$$

as $N \to \infty$.

Exercise 10.4.8

(i) We note that the position of the players is symmetric and that in each case, if the double is accepted, the player who doubles moves from a player who can double at will to a player who cannot double facing an opponent who can double at will. Thus the decision is always the same.

(ii) Suppose the players are at M. Let p be the probability that the first player wins if doubling is prohibited and q the probability that the players will reach $-M$ before the game ends (again doubling being prohibited).

$$\begin{aligned} p &= \Pr(\text{players never pass through } -M) \\ &\qquad + \Pr(\text{players pass through } -M \text{ and first player wins}) \\ &= (1-q) + q(1-p) = 1 - qp \end{aligned}$$

Thus, assuming that this is the first double and the second player accepts, we know that there is a probability $1-q$ that the first player will win (so the game will have value -2 to the second player) without passing through $-M$ and a probability $1-q$ that the game will pass through $-M$, at which point the game will have value -2 to the first player and so value 2 the second. Thus

$$\begin{aligned} -1 &= \text{expected value for the second player if accepts} \\ &= -2(1-q) + 2q \end{aligned}$$

Thus $q = 1/4$ and $p = 4/5$.

(iii) Observe that the players are playing a standard heads tails, so if p_r is the probability that the first player wins from the point r we have

$$p_r = \tfrac{1}{2}p_{r-1} + \tfrac{1}{2}p_{r+1}$$

and $p_N = 1$, $p_{-N} = 0$ and by standard calculations

$$p_{r+1} = \frac{r+N}{2N}$$

so, if N is divisible by 5 we have critical value $M = 3N/5$.

Exercise 10.4.9

(i) Write M for the total number of coins involved. Let e_r be the expected time to the end if Rosencrantz has fortune r. With probability $1/2$ his fortune is r after one throw, With probability $1/4$ his fortune is $r-1$ after one throw and with probability $1/4$ his fortune is $r+1$ $[1 \leq r \leq M-1]$. Thus

$$e_r = 1 + \frac{1}{4}(e_{r-1} + 2e_r + e_{r+1})$$

ie

$$e_{r+1} - 2e_r + e_{r-1} = 4.$$

A particular solution is $e_r = Ar^2$ with $A = 2$. Thus

$$e_r = Br + C + 2r^2.$$

Since $e_0 = e_M = 0$ we have $C = 0$ and $B = -2M$. Thus

$$e_r = 2(r^2 - Mr) = 2rg.$$

(ii) The probability that any man wins a round is the probability that the other two throw the opposite ie $1/4$. Thus we have

$$e_{hrg} = 1 + \frac{1}{4}(e_{(h+2)(r-1)(g-1)} + e_{(h-1)(r+2)(g-1)} + e_{(h-1)(r-1)(g+2)} + e_{hrg}$$

so

$$\bigstar \quad 4 = e_{(h+2)(r-1)(g-1)} + e_{(h-1)(r+2)(g-1)} + e_{(h-1)(r-1)(g+2)} - 3e_{hrg}\big).$$

Further $e_{0uv} = e_{u0v} = e_{uv0} = 0$.

We try

$$e_{hrg} = Ahrg$$

which certainly satisfies the boundary conditions.

The equation $\bigstar$ is satisfied if and only if

$$\begin{aligned} 4 &= -A\big((h+2)(r-1)(g-1) + (h-1)(r+2)(g-1) \\ &\qquad + (h-1)(r-1)(g+2) - 3hrg\big) \\ &= A\big(3(h+r+g) - 6\big) \end{aligned}$$

ie

$$A = \frac{4}{3N-6}$$

where $N = h + r + g$.

Since $\bigstar$ has at most one solution satisfing the boundary condition and we have found a solution, this must be the unique soltion. We have

$$e_{hrg} = hrg\frac{4}{3N-6}.$$

If they start with 100 coins each they are likely to reach England before they finish the game.

Exercise 10.4.10

(i) Jack will be paid j with probability p_j if $j \geq m$. With probability $\sum_{j=1}^{m-1} p_j$ he will take the cow home and with probability q the cow will not die and he will be in exactly the same position with expected gain e_m. Thus

$$e_m = \sum_{j=m}^{n} jp_j + \left(\sum_{j=1}^{m-1} p_j\right) qe_m$$

so

$$\left(1 - q\sum_{j=1}^{m-1} p_j\right) e_m = \sum_{j=m}^{n} jp_j$$

as stated.

We claim that e_m increases and then decreases. Matthew Towers gives the following argument. Observe that this statement is equivalent to saying that $e_{m+1} \leq e_m$ implies $e_{m+2} \leq e_{m+1}$ and this is equivalent to saying that

$$p_m\left(m - q\left(m\sum_{j=0}^{m} p_j + \sum_{j=m+1}^{n} jp_j\right)\right) \geq 0$$

$$m - q\left(m\sum_{j=0}^{m} p_j + \sum_{j=m+1}^{n} jp_j\right) \geq 0$$

implies

$$m + 1 - q\left(\sum_{j=0}^{m} p_j + m\sum_{j=0}^{m} p_j + \sum_{j=m+1}^{n} jp_j\right) \geqslant 0$$

and this implication follows from the fact that

$$q\sum_{j=0}^{m} p_j \leq 1.$$

Let us write $A_m = \sum_{j=m}^{n} jp_j$ and $B_m = 1 - q\sum_{j=1}^{m-1} p_j$. Then

$$\begin{aligned} e_{m+1} - e_m &= \frac{A_{m+1}}{B_{m+1}} - \frac{A_m}{B_m} \\ &= \frac{A_m - mp_m}{B_m - qp_m} - \frac{A_m}{B_m} \\ &= \frac{(qA_m - mB_m)p_m}{(B_m - qp_m)B_m}. \end{aligned}$$

Thus $e_{m+1} - e_m \geq 0$ if $qA_m - mB_m \geq 0$ and $e_{m+1} - e_m \leq 0$ if $qA_m - mB_m \leq 0$.

But qA_m is a decreasing and mB_m is an increasing sequence so $e_{m+1}-e_m$ is a decreasing sequence and the result follows.

Jack should choose $m = m_0$.

(ii) If he has to take his cow home he incurs an extra cost of k so his expected gain e_m is now

$$e_m = \sum_{j=m}^{n} jp_j + \left(\sum_{j=1}^{m-1} p_j\right)(-k + qe_m)$$

so

$$e_m = \frac{\sum_{j=m}^{n} jp_j - k\sum_{j=1}^{m-1} p_j}{1 - q\sum_{j=1}^{m-1} p_j}.$$

Let us write $A_m = \sum_{j=m}^{n} jp_j - k\sum_{j=1}^{m-1} p_j$ and $B_m = 1 - q\sum_{j=1}^{m-1} p_j$. Then

$$\begin{aligned}
e_{m+1} - e_m &= \frac{A_{m+1}}{B_{m+1}} - \frac{A_m}{B_m} \\
&= \frac{A_m - (m+k)p_m}{B_m - qp_m} - \frac{A_m}{B_m} \\
&= \frac{(qA_m - (m+k)B_m)p_m}{(B_m - qp_m)B_m}.
\end{aligned}$$

But qA_m is a decreasing and $(m+k)B_m$ is an increasing sequence so $e_{m+1} - e_m$ is a decreasing sequence. Thus there is an integer m_0 with $0 \leq m_0 \leq n$ such that $e_{m-1} \leq e_m$ when $1 \leq m \leq m_0$ and $e_m \leq e_{m+1}$ when $m_0 \leq m \leq n-1$.

Note that in case (i) (except in the trivial cases $p_0 = 1$ or $q = 1$) Jack will have $m_0 \geq 1$ but in case (ii) Jack may be happy to give the cow away (this is obvious if $k > n$). We have assumed that beans are traded in integral multiples but the ideas clearly work more generally.

Exercise 10.4.11

The computation of V_0 is essentially that of the onesided duel.

$$V_0 = \sum_{j=0}^{\infty} j \Pr(A_{-j} \text{ first unoccupied space})$$

so

$$V_0 = q(1 + V_0)$$

and $V_0 = q/p$.

If I use my plan with $m = n$ then, with probability q, the nth state is free and I walk n units and with probability p it is occupied and my expected walk is that obtained by taking $m = n - 1$. Thus

$$V_n = pn + qV_{n-1}.$$

If we look for a solution of

$$u_n = pn + qu_{n-1}$$

with $u_n = Bn + C$, we obtain

$$Bn + C = pn + qB(n-1) + qC$$

so $B = 1$, $C = -q/p$. By inspection $u_n = n - q/p$ is indeed a particular solution. Thus, the general solution is

$$u_n = Aq^n + n - \frac{q}{p}.$$

We now know that

$$V_n = Aq^n + n - \frac{q}{p}$$

for some constant A. Since $V_0 = q/p$ $A = (2q/p)$ so

$$V_n = n + \frac{(2q^n - 1)q}{p}$$

for $n \geq 0$ as stated.

Thus

$$V_{n-1} - V_n = -1 + 2q\frac{q^{n-1} - q^n}{p} = 2q^n - 1$$

and

$$V_n - V_{n-1} \begin{cases} \geq 0 & \text{if } 2q^n - 1 \geq 0 \\ \leq 0 & \text{if } 2q^n - 1 \leq 0 \end{cases}$$

so V_n is minimised when n takes its largest value with

$$n \leq \frac{\log(1/2)}{\log q}.$$

EXERCISE 10.5.1

(i) We have

$$\begin{aligned}\Pr(\text{Law loses}) &= \Pr(\text{double six on a particular throw})^6\\ &= \left(\frac{1}{36}\right)^2 \approx 4.6\times 10^{-10}.\end{aligned}$$

(ii) About 2.2×10^9 seconds.

(iii) He has paid 1 shilling for a bet worth approximately

$$4.6\times 10^{-10}\times 10^3\times 20 = 9.2\times 10^{-4}\approx 10^{-3}$$

shillings.

