

# Topics in Analysis

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**Small print** This course is not intended as a foundation for further courses, so the lecturer is allowed substantial flexibility. Exam questions will be set on the course as given (so, if I do not have time to cover everything in these notes, the topics examined will be those actually lectured). Unless there is more time to spare than I expect, I will not talk about topological spaces but confine myself to ordinary Euclidean spaces  $\mathbb{R}^n$ ,  $\mathbb{C}$  and complete metric spaces. Some results will be marked with a ★ because most students will have met the ideas before. The proofs of these results will not be given in lectures but I intend to give additional classes and I will be happy to discuss these or anything else to do with the course during the classes.

I have sketched solutions to the exercises which should be available for supervisors from the departmental secretaries or in tex, ps, pdf and dvi format by e-mail.

I should **very much** appreciate being told of any corrections or possible improvements to these notes and might even part with a small reward to the first finder of particular errors. These notes are written in L<sup>A</sup>T<sub>E</sub>X 2<sub>ε</sub> and should be available in tex, ps, pdf and dvi format from my home page

<http://www.dpmms.cam.ac.uk/~twk/>

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## 1 Metric spaces

This section is devoted to fairly formal preliminaries. Things get more interesting in the next section and the course gets fully under way in the third. Both those students who find the early material worryingly familiar and those who find it worryingly unfamiliar are asked to suspend judgement until then.

Most Part II students will be familiar with the notion of a metric space.

**Definition 1.1.** *Suppose that  $X$  is a non-empty set and  $d : X^2 \rightarrow \mathbb{R}$  is a function such that*

(i)  $d(x, y) \geq 0$  for all  $x, y \in X$ .

(ii)  $d(x, y) = 0$  if and only if  $x = y$ .

(iii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .

(iv)  $d(x, y) + d(y, z) \geq d(x, z)$  for all  $x, y, z \in X$ . (This is called the 'triangle inequality' after the result in Euclidean geometry that the sum of the lengths of two sides of a triangle is at least as great as the length of the third side.)

Then we say that  $d$  is a metric on  $X$  and that  $(X, d)$  is a metric space.

You should imagine the lecturer muttering under his breath

- (i) Distances are always positive.
- (ii) Two points are zero distance apart if and only if they are the same point.
- (iii) The distance from  $A$  to  $B$  is the same as the distance from  $B$  to  $A$ .
- (iv) The distance from  $A$  to  $B$  via  $C$  is at least as great as the distance from  $A$  to  $B$  directly.

For most of the course we shall be concerned with metrics which you already know well.

**Lemma 1.2.** ★ (i) Consider  $\mathbb{R}^n$ . If we take  $d$  to be ordinary Euclidean distance

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left( \sum_{j=1}^n |x_j - y_j|^2 \right)^{1/2},$$

then  $(\mathbb{R}^n, d)$  is a metric space. We refer to this space as Euclidean space.

(ii) Consider  $\mathbb{C}$ . If we take  $d(z, w) = |z - w|$ , then  $(\mathbb{C}, d)$  is a metric space.

*Proof.* Left as an exercise. (Exercise 17.1.) □

However, the notion of a metric is more general. Here are two very simple examples for you to check.

**Exercise 1.3.** ★ (i) Let  $X$  be a non-empty set. If we write

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y, \end{cases}$$

then  $(X, d)$  is a metric space. (We call  $d$  the discrete metric but you will not be called on to remember this.)

(ii) Consider  $X = \{0, 1\}^n$ . If we take

$$d(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^n |x_j - y_j|,$$

then  $(X, d)$  is a metric space. (If you do the Codes and Cryptography course you will recognise this as the Hamming metric, but you are not called on to remember this for this course.)

The next definitions work in any metric space but, if you have not met them before, you can concentrate on what they mean for ordinary Euclidean space.

**Definition 1.4.** *If  $(X, d)$  is a metric space  $x_n \in X$ ,  $x \in X$  and  $d(x_n, x) \rightarrow 0$ , then we say that  $x_n \xrightarrow[d]{} x$  as  $n \rightarrow \infty$ .*

**Definition 1.5.** *If  $(X, d)$  is a metric space  $x \in X$  and  $r > 0$ , then we write*

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

*and call  $B(x, r)$  the open ball of radius  $r$  with centre  $x$ .*

**Definition 1.6.** *Let  $(X, d)$  be a metric space.*

*(i) We say that a subset  $E$  of  $X$  is closed if, whenever  $x_n \in E$  and  $x_n \xrightarrow[d]{} x$ , it follows that  $x \in E$ .*

*(ii) We say that a subset  $U$  of  $X$  is open if, whenever  $u \in U$ , we can find a  $\delta > 0$  such that  $B(u, \delta) \subseteq U$ .*

**Exercise 1.7. ★** *(i) Show that, if  $x \in X$  and  $r > 0$ , then  $B(x, r)$  is open in the sense of Definition 1.6.*

*(ii) Show that, if  $x \in X$  and  $r > 0$ , then the set*

$$\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$$

*is closed. (Naturally enough, we call  $\bar{B}(x, r)$  a closed ball but you do not have to remember this.)*

We recall (without proof) the following important results from 1A.

**Theorem 1.8. [Cauchy criterion]** *A sequence  $a_n \in \mathbb{R}$  converges if and only if, given  $\epsilon > 0$ , we can find an  $N(\epsilon)$  such that  $|a_n - a_m| < \epsilon$  for all  $n, m \geq N(\epsilon)$*

**Theorem 1.9. [Axiom of Archimedes]** *If we work in  $\mathbb{R}$ ,  $1/n \rightarrow 0$  as  $n \rightarrow \infty$  through integer values.*

Although we shall use the Axiom of Archimedes implicitly from time to time we shall not worry about it.

More importantly, from the point of view of the course, the Cauchy criterion leads us to the following definitions.

**Definition 1.10.** *If  $(X, d)$  is a metric space, then a sequence  $(a_n)$  with  $a_n \in X$  is called a Cauchy sequence if, given  $\epsilon > 0$ , we can find an  $N(\epsilon)$  such that  $d(a_n, a_m) < \epsilon$  for all  $n, m \geq N(\epsilon)$ .*

**Definition 1.11.** We say that a metric space  $(X, d)$  is complete if every Cauchy sequence converges.

**Exercise 1.12. ★** Show that if  $(X, d)$  is a metric space (complete or not), then every convergent sequence is Cauchy.

We note the following useful remarks.

**Lemma 1.13.** Let  $(X, d)$  be a metric space.

(i) If a Cauchy sequence  $x_n$  in  $(X, d)$  has a convergent subsequence with limit  $x$  then  $x_n \rightarrow x$ .

(ii) Let  $\epsilon_n > 0$  and  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . If  $(X, d)$  has the property that, whenever  $d(x_n, x_{n+1}) < \epsilon_n$  for  $n \geq 1$ , it follows that the sequence  $x_n$  converges, then  $(X, d)$  is complete.

Lemma 1.13 (ii) is most useful when we have  $\sum_{n=1}^{\infty} \epsilon_n$  convergent, for example if  $\epsilon_n = 2^{-n}$ .

**Exercise 1.14. ★** Suppose that  $(X, d)$  is a metric space and  $Y$  is a non-empty subset of  $X$ .

(i) Show that, if  $d_Y(a, b) = d(a, b)$  for all  $a, b \in Y$ , then  $(Y, d_Y)$  is a metric space.

(ii) Show that, if  $(X, d)$  is complete and  $Y$  is closed in  $(X, d)$ , then  $(Y, d_Y)$  is complete.

(iii) Show that, if  $(Y, d_Y)$  is complete, then (whether  $(X, d)$  is complete or not)  $Y$  is closed in  $(X, d)$ .

We now come to our first real theorem.

**Theorem 1.15.** The Euclidean space  $\mathbb{R}^m$  with the usual metric is complete.

We shall usually prove such theorems in the case  $n = 2$  and remark that the general case is similar.

## 2 Compact sets in Euclidean Space

In Part 1A we showed that any bounded sequence had a convergent subsequence. We shall use lion hunting to prove the result for  $\mathbb{R}^m$ . (Henceforth, when we talk about  $\mathbb{R}^m$  or  $\mathbb{C}$ , we shall assume the usual metric unless we specifically say otherwise.)

**Theorem 2.1.** If  $K > 0$  and  $\mathbf{x}_r \in \mathbb{R}^m$  satisfies  $\|\mathbf{x}_r\| \leq K$  for all  $r$ , then we can find an  $\mathbf{x} \in \mathbb{R}^m$  and  $r(k) \rightarrow \infty$  such that  $\mathbf{x}_{r(k)} \rightarrow \mathbf{x}$  as  $k \rightarrow \infty$ .

**Exercise 2.2.** ★ Prove Theorem 2.1 by using the case  $m = 1$  proved in 1A and the kind of argument we used to prove Theorem 1.15.

We now prove a very useful theorem.

**Theorem 2.3.** (i) If  $E$  is closed bounded subset of  $\mathbb{R}^m$ , then any sequence  $\mathbf{x}_r \in E$  has a subsequence with a limit in  $E$ .

(ii) Conversely, if  $E$  is a subset of  $\mathbb{R}^m$  with the property that any sequence  $\mathbf{x}_r \in E$  has a subsequence with a limit in  $E$ , then  $E$  is closed and bounded.

We often refer to closed bounded subsets of  $\mathbb{R}^m$  as compact sets. (The reader is warned that, in general metric spaces, ‘closed and bounded’ does not mean the same thing as ‘compact’ (see, for example, Exercise 18.13). If we deal with the even more general case of topological spaces we have to distinguish between compactness and sequential compactness<sup>1</sup>. We shall only talk about compact sets in  $\mathbb{R}^m$ .)

The reader will be familiar with definitions of the following type.

**Definition 2.4.** If  $(X, d)$  and  $(Y, \rho)$  are metric spaces and  $E \subseteq X$  we say that a function  $f : E \rightarrow Y$  is continuous if, given  $\epsilon > 0$  and  $x \in E$ , we can find a  $\delta(x, \epsilon) > 0$  such that, whenever  $z \in E$  and  $d(z, x) < \delta(x, \epsilon)$ , we have  $\rho(f(z), f(x)) < \epsilon$ .

Even if the reader has not met the general metric space definition, she will probably have a good idea of the properties of continuous functions  $f : E \rightarrow \mathbb{R}^n$  when  $E$  is a subset of  $\mathbb{R}^m$ .

We now prove that the continuous image of a compact set is compact.

**Theorem 2.5.** If  $E$  is a compact subset of  $\mathbb{R}^m$  and  $f : E \rightarrow \mathbb{R}^n$  is continuous, then  $f(E)$  is a compact subset of  $\mathbb{R}^n$ .

At first sight Theorem 2.5 seems too abstract to be useful but it has an immediate corollary.

**Theorem 2.6.** If  $E$  is a compact subset of  $\mathbb{R}^m$  and  $f : E \rightarrow \mathbb{R}$  is continuous, then there exist  $\mathbf{a}, \mathbf{b} \in E$  such that

$$f(\mathbf{a}) \geq f(\mathbf{x}) \geq f(\mathbf{b})$$

for all  $\mathbf{x} \in E$ .

Thus a continuous real valued function on a compact set is bounded and attains its bounds.

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<sup>1</sup>If you intend to climb Everest you need your own oxygen supply. If you intend to climb the Gog Magogs you do not.

**Exercise 2.7.** ★ Deduce the theorem in 1A which states that if  $f : [c, d] \rightarrow \mathbb{R}$  is continuous, then  $f$  is bounded and attains its bounds.

Theorem 2.6 gives a neat proof of the fundamental theorem of algebra (which, contrary to its name, is a theorem of analysis).

**Theorem 2.8. [Fundamental Theorem of Algebra]** *If we work in the complex numbers, every polynomial has a root.*

The reader will probably have seen, but may well have forgotten, the contents of the next exercise.

**Exercise 2.9.** ★ *We work in the complex numbers.*

(i) Use induction on  $n$  to show that, if  $P$  is a polynomial of degree  $n$  and  $a \in \mathbb{C}$ , then there exists a polynomial  $Q$  of degree  $n - 1$  and an  $r \in \mathbb{C}$  such that

$$P(z) = (z - a)Q(z) + r.$$

(ii) Deduce that, if  $P$  is a polynomial of degree  $n$  with root  $a$ , then there exists a polynomial  $Q$  of degree  $n - 1$  such that

$$P(z) = (z - a)Q(z).$$

(iii) Use induction and the fundamental theorem of algebra to show that every polynomial  $P$  of degree  $n$  can be written in the form

$$P(z) = A \prod_{j=1}^n (z - a_j)$$

with  $A, a_1, a_2, \dots, a_n \in \mathbb{C}$  and  $A \neq 0$ .

(iv) Show that, if  $P$  is a polynomial of degree at most  $n$  which vanishes at  $n + 1$  points, then  $P$  is the zero polynomial.

### 3 Laplace's equation

We need a preliminary definition.

**Definition 3.1.** *Let  $(X, d)$  be a metric space and  $E$  a subset of  $X$ .*

(i) *The interior of  $E$  is the set of all points  $x$  such that there exists a  $\delta > 0$  (depending on  $x$ ) such that  $B(x, \delta) \subseteq E$ . We write  $\text{Int } E$  for the interior of  $E$ .*

(ii) *The closure of  $E$  is the set of points  $x$  in  $X$  such that we can find  $e_n \in E$  with  $e_n \xrightarrow{d} x$ . We write  $\text{Cl } E$  for the closure of  $E$ .*

(iii) *The boundary of  $E$  is the set  $\partial E = \text{Cl } E \setminus \text{Int } E$ .*

**Exercise 3.2.** ★ (i) Show that  $\text{Int } E$  is open. Show also that, if  $V$  is open and  $V \subseteq E$ , then  $V \subseteq \text{Int } E$ .

(ii) Show that  $\text{Cl } E$  is closed. Show also that, if  $F$  is closed and  $F \supseteq E$ , then  $F \supseteq \text{Cl } E$ .

(Thus  $\text{Int } E$  is the largest open set contained in  $E$  and  $\text{Cl } E$  is the smallest closed set containing  $E$ .)

(iii) Show that  $\partial E$  is closed.

(iv) Suppose that we work in  $\mathbb{R}^m$  with the usual metric. Show that if  $E$  is bounded, then so is  $\text{Cl } E$ .

Recall that, if  $\phi$  is a real valued function in  $\mathbb{R}^m$  with sufficiently many derivatives, then we write

$$\nabla^2 \phi = \sum_{j=1}^m \frac{\partial^2 \phi}{\partial x_j^2}.$$

The next few results concern solutions of Laplace's equation

$$\nabla^2 \phi = 0.$$

**Lemma 3.3.** Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^m$  and  $\epsilon > 0$ . Suppose  $\phi : \text{Cl } \Omega \rightarrow \mathbb{R}$  is continuous and satisfies

$$\nabla^2 \phi > \epsilon$$

on  $\Omega$ . Then  $\phi$  attains its maximum on  $\partial \Omega$ .

**Theorem 3.4.** Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^m$ . Suppose  $\phi : \text{Cl } \Omega \rightarrow \mathbb{R}$  is continuous on  $\text{Cl } \Omega$  and satisfies

$$\nabla^2 \phi = 0$$

on  $\Omega$ . Then  $\phi$  attains its maximum on  $\partial \Omega$ .

**Exercise 3.5.** ★ Let  $\Omega$  be a bounded open subset of  $\mathbb{C}$ . Suppose that

$$f : \text{Cl } \Omega \rightarrow \mathbb{C}$$

is continuous and  $f$  is analytic on  $\Omega$ . Recall that the real and imaginary parts of  $f$  satisfy Laplace's equation on  $\text{Int } \Omega$ . Show that  $|f|$  attains its maximum on  $\partial \Omega$ .

[Hint: Why can we assume that  $\Re f(z_0) = |f(z_0)|$  at any particular point  $z_0$ ?]

**Theorem 3.6.** *Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^m$ . Suppose that the functions  $\phi, \psi : \text{Cl}\Omega \rightarrow \mathbb{R}$  are continuous and satisfy*

$$\nabla^2\phi = 0, \quad \nabla^2\psi = 0$$

*on  $\Omega$ . Then, if  $\phi = \psi$  on  $\partial\Omega$ , it follows that  $\phi = \psi$  on  $\text{Cl}\Omega$ .*

You proved a version of Theorem 3.6 in Part 1A but only for ‘nice boundaries’ and functions that behaved ‘nicely’ near the boundary.

In Part 1A you assumed that you could always solve Laplace’s equation. The next exercise (which will be done by the lecturer in the lecture) shows that we must think more carefully about this.

**Exercise 3.7.** (i) *Let*

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 : 0 < \|\mathbf{x}\| < 1\}.$$

*Show that  $\Omega$  is open and*

$$\partial\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\} \cup \{\mathbf{0}\}.$$

(ii) *Suppose that  $\phi : \text{Cl}\Omega \rightarrow \mathbb{R}$  is continuous and satisfies*

$$\nabla^2\phi = 0$$

*on  $\Omega$  together with the boundary conditions*

$$\phi(\mathbf{x}) = \begin{cases} 0 & \text{if } \|\mathbf{x}\| = 1, \\ 1 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

*Use the uniqueness of solutions of Laplace’s equation to show that  $\phi$  must be radially symmetric in the sense that*

$$\phi(\mathbf{x}) = f(\|\mathbf{x}\|)$$

*for some function  $f : [0, 1] \rightarrow \mathbb{R}$ .*

(iii) *Show that*

$$\frac{d}{dr}(rf(r)) = 0$$

*for  $0 < r < 1$  and deduce that  $f(r) = A + B \log r$  [ $0 < r < 1$ ] for some constants  $A$  and  $B$ .*

(iv) *Conclude that the function  $\phi$  described in (ii) can not exist.*

Exercise 3.7 can be criticised on two counts. The first is that we have not made quite clear how well behaved  $\phi$  must be. Our arguments works provided  $\phi$  is twice continuously differentiable and it can be shown that even this condition can be dropped. The second is that  $\partial\Omega$  is a little odd. Lebesgue produced an example in  $\mathbb{R}^3$  (the Lebesgue thorn) which is not subject to this objection.

## 4 Fixed points

In Part 1A we proved the intermediate value theorem.

**Theorem 4.1.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous function and  $f(a) \geq c \geq f(b)$ , then we we can find a  $y \in [a, b]$  such that  $f(y) = c$ .*

We then used it to the following very pretty fixed point theorem.

**Theorem 4.2.** *If  $f : [a, b] \rightarrow [a, b]$  is a continuous function then we we can find a  $w \in [a, b]$  such that  $f(w) = w$ .*

Notice that we can reverse the implication and use Theorem 4.2 to prove Theorem 4.1. (See Exercise 4.9.)

The object of this section is to extend the fixed point theorem to two dimensions.

**Theorem 4.3.** *Let  $\bar{D} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1\}$ . If  $f : \bar{D} \rightarrow \bar{D}$  is a continuous function, then we we can find a  $\mathbf{w} \in \bar{D}$  such that  $f(\mathbf{w}) = \mathbf{w}$ .*

We need some preliminary remarks.

**Exercise 4.4.** *Let  $(X, d)$  be a metric space. Then  $E$  is open if and only if  $X \setminus E$  is closed.*

**Exercise 4.5.** *Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces and let  $f : X \rightarrow Y$  be a function.*

*(i)  $f$  is continuous if and only if  $f^{-1}(U)$  is open whenever  $U$  is.*

*(ii)  $f$  is continuous if and only if  $f^{-1}(E)$  is closed whenever  $E$  is.*

**Lemma 4.6.** *If  $(X, d)$  is a metric space and  $A$  is a closed subset of  $X$  let us write*

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

*Then  $d(x, A) = 0$  implies  $x \in A$ . Further the map  $x \mapsto d(x, A)$  is continuous.*

We start by showing that a number of interesting statements are equivalent. We write

$$\bar{D} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1\}.$$

Let  $\mathbf{a}_1$ ,  $\mathbf{a}_2$  and  $\mathbf{a}_3$  be unit vectors making angles of  $\pm 2\pi/3$  with each other. We take  $T$  to be the closed triangle with vertices  $\mathbf{a}_1$ ,  $\mathbf{a}_2$  and  $\mathbf{a}_3$  and sides  $I$ ,  $J$  and  $K$ .