(iv) No. The Kelly criterion deals with the long run and even if we bet every second of a lifetime we will not have made enough bets for the appropriate Tchebychev inequality to give any information.

Exercise 10.5.2

The probability of getting the right numbers in some fixed order is
$$\frac{1}{49}\times\frac{1}{48}\times\frac{1}{47}\times\frac{1}{46}\times\frac{1}{45}\times\frac{1}{44}$$
and there are 6! ways of obtaining the right numbers in any order so the probability of winning is
$$\begin{aligned}6!\times\frac{1}{49}\times\frac{1}{48}\times\frac{1}{47}\times\frac{1}{46}\times\frac{1}{45}\times\frac{1}{44}&=\frac{1}{49}\times\frac{1}{47}\times\frac{1}{46}\times\frac{1}{3}\times\frac{1}{44}\\&=\frac{1}{13983816}.\end{aligned}$$

Exercise 10.5.4

Observe that

$$\frac{d}{da}ae^{-a} = e^{-a} - ae^{-a} = (1-a)e^{-a}$$

so ae^{-a} increases with a until $a = 1$ and then decreases. Thus ae^{-a} is maximised by taking $a = 1$.

Exercise 10.5.5

Assuming that N is so large that we may take

$$\Pr(\text{exactly } m \text{ winners}) = \frac{1}{m!}a^m e^{-a},$$

we must seek to maximise $f(a) = a^m e^{-a}$.

Now

$$f'(a) = (m - a)a^{m-1}e^{-a}$$

so f increases as a increases from 0 to m and then decreases. We should take $a = m$.

Exercise 10.5.6

Let X_j be the sum paid out to the jth participant The expected value of the total sum paid out is

$$\begin{aligned}\mathbb{E}\sum_{j=1}^{N} X_j &= \sum_{j=1}^{N} \mathbb{E}X_j = N\mathbb{E}X_1 \\ &= N \times \frac{N}{2m} = \frac{N}{2}.\end{aligned}$$

Exercise 10.5.8

Since
$$\sum_{r=2m+1}^{\infty} \frac{m^r}{r!} \leq 2u_{m+1}$$
it is sufficient to find a k so that
$$\sum_{r=2m+k+1}^{\infty} \frac{m^r}{r!} \leq \frac{u_{m+1}}{200}.$$
But arguing as in Lemma 10.5.7, we have
$$\sum_{r=2m+k+1}^{\infty} \frac{m^r}{r!} \leq u_{2m+1} \sum_{r=k+1}^{\infty} 2^{-r} = 2^{-k} u_{2m+1}$$
so $k = 8$ will certainly do.

Exercise 10.5.9

$$e^{-5}\frac{5^{11}}{11!} \approx 0.0082$$
$$e^{-10}\frac{10^{21}}{21!} \approx 0.00089$$

Exercise 10.5.10

(i) We did the case $-1/2 < x \leq 0$ in Exercise 10.2.6. Now we consider $0 \leq x < 1/2$.

If $g(x) = x - \log(1+x)$, then

$$g'(x) = 1 - \frac{1}{1+x} > 0$$

for $0 \leq x$ so $g(x) \geq g(0) = 0$ and

$$\log(1+x) \leq x$$

for all $x \geq 0$.

If $f(x) = x^2 + \log(1+x) - x$, then

$$f'(x) = 2x - 1 + \frac{1}{1+x} = \frac{2x^2+x}{1+x} > 0$$

for $0 \leq x$ so $f(x) \geq f(0) = 0$ and

$$\log(1+x) \geq x - x^2$$

for all $x \geq 0$.

The result follows.

(ii) By (i) we know that If $n \geq 2|a|^{-1}$

$$\left|\frac{a}{n} - \log\left(1+\frac{a}{n}\right)\right| \leq \left(\frac{a}{n}\right)^2$$

so

$$\left|a - n\log\left(1+\frac{a}{n}\right)\right| \leq \frac{a^2}{n} \to 0$$

as $n \to \infty$.

We have shown that

$$n\log\left(1+\frac{a}{n}\right) \to \log a$$

so taking exponentials

$$\left(1+\frac{a}{n}\right)^n \to e^a$$

as $n \to \infty$.

(iii) If

$$\sum_{r=1}^{n-1} r^2 \leq \frac{n^3}{3}$$

then

$$\begin{aligned}\sum_{r=1}^{n} r^2 &\leq \frac{n^3}{3} + n^2\\ &\leq \frac{1}{3}(n^3+3n^2)\\ &\leq \frac{1}{3}(n^3+3n^2+3n+1)\\ &= \frac{(n+1)^3}{3}\end{aligned}$$

so, since

$$\sum_{r=1}^{1} r^2 = 1 \leq \frac{8}{3} = \frac{(1+1)^3}{3},$$

the result follows by induction.

[Or we could use known formulae for $\sum_{r=1}^{n} r^2$.]

(iv) Provided that n is so large that $N(n)|a|/n < 1/2$, we have

$$\left|\frac{ra}{n} - \log\left(1+\frac{ra}{n}\right)\right| \leq \left(\frac{ra}{n}\right)^2$$

for all $1 \leq r \leq N(n)$ and so

$$\left|\sum_{r=1}^{N(n)} \frac{ra}{n} - \sum_{r=1}^{N(n)} \log\left(1+\frac{ra}{n}\right)\right| \leq \sum_{r=1}^{N(n)} \left(\frac{ra}{n}\right)^2$$

so

$$\left|\frac{N(n)(N(n)+1)a}{2n} - \sum_{r=1}^{N(n)} \log\left(1+\frac{ra}{n}\right)\right| \leq \frac{(N(n)+1)^3a^2}{3n^2} \leq \frac{N(n)^3a^2}{n^2}.$$

Thus

$$\begin{aligned}\left|(1+N(n)^{-1})a - \frac{2}{N(n)^2}\sum_{r=1}^{N(n)} \log\left(1+\frac{ra}{n}\right)\right| &\leq \frac{(N(n)+1)^3a^2}{3n^2}\\ &\leq \frac{2N(n)a^2}{n} \to 0.\end{aligned}$$

as $n \to \infty$ and so

$$\frac{2}{N(n)^2}\sum_{r=1}^{N(n)} \log\left(1+\frac{ra}{n}\right) \to a$$

as $n \to \infty$. Taking exponentials we deduce that

$$\left[\prod_{r=1}^{N(n)} \left(1+\frac{ra}{n}\right)\right]^{2/N(n)^2} \to e^a.$$

(v) We have

$$\begin{aligned}\Pr(\text{some common birthday} &= 1 - \Pr(\text{no common birthday})\\ &= 1 - \prod_{r=1}^{m}\left(1 - \frac{r}{365}\right).\end{aligned}$$

If m is not large, then taking $a = -1$, $n = 365$, $N(n) = m$ in (iv) we have

$$\Pr(\text{some common birthday}) \approx 1 - e^{aN(n)^2/2} = 1 - e^{-m^2/365}.$$

(vi) Thus the approximate number required for the probability of joint birthdays to be at least $1/2$ is given by

$$m^2/(2 \times 365) = \log 2$$

so $m \approx 22.49$. This suggests 23 people are required.

Exact calculation shows that with 22 people the probability of coincidence is about .48 (and certainly less than $1/2$) and with 23 people the probability of coincidence is about .51 (and certainly more than $1/2$).

(vii) We take a 365 day year.

$$\begin{aligned}\Pr(\text{some common birth-hour}) &= 1 - \Pr(\text{no common birth-hour})\\ &= 1 - \prod_{r=1}^{m}\left(1 - \frac{r}{365 \times 24}\right).\end{aligned}$$

Proceeding as in (vi) and (vii) with $a = -1$, $n = 365 \times 24$, $N(n) = m$ we see that the approximate number required for the probability of joint birth-hours to be at least $1/2$ is given by

$$m^2/(2 \times 365 \times 24) = \log 2$$

so $m \approx 110.19$. This suggests 111 people are required.

(viii) This is just the birthday problem again. The probability of two people getting the same password is approximately $1 - e^{-m^2/2n}$ and to make this small we need $m^2/2n$ very small.

Exercise 10.5.11

(i) This is just the basic case of the AM/GM inequality. A direct proof is as follows.

$$\left(\frac{x+y}{2}\right)^2 - xy = \left(\frac{x-y}{2}\right)^2 \geq 0$$

with equality if and only if $x = y$.

(ii) If we specify who is to have which birthday the probability is $p_{j(1)}p_{j(2)}\dots p_{j(n)}$. But there are $n!$ ways of assigning birthdays in this pattern so

$$\begin{aligned}&\Pr(\text{together the } n \text{ people have birthdays on days } j(1),\, j(2),\, \dots,\, j(n))\\ &\qquad\qquad = n! p_{j(1)}p_{j(2)}\dots p_{j(n)}.\end{aligned}$$

and summing over all sets of n distinct birthdays

$$\begin{aligned}&\Pr(\text{the } n \text{ people do not share birthdays})\\ &\qquad\qquad = n! \sum_{1\leq j(1)<j(2)<\dots<j(n)\leq N} p_{j(1)}p_{j(2)}\dots p_{j(n)}.\end{aligned}$$

(iii) Thus collecting terms

$$\begin{aligned}&\sum_{1\leq j(1)<j(2)<\dots<j(n)\leq N} p_{j(1)}p_{j(2)}\dots p_{j(n)}\\ &\qquad = p_1p_2 \sum_{3\leq j(3)<\dots<j(n)\leq N} p_{j(1)}p_{j(2)}\dots p_{j(n)}\\ &\qquad + (p_1+p_2) \sum_{3\leq j(2)<\dots<j(n)\leq N} p_{j(1)}p_{j(2)}\dots p_{j(n)}\\ &\qquad + \sum_{3\leq j(1)<j(2)<\dots<j(n)\leq N} p_{j(1)}p_{j(2)}\dots p_{j(n)}.\end{aligned}$$