(For those who insist on things being spelt out

$$T = \{\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \lambda_3 \mathbf{a}_3 : \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_j \geq 0\}$$

but though such ultra precision has its place, that place is not this course.)

**Lemma 4.7.** *The following five statements are equivalent.*

(i) *If  $f : \bar{D} \rightarrow \bar{D}$  is continuous, then we can find a  $\mathbf{w} \in \bar{D}$  such that  $f(\mathbf{w}) = \mathbf{w}$ .*

(ii) *There does not exist a continuous function  $g : \bar{D} \rightarrow \partial D$  with  $g(\mathbf{x}) = \mathbf{x}$  for all  $\mathbf{x} \in \partial D$ . (We say that there is no retraction mapping from  $D$  to  $\partial D$ .)*

(iii) *We work in  $\mathbb{C}$  and take  $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ . If*

$$\begin{aligned} \tilde{I} &= \{e^{i\theta} : 0 \leq \theta \leq 2\pi/3\}, \quad \tilde{J} = \{e^{i\theta} : 2\pi/3 \leq \theta \leq 4\pi/3\} \\ &\text{and } \tilde{K} = \{e^{i\theta} : 4\pi/3 \leq \theta \leq 2\pi\} \end{aligned}$$

*there does not exist a continuous function  $k : D \rightarrow \partial D$  with*

$$k(z) \in \tilde{I} \text{ for all } z \in \tilde{I}, \quad k(z) \in \tilde{J} \text{ for all } z \in \tilde{J}, \quad k(z) \in \tilde{K} \text{ for all } z \in \tilde{K}.$$

(iv) *There does not exist a continuous function  $h : T \rightarrow \partial T$  with*

$$h(\mathbf{x}) \in I \text{ for all } \mathbf{x} \in I, \quad h(\mathbf{x}) \in J \text{ for all } \mathbf{x} \in J, \quad h(\mathbf{x}) \in K \text{ for all } \mathbf{x} \in K.$$

(v) *If  $A, B$  and  $C$  are closed subsets of  $T$  with  $A \supseteq I, B \supseteq J$  and  $C \supseteq K$  and  $A \cup B \cup C = T$ , then  $A \cap B \cap C \neq \emptyset$ .*

We shall prove statement (v) from which the remaining statements will then follow. The key step is Sperner's Lemma.

**Lemma 4.8.** *Consider a triangle  $DEF$  divided up into a triangular grid. If all the vertices of the grid are coloured red, green or blue and every vertex on the side  $DE$  of the big triangle (with the exception of  $E$ ) are coloured red, every vertex of  $EF$  (with the exception of  $F$ ) green and every vertex of  $FD$  (with the exception of  $D$ ) blue then there is a triangle of the grid all of whose vertices have different colours.*

We can now prove Theorem 4.3.

The following pair of exercises may be helpful in thinking about the arguments of this section.

**Exercise 4.9.** *The following four statements are equivalent.*

(i) *If  $f : [0, 1] \rightarrow [0, 1]$  is continuous, then we can find a  $w \in [0, 1]$  such that  $f(w) = w$ .*

(ii) *There does not exist a continuous function  $g : [0, 1] \rightarrow \{0, 1\}$  with  $g(0) = 0$  and  $g(1) = 1$ . (In topology courses we say that  $[0, 1]$  is connected.)*

(iii) *If  $A$  and  $B$  are closed subsets of  $[0, 1]$  with  $0 \in A, 1 \in B$  and  $A \cup B = [0, 1]$  then  $A \cap B \neq \emptyset$ .*

(iv) *If  $h : [0, 1] \rightarrow \mathbb{R}$  is continuous and  $h(0) \leq c \leq h(1)$ , then we can find a  $y \in [0, 1]$  such that  $g(y) = c$ .*

**Exercise 4.10.** *Suppose that we colour the points  $r/n$  red or blue [ $r = 0, 1, \dots, n$ ] with 0 red and 1 blue. Show that there are a pair of neighbouring points  $u/n, (u + 1)/n$  of different colours. Use this result to prove statement (iii) of Exercise 4.9.*

In our proof of the fixed point theorem we wandered between the disc and the triangle and it is clear that, to some extent, the shape of the region does not matter.

**Theorem 4.11. [Brouwer's fixed point theorem]** *Let  $D$  be the unit closed disc. If  $E$  is a set in  $\mathbb{R}^2$  with the property that there exists a bijection  $\phi : D \rightarrow E$  such that  $\phi$  and  $\phi^{-1}$  are continuous (in more highbrow language,  $D$  and  $E$  are homeomorphic). Then, if  $f : E \rightarrow E$  is continuous, there exists a  $\mathbf{w} \in E$  with  $f(\mathbf{w}) = \mathbf{w}$ .*

A similar result holds in higher dimensions, but, although the proof runs along similar lines, it requires one or two additional ideas.

**Exercise 4.12.** *Suppose that  $A = (a_{ij})$  is a  $3 \times 3$  matrix such that  $a_{ij} \geq 0$  for all  $1 \leq i, j \leq 3$  and  $\sum_{i=1}^3 a_{ij} = 1$  for all  $1 \leq j \leq 3$ . Let*

$$T = \{\mathbf{x} \in \mathbb{R}^3 : x_j \geq 0 \text{ for all } j \text{ and } x_1 + x_2 + x_3 = 1\}.$$

*By considering the effect of  $A$  on  $T$ , show that  $A$  has an eigenvector lying in  $T$  with eigenvalue 1.*

If you have not done the 1B Markov chains course Exercise 4.12 may appear somewhat artificial. However, if you have done that course, you will see that is not.

**Exercise 4.13.** *(Only for those who understand the terms used. This is not part of the course.) Use the argument of Exercise 4.12 to show that every 3 state Markov chain has an invariant measure. (Remember that in Markov chains 'the  $i$ 's and  $j$ 's swap places'.) What result can you obtain under the assumption that Brouwer's theorem holds in higher dimensions?*

## 5 Non-zero sum games

It is said that converting the front garden of a house into a parking place raises the value of a house but lowers the value of the other houses in the road. Once everybody has done the conversion the value of each house is lower than before the process started.

Let us make a simple model of such a situation involving just two people with just two choices to see what we can say about it.

Suppose that Albert has the choice of doing  $A_1$  or  $A_2$  and Bertha the choice of doing  $B_1$  or  $B_2$ . If  $A_i$  and  $B_j$  occur, then Albert gets  $a_{ij}$  units and Bertha gets  $b_{ij}$  units. If you went to the 1B course you learnt how to deal with the case when  $a_{ij} = -b_{ij}$  (this is called zero-sum case since  $a_{ij} + b_{ij} = 0$  and Albert's loss is Bertha's gain). Albert and Bertha agree that Albert will choose  $A_i$  with probability  $p_i$  and Bertha will choose  $B_j$  with probability  $q_j$ . The expected value of the arrangement to Albert is

$$A(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} p_i q_j$$

and the expected value of the arrangement to Bertha

$$B(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^2 \sum_{j=1}^2 b_{ij} p_i q_j$$

So far so good, but we can imagine a situation in which Albert chooses  $\mathbf{p}$  and then Bertha chooses  $\mathbf{q}$  but, now knowing Bertha's choice, Albert changes his choice to  $\mathbf{p}'$  and then, knowing Albert's new choice, Bertha changes to  $\mathbf{q}'$  and then . . . . The question we ask ourselves is whether there is a 'stable choice' of  $\mathbf{p}$  and  $\mathbf{q}$  such that neither party can do better by unilaterally choosing a new value. This question is answered by a remarkable theorem of Nash.

**Theorem 5.1.** *Suppose  $a_{ij}$  and  $b_{ij}$  are real numbers. Let  $E = \{(p, q) : 1 \geq p, q \geq 0\}$ , set  $p_1 = p, p_2 = 1 - p, q_1 = q, q_2 = 1 - q$*

$$A(p, q) = \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} p_i q_j, \text{ and } B(p, q) = \sum_{i=1}^2 \sum_{j=1}^2 b_{ij} p_i q_j.$$

*Then we can find  $(p^*, q^*) \in E$  such that*

$$B(p^*, q^*) \geq B(p^*, q) \text{ for all } (p^*, q) \in E$$

*and*

$$A(p^*, q^*) \geq A(p, q^*) \text{ for all } (p, q^*) \in E.$$

The interested reader should have no difficulty in convincing herself (given Brouwer's fixed point theorem in the appropriate dimension) that the result can be extended to many participants with many choices to state that there

is always a choice of probabilities such that *no single participant* has an incentive to change their choice<sup>2</sup>.

Unfortunately the stable points need not be unique. Suppose that Albert and Bertha have to choose scissors or paper. If they both choose scissors they get £1 each. If they both choose paper they get £2 each but if they disagree they get nothing. It is clear that the points corresponding to ‘both choose paper’ and ‘both choose scissors’ are stable. The same is true if Albert gets £1 and Bertha £2 if they both choose scissors whilst, if they both choose paper the payments are reversed.

It is also clear from examples like the one that began this section that even if the stable point is unique it may be unpleasant for all concerned<sup>3</sup>. However this is not the concern of the mathematician.

## 6 Dividing the pot

Faced with problems like those of the previous section the young and tender hearted often ask ‘Why not cooperate?’ It is, of course, true that under certain conditions people are willing to cooperate but, even if these conditions are met, the question remains of how to divide up the gains due to cooperation.

**Exercise 6.1.** *Consider two rival firms A and B engaged in an advertising war. So long as the war continues, the additional costs of advertising mean that the larger firm A loses 3 million pounds a year and the smaller firm B loses 1 million pounds a year. If they can agree to cease hostilities then A will make 8 million a year and B will make 1 million a year. How much should A pay B per year to achieve this end<sup>4</sup>?*

Nash produced a striking answer to this question. There are objections to his argument but I hope the reader will agree with me that it is a notable contribution.

In order to examine his answer we need to introduce the notion of a convex set.

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<sup>2</sup>However, this applies only to single participants. There may be an incentive for two or more participants (if they can agree) to change their choices jointly.

<sup>3</sup>McNamara, the US Defence Secretary at the time, was of the opinion that, during the Cuban crisis, all the participants behaved in a perfectly rational manner and only good luck prevented a full scale nuclear war.

<sup>4</sup>The reader may feel that it would be very difficult for rival firms to come to an agreement in this way. In fact, it appears to be so easy that most countries have strict laws against such behaviour.

**Definition 6.2.** A subset  $E$  of  $\mathbb{R}^m$  is convex if, whenever  $\mathbf{u}, \mathbf{v} \in E$  and  $1 \geq p \geq 0$ , we have

$$p\mathbf{u} + (1 - p)\mathbf{v} \in E.$$

Nash considers a situation in which  $m$  players must choose a point  $\mathbf{x} \in E$  where  $E$  is a closed, bounded, convex set in  $\mathbb{R}^m$ . The value of the outcome to the  $j$ th participant is  $x_j$ . To see why it is reasonable to take  $E$  convex suppose that the participants can choose two points  $\mathbf{u}$  and  $\mathbf{v}$ . The participants can agree among themselves to toss a suitable coin and choose  $\mathbf{u}$  with probability  $p$  and  $\mathbf{v}$  with probability  $1 - p$ . The expected value of the outcome to the  $j$ th participant is  $pu_j + (1 - p)v_j$ , that is to say, the value of the  $j$ th component of  $p\mathbf{u} + (1 - p)\mathbf{v}$ .

The participants also know a point  $\mathbf{s} \in E$  (the status quo) which will be the result if they can not agree on any other point.

Nash argues that a best point  $\mathbf{x}^*$  if it exists must have the following properties.

(1)  $x_j^* \geq s_j$  for all  $j$ . (Everyone must be at least as well off as if they failed to agree.)

(2) (Pareto Optimality) If  $\mathbf{x} \in E$  and  $x_j \geq x_j^*$  for all  $j$ , then  $\mathbf{x} = \mathbf{x}^*$ . (If there is a choice which makes some strictly better off and nobody worse off, then the participants should take it.)

(3) (Independence of irrelevant alternatives.) Suppose  $E'$  is a closed bounded convex set with  $E' \supseteq E$  and  $\mathbf{x}^{**}$  is a best point for  $E'$ . Then, if  $\mathbf{x}^{**} \in E$  it follows that  $\mathbf{x}^{**}$  is a best point for  $E$ .

(4) If  $E$  is symmetric (that is, if whenever  $\mathbf{x} \in E$  and  $y_1, y_2, \dots, y_m$  is some rearrangement of  $x_1, x_2, \dots, x_m$ , then  $\mathbf{y} \in E$ ), then  $x_1^* = x_2^* = \dots = x_m^*$ . This corresponds to our beliefs about 'fairness'.

(5) Our final assumption is that we must treat the poor man's penny with the same respect as the rich man's pound. Suppose that  $\mathbf{x}^*$  is a best point for  $E$ . If we change coordinates and consider

$$E' = \{\mathbf{y} : y_j = a_j x_j + b_j \text{ for } 1 \leq j \leq m \text{ and } \mathbf{x} \in E\}$$

with  $a_j > 0$ , then  $\mathbf{y}^*$  with  $y_j^* = a_j x_j^* + b_j$  is a best point for  $E'$ .

**Exercise 6.3.** Show that the  $E'$  defined above is closed bounded and convex.

There is no difficulty in remembering these conditions since they each play a particular role in the proof. If the reader prefers only to deal with the case  $m = 2$ , she will lose nothing of the argument<sup>5</sup>.

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<sup>5</sup>And no marks in the exam

**Theorem 6.4.** *Suppose that we agree to the Nash conditions. If  $E$  is closed bounded convex set in  $\mathbb{R}^m$ ,  $\mathbf{s}$  is the status quo point and the function  $f : E \rightarrow \mathbb{R}$  given by*

$$f(\mathbf{x}) = \prod_{j=1}^m (x_j - s_j)$$

*has a maximum (in  $E$ ) with  $x_j - s_j \geq 0$  at  $\mathbf{x}^*$  then  $\mathbf{x}^*$  is the unique best point.*

We complete our discussion with the following lemma.

**Lemma 6.5.** *If  $E$  is closed bounded convex set in  $\mathbb{R}^m$ ,  $\mathbf{s} \in E$  and the function  $f : E \rightarrow \mathbb{R}$  given by*

$$f(\mathbf{x}) = \prod_{j=1}^m (x_j - s_j),$$

*then there is a unique point in  $E$  with  $x_j - s_j \geq 0$  where  $f$  attains a maximum.*

‘There is no patent for immortality under the the moon’ but I suspect that Nash’s results will be remembered long after the last celluloid copy of *A Beautiful Life* has crumbled to dust.

The book *Games, Theory and Applications* [5] by L. C. Thomas maintains a reasonable balance between the technical and non-technical and would make a good port of first call if you wish to learn more along these lines.

## 7 Approximation by polynomials

It is a guiding idea of both the calculus and of numerical analysis that ‘well behaved functions look like polynomials’. Like most guiding principles it needs to be used judiciously.

If asked to justify it, we might mutter something about Taylor’s Theorem, but Cauchy produced the following example to show that this is not sufficient.

**Exercise 7.1.** *Let  $E : \mathbb{R} \rightarrow \mathbb{R}$  be defined by*

$$E(t) = \begin{cases} \exp(-1/t^2) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

*(i)  $E$  is infinitely differentiable, except, possibly, at 0, with*

$$E^{(n)}(t) = P_n(1/t)E(t)$$

*for all  $t \neq 0$  for some polynomial  $P_n$ .*

(ii)  $E$  is infinitely differentiable everywhere with

$$E^{(n)}(0) = 0.$$

(iii) We have

$$E(t) \neq \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!} t^n$$

for all  $t \neq 0$ .

We could also mutter something like ‘interpolation’. The reader probably knows all the facts given in the next lemma.

**Lemma 7.2.** ★ Let  $x_0, x_1, \dots, x_n$  be distinct points of  $[a, b]$ .

(i) If  $f : [a, b] \rightarrow \mathbb{R}$  then there is at most one polynomial of degree at most  $n$  with  $P(x_j) = f(x_j)$  for  $0 \leq j \leq n$ .

(ii) Write

$$e_j(t) = \prod_{k \neq j} \frac{t - x_k}{x_j - x_k}.$$

If  $f : [a, b] \rightarrow \mathbb{R}$  then

$$P = \sum_{j=0}^n f(x_j) e_j$$

is a polynomial of degree at most  $n$  with  $P(x_j) = f(x_j)$  for  $0 \leq j \leq n$ .

(iii) In the language of vector spaces, the  $e_j$  form a basis for the vector space of polynomials of degree  $n$  or less.

However polynomials can behave in rather odd ways.

**Theorem 7.3.** There exist polynomials  $T_n$  of degree  $n$  and  $U_{n-1}$  of degree  $n - 1$  such that

$$T_n(\cos \theta) = \cos n\theta$$

for all  $\theta$  and

$$U_{n-1}(\cos \theta) = \frac{\sin n\theta}{\sin \theta}$$

for  $\sin \theta \neq 0$ ,  $U_{n-1}(0) = n$ .

The coefficient of  $t^n$  in  $T_n$  is  $2^{n-1}$  for  $n \geq 1$ .

We call  $T_n$  the Chebychev<sup>6</sup> polynomial of degree  $n$ . The  $U_n$  are called Chebychev polynomials of the second kind. Looking at the Chebychev polynomials of the second kind, we see that we can choose a well behaved function

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<sup>6</sup>Or Tchebychev, hence the  $T$ .

$f$  which is well behaved at  $n + 1$  reasonably well spaced points but whose  $n$ th degree interpolating polynomial is very large at some other point. It can be shown (though this is harder to prove) that this kind of thing can happen however we choose our points of interpolation.

A little thought shows that we are not even sure what it means for one function to look like another. It is natural to interpret  $f$  looks like  $g$  as saying that  $f$  and  $g$  are close in some metric. However there are a number of ‘obvious’ metrics.

**Exercise 7.4.** Show that the following define metrics on the space  $C([0, 1])$  of continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ .

$$(i) \ d_1(f, g) = \|f - g\|_1 = \int_0^1 |f(t) - g(t)| dt.$$

$$(ii) \ d_2(f, g) = \|f - g\|_2 = \left( \int_0^1 |f(t) - g(t)|^2 dt \right)^{1/2}.$$

$$(iii) \ d_3(f, g) = \|f - g\|_\infty = \sup_{t \in [0, 1]} |f(t) - g(t)|.$$

Show that

$$\|f - g\|_\infty \geq \|f - g\|_1 \geq \|f - g\|_2.$$

[Hint: You may find the Cauchy–Schwartz inequality for integrals useful in at least two places.]

Let

$$f_n(t) = \begin{cases} (1 - nt) & \text{for } 0 \leq t \leq 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

Compute  $\|f_n\|_\infty / \|f_n\|_1$  and  $\|f_n\|_1 / \|f_n\|_2$ . Comment.

Each of these metrics has its advantages and all are used in practice. We shall concentrate on the metric  $d_3$ . We quote the following result from a previous course (where it is known as the General Principle of Uniform Convergence).

**Theorem 7.5.** If  $[a, b]$  is a closed interval and  $C([a, b])$  is the space of continuous functions on  $[a, b]$  then the uniform metric

$$d(f, g) = \|f - g\|_\infty$$

is complete.

We have now obtained a precise question. If  $f$  is a continuous function can we find polynomials which are arbitrarily close in the uniform norm? This question was answered in the affirmative by Weierstrass in a paper published when he was 70 years old. Since then, several different proofs have been discovered. We present one due to Bernstein based on probability theory.

Before that, we need a definition and theorem which the reader will have met in a simpler form earlier.

**Definition 7.6.** Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. We say that a function  $f : X \rightarrow Y$  is uniformly continuous if, given  $\epsilon > 0$ , we can find a  $\delta > 0$  such that  $\rho(f(a), f(b)) < \epsilon$  whenever  $d(a, b) < \delta$ .

**Theorem 7.7.** If  $E$  is a bounded closed set in  $\mathbb{R}^m$  and  $f : E \rightarrow \mathbb{R}^p$  is continuous, then  $f$  is uniformly continuous.

Although we require probability theory we only need deal with the simplest case of a random variable taking a finite number of values and, if the reader wishes, she need only prove the next result in that case.