(iv) If q_j and p_j are as specified

$$\begin{aligned}
\sum_{1\leq j(1)<j(2)<\dots<j(n)\leq N} & p_{j(1)}p_{j(2)}\dots p_{j(n)} \\
&= p_1p_2 \sum_{3\leq j(3)<\dots<j(n)\leq N} p_{j(1)}p_{j(2)}\dots p_{j(n)} \\
&+ (p_1+p_2) \sum_{3\leq j(2)<\dots<j(n)\leq N} p_{j(1)}p_{j(2)}\dots p_{j(n)} \\
&+ \sum_{3\leq j(1)<j(2)<\dots<j(n)\leq N} p_{j(1)}p_{j(2)}\dots p_{j(n)} \\
&= p_1p_2 \sum_{3\leq j(3)<\dots<j(n)\leq N} q_{j(1)}q_{j(2)}\dots q_{j(n)} \\
&+ (p_1+p_2) \sum_{3\leq j(2)<\dots<j(n)\leq N} p_{j(1)}q_{j(2)}\dots q_{j(n)} \\
&+ \sum_{3\leq j(1)<j(2)<\dots<j(n)\leq N} q_{j(1)}q_{j(2)}\dots q_{j(n)} \\
&\leq q_1q_2 \sum_{3\leq j(3)<\dots<j(n)\leq N} q_{j(1)}q_{j(2)}\dots q_{j(n)} \\
&+ (q_1+q_2) \sum_{3\leq j(2)<\dots<j(n)\leq N} q_{j(1)}q_{j(2)}\dots q_{j(n)} \\
&= \sum_{1\leq j(1)<j(2)<\dots<j(n)\leq N} q_{j(1)}q_{j(2)}\dots q_{j(n)}
\end{aligned}$$

with equality if and only if $p_1 = p_2$.

This gives the required result.

(v) If it is not true that $p_j = 1/n$ for all j then we can find $r \neq s$ with $p_r \neq p_s$. Applying (iv) we can find another set of birthday probabilities such that coincidences are strictly less probable.

Thus if there is a set of probabilities which minimises the probability of coincidence, this must be $p(j) = 1/N$ for all j.

Exercise 10.5.12

If k is small and we book $500+k$ people the probability that there will be j cancellations (with j also small) is approximately

$$\frac{1}{j!}\left(\frac{500+k}{500}\right)^j \exp\left(-\frac{500+k}{500}\right) \approx \frac{1}{j!}e^{-1}.$$

Thus our expected extra profit if we overbook by k is approximately.

$$\begin{aligned} u_k &= 100k - 200\sum_{r=0}^{k}(k-r)\Pr(r \text{ cancellations}) \\ &= 100k - 200\sum_{r=0}^{k}\frac{1}{r!}(k-r)e^{-1} \end{aligned}$$

Now

$$u_k - u_{k-1} = 100 - 200\sum_{r=0}^{k}\frac{1}{r!}e^{-1} = v_k$$

say. We observe that $v_{k+1} - v_k < 0$ and $v_k \to -100$ as $k \to \infty$. Thus u_k increases to a maximum and then decreases. Now

$$u_1 - u_0 = 100 - 200e^{-1} \approx 26.4 > 0$$

and

$$u_2 - u_1 = 100 - 200 \times 2e^{-1} \approx -47.1$$

so we should make 501 bookings.

Exercise 10.5.13

The probability that a random combination will not contain 1 to 5 is

$$\frac{44}{49} \times \frac{43}{48} \times \frac{42}{47}\frac{41}{46} \times \frac{40}{45} \times \frac{38}{44} \approx .49$$

so about half the combinations are used. In about half the draws a 'forbidden number will turn up' and no one will get a prize. In about half the draws there will be no forbidden draws and the result will be those of lottery in which the number of possible out comes has been more or less halved.

In the notation used in this section instead of the chance of any single participant winning being m/N then (in non-forbidden weeks) the chance of any single participant winning being $2m/n$.

$$\frac{\Pr(r \text{ winners in non-forbidden week})}{\Pr(r \text{ winners in old scheme})} = \frac{\big((2m)^r/r!\big)e^{-2m}}{\big(m^r/r!\big)e^{-m}} = 2^r e^{-m}$$

and, as one would expect there is a much higher chance of multiple prize winners.

In cases (C) and (D) the actions of others do not influence the prize. Whatever you do you have the same expectation of about half a pound.

Now look at (A) and (B). In this case if you do (b) or (c) you are guaranteed to be sole winner so you are better off than the others. Cases (b) and (c) offer the same chance of winning. (If there were other smaller prizes offered you might be better under (b) but that is a different question.)

In case (d) you have the same chance of winning with k others [$k = 0, 1, 2, \dots$] as everybody else. If you do (a) you have probability about 1/2 of ending up in a case like (b) or (c) and a probability about 1/2 of ending up in a case like (d). You are better off than under (d) but worse off than under (b) or (c).

However, in case (A) if you choose (d) and if there is then a probability p that you will win, it will then be the case that the probability that you will win and not share is not far from $6p/7$. Thus although you are better off doing (b) or (c) you are not much better off. If you do (b) or (c) you have expectation of about half a pound.

In case (B) it is clear that (b) and (c) are much better than (d). The twenty eight million other players have bought tickets in a lottery where with probability about 1/2 there is one prize (shared if necessary) of fourteen million pounds. Each of their tickets is worth 1/4 pounds. You have bought one ticket with probability about one in fourteen million you get a prize of fourteen million. Your ticket is worth one pound. If there is no roll over of prize money the organisers are delighted since they have been given twenty eight million and have probability about 1/2 of having to disburse fourteen million.

The value of your ticket if you do (a) is 5/8 pounds since you have probability 1/2 of gaining a ticket worth 1 and probability 1/2 of gaining a ticket worth 1/4.

Exercise 10.6.1*

Exercise 10.6.2*

Exercise 10.6.3*

Exercise 11.1.1

(i) I toss r coins. The probability that they all come down heads is 2^{-r} and I stop, otherwise my expected time from then on is u_r. Thus

$$u_r = r + (1 - 2^{-r})u_r$$

so

$$2^{-r}u_r = r$$

and

$$u_r = r2^{-r}.$$

(ii) This is similar to (i) except that the probability that they come down all heads or all tails is is $2^{-r} + 2^{-r} = 2^{-r+1}$ and the argument as before gives $v_r = r2^{-r+1}$.

(iii) $e_1 = u_1$ so $e_1 = 1 + 2^{-1}e_1$ and $e_1 = 2 = 2^{1+1} - 2$. The expected time to reach r heads is e_r. Once I am at r heads then after one further throw with probability $1/2$ I reach $r + 1$ heads or with probability $1/2$ I must start again and expect to take e_{r+1} to reach $r + 1$ heads. Thus

$$e_{r+1} = e_r + 1 + \tfrac{1}{2}e_{r+1}$$

and

$$e_{r+1} - 2e_r = 2.$$

Using induction or solving the difference equation we get $e_r = 2^{r+1} - 2$.

(iii) Let g_r be the expected time to throw r heads or tails starting from one head. Observe that by symmetry g_r is the expected time to throw r heads or tails starting from one tail. Further $f_r = 1 + g_r$ (since we must throw heads or tails in our first throw) and $g_1 = 0$. The expected time to reach r starting from one head is g_r. Once I am at r, then after one further throw with probability $1/2$ I reach $r + 1$ same or with probability $1/2$ I must start again with one coin already thrown so my expected time from then to $r + 1$ is g_{r+1}. Thus

$$g_{r+1} = g_r + 1 + \tfrac{1}{2}g_{r+1}$$

and

$$g_{r+1} - 2g_r = 2.$$

Solving the difference equation we get

$$g_r = A2^r - 2.$$

Since $g_1 = 0$ we have $A = 1$, $g_r = 2^r - 2$ and $f_r = 2^r - 1$.

[Or we could argue that half are all heads and half all tails so $e_r = 2f_r$.]

(iv) In (i) if a tails appears you have to go on throwing until you have done a full r tosses before you start again looking for r in a row. In (iii) you start again immediately after the next throw.

Exercise 11.2.1*

Exercise 11.2.2

number of heads out of 20	0	1	2	3	4	5	6	7
probability if $p = 1/20$	.122	.270	.285	.190	.090	.032	.009	.002

number of heads out of 20	0	1	2	3	4	5	6	7
probability if $p = 1/2$	.000	.000	.000	.001	.005	.012	.029	.058

If we reject on 5 heads or less we will reject good coins with probability about .02 and accept bad coins with probability about .01. (But we might choose differently if we wished to have very small probability of rejecting good coins or if we wished to have very small probability of accepting bad coins.)

Exercise 11.3.2

Completing the square,

$$ns^2 - as = n(s^2 - n^{-1}as) = n\left(s - \frac{a}{2n}\right)^2 - \frac{a^2}{4n}$$

which is minimised by taking $s = a/(2n)$.

Exercise 11.3.3

(i) We have

$$\frac{e^t + e^{-t}}{2} = \frac{1}{2}\left(\sum_{r=0}^{\infty}\frac{t^r}{r!} + \sum_{r=0}^{\infty}(-1)^r\frac{t^r}{r!}\right)$$
$$\sum_{r=0}^{\infty}\frac{t^{2r}}{(2r)!} \geq 1$$

and

$$\begin{aligned}\frac{e^t + e^{-t}}{2} &= \sum_{r=0}^{\infty}\frac{t^{2r}}{(2r)!}\\ &\leq \sum_{r=0}^{\infty}\frac{t^{2r}}{2^r(r!)}\\ &= e^{t^2/2}.\end{aligned}$$

(ii) If $f(x) = -e^{sx}$ then $f''(x) = -s^2e^{sx} < 0$ for all x so by concavity

$$f(x) = f\left(\frac{1-x}{2}\times(-1) + \frac{1+x}{2}\times 1\right) \geq \frac{1-x}{2}f(-1) + \frac{1-x}{2}f(1)$$

that is to say

$$e^{sx} \leq \frac{1-x}{2}e^{-s} + \frac{1+x}{2}e^s$$

for all s and all $|x| \leq 1$.