**Theorem 7.8. [Chebychev's inequality]** If  $X$  is a real valued bounded random variable, then, writing

$$\sigma^2 = \text{var } X = \mathbb{E}(X - \mathbb{E}X)^2,$$

we have

$$\Pr(|X - \mathbb{E}X| \geq a) \leq \frac{\sigma^2}{a^2}.$$

**Exercise 7.9.** Suppose we only deal with the case when  $X$  takes a finite number of values. State and prove the algebraic inequality corresponding to Chebychev's inequality without explicitly mentioning probability. Do you think there is any advantage in doing this?

**Theorem 7.10. [Bernstein]** Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous. Let  $X_1, X_2, \dots, X_n$  be independent identically distributed random variables with  $\Pr(X_r = 0) = 1 - t$  and  $\Pr(X_r = 1) = t$  (think of tossing a biased coin). Let

$$Y_n(t) = \frac{X_1 + X_2 + \dots + X_n}{n}$$

and let

$$p_n(t) = \mathbb{E}f(Y_n(t)).$$

Then

(i)  $p_n$  is polynomial of degree  $n$ . Indeed,

$$p_n(t) = \sum_{j=0}^n \binom{n}{j} f(j/n) t^j (1-t)^{n-j}.$$

(ii)  $\|p_n - f\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ .

## 8 Best approximation by polynomials

Bernstein's theorem gives an explicit approximating polynomial but not necessarily the best. Chebychev was very interested in this problem and gave a way of telling when we do have a best approximation.

**Theorem 8.1.** [The Chebychev equiripple criterion] *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function and  $P$  a polynomial of degree at most  $n-1$ . Suppose that we can find  $a \leq a_0 < a_1 < \dots < a_n \leq b$  such that, writing  $\sigma = \|f - P\|_\infty$  we have either*

$$f(a_j) - P(a_j) = (-1)^j \sigma \text{ for all } 0 \leq j \leq n$$

or

$$f(a_j) - P(a_j) = (-1)^{j+1} \sigma \text{ for all } 0 \leq j \leq n.$$

Then  $\|P - f\|_\infty \leq \|Q - f\|_\infty$  for all polynomials  $Q$  of degree  $n-1$  or less.

We apply this to find the polynomial of degree  $n-1$  which gives the best approximation to  $t^n$  on  $[-1, 1]$ .

**Theorem 8.2.** *Write  $S_n(t) = t^n - 2^{1-n}T_n(t)$  where  $T_n$  is the Chebychev polynomial of degree  $n$ . Then (if  $n \geq 1$ )*

$$\sup_{t \in [-1, 1]} |t^n - Q(t)| \geq \sup_{t \in [-1, 1]} |t^n - S_n(t)| = 2^{1-n}$$

for all polynomials  $Q$  of degree  $n-1$ .

**Corollary 8.3.** *We work on  $[-1, 1]$ .*

(i) *If  $P(t) = \sum_{j=0}^n a_j t^j$  is a polynomial of degree  $n$  with  $|a_n| \geq 1$  then  $\|P\|_\infty \geq 2^{-n+1}$ .*

(ii) *We can find  $\epsilon(n) > 0$  with the following property. If  $P(t) = \sum_{j=0}^n a_j t^j$  is a polynomial of degree  $n$  and  $|a_k| \geq 1$  for some  $n \geq k \geq 0$  then  $\|P\|_\infty \geq \epsilon(n)$ .*

We can now use a compactness argument to prove that there does exist a best approximation.

**Theorem 8.4.** *If  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function there exists a polynomial  $P$  of degree at most  $n$  such that  $\|P - f\|_\infty \leq \|Q - f\|_\infty$  for all polynomials  $Q$  of degree  $n$  or less.*

We have only shown that the Chebychev criterion is a sufficient condition. However, it can be shown that it is also a necessary one. The proof is given in Exercise 19.5 but is not part of the course.

## 9 Gaussian integration

How should we attempt to estimate  $\int_a^b f(x) dx$  if we only know  $f$  at certain points. One rather naive approach is to find the interpolating polynomial for those points and integrate that. This leads rapidly, via Lemma 7.2, to the following result.

**Lemma 9.1.** ★ *Let  $x_0, x_1, \dots, x_n$  be distinct points of  $[a, b]$ . Then there are unique real numbers  $A_0, A_1, \dots, A_n$  with the property that*

$$\int_a^b P(x) dx = \sum_{j=0}^n A_j P(x_j)$$

*for all polynomials of degree  $n$  or less.*

However our previous remarks about interpolating polynomials suggest, and experience confirms, that it may not be wise to use the approximation

$$\int_a^b f(x) dx \approx \sum_{j=0}^n A_j f(x_j)$$

even when  $f$  well behaved. In the particular case when the interpolation points are equally spaced computation suggests that as the number of points used increases the  $A_j$  begin to vary in sign and become large in absolute value. It can be shown that this actually the case and that this means that the approximation can actually get worse as the number of points increase.

It is rather surprising that there is a choice of points which avoids this problem. Let us recall some results from vector space theory.

**Lemma 9.2.** ★ *Consider the space  $C([-1, 1])$  of continuous functions on  $[-1, 1]$ .*

*(i)  $C([-1, 1])$  is a vector space.*

*(ii) The definition*

$$(f, g) = \int_{-1}^1 f(t)g(t) dt$$

*gives an inner product on  $C([-1, 1])$ .*

**Lemma 9.3.** ★ [Gramm–Schmidt] *Let  $V$  be a vector space with an inner product. Suppose that  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$  are orthonormal and  $\mathbf{f}$  is not in their linear span. If we set*

$$\mathbf{v} = \mathbf{f} - \sum_{j=1}^n (\mathbf{f}, \mathbf{e}_j) \mathbf{e}_j$$

*we know that  $\mathbf{v} \neq \mathbf{0}$  and that, setting  $\mathbf{e}_{n+1} = \|\mathbf{v}\|^{-1} \mathbf{v}$  the vectors  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+1}$  are orthonormal.*

Lemma 9.3 enables us to make the following definition.

**Definition 9.4.** *The Legendre polynomials  $p_n$  are the polynomials given by the following conditions<sup>7</sup>*

(i)  $p_n$  is a polynomial of degree  $n$  with positive leading coefficient.

$$(ii) \int_{-1}^1 p_n(t)p_m(t) dt = \delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 9.5.** *The  $n$ th Legendre polynomial  $p_n$  has  $n$  distinct roots all lying in  $(-1, 1)$ .*

Gauss had the happy idea of choosing the evaluation points to be the roots of a Legendre polynomial.

**Theorem 9.6.** (i) *If  $\alpha_1, \alpha_2, \dots, \alpha_n$  are the  $n$ th roots of the  $n$ th Legendre polynomial  $p_n$  and the  $A_j$  are chosen so that*

$$\int_{-1}^1 P(x) dx = \sum_{j=0}^n A_j P(\alpha_j)$$

*for every polynomial  $P$  of degree  $n - 1$  or less, then, in fact*

$$\int_{-1}^1 Q(x) dx = \sum_{j=0}^n A_j Q(\alpha_j)$$

*for every polynomial  $Q$  of degree  $2n - 1$  or less.*

(ii) *If  $\beta_j \in [-1, 1]$  and  $B_j$  are such that*

$$\int_{-1}^1 Q(x) dx = \sum_{j=0}^n B_j Q(\beta_j)$$

*for every polynomial  $Q$  of degree  $2n - 1$  or less, then the  $\beta_j$  are the roots of the  $n$ th Legendre polynomial.*

Theorem 9.6 looks impressive but it is the next result which really shows how good Gauss' idea is.

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<sup>7</sup>There are various other definitions but they all give polynomials of the form  $b_n p_n$ . The only difference is in the choice of  $b_n$ . As may be seen from Exercise 19.7 our choice is not very convenient.

**Theorem 9.7.** *We continue with the notation of Theorem 9.6.*

(i)  $A_j \geq 0$  for each  $1 \leq j \leq n$ .

(ii)  $\sum_{j=1}^n A_j = 2$ .

(iii) If  $f : [-1, 1] \rightarrow \mathbb{R}$  is continuous and  $P$  is any polynomial of degree at most  $2n - 1$ , then

$$\left| \int_{-1}^1 f(x) dx - \sum_{j=0}^n A_j f(\alpha_j) \right| \leq 4 \|f - P\|_\infty.$$

(iv) Write  $G_n f$  for the estimate of  $\int_{-1}^1 f(t) dt$  obtained using Gauss' idea with the  $n$ th Legendre polynomial. Then, if  $f$  is continuous on  $[-1, 1]$ ,

$$G_n(f) \rightarrow \int_{-1}^1 f(t) dt$$

as  $n \rightarrow \infty$ .

## 10 Distance and compact sets

This section could come almost anywhere in the notes, but provides some helpful background to the section on Runge's theorem.

**Lemma 10.1.** *If  $E$  is a non-empty compact set in  $\mathbb{R}^m$  and  $\mathbf{a} \in \mathbb{R}^m$  then there is a point  $\mathbf{e} \in E$  such that*

$$\|\mathbf{a} - \mathbf{e}\| = \inf_{\mathbf{x} \in E} \|\mathbf{a} - \mathbf{x}\|.$$

We write  $d(\mathbf{a}, E) = \inf_{\mathbf{x} \in E} \|\mathbf{a} - \mathbf{x}\|$ .

**Exercise 10.2.** (i) *Give an example to show that the point  $\mathbf{e}$  in Lemma 10.1 need not be unique.*

(ii) *Show that, if  $E$  is convex,  $\mathbf{e}$  is unique.*

**Lemma 10.3.** (i) *If  $E$  and  $F$  are non-empty compact sets in  $\mathbb{R}^m$ , then there exist points  $\mathbf{e} \in E$  and  $\mathbf{f} \in F$  such that*

$$\|\mathbf{e} - \mathbf{f}\| = \inf_{\mathbf{y} \in E} d(\mathbf{y}, F).$$

(ii) *The result in (i) remains true when  $E$  is compact and non-empty and  $F$  is closed and non-empty.*

(iii) *The result in (i) may fail when  $E$  and  $F$  are closed and non-empty.*

Let us write  $\tau(E, F) = \inf_{\mathbf{y} \in E} d(\mathbf{y}, F)$ .