(iii) Using (ii), we have, since $|Y| \leq 1$,

$$e^{sY} \leq \frac{1-Y}{2}e^{-s} + \frac{1+Y}{2}e^s$$

so taking expectations,

$$\begin{aligned}\mathbb{E}e^{sY} &\leq \frac{1-\mathbb{E}Y}{2}e^{-s} + \frac{1+\mathbb{E}Y}{2}e^s\\ &= \frac{e^s + e^{-s}}{2} \leq e^{-s^2/2}.\end{aligned}$$

(iv) If $a = 0$ the result is trivial. If $a > 0$ set $Y = X/a$ so $|Y| \leq 1$ and

$$\mathbb{E}e^{tY} \leq e^{-t^2/2}.$$

Now set $t = sa$ to recover

$$\mathbb{E}e^{sX} \leq e^{a^2s^2/2}.$$

(v) Since the X_j are independent, so are the e^{sX_j} and, using (iv),

$$\mathbb{E}e^{s\sum_{j=1}^n X_j} = \mathbb{E}\prod_{j=1}^n e^{sX_j} = \prod_{j=1}^n \mathbb{E}e^{sX_j} \leq \prod_{j=1}^n \mathbb{E}e^{a_j^2 s^2/2} = As^2/2.$$

(vi) Set $Y = \sum_{j=1}^n X_j$. If $s > 0$ so $y \mapsto e^{ys}$ is increasing, we have

$$\Pr(Y \geq y) \leq \frac{\mathbb{E}e^{s\Pr(Y\geq y)}}{e^{sy}} \leq \exp\left(\frac{As^2}{2} - sy\right)$$

and setting $s = A/y$ we obtain

$$\Pr(Y \geq y) \leq \exp\left(-\frac{y^2}{2A}\right).$$

Replacing X_j by $-X_j$ or repeating the argument we have

$$\Pr(Y \leq -y) \leq \exp\left(-\frac{y^2}{2A}\right)$$

and so, combining our two results

$$\Pr(|Y| \leq y) \leq 2\exp\left(\frac{y^2}{2A}\right).$$

(vii) Set $Z_j = 1$ if the jth toss is heads and $Z_j = 0$ if the jth toss is tails. If we set

$$X_j = Z_j - p$$

then $\mathbb{E}X_j = 0$ and $|X_j| \leq p$ so

$$\begin{aligned}\Pr(|Y_n - np| \geq Kn^{1/2}) &= \Pr\left(\left|\sum_{j=1}^n X_j\right| \leq Kn^{1/2}\right)\\ &\leq 2\exp\left(-\frac{(Kn^{1/2})^2}{2np^2}\right)\\ &\leq 2\exp\left(-\frac{K^2}{2p^2}\right)\end{aligned}$$

for all $K > 0$.

If $1/2 \geq p \geq 0$ then since

$$Y_n - np = -\big((n - Y_n) - n(1-p)\big)$$

we can reverse the roles of heads and tails to get

$$\Pr(|Y_n - np| \geq Kn^{1/2}) \leq 2\exp\left(-\frac{K^2}{2(1-p)^2}\right).$$

Exercise 11.3.4

We have

$$\begin{aligned}\Pr\big(|Y_n - nq| \le n\epsilon\big) &\le \Pr\big(|Y_n - np| \ge n\epsilon\big)\\ &\le 2\exp\big(-n\epsilon^2/2).\end{aligned}$$

Exercise 11.4.1

The symmetry of our demands suggest we try for n odd and accept the drug if number of successes exceeds 0 and reject if number of successes less than 0.

In the notation of Exercise 11.3.3 (vii) with $p = .55$ we want

$$\Pr(Y_n - .5n < 0) \leq .05$$

ie

$$\Pr(Y_n - .55n < -.05n) \leq .05$$

so setting, $K = .05n^{1/2}$,we see that the result of Exercise 11.3.3 (vii) guarantees this if

$$\exp\left(-\frac{(.05n^{1/2})^2}{2(.55)^2}\right) \leq .05$$

ie if

$$\exp(-n/(2 \times 11^2), \leq .05$$

ie

$$n \geq 2 \times 11^2 \log 20$$

so taking $n = 725$ will give the desired outcome.

Exercise 11.4.2

If we have an excess of r with $-a+1 \leq r \leq a-1$ and make a further throw then with probability p we will then have an excess $r+1$ and the probability of acceptance will be u_{r+1} and with probability q we will then have an excess $r-1$ and the probability of acceptance will be u_{r-1}. Thus

$$u_r = qu_{r-1} + pu_{r+1}.$$

If $r = a$ then we accept so $u_a = 1$. If $r = -a$ we reject so $u_{-a} = 0$.

We have

$$(pE^2 - E + qI)u_r = 0$$

so

$$(E - I)(pE - qI)u_r = 0$$

and

$$u_r = A + B\left(\frac{q}{p}\right)^r$$

for some A and B. Setting $r = a$ and $r = -a$ we get

$$1 = A + B\left(\frac{q}{p}\right)^a$$

$$0 = A + B\left(\frac{q}{p}\right)^{-a}$$

so subtracting we have

$$1 = B\left(\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^{-a}\right).$$

Thus

$$u_r = \frac{\left(\frac{q}{p}\right)^r - \left(\frac{q}{p}\right)^{-a}}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^{-a}}$$

Setting $r = 0$ we see that the probability of acceptance is

$$\begin{aligned} u_0 &= \frac{1 - \left(\frac{q}{p}\right)^{-a}}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^{-a}} \\ &= \frac{1}{\left(\frac{q}{p}\right)^a}\frac{1}{1 + \left(\frac{q}{p}\right)^{-a}} \\ &\frac{1}{1 + \left(\frac{q}{p}\right)^a} \end{aligned}$$

The probability of acceptance increases as p increase so the probability of rejection is smaller than 0.05 for $p \geq .55$ is given by

$$\frac{1}{1+\left(\frac{.45}{.55}\right)^a} \geq .95$$

that is to say

$$1+\left(\frac{9}{11}\right)^a \leq \frac{20}{19}$$

ie

$$\left(\frac{9}{11}\right)^a \leq \frac{1}{19}$$

ie

$$a \geq \frac{\log 19}{\log(11/9)} \approx 14.7$$

We should take $a = 15$. By symmetry this will also ensure that the probability of acceptance is less than 0.05 if $p \leq .45$.

If $a = b$ then fixing a to give a certain probability of acceptance for given $p = p'$ also fixes the probability of rejection. for given $p = p''$. If we can chose b as well we can choose both probabilities.

Exercise 11.4.3

If the excess is r then, if $a-1 \geq r \geq 1-a$, making one further trial there is a probability $1/2$ we throw a head, so we now expect to wait e_{r+1} to the end of the trial, and a probability $1/2$ we throw a tail, so we now expect to wait e_{r-1} to the end of the trial. If $r=-a$ or $r=a$, the number of further throws required is 0 so $e_a = e_{-a} = 0$. Thus

$$e_r = 1 + \tfrac{1}{2}e_{r-1} + \tfrac{1}{2}e_{r+1} \text{ for } -a+1 \leq r \leq a-1,$$

and $e_a = e_{-a} = 0$.

A complementary solution of our difference equation

$$(E-I)^2 u_r = -1$$

is $u_r = Cr^2$ with

$$C\big((r+1)^2 - 2r^2 + (r-1)^2\big) = -2$$

ie $C=-1$. Thus

$$e_r = A + Br - r^2.$$

Since $e_a = u_{-a}$, $B=0$. Since $u_a = 0$ $A = a^2$. Thus

$$e_r = a^2 - r^2$$

and the expected number of trials when we start is a^2.

If $a = 15$ this gives 225 expected trials.

Exercise 11.4.4

(i) Write $e_r = e_r(p)$. If $-a+1 \leq r \leq a-1$, then if the excess of successes is r then, after 1 further trial, with with probability p we will have an excess $r+1$ and the expected number of trials remaining will be e_{r+1} while with probability q we will have an excess $r-1$ and the expected number of trials remaining will be e_{r-1}. Thus

$$e_r = 1 + qe_{r-1} + pe_{r+1}$$

If the excess is a or $-a$, no further trials are required so $e_a = e_{-a}$.

We have

$$(pE^2 - E + qI)e_r = 1$$

or

$$(pE - q)(E-1)e_r = 1.$$

We seek a particular solution of the for $e_r = Kr$ obtaining $K = (p-q)^{-1}$. Thus

$$e_r = A + B\frac{q}{p}^r + \frac{r}{p-q}.$$

Setting $r = a$ and $r = -a$ we obtain

$$0 = A + B\frac{q}{p}^a + \frac{a}{p-q}$$
$$0 = A + B\frac{q}{p}^{-a} - \frac{a}{p-q}$$

Thus

$$e_r(p) = \frac{2a}{p-q}\left(\frac{1-\left(\frac{q}{p}\right)^{r+a}}{\left(1-\left(\frac{q}{p}\right)^{2a}\right)}\right) - \frac{r+a}{p-q}.$$

Setting $r = 0$ we see that the expected number of trials required by our test is

$$\begin{aligned}
e_0 &= \frac{2a}{p-q}\left(\frac{1-\left(\frac{q}{p}\right)^{a}}{\left(1-\left(\frac{q}{p}\right)^{2a}\right)}\right) - \frac{a}{p-q} \\
&= \frac{a}{p-q}\left(\left(\frac{2}{\left(1+\left(\frac{q}{p}\right)^{a}\right)}\right) - 1\right) \\
&= \frac{a}{p-q}\left(\frac{1-\left(\frac{q}{p}\right)^{a}}{1+\left(\frac{q}{p}\right)^{a}}\right).
\end{aligned}$$

(ii) If
$$f(x) = \left(\frac{\frac{1}{2} - x}{\frac{1}{2} + x}\right)^a$$
then
$$f'(x) = a\left(\frac{\frac{1}{2} - x}{\frac{1}{2} + x}\right)^{a-1}\left(-\frac{1}{\frac{1}{2} + x} - \frac{\frac{1}{2} - x}{(\frac{1}{2} + x)^2}\right)$$
and
$$f'(0) = -4a$$
Thus setting $p = \frac{1}{2} + \delta$ so $q = \frac{1}{2} - \delta$
$$\frac{a}{p-q}\left(1 - \left(\frac{q}{p}\right)^a\right) = \frac{a}{2\delta}\big(f(0) - f(\delta)\big) \to -\frac{a}{2}f'(0) = 2a^2$$
and
$$R(p, a) \to \frac{2a^2}{2} = a^2 = R(\tfrac{1}{2}, a)$$
as $p \to 0$.

(iii) We have
$$a^1 R(p, a) = \frac{1}{p-q}\left(\frac{1 - \left(\frac{q}{p}\right)^a}{1 + \left(\frac{q}{p}\right)^a}\right) \to \frac{1}{p-q}\frac{1-0}{1+0} = \frac{1}{p-q}$$
as $a \to \infty$. By the law of large numbers, we expect that if we have a large number of trials N (which we must have if a is large), then roughly Np will be success and Nq failures (with high probability) so if we finish after N trials
$$a \approx Np - Nq$$
and we expect $N \approx a(p-q)^{-1}$.

If $p < 1/2$
$$a^{-1}R(p, a) \to \frac{1}{q-p}$$
as $a \to \infty$.

(iii) We have $a = 15$, $p = 2/3$, $q = 1/3$ so
$$R(p, a) = \frac{15}{1/3}\frac{2^{15} - 1}{2^{15} + 1} \approx 45.$$

Exercise 12.1.1

If $A \geq 0$ then setting $u = 1$, $v = 0$ we see that the economists' condition fails. Thus the economists' condition implies $A < 0$. Similarly the economists' condition implies $C < 0$. If A, $C < 0$

$$Av^2 - 2Buv + Cu^2 = A(v - uBA^{-1})^2 + (C - B^2A^{-1})u^2.$$

Setting $u = 1$, $v = BA^{-1}$ we see that the economists' condition implies $C - B^2A^{-1} < 0$, ie $CA - B^2 > 0$ so $CA > B^2$.

Conversely, if A, $C < 0$ and $CA > B^2$ then

$$Av^2 - 2Buv + Cu^2 \geq 0$$

implies

$$A(v - uBA^{-1})^2 + (C - B^2A^{-1})u^2 \geq 0$$

so $v - uBA^{-1} = 0$ and $u = 0$ whence $u = v = 0$. Thus the economists' condition holds.

Exercise 12.1.2

We require $x_0 = y_0 > 0$, $a, b > 0$, $u, v > 0$, $B^2 < AC$, $av - bu = 0$, $A, C < 0$. We could take $a = b = x_0 = y_0 = u = v = 1$, $A = C = -2$ $B = 1$.

Exercise 12.1.3

We have

$$\bigstar \qquad u\triangle x + v\triangle y = \triangle c$$

and

$$\tilde{a}v - \tilde{b}u = 0$$

where

$$\tilde{a} = a + A\triangle x + B\triangle y, \text{ and } \tilde{b} = b + B\triangle x + C\triangle y.$$

Thus

$$(a + A\triangle x + B\triangle y)v - (b + B\triangle x + C\triangle y)u = 0.$$

But $au - bv = 0$ so

$$(Av - Bu)\triangle x - (Bv - Cu)\triangle y = 0$$

Using $\bigstar$, this gives

$$(Av + Bu)(\triangle c - v\triangle y) - u(Bv + Cu)\triangle y = 0$$

so

$$(Av^2 - 2Buv + Cu^2)\triangle y = (Av + Bu)\triangle c$$

ie

$$\triangle y = \frac{Av + Bu}{Av^2 - 2Buv + Cu^2}\triangle c.$$

y will decrease as c increases if $Av + Bu > 0$.

Exercise A.1

(ii) Rescale y axis by a and x axis by a^{-1}. If $f(x) = 1/x$ the graph is unchanged.

(iii) Make the change of variable $s = a^{-1}x$ to obtain

$$\int_a^b \frac{1}{x}\,dx = \int_1^{b/a} \frac{1}{as}\,ads = \int_1^{b/a} \frac{1}{s}\,ds.$$

(iv) We have, using (iii),

$$\begin{aligned}
\log uv &= \int_1^{uv} \frac{1}{x}\,dx \\
&= \int_1^{u} \frac{1}{x}\,dx + \int_u^{uv} \frac{1}{x}\,dx \\
&= \int_1^{u} \frac{1}{x}\,dx + \int_1^{v} \frac{1}{x}\,dx \\
&= \log u + \log v.
\end{aligned}$$

EXERCISE A.2

(ii) By the fundamental theorem of the calculus (looked at in (i)) log is differentiable and

$$\log'(t) = \frac{d}{dt}\int_1^t \frac{1}{x}\,dx = \frac{1}{t}.$$

Exercise A.3

(i) Since $\log'(t) = 1/t > 0$, log is a strictly increasing function.

(ii) Observe that
$$\log 1 = \int_1^1 \frac{1}{x}\, dx = 0.$$

(iii) Since $2 > 1$, we have $\log 2 > \log 1 = 0$ so
$$\log 2^n = n \log 2 \to \infty.$$

(iv) Since log is increasing (iii) tells us that
$$\log x \to \infty$$
as $x \to \infty$.

(v) Since $1/x \to \infty$ as $x \to 0+$
$$\log x = -\log(1/x) \to -\infty$$
as $x \to \infty$.

(vi) We have
$$\frac{d^2}{dx^2} \log x = \frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2} < 0.$$

Exercise A.4

(iii) By part (ii), exp is differentiable with

$$\frac{d}{dx}\exp x = \frac{d}{dx}\log^{-1} x = \frac{1}{\log' x} = \frac{1}{x^{-1}} = x.$$

(iv) An appropriate answer is that log is defined only on the strictly positive reals. But there are other ways of looking at this.

Exercise A.5

(i) We have

$$\log\bigl((\exp x)(\exp y)\bigr) = \log\exp x + \log\exp y = x + y,$$

so applying exp to both sides yields

$$(\exp x)(\exp y) = \exp(x + y).$$

(ii) Observe that $\exp' x = \exp x > 0$.

(iii) $\log\exp x = x \to \infty$ as $x \to \infty$ so $\exp x = x \to \infty$ as $x \to \infty$.

(iv) $\exp x = 1/(\exp -x) \to 0$ as $x \to -\infty$.

Exercise A.6

(i) $\exp n \log x = \exp \log x^n = x^n$. (If you need a more detailed proof, use induction.)

(ii) If $n = 0$, $\exp 0 \log x = \exp 0 = 1$. If $n < 0$,

$$x^{-n}(\exp n \log x) = \exp(-n \log x) \exp(n \log x) = \exp 0 = 1$$

so $\exp n \log x = x^n$.

(iii) We have

$$\left(\exp\left(\frac{m}{n}\log x\right)\right)^n = \exp n\left(\frac{m}{n}\log x\right) = \exp m \log x = x^m$$

and so

$$\exp\left(\frac{m}{n}\log x\right) = x^{m/n}$$

where $x^{m/n}$ has its standard elementary meaning.

(iv) We have

$$e^x = \exp(a \log e) = \exp(a \log \exp 1) = \exp(a \times 1) = \exp a.$$

Exercise A.7

(i) Observe that

$$\begin{aligned}(xy)^a &= \exp(a \log xy) = \exp\big(a(\log x + \log y)\big)\\ &= \exp(a \log x + a \log y) = (\exp a \log x)(\exp a \log y) = x^a y^a.\end{aligned}$$

(ii) Observe that

$$\begin{aligned}x^{a+b} &= \exp\big((a+b)\log x\big) = \exp(a \log x + b \log x)\\ &= \exp(a \log x)\exp(b \log x) = x^a x^b.\end{aligned}$$

(iii) Observe that

$$x^{ab} = \exp(ab \log x) = \exp\big(a(b \log x)\big) \quad = \exp\big(a \log(x^b)\big) = (x^b)^a.$$

Thus

$$x^{ab} = x^{ba} = (x^a)^b.$$

Exercise A.8

(i) We have

$$\frac{\log(1+h)}{h} = \frac{\log(1+h) - \log 1}{h} \to \log' 1 = 1$$

as $h \to 0$.

(ii) Taking $h = a/n$ in (i), we get

$$\frac{\log(1 + a/n)}{a/n} \to 1$$

as $n \to \infty$.

(iii) Multiplying through by a and simplifying, we get

$$n \log(1 + a/n) \to a$$

as $n \to \infty$.

(iv) Taking the exponential of both sides we get

$$\left(1 + \frac{a}{n}\right)^n \to e^a$$

as $n \to \infty$.

(v) We want

$$\left(1 + \frac{x}{100}\right)^n = 2$$

so

$$n \log\left(1 + \frac{x}{100}\right) = \log 2.$$

If x is small, part (i) gives

$$\frac{x}{100} \approx \log 2$$

so

$$n \approx \frac{100 \times \log 2}{x} \approx \frac{69}{n} \approx \frac{72}{n}$$

(note that 72 has many small factors).

As an example suppose $x = 4$. The rule of 72 gives a doubling time of $72/4 = 18$ years. In fact

$$\left(1 + \frac{4}{100}\right)^{18} \approx 2.03$$

which is not bad.