**Exercise 10.4.** Give an example to show that the points  $\mathbf{e}$  and  $\mathbf{f}$  in Lemma 10.3 need not be unique.

Let us recall the definition of a metric.

**Definition 1.1.** Suppose that  $X$  is a non-empty set and  $d : X^2 \rightarrow \mathbb{R}$  is a function such that

- (i)  $d(x, y) \geq 0$  for all  $x, y \in X$ .
- (ii)  $d(x, y) = 0$  if and only if  $x = y$ .
- (iii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .
- (iv)  $d(x, y) + d(y, z) \geq d(x, z)$  for all  $x, y, z \in X$ .

Then we say that  $d$  is a metric on  $X$  and that  $(X, d)$  is a metric space.

**Exercise 10.5.** Show that, if we consider the space  $\mathcal{K}$  of non-empty compact sets in  $\mathbb{R}^m$ , then  $\tau$  obeys conditions (i) and (iii) for a metric but not conditions (ii) and (iv).

Show that, if  $E, F \in \mathcal{K}$ , then  $\tau(E, F) = 0$  if and only if  $E \cap F \neq \emptyset$ .

Since  $\tau$  does not provide a satisfactory metric on  $\mathcal{K}$ , we try some thing else. If  $E$  and  $F$  are compact sets in  $\mathbb{R}^m$ , let us set  $\sigma(E, F) = \sup_{\mathbf{y} \in E} d(\mathbf{y}, F)$ .

**Exercise 10.6.** Suppose that  $E$  and  $F$  are non-empty compact sets. Show that there exists an  $\mathbf{e} \in E$  such that  $d(\mathbf{e}, F) = \sigma(E, F)$ .

**Exercise 10.7.** Show that, if we consider the space  $\mathcal{K}$  of non-empty compact sets in  $\mathbb{R}^m$ , then  $\sigma$  obeys condition (i) for a metric but not conditions (ii) and (iii).

Show that  $\sigma(E, F) = 0$  if and only if  $E \subseteq F$ .

However,  $\sigma$  does obey the triangle inequality.

**Lemma 10.8.** If  $E, F$  and  $G$  are non-empty compact sets then

$$\sigma(E, G) \leq \sigma(E, F) + \sigma(F, G).$$

This enables us to define the Hausdorff metric  $\rho$ .

**Definition 10.9.** If  $E$  and  $F$  are non-empty compact subsets of  $\mathbb{E}$ , we set

$$\rho(E, F) = \sigma(E, F) + \sigma(F, E),$$

that is to say,

$$\rho(E, F) = \sup_{\mathbf{e} \in E} \inf_{\mathbf{f} \in F} \|\mathbf{e} - \mathbf{f}\| + \sup_{\mathbf{f} \in F} \inf_{\mathbf{e} \in E} \|\mathbf{e} - \mathbf{f}\|.$$

**Theorem 10.10.** *The Hausdorff metric  $\rho$  is indeed a metric on the space  $\mathcal{K}$  of non-empty compact subsets of  $\mathbb{R}^m$ .*

Indeed, we can say something even stronger.

**Theorem 10.11.** *The Hausdorff metric  $\rho$  is a complete metric on the space  $\mathcal{K}$  of non-empty compact subsets of  $\mathbb{R}^m$ .*

## 11 Runge's theorem

Weierstrass' theorem tells us that every continuous real valued function on  $[a, b]$  can be uniformly approximated by polynomials. Does a similar theorem hold for complex variable?

Cauchy's theorem enables us to answer with a resounding no.

**Example 11.1.** *Let  $\bar{D} = \{z : |z| \leq 1\}$  and define  $f : \bar{D} \rightarrow \mathbb{C}$  by*

$$f(z) = \bar{z}.$$

*If  $P$  is any polynomial, then*

$$\sup_{z \in \bar{D}} |f(z) - p(z)| \geq 1.$$

After looking at this example the reader may recall the following theorem (which is not, however part of this course).

**Theorem 11.2.** *If  $\Omega$  is an open subset of  $\mathbb{C}$  and  $f_n : \Omega \rightarrow \mathbb{C}$  is analytic, then if  $f_n \rightarrow f$  uniformly on  $\Omega$  (or, more generally, if  $f_n \rightarrow f$  uniformly on each compact subset of  $\Omega$ ) then  $f$  is analytic.*

We might now conjecture that every analytic function on a well behaved set can be uniformly approximated by polynomials. Cauchy's theorem again shows that the matter is not straightforward.

**Example 11.3.** *Let  $\bar{T} = \{z : 1/2 \leq |z| \leq 2\}$  and define  $f : \bar{T} \rightarrow \mathbb{C}$  by*

$$f(z) = \frac{1}{z}.$$

*If  $P$  is any polynomial then*

$$\sup_{z \in \bar{T}} |f(z) - p(z)| \geq 1.$$

Thus the best we can hope for is a theorem that tells us that every analytic function on a suitable set ‘without holes’ can be uniformly approximated by polynomials.

The reader may ask if this does not follow from Taylor’s theorem in complex variables. To see that it does not we recall various earlier results. (The proofs are not part of the course but you are expected to know the results.)

**Lemma 11.4.** *If  $a_j \in \mathbb{C}$ , then there exists an  $R \in [0, \infty]$  (with suitable conventions when  $R = \infty$ ) such that  $\sum_{j=0}^{\infty} a_j z^j$  converges for  $|z| < R$  and diverges for  $|z| > R$ .*

We call  $R$  the radius of convergence of  $\sum_{j=0}^{\infty} a_j z^j$ .

**Lemma 11.5.** *If  $\sum_{j=0}^{\infty} a_j z^j$  has radius of convergence  $R$  and  $R' < R$  then  $\sum_{j=0}^{\infty} a_j z^j$  converges uniformly for  $|z| \leq R'$ .*

**Lemma 11.6.** *Suppose that  $\sum_{j=0}^{\infty} a_j z^j$  has radius of convergence  $R$  and that  $\sum_{j=0}^{\infty} b_j z^j$  has radius of convergence  $R'$ . If there exists an  $R''$  with  $0 < R'' < R, R'$  such that*

$$\sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} b_j z^j$$

for all  $|z| < R''$ , then  $a_j = b_j$  for all  $j$ .

By a careful use of Taylor’s theorem Lemma 11.6 can be used to give the following extension. (The proof is not part of the course but, again, you are expected to know the result.)

**Lemma 11.7.** *Suppose that  $f, g : B(w, r) \rightarrow \mathbb{C}$  are analytic. If there is a non-empty open subset  $U$  of  $B(w, r)$  such that  $f(z) = g(z)$  for all  $z \in U$ , it follows that  $g = f$ .*

**Exercise 11.8.** (i) *If  $w \neq 0$  show that we can find a power series  $\sum_{j=0}^{\infty} a_j (z - w)^j$  with radius of convergence  $|w|$  such that*

$$z^{-1} = \sum_{j=0}^{\infty} a_j (z - w)^j$$

for all  $|z - w| < |w|$ .

(ii) *Let*

$$\Omega = \{z : |z| < 1\} \setminus \{z : |\Im z| \leq 10^{-2}, \Re z \geq -10^{-2}\}.$$

Show that  $\Omega$  is open and bounded and  $f(z) = 1/z$  defines a bounded analytic function on  $\Omega$  but we can not find  $z_0$  and  $b_j$  such that

$$z^{-1} = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

for all  $z \in \Omega$ .

Let us see what Taylor's theorem actually says. (I will probably run through the proof to remind the audience of some of the methods of complex analysis.)

**Theorem 11.9.** *Suppose that  $\Omega$  is a open set in  $\mathbb{C}$  and  $f : \Omega \rightarrow \mathbb{C}$  is analytic. If the open disc*

$$B(z_0, \delta) = \{z : |z - z_0| < \delta\}$$

*lies in  $\Omega$ , then we can find  $a_j \in \mathbb{C}$  such that  $\sum_{j=0}^{\infty} a_j z^j$  has radius of convergence at least  $\delta$  and*

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j = f(z)$$

for all  $z \in B(z_0, \delta)$ .

Thus Taylor's theorem for analytic functions says (among other things) that an analytic function can be *locally* approximated uniformly by polynomials. Runge's theorem asserts that (under certain conditions) an analytic function can be *globally* approximated uniformly by polynomials.

**Theorem 11.10. [Runge's theorem]** *Suppose that  $\Omega$  is an open set and  $f : \Omega \rightarrow \mathbb{C}$  is analytic. Suppose that  $K$  is a compact set with  $K \subseteq \Omega$  and  $\mathbb{C} \setminus K$  path connected. Then given any  $\epsilon > 0$ , we can find a polynomial  $P$  with*

$$\sup_{z \in K} |f(z) - P(z)| < \epsilon.$$

I shall make a number of remarks before moving on to the proof. The first is that (as might be expected) Theorem 11.10 is the simplest of a family of results which go by the name of Runge's theorem. However, I think that it is fair to say that, once the proof of this simplest case is understood, both the proofs and the meanings of the more general theorems are not hard to grasp.

The second remark is that the statement ' $\mathbb{C} \setminus K$  path connected' is a 'no holes' condition.

**Definition 11.11.** An open set  $U \subseteq \mathbb{C}$  is path connected if, given  $z_0, z_1 \in U$ , we can find a continuous map  $\gamma : [0, 1] \rightarrow U$  with  $\gamma(0) = z_0$  and  $\gamma(1) = z_1$ .

The reader will lose very little understanding<sup>8</sup> if she concentrates on the example of Runge's theorem in which

$$K = \{z, : |z| \leq 1\}, \quad \Omega = \{z, : |z| \leq r\}$$

(where  $r > 1$ ).

Our proof of Runge's theorem splits into several steps.

**Lemma 11.12.** Suppose that  $K$  is a compact set with  $K \subseteq \Omega$ . Then we can find a finite set of piecewise linear contours  $C_m$  lying entirely within  $\Omega \setminus K$  such that

$$f(z) = \sum_{m=1}^M \int_{C_m} \frac{f(w)}{w - z} dw$$

whenever  $z \in K$  and  $f : \Omega \rightarrow \mathbb{C}$  is analytic.

It is worth making the following observation explicit.

**Lemma 11.13.** With the notation and conditions of Lemma 11.12, we can find a  $\delta > 0$  such that  $|z - w| \geq \delta$  whenever  $z \in K$  and  $w$  is a point of one of the contours  $C_m$ .