If we take $x = 8$ we have

$$\left(1 + \frac{8}{100}\right)^9 \approx 2.$$

If we take $x = 24$ (a rate more often on the lips of snake oil salesmen than respectable bankers), then

$$\left(1 + \frac{24}{100}\right)^3 \approx 1.91.$$

Exercise A.9

(i) Observe that

$$(1-t)(1+t+t^2+\dots+t^n) = 1-t^{n+1}$$

so, dividing by $(1-t)$ and rearranging, we get

$$\frac{1}{1-t} = 1+t+\dots+t^n+\frac{t^{n+1}}{1-t}.$$

Integrating both sides, gives

$$\int_0^x \frac{1}{1-t}\,dt = \int_0^x \left(1+t+\dots+t^n+\frac{t^{n+1}}{1-t}\right)\,dt$$

$$x+\frac{x^2}{2}+\dots+\frac{x^n}{n}+R_n(x)$$

where

$$R_n(x) = \int_0^x \frac{t^{n+1}}{1-t}\,dt.$$

But making the substitution $s = 1-t$ gives

$$\int_0^x \frac{1}{1-t}\,dt = -\int_1^{1-x} \frac{1}{s}\,ds = -\log(1-x),$$

so

$$-\log(1-x) = x+\frac{x^2}{2}+\dots+\frac{x^n}{n}+R_n(x).$$

(ii) If $|t| \leq |x| < 1$, then $|1-t| \geq 1-|t| \geq 1-|x|$ so

$$\left|\frac{t^{n+1}}{1-t}\right| \leq \frac{|x|^{n+1}}{|1-t|} \leq \frac{|x|^{n+1}}{1-|x|}$$

and so

$$|R_n(x)| \leq \left|\int_0^x \frac{|x|^{n+1}}{1-|x|}\,dt\right| \leq \frac{|x|^{n+2}}{1-|x|} \to 0$$

as $n \to \infty$.

(iii) Combining (i) and (ii),

$$x+\frac{x^2}{2}+\dots+\frac{x^n}{n} = -\log(1-x)-R_n(x) \to -\log(1-x)$$

as $n \to \infty$. In other words,

$$-\log(1-x) = x+\frac{x^2}{2}+\dots+\frac{x^n}{n}+\dots$$

for all $|x| < 1$.

Setting $x = -y$, we obtain

$$\log(1+y) = y-\frac{y^2}{2}+\dots+\frac{(-1)^{n+1}y^n}{n}+\dots$$

for all $|y| < 1$.

Exercise A.10

(i) Set

$$h_+(x) = \begin{cases} h(x) & \text{if } h(x) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h_-(x) = \begin{cases} h(x) & \text{if } h(x) \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \left| \int_a^b h(x)\,dx \right| &= \left| \int_a^b h_+(x) + h_-(x)\,dx \right| \\ &= \left| \int_a^b h_+(x)\,dx + \int_a^b h_-(x)\,dx \right| \\ &\leq \max\left(\int_a^b h_+(x)\,dx, - \int_a^b h_-(x)\,dx \right) \\ &= \max\left(\int_a^b h_+(x)\,dx, \int_a^b \big(- h_-(x) \big)\,dx \right). \end{aligned}$$

But

$$0 \leq -h_-(x) \leq g(x)$$

so

$$\int_a^b \big(- h_-(x) \big)\,dx \leq \int_a^b g(x)\,dx.$$

Similarly

$$\int_a^b h_+(x)\,dx \leq \int_a^b g(x)\,dx$$

so

$$\left| \int_a^b h(x)\,dx \right| \leq \int_a^b g(x)\,dx.$$

(ii) If $1 \leq r \leq n$ and

$$f^{(r)}(x) \leq A \frac{|x|^{n-r}}{(n-r)!}$$

then, using part (i),

$$\begin{aligned}|f^{(r-1)}(x)| &= \left|\int_0^x f^{(r)}(t)\,dt\right| \\ &\leq \left|\int_0^x A\frac{|x|^{n-r}}{(n-r)!}\,dt\right| \\ &= \int_0^{|x|} A\frac{x^{n-r}}{(n-r)!}\,dt \\ &= A\frac{|x|^{n-r+1}}{(n-r+1)!}.\end{aligned}$$

Thus, by induction,

$$|f(x)| \leq A\frac{|x|^n}{n!}$$

for all $|x| \leq X$.

(iii) Since exp is increasing

$$0 < \exp x \leq \exp X$$

for all $x \leq X$.

Now set

$$f(x) = \exp x - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{(n-1)!}\right).$$

By induction

$$f^{(r)}(x) = \exp x - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-r-1}}{(n-r-1)!}\right)$$

for $0 \leq r \leq n-1$ and

$$f^{(n)}(x) = \exp x.$$

Thus $|f^{(n)}(x)| \leq \exp X$ for all $|x| \leq X$ and $f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = 0$. By part (ii)

$$|f(x)| \leq \exp X \frac{|x|^n}{n!}$$

for all $|x| \leq X$.

(iv) Thus

$$\left|\exp x - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{(n-1)!}\right)\right| \leq e^X\frac{|x|^n}{n!} \to 0$$

for all $|x| \leq X$. But X was arbitrary so

$$\left|\exp x - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{(n-1)!}\right)\right| \to 0$$

as $n \to \infty$ and this is the required result.

Exercise A.11

We use the following form of Taylor's theorem, obtained by integrating by parts and true if f is $n+1$ times continuously differentiable.

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)x^j}{j!} + R_n(f,x)$$

where

$$R_n(f,x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n\,dt.$$

If $f(x) = \exp(x)$ then $f^{(r)}(x) = \exp x$ so

$$\exp x = \sum_{j=0}^{n} \frac{x^j}{j!} + R_n(x)$$

where

$$|R_n(x)| = \left|\frac{1}{n!}\int_0^x \exp(t)(x-t)^n\,dt\right| \leq \frac{1}{n!}|x|^n \exp(|x|) \to 0$$

(observe $(|x|^{n+1}/(n+1)!)/(|x|^n/n!) = |x|/(n+1) \to 0$. Thus

$$\exp x = \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

If $f(x) = \log(1-x)$ then $f'(x) = -(1-x)^{-1}$ for $|x| < 1$ and, by induction,

$$f^{(r)}(x) = -\frac{(r-1)!}{(1-x)^r}$$

for all $r \geq 1$. Thus if $|x| < 1$

$$\log(1-x) = -\sum_{j=0}^{n} \frac{x^j}{j} + R_n(x)$$

where

$$|R_n(x)| = \left|\int_0^x \frac{(x-t)^n}{1-t)^{n+1}}\,dt\right|.$$

Now, if $0 \leq t \leq x$,

$$0 \leq \frac{x-t}{1-t} = 1 - \frac{1-x}{1-t} \leq 1-(1-x) = x = |x|$$

and, if $x \leq t \leq 0$,

$$\left|\frac{x-t}{1-t}\right| \leq \frac{|x|-|t|}{1-|t|} \leq |x|.$$

Thus

$$|R_n(x)| \leq \int_0^{|x|} |x|^n \frac{1}{1-|x|}\,dt \leq \frac{|x|^{n+1}}{1-|x|} \to 0$$

as $n \to \infty$. Thus

$$\log(1-x) = -\sum_{j=0}^{\infty} \frac{x^j}{j!}$$

for $|x| < 1$.

Exercise A.12

(i) We have

$$a^{\log_a x} = \exp\left(\log a \times \frac{\log x}{\log a}\right) = \exp \log x = x.$$

(ii) We have

$$\log_a xy = \frac{\log xy}{\log a} = \frac{\log x + \log y}{\log a} = \frac{\log x}{\log a} + \frac{\log y}{\log a} = \log_a x + \log_a y.$$

(iii) We have

$$\log_a x^k = \frac{\log x^k}{\log a} = k\frac{\log x}{\log a} = k \log_a x.$$

(iv) We have

$$\log_e x = \frac{\log x}{\log e} = \frac{\log x}{1} = \log x.$$

(v) We have

$$\log_a b \log_b a = \frac{\log b}{\log a} \times \frac{\log a}{\log b} = 1.$$

Exercise A.13

(i) We have

$$\log_{10} 5 = \log_{10} 10 - \log_{10} 2 \approx 1 - 0.301029996 \approx .699$$
$$\log_{10} 6 = \log_{10} 3 + \log_{10} 2 \approx 0.477121255 + 0.301029996 \approx .778$$

correct to three decimal places.

Now

$$\log_{10}(3^{100}/10^{47}) = 100 \log_{10} 3 - 47 \approx .712$$

correct to three decimal places so

$$\log_{10} 5 < \log_{10}(3^{100}/10^{47}) < \log_{10} 6$$

whence, since $\log_{10}$ is increasing,

$$5 < 3^{100}/10^{47} < 6.$$

Thus

$$5 \times 10^{47} < 3^{100} < 6 \times 10^{47}.$$

and the first digit of 3^{100} is 5.

(ii) We have

$$\log_{10} 2^{1000} = 301 + .029$$

correct to three decimal places so

$$10^{301} < 2^{1000} < 2 \times 10^{301}$$

and the first digit of 2^{1000} is 1.

We have

$$\log_{10} 2^{10\,000} < 3010 + .300 < 3010 + \log_{10} 2$$

so

$$10^{3010} < 2^{10\,000} < 2 \times 10^{3010}$$

so the first digit of $2^{10\,000}$ is 1.

We have

$$30103 > \log_{10} 2^{100\,000} > 30102{+}999 > 30102{+}2\log_{10} 3 = 30102{+}\log_{10} 9$$

so

$$10^{30103} > 2^{100\,000} > 9 \times 10^{30102}$$

and the first digit of $2^{100\,000}$ is 9.

(iii) We have

$$2^{100\,000} \equiv (2^{10})^5 \equiv 4^5 = \equiv 2^{10} = 4 \mod 10$$

so the last digit of $2^{100\,000}$ is 4.

We have

$$3^{100} \equiv (3^2)^{50} \equiv (-1)^{50} = \equiv 1 \mod 10$$

so the last digit of 3^{100} is 1.

EXERCISE B.1

$$\begin{aligned}
\Pr(\text{at least one shows 1}) &= 1 - \Pr(\text{none show 1}) \\
&= 1 - \Pr(\text{single die does not show 1})^3 \\
&= 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}
\end{aligned}$$

Exercise B.2

(i) If $y = x + d$ then $x = y - d$ and

$$x^n + a_1 x^{n-1} + \cdots + a_0 = (y-d)^n + a_1(y-d)^{n-1} + \cdots + a_0$$
$$y^n + (a_1 - dn)y^{n-1} + b_2 y^{n-2} + b_3 y^{n-3} + \cdots + b_n$$

for appropriate b_j.

Taking $d = a_1/n$ we have

$$x^n + a_1 x^{n-1} + \cdots + a_0 = y^n + b_2 y^{n-2} + b_3 y^3 + \cdots + b_n$$

so $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ if and only if $y^n + b_2 y^{n-2} + \cdots + b_n = 0$.

Suppose

$$x^3 + ax^2 + bx + c = 0.$$

Setting $y = x - a/3$ we obtain

$$y^3 + Bx + C = 0.$$

If α is a root of the new equation (which we can solve) then $\alpha + a/3$ is a root of the old and vice-versa.

(ii) By long division (or you can establish the result by induction on n if you prefer).

$$x^n + a_1 x^{n-1} + \cdots + a_n = (x - \alpha)(x^{n-1} + b_1 x^{n-2} + \cdots + b_0) + R$$

for some b_j and some R. Write

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n,\ Q(x) = x^{n-1} + b_1 x^{n-2} + \cdots + b_0.$$

Since

$$P(x) = (x - \alpha)Q(x) + R$$

it follows that if α is a root of P,

$$0 = 0 + R$$

so $R = 0$ and

$$P(x) = (x - \alpha)Q(x)$$

Thus

$$\begin{aligned} \beta \text{ is a root of } P &\Leftrightarrow P(\beta) = 0 \\ &\Leftrightarrow Q(\beta) = 0 \text{ and/or } \beta = \alpha \\ &\Leftrightarrow \beta \text{ is a root of } Q \text{ and/or } \beta = \alpha. \end{aligned}$$

(iii) By the arguments of (ii)

$$x^n + a_1 x^{n-1} + \cdots + a_n = \prod_{j=1}^{n} (x - \alpha_j)$$

where the α_j are the roots. Thus

$$x^n + a_1 x^{n-1} + \cdots + a_n = x^n - \sum_{j=1}^{n} \alpha_j x^{n-1} + \cdots + (-1)^n \prod_{j=1}^{n} \alpha_j$$

so, equating coefficients,

$$a_1 = -\sum_{j=1}^{n} \alpha_j, \;\; a_n = (-1)^n \prod_{j=1}^{n} \alpha_j.$$

Exercise B.3

We wish to find x and y such that $x + y = 10$ and $xy = 40$.

The first equation yields $y = 10 - x$ so substituting in the second we have
$$10x - x^2 = 40$$
or
$$x^2 - 10x + 40 = 0$$
so, using the standard formula
$$x = \frac{10 \pm \sqrt{1}00 - 400}{2} = 5(1 \pm 3^{1/2}i)$$
and
$$y = 10 - x = 5(1 \pm 3^{1/2}i).$$
Thus either $x = 5(1 + 3^{1/2}i)$, $y = 5(1 - 3^{1/2}i)$ or $x = 5(1 - 3^{1/2}i)$, $y = 5(1 + 3^{1/2}i)$. By inspection these are solutions.

Exercise B.4

By Exercise B.1, the probability of getting at least one 1 on any particular throw is 91/216. Thus, by independence, the probability of getting least one 1 on each of three throws is

$$\left(\frac{91}{216}\right)^3 \approx \frac{1}{13.373}$$

so the probability is a little less than 1/13 and the odds a little greater than 12 to 1.

Exercise B.5

If we consider the 216 ways in which three dice can land we see that nine can be produced by the following combinations of dice in any order

$$\begin{array}{l}
1,\ 2,\ 6 \text{ appearing in } 6 \text{ ways}\\
1,\ 3,\ 5 \text{ appearing in } 6 \text{ ways}\\
1,\ 4,\ 4 \text{ appearing in } 3 \text{ ways}\\
2,\ 2,\ 5 \text{ appearing in } 3 \text{ ways}\\
2,\ 3,\ 4 \text{ appearing in } 6 \text{ ways}\\
3,\ 3,\ 3 \text{ appearing in } 1 \text{ ways}
\end{array}$$

so that nine can appear in 25 ways.

On the other hand ten can be produced by the following combinations of dice in any order

$$\begin{array}{l}
1,\ 3,\ 6 \text{ appearing in } 6 \text{ ways}\\
1,\ 4,\ 5 \text{ appearing in } 6 \text{ ways}\\
2,\ 2,\ 6 \text{ appearing in } 3 \text{ ways}\\
2,\ 3,\ 5 \text{ appearing in } 6 \text{ ways}\\
2,\ 4,\ 4 \text{ appearing in } 3 \text{ ways}\\
3,\ 3,\ 4 \text{ appearing in } 3 \text{ ways}
\end{array}$$

so that ten can appear in 27 ways.

Thus nine has probability $p = 25/216$ and ten $q = 27/216$.

If we throw dice until either 9 or 10 comes up then the mathematics of the one-sided duel shows that the bettor on 10 has a probability

$$\frac{q}{p+q} \approx 0.519$$

so an expected return on a stake of 1 unit of .038. Again I wonder if the query arose directly from play. (It seems to me more likely that people knew that 'numbers closer to the average are thrown more often'.)

Exercise B.6

(i) If $f(x) = x^3 + ax - b$ then $f'(x) = 3x^2 + a \geq a > 0$ for all x so f is strictly increasing. Since $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$, the function f has exactly one real zero.

(ii) If we set $x = u^{1/3} + v^{1/3}$, then

$$\begin{aligned} x^3 + ax &= u + v + 3u^{1/3}v^{2/3} + 3u^{2/3}v^{1/3} + au^{1/3} + av^{1/3} \\ &= u + v + (3u^{1/3}v^{1/3} + a)(u^{1/3} + v^{1/3}). \end{aligned}$$

(iii) If

$$\begin{aligned} uv &= -\frac{a^3}{27} \\ u + v &= b. \end{aligned}$$

then

$$x^3 + ax = b + 0 \times (u^{1/3} + v^{1/3}) = b.$$

(iv) Observe that, if u and v are as stated, $u = v - b$ and

$$-\frac{a^3}{27} = uv = v(v - b) = v^2 - bv$$

so v and (by symmetry u) are the of roots

$$t^2 - bt - \frac{a^3}{27} = 0.$$

Since $u \neq v$ they are the distinct roots.

(v) Solving the quadratic of (iv), we get roots

$$\frac{b \pm \sqrt{b^2 + \frac{4a^3}{27}}}{2}$$

so $x^3 + ax = b$ has the root

$$\left(\frac{b + \sqrt{b^2 + \frac{4a^3}{27}}}{2}\right)^{1/3} + \left(\frac{b - \sqrt{b^2 + \frac{4a^3}{27}}}{2}\right)^{1/3}.$$

(vi) Note that uv must be real. Thus the only allowable possibilities are

$$\left(\frac{b+\sqrt{b^2+\frac{4a^3}{27}}}{2}\right)^{1/3}+\left(\frac{b-\sqrt{b^2+\frac{4a^3}{27}}}{2}\right)^{1/3},$$

$$\omega\left(\frac{b+\sqrt{b^2+\frac{4a^3}{27}}}{2}\right)^{1/3}+\omega^2\left(\frac{b-\sqrt{b^2+\frac{4a^3}{27}}}{2}\right)^{1/3},$$

$$\omega^2\left(\frac{b+\sqrt{b^2+\frac{4a^3}{27}}}{2}\right)^{1/3}+\omega\left(\frac{b-\sqrt{b^2+\frac{4a^3}{27}}}{2}\right)^{1/3}.$$

It is easy to check directly (or by looking at $u+v$) that these are all solutions.

(vi) By Exercise B.2 it suffices to solve

$$z^3+az=b.$$

Our earlier manipulations remain valid (though we must be careful about taking cube roots).

(vii) and (viii) See for example Ian Stewart's excellent *Galois Theory*.

Exercise C.1

(i) We have

$$\begin{aligned}\Pr(\text{at least one six in four tosses}) &= 1 - \Pr(\text{no six in four tosses})\\ &= 1 - \Pr(\text{no six in one toss})^4\\ &= 1 - \left(\frac{5}{6}\right)^4 \approx .5177\end{aligned}$$

His expected winnings if he stakes one unit is about .035. This is a house advantage of $3\frac{1}{2}\%$. A casino would certainly be happy to run such a game but I think it would be very hard work for an individual to make it pay whilst retaining the appearance of a gentleman.

(ii) We have

$$\begin{aligned}&\Pr(\text{at least one double six in 24 tosses})\\ &\qquad = 1 - \Pr(\text{no double six in 24 tosses})\\ &\qquad = 1 - \Pr(\text{no double six in one toss})^{24}\\ &\qquad = 1 - \left(\frac{35}{36}\right)^{24} \approx 0.4914\end{aligned}$$

If you stake one unit at evens against a double six appearing your expected winnings are about .017. Of course, it would be possible for an individual to lose or make a fortune making this bet, just as it is possible for an individual to lose or make a fortune betting heads and tails, but it would take a very long run of games and very accurate record keeping to detect the house advantage.

(iii) We have

$$\begin{aligned}&\Pr(\text{at least one double six in 24 tosses})\\ &\qquad = 1 - \Pr(\text{no double six in 24 tosses})\\ &\qquad = 1 - \Pr(\text{no double six in one toss})^{24}\\ &\qquad = 1 - \left(\frac{35}{36}\right)^{25} \approx 0.5055\end{aligned}$$

The house advantage is about 1%.

Exercise C.2

Writing $N = N(p)$, we want

$$\begin{aligned}\frac{1}{2} &\approx \text{Pr(at least one in } N \text{ goes)}\\ &= 1 - Pr(\text{none in } N \text{ goes})\\ &= 1 - Pr(\text{not in 1 go})^N\\ &= 1 - (1-p)^N\end{aligned}$$

so $(1-p)^N \approx 1/2$, Taking logarithms and using the standard approximation of Exercise A.8, we get

$$-\log 2 = \log\frac{1}{2} = N\log(1-p) \approx -Np$$

so

$$N(p) \approx \frac{\log 2}{\log p}.$$

As p gets smaller all our approximations improve.