We use Lemma 11.12 to prove the following result which takes us closer to our goal.

**Lemma 11.14.** Suppose that  $K$  is a compact set with  $K \subseteq \Omega$ . Then given any analytic  $f : \Omega \rightarrow \mathbb{C}$  and any  $\epsilon > 0$  we can find an integer  $N$ , complex numbers  $A_1, A_2, \dots, A_N$  and  $\alpha_1, \alpha_2, \dots, \alpha_N \in \Omega \setminus K$  such that

$$\left| f(z) - \sum_{n=1}^N \frac{A_n}{z - \alpha_n} \right| < \epsilon$$

for all  $z \in K$ .

Thus Runge's theorem follows at once from the following special case.

**Lemma 11.15.** Suppose that  $K$  is a compact set and  $\mathbb{C} \setminus K$  path connected. Then, given any  $\alpha \notin K$  and any  $\epsilon > 0$ , we can find a polynomial  $P$  with

$$\left| P(z) - \frac{1}{z - \alpha} \right| < \epsilon$$

for all  $z \in K$ .

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<sup>8</sup>And, provided she does not twist the examiner's nose, few marks in the exam.

Let us make a temporary definition.

**Definition 11.16.** Let  $K$  be a compact set in  $\mathbb{C}$ . We write  $\Lambda(K)$  for the set of points  $\alpha \notin K$  such that, given any  $\epsilon > 0$ , we can find a polynomial  $P$  with

$$\left| P(z) - \frac{1}{z - \alpha} \right| < \epsilon$$

for all  $z \in K$ .

A series of observations about  $\Lambda(K)$  brings the proof of Runge's theorem to a close.

**Lemma 11.17.** Let  $K$  be a compact set in  $\mathbb{C}$ . Then there exists an  $R$  such that  $|\alpha| > R$  implies  $\alpha \in \Lambda(K)$ .

**Lemma 11.18.** Let  $K$  be a compact set in  $\mathbb{C}$ . If  $\alpha \in \Lambda(K)$  and  $|\alpha - \beta| < d(\alpha, K)$  then  $\beta \in \Lambda(K)$ .

**Lemma 11.19.** Suppose that  $K$  is a compact set in  $\mathbb{C}$  and  $\mathbb{C} \setminus K$  is path connected. Then  $\Lambda(K) = \mathbb{C} \setminus K$ .

Since Lemma 11.18 is equivalent to Lemma 11.15, this completes the proof of our version of Runge's theorem.

It is natural to ask if the condition of uniform convergence can be dropped in Theorem 11.2. We can use Runge's theorem to show that it can not.

**Example 11.20.** Let  $D = \{z : |z| < 1\}$  and define  $f : D \rightarrow \mathbb{C}$  by

$$f(re^{i\theta}) = r^{3/2}e^{3i\theta/2}$$

for  $r \geq 0$  and  $0 < \theta \leq 2\pi$  (so that  $f$  is not even continuous). Then we can find a sequence of polynomials  $P_n$  such that  $P_n(z) \rightarrow f(z)$  as  $n \rightarrow \infty$  for all  $z \in D$ .

## 12 Odd numbers

According to Von Neumann<sup>9</sup> 'In mathematics you don't understand things. You just get used to them.' The real line is one of the most extraordinary

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<sup>9</sup>This quotation is widely attributed to him and sounds characteristic but I would be glad to have a proper source.

objects in mathematics<sup>10</sup>. A single apparently innocuous axiom (‘every increasing bounded sequence has a limit’ or some equivalent formulation) calls into being an indescribably complicated object.

We know from 1A that  $\mathbb{R}$  is uncountable (a different proof of this fact will be given later in Corollary 13.7). But, if we have a finite alphabet of  $n$  symbols (including punctuation), then we can only describe at most  $n^m$  real numbers in phrases exactly  $m$  symbols long. Thus the collection of describable real numbers is the countable union of finite (so countable) sets so (quoting 1A again) countable! We find ourselves echoing Newton.

I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me. [*Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton* Brewster (Volume II. Ch. 27)]

Let us look at some of the prettier shells.

**Theorem 12.1.** *The number  $e$  is irrational.*

**Theorem 12.2.** *The number  $\pi$  is irrational.*

Our proof of Theorem 12.2 depends on the following lemma.

**Lemma 12.3.** *Let us write*

$$S_n(x) = \frac{1}{2^n n!} \int_0^x (x^2 - t^2)^n \cos t \, dt.$$

*Then  $S_n(x) = q_n(x) \sin x - p_n(x) \cos x$  where  $p_n$  and  $q_n$  satisfy the recurrence relations*

$$\begin{aligned} p_n(x) &= (2n - 1)p_{n-1}(x) - x^2 p_{n-2}(x), \\ q_n(x) &= (2n - 1)q_{n-1}(x) - x^2 q_{n-2}(x) \end{aligned}$$

*and  $p_0(x) = 0$ ,  $q_0(x) = 1$ ,  $p_1(x) = x$ ,  $q_1(x) = 1$*

Faced with a proof like that of Theorem 12.2 the reader may cry ‘How did you think of looking at  $S_n(x)$ ?’ The first, though not very helpful, answer is ‘I did not, I learnt it from someone else<sup>11</sup>.’ The second answer is that,

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<sup>10</sup>I am being modest on behalf of analysis, I suspect the real line is the most extraordinary object in mathematics.

<sup>11</sup>Professor Gowers in this case and he, in turn, learnt it from someone else.

even admitting that we could not have thought of it in a thousand years, once we are presented with the argument we can see a path (not the actual one<sup>12</sup>) by which it might have been thought of. We are all aware of the use of repeated integration to evaluate

$$\int_0^{\pi/2} x^n \cos x \, dx$$

and that the result is polynomial in  $\pi/2$  with integral coefficients. Obviously if  $P$  is a polynomial in  $x$  with integral coefficients  $\int_0^{\pi/2} P(x) \cos x \, dx$  is still a polynomial in  $\pi/2$  with integral coefficients. Thus we could apply the argument of Theorem 12.2 provided only that we could make  $\int_0^{\pi/2} P(x) \cos x \, dx$  sufficiently small. It might also occur to us (though this would probably come later) that the argument would also work in the more general case when  $P$  is a polynomial in  $x$  with coefficients integral multiples of powers of  $\pi/2$ , since it will remain true that  $\int_0^{\pi/2} P(x) \cos x \, dx$  is a polynomial in  $\pi/2$  with integral coefficients. After this we start experimenting with various  $P$ 's (including  $x^n$  and  $(\pi/2 - x)^n$ ). Repeated trials could well<sup>13</sup> lead us to  $S_n(x)$  or to some other suitable polynomial.

For what it is worth, I think the restriction 'candidates will not be required to quote elaborate formula from memory' ought to mean that the examiners remind you of the formula for  $S_n$  in a question that requires it. However, it is also my opinion that examiners like umpires are always right.

It may be worth remembering that, after 300 years we still do not know if Euler's constant

$$\gamma = \lim_{N \rightarrow \infty} \left( \sum_{n=0}^N \frac{1}{n} - \log N \right)$$

is irrational or not.

If we think of the rationals as 'the best understood numbers' then the algebraic numbers can be thought of as 'the next best understood numbers'.

**Definition 12.4.** *We say that a real number  $\alpha$  is algebraic if it is a zero of a polynomial with integer coefficients. Real numbers which are not algebraic are called transcendental.*

**Exercise 12.5.** *Show that a real number  $\alpha$  is algebraic if and only if it is a zero of a polynomial with rational coefficients.*

<sup>12</sup>The argument seems to have its roots in Lambert's original proof which used the continued fraction expansion of  $\tan x$  discussed in Theorem 15.10.

<sup>13</sup>'Computation is the way to truth.' Of course, the opposite is also true.

**Lemma 12.6.** ★ *The algebraic numbers are countable.*

Since the reals are uncountable, this shows that transcendental numbers exist.

The argument just given (which you saw in 1A) is due to Cantor. It is beautiful but non-constructive. It tells us that transcendental numbers exist (indeed that uncountably many transcendental numbers exist) without showing us any.

The first proof that transcendentals exist is due to Liouville. It is longer but actually produces particular examples.

**Theorem 12.7.** [Liouville] *Suppose  $\alpha$  is an irrational root of the equation*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$

*where  $a_j \in \mathbb{Z}$  [ $0 \leq j \leq n$ ] and  $a_n \neq 0$ . Then there is a constant  $c > 0$  (depending on the  $a_j$ ) such that*

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^n}$$

*for all  $p, q \in \mathbb{Z}$  with  $q \neq 0$*

We say that ‘irrational algebraic numbers are not well approximated by rationals’.

**Theorem 12.8.** *The number*

$$\sum_{n=0}^{\infty} \frac{1}{10^{n!}}$$

*is transcendental.*

**Exercise 12.9.** *By considering*

$$\sum_{n=0}^{\infty} \frac{b_n}{10^{n!}}$$

*with  $b_j \in \{1, 2\}$ , give another proof that the set of transcendental numbers is uncountable.*

As might be expected, it turned out to be very hard to show that particular numbers are transcendental. Hermite proved that  $e$  is transcendental and Lindemann adapted Hermite’s method to show that  $\pi$  is transcendental (and so the circle can not be squared). Alan Baker has contributed greatly to this field, and his book *Transcendental number theory* [1] contains accessible<sup>14</sup> proofs of the transcendence of  $e$  and  $\pi$ .

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<sup>14</sup>But miraculous.

## 13 The Baire category theorem

The following theorem turns out to be much more useful than its somewhat abstract formulation makes it appear.

**Theorem 13.1.** [The Baire category theorem] *If  $(X, d)$  is a complete metric space and  $U_1, U_2, \dots$  are open sets whose complements have empty interior, then*

$$\bigcap_{j=1}^{\infty} U_j \neq \emptyset.$$

Taking complements, gives the following equivalent form.

**Theorem 13.2.** *If  $(X, d)$  is a complete metric space and  $F_1, F_2, \dots$  are closed sets with empty interior, then*

$$\bigcup_{j=1}^{\infty} F_j \neq X.$$

Baire's theorem has given rise to the following standard definitions<sup>15</sup>.