Thus if p and q are small. Using the results of Exercise C.1 (and the observation that $1-(5/6)^3 \leq .42 < 1/2$) we know that $N(1/6) = 4$ and $N(1/36) = 25$

Now

$$\frac{\log 2}{1/6} \approx 4.159$$

and

$$\frac{\log 2}{1/36} \approx 24.953$$

so the estimates are good.

Exercise C.3

(i) We have

$$\begin{aligned}&\Pr(\text{second side wins stake})\\&\qquad=\Pr(\text{second side wins innings three times in a row})\\&\qquad=(1/2)^3=1/8.\end{aligned}$$

Thus the expected winnings of the second team are $22/8$ ducats.

(ii) Suppose team A wins an innings with probability p and team B wins an innings with probability q where $p+q=1$. Team A wins if they win $r+1$ innings before player B wins $s+1$ innings. Otherwise team B wins. If the winner gets K what are the expected winnings of A?

We answer as follows.

$$\begin{aligned}&\Pr(A\text{ wins stake})\\&=\sum_{j=1}^{s}\Pr(A\text{ wins }r\text{ ins and }B\text{ wins }j\text{ ins, then }A\text{ wins an in})\\&=\sum_{j=0}^{s}\Pr(A\text{ wins }r\text{ ins and }B\text{ wins }j\text{ ins})\Pr(A\text{ wins an in})\\&=p^{r+1}\sum_{j=0}^{s}\binom{r+j}{j}q^j\end{aligned}$$

Thus A has expected winnings

$$Kp^{r+1}\sum_{j=0}^{s}\binom{r+j}{j}q^j.$$

Exercise C.4

(i) The probability of A winning in a single throw is $r = 5/36$. The probability of A winning in two throws is

$$p = 1 - \Pr(\text{A loses both}) = 1 - \left(\frac{31}{36}\right)^2 = \frac{335}{1296}.$$

The probability of B winning in a single throw is $r = 6/36$. The probability of A winning in two throws is

$$q = 1 - \Pr(\text{B loses both}) = 1 - \left(\frac{30}{36}\right)^2 = \frac{396}{1296}.$$

If A fails to win on his first throw we have a two sided duel with B starting so B has probability of winning

$$\frac{q}{q+p-qp}$$

and B's probability of winning the game is

$$\begin{aligned}(1-r)\frac{q}{q+p-qp} &= \frac{31}{36} \times \frac{396 \times 1296}{(335+396)\times 1296 - (335 \times 396)}\\ &= \frac{31 \times 396}{(335+396)\times 36 - (335 \times 11)}\\ &= \frac{12276}{22631}\end{aligned}$$

so A's probability of winning is

$$1 - \frac{12276}{22631} = \frac{10355}{22631}$$

and the ratio is as stated. (In those days men were men when it came to arithmetic.)

(ii) (1) We observe that

$$\begin{aligned}\Pr(A \text{ wins}) &= \Pr(A \text{ wins on 1st draw})\\ &\quad + \Pr(A \text{ wins on 4th draw}) + \Pr(A \text{ wins on 7th draw})\\ &= \frac{4}{12} + \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10} \times \frac{4}{9}\\ &\qquad + \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \times \frac{4}{6}\\ &= \frac{165+56+10}{11 \times 5 \times 9} = \frac{77}{165}\end{aligned}$$

$$\begin{aligned}
\Pr(\text{B wins}) &= \Pr(B \text{ wins on 2nd draw})\\
&\qquad + \Pr(B \text{ wins on 5th draw})\\
&\qquad + \Pr(B \text{ wins on 8th draw})\\
&= \frac{8}{12}\times\frac{4}{11}+\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\\
&\quad +\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\\
&\quad +\frac{4}{8}\times\frac{3}{7}\times 26\times\frac{4}{5}\\
&= \frac{120+35+4}{11\times 5\times 9}=\frac{53}{165}
\end{aligned}$$

Thus the ratio of probabilities is $77 : 53 : 35$.

(2) Each player has probability $1/3$ of winning a go. If p_A is the probability of A winning, p_B of B and p_C of C then considering the result of the first trial

$$\begin{aligned}
p_A &= \tfrac{1}{3}+\tfrac{2}{3}p_C\\
p_B &= \tfrac{2}{3}p_A\\
p_C &= \tfrac{2}{3}p_B
\end{aligned}$$

Thus $p_A = \frac{1}{3}+(\frac{2}{3})^3 p_A$ and

$$27p_A = 9 + 8p_A.$$

If follows that $p_A = 9/19$, $p_B = 6/19$ and $p_C = 4/19$.

Thus the ratio of probabilities is $9 : 6 : 4$.

(3) We say that the game goes to the kth round if A, B and C have each drawn $k-1$ counters. The kth round consists of the draws then made by A and then (if A fails) by B and then by C if B fails.

We observe that if we reach the kth round A has probability

$$p_A(k) = \frac{4}{13-k}$$

of winning in that round, B has probability

$$p_B(k) = (1-p_A(k))\frac{4}{13-k} = \frac{9-k}{13-k}\frac{4}{13-k}$$

of wiining that round and C has probability

$$p_C(k) = (1-p_A(k))(1-p_B(k))\frac{4}{13-k} = \left(\frac{9-k}{13-k}\right)^2\frac{4}{13-k}$$

and the probability that we go to the next round is

$$P_X(k) = \left(\frac{9-k}{13-k}\right)^3.$$

We note that $p_A(9) = 1$, $p_B(9) = p_C(9) = 0$.

The probability that we reach the kth round is thus given by $P_Y(k)$ where $P_Y(1) = 1$ and

$$P_Y(k) = \prod_{j=1}^{k-1} P_X(k) = \left(\prod_{j=1}^{k-1} \frac{9-k}{13-k}\right)^3.$$

The probability of A winning is

$$\sum_{k=1}^{9} P_Y(k)\frac{4}{13-k},$$

the probability of B winning is

$$\sum_{k=1}^{8} P_Y(k)\frac{9-k}{13-k}\frac{4}{13-k},$$

and the probability of C winning is

$$\sum_{k=1}^{8} P_Y(k)\left(\frac{9-k}{13-k}\right)^2\frac{4}{13-k}.$$

(iii) The probability of a particular correct hand is

$$p = \frac{40}{40} \times \frac{30}{39} \times \frac{20}{38} \times \frac{10}{37} = \frac{1000}{13 \times 19 \times 37} = \frac{1000}{9139}$$

so the proportion of their chances is

$$\frac{p_A}{1-p_A} = \frac{1000}{8139}.$$

(iv) The probability of any particular arrangement such as

$$BWBWBWB$$

is

$$q = \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \times \frac{2}{6}$$

There are

$$\binom{7}{3} = 35$$

different arrangements so the probability of A winning is

$$35q = 35 \times \times \frac{1}{11} \times \frac{1}{9} = \frac{35}{99}.$$

The proportion of chances is 35 to 64.

If we allow A to win if he gets 3 or more white then we must consider the probability that he gets 4 white. The probability of a particular 4 white hand is

$$p = \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} \times \frac{1}{9} = \frac{1}{11} \times \frac{1}{5} \times \frac{1}{9}$$

and there are

$$\binom{7}{3} = 35$$

different arrangements so the probability of A winning with 4 white pieces

$$35q = \frac{7}{99}.$$

Thus A's probability of winning with three or more white counters is

$$\frac{35}{99} + \frac{7}{99} = \frac{42}{99}$$

and the proportion of chances is 42 to 57 ie 14 to 19.

(v) Enumerating we can throw 11 by throwing

1, 4, 6 in some order (6 ways)

1, 5, 5 in some order (3 ways)

2, 3, 6 in some order (6 ways)

2, 4, 5 in some order (6 ways)

3, 3, 5 in some order (3 ways)

3, 4, 4 in some order (3 ways).

Thus we have probability $p = (6+3+6+6+3+3) \times 6^{-3} = 27 \times 6^{-3}$ of throwing 11.

Enumerating we can throw 14 by throwing

2, 6, 6 in some order (3 ways)

3, 5, 6 in some order (6 ways)

4, 4, 6 in some order (3 ways)

4, 5, 5 in some order (3 ways)

Thus we have probability $q = (3 + 6 + 3 + 3) \times 6^{-3} = 15 \times 6^{-3}$ of throwing 11.

If u_n is the probability that A will win if he has n pieces of money then $u_{24} = 1$, $u_0 = 0$ and by considering the effect of one throw

$$u_n = pu_{n+1} + (1 - p - q)u_n + qu_{n-1}$$

for $1 \leq n \leq 23$. Thus

$$pu_{n+1} - (p+q)u_n + qu_{n-1} = 0.$$

Now we solve our difference equation by observing that

$$pm^2 - (p+q)m + q = (m-1)(pm-q)$$

so

$$u_n = A + B(q/p)^n$$

for some constants A and B.

Since $u_0 = 0$, $B = -A$ so

$$u_n = A(1 - (p/q)^n)$$

and, using the fact that $u_{24} = 1$.

$$u_n = \frac{(1 - (p/q)^n}{(1 - (p/q)^{24}}$$

In particular

$$u_{12} = \frac{1}{1 + (p/q)^{12}}.$$

Thus A probability of winning is

$$\frac{15^{12}}{15^{12} + 27^{12}}$$

and the ratio of A's chance to B's chances is

$$\left(\frac{15}{27}\right)^{12} = \left(\frac{5}{9}\right)^{12} = \frac{244140625}{282429536481}.$$

Exercise C.5

(i) Take $X_j = 1$ (heads) if Y_j is even, $X_j = -1$ (tails) if Y_j is odd.

(iii) The expected time to return to zero is infinite.

(v) The Kelly method is scale invariant (if you double your fortune, you double your bet) so the bumpiness is unaffected by the size of your fortune.

If $\alpha > 1$ then there will be an $\alpha' < 1$ which gives the same long term expected rate of increase. The ride with α' is less bumpy than the ride with α.