**Definition 13.3.** *A set in a metric space is said to be nowhere dense if its closure has empty interior. A set in a metric space is said to be of first category if it is the countable union of nowhere dense sets. Any set which is not of first category is said to be of second category.*

**Lemma 13.4.** (i) *If  $(X, d)$  is a complete metric space and  $E$  is a subset of first category, then  $E \neq X$ .*

(ii) *The countable union of sets of first category is of first category.*

We need one more definition.

**Definition 13.5.** (i) *If  $(X, d)$  is a metric space, we say that a point  $x \in X$  is isolated if we can find a  $\delta > 0$  such that  $B(x, \delta) = \{x\}$ .*

(ii) *If  $(X, d)$  is a metric space, we say that a subset  $E$  of  $X$  contains no isolated points if, whenever  $x \in E$  and  $\delta > 0$ , we have  $B(x, \delta) \cap E \neq \{x\}$ .*

**Theorem 13.6.** *A complete metric space without isolated points is uncountable.*

**Corollary 13.7.** *The real numbers are uncountable.*

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<sup>15</sup>Your lecturer thinks that, whilst the concepts defined are very useful, the nomenclature is particularly unfortunate.

The proof we have given here is much closer to Cantor's original proof than that given in 1A. It avoids the use of extraneous concepts like decimal representation.

Here is another corollary of Theorem 13.6.

**Corollary 13.8.** *A non-empty closed subset of  $\mathbb{R}$  without isolated points is uncountable.*

Do there exist nowhere dense closed subsets of  $\mathbb{R}$  with no isolated points<sup>16</sup>? We shall answer this question by applying Baire's theorem in the context of the Hausdorff metric.

**Lemma 13.9.** *Consider the space of compact subsets of  $\mathbb{R}$  with the Hausdorff metric  $\rho$ .*

(i) *The set  $\mathcal{E}_k$  of compact sets  $E$  such that there exists an  $x \in E$  with  $B(x, 1/k) \cap E = \{x\}$  is closed and nowhere dense.*

(ii) *The set  $\mathcal{E}$  of compact sets with an isolated point is of first category.*

**Lemma 13.10.** *Consider the space  $\mathcal{K}$  of compact subsets of  $\mathbb{R}$  with the Hausdorff metric  $\rho$ .*

(i) *The set  $\mathcal{F}_{j,k}$  of compact sets  $F$  such that there exists an  $F \supseteq [j/k, (j+1)/k]$  is closed and nowhere dense.*

(ii) *The set  $\mathcal{E}$  of compact sets with non empty interior is of first category.*

**Theorem 13.11.** *The set of compact sets with empty interior and no isolated points is of second category in the space  $\mathcal{K}$  of compact subsets of  $\mathbb{R}$  with the Hausdorff metric  $\rho$ .*

The following example provides a background to our next use of Baire category.

**Exercise 13.12.** (i) *Show that we can find continuous functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  such that  $f_n(x) \rightarrow 0$  for each  $x \in [0, 1]$  but*

$$\sup_{t \in [0, 1]} f_n(t) \rightarrow \infty$$

as  $n \rightarrow \infty$ .

[Hint: Witch's hat.]

(ii) *Let  $N$  be a strictly positive integer. Show that we can find continuous functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  such that  $f_n(x) \rightarrow 0$  for each  $x \in [0, 1]$  but*

$$\sup_{t \in [r/N, (r+1)/N]} f_n(t) \rightarrow \infty$$

for each  $0 \leq r \leq N - 1$  as  $n \rightarrow \infty$ .

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<sup>16</sup>Such sets are called perfect. If they make pretty pictures they are called fractals. You do not have to remember either name.

In spite of the previous example we have the following remarkable theorem.

**Theorem 13.13.** *Suppose that we have a sequence of continuous functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  such that  $f_n(x) \rightarrow 0$  for each  $x \in [0, 1]$  as  $n \rightarrow \infty$ . Then we can find an interval  $(a, b) \subseteq [0, 1]$  and an  $M > 0$  such that*

$$|f_n(t)| \leq M$$

for all  $t \in (a, b)$  and all  $n \geq 1$ .

## 14 Continued fractions

One problem with preparing notes in advance is that material may take more or less time to cover. In the rest of the course I will cover some or all of the remaining parts of these notes, but the choice of topics and the order that I follow will depend on the time remaining.

We are used to writing real numbers as decimals, but there are other ways of specifying real numbers which may be more convenient. The oldest of these is the method of continued fractions. Suppose that  $x$  is irrational and  $1 \geq x > 0$ . We know that there is a strictly positive integer  $N(x)$  such that

$$\frac{1}{N(x)} \geq x > \frac{1}{N(x) + 1}$$

so we can write

$$x = \frac{1}{N(x) + T(x)}$$

where  $T(x)$  is irrational and  $1 \geq T(x) > 0$ . We can do the same things to  $T(x)$  as we did to  $x$ , obtaining

$$T(x) = \frac{1}{N(T(x)) + T(T(x))}$$

and so, using the standard notation for composition of functions,

$$x = \frac{1}{N(x) + \frac{1}{NT(x) + T^2(x)}}.$$

The reader<sup>17</sup> will have no difficulty in proceeding to the next step and obtaining

$$x = \frac{1}{N(x) + \frac{1}{NT(x) + \frac{1}{NT^2(x) + T^3(x)}}},$$

and so on indefinitely. We write

$$x = \frac{1}{N(x) + \frac{1}{NT(x) + \frac{1}{NT^2(x) + \frac{1}{NT^3(x) + \dots}}}}.$$

If  $y$  is a general irrational number, we can take  $x = y - [y]$  and write

$$y = [y] + \frac{1}{N(x) + \frac{1}{NT(x) + \frac{1}{NT^2(x) + \frac{1}{NT^3(x) + \dots}}}}.$$

We can do the same thing if  $y$  is rational, but we must allow for the possibility that the process does not continue indefinitely. It is instructive to carry out the process in a particular case.

**Exercise 14.1.** *Carry out the process outlined above for  $100/37$ . Carry out the process for the rational of your choice.*

Once we have done a couple of examples it is clear what is going on.

**Lemma 14.2.** *(i) Suppose that  $0 < x < 1$ ,  $x = b_0/a_0$  with  $a_0$  and  $b_0$  coprime strictly positive integers and  $a_0 \geq 2$ . Then*

$$x = \frac{1}{k_1 + \frac{b_1}{a_1}}$$

with

$$b_0 = a_1, \quad a_0 = k_1 a_1 + b_1$$

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<sup>17</sup>As opposed to typesetters; this sort of thing turned their hair prematurely grey.

for some positive integer  $k_1$ . Further  $a_1$  and  $b_1$  are coprime strictly positive integers with  $0 < b_1/a_1 < 1$ .

(ii) If  $y$  is a rational number, its continued fraction expansion (obtained by the method described above) terminates.

**Exercise 14.3.** Show that  $\sqrt{2}$  has continued fraction expansion

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

Deduce that  $\sqrt{2}$  is irrational.

If we look at a random variable with the uniform distribution on  $[0, 1]$  then the successive terms in the decimal expansion of  $X$  will be independent and will take the value  $j$  with probability  $1/10$  [ $0 \leq j \leq 9$ ].

**Exercise 14.4.** (This easy exercise formalises the remark just made.) If  $x \in [0, 1)$  let us write  $Dx = 10x - [10x]$  (in other words,  $Dx$  is the fractional part of  $10x$ ) and  $Nx = [10x]$ ). Show that

$$x = 10^{-1}(Dx + Nx) = 10^{-1}(Dx + 10^{-1}(D^2x + NDx)) = \dots$$

and write down the next term in the chain of equalities explicitly.

If  $X$  is a random variable with uniform distribution on  $[0, 1]$  show that  $NX, ND^2X, ND^4X, \dots$  are independent and

$$\Pr(ND^kX = j) = 1/10$$

for  $0 \leq j \leq 9$ .

Gauss made the following observation.

**Lemma 14.5.** Suppose that  $X$  is a random variable on  $[0, 1]$  with density function

$$f(x) = \left( \frac{1}{\log 2} \right) \frac{1}{1+x}.$$

Then  $T(X)$  is a random variable with the same density function.

**Corollary 14.6.** *Suppose that  $X$  is a random variable on  $[0, 1]$  with density function*

$$f(x) = \left(\frac{1}{\log 2}\right) \frac{1}{1+x}.$$

*Then*

$$\Pr(NT^m X = j) = \frac{1}{\log 2} \int_{1/(j+1)}^{1/j} \frac{dx}{1+x} = \log \left( \frac{(j+1)^2}{(j+1)^2 - 1} \right).$$

With a little extra work (which we shall not do) we can show that, if  $X$  has the density suggested by Gauss, then  $NX$ ,  $NTX$ ,  $NT^2X$ , ... are independent random variables all with the same probability distribution. It is also not hard to guess, and not very hard to prove, that if  $X$  is uniformly distributed, then

$$\Pr(NT^m X = j) \rightarrow \log \left( \frac{(j+1)^2}{(j+1)^2 - 1} \right)$$

as  $m \rightarrow \infty$ , but we shall not take the matter further.

## 15 Continued fractions (continued)

In the previous section we showed how to compute the continued fraction associated with a real number  $x$ , but we did not really consider what exact meaning was to be assigned to the result. In this section we show that continued fractions really do what we might hope they do.

**Definition 15.1.** *If  $a_1, a_2, \dots$  is a sequence of strictly positive integers we call*

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}$$

*the associated continued fraction.*

**Lemma 15.2.** (i) We use the notation of Definition 15.1. For appropriate choices of the integers  $r_j$  and  $s_j$ , we have

$$\frac{r_k}{s_k} = a_k + \frac{1}{a_{k+1} + \frac{1}{a_{k+2} + \frac{1}{a_{k+3} + \frac{1}{a_{k+4} + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}}}$$

and

$$\begin{pmatrix} r_k & s_k \end{pmatrix} = \begin{pmatrix} r_{k+1} & s_{k+1} \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$$

(ii) For appropriate choices of the integers  $p_j$  and  $q_j$ , we have

$$\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}}}$$

and

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

**Theorem 15.3.** Choose  $p_j$  and  $q_j$  as in Lemma 15.2.

- (i)  $p_k q_{k-1} - q_k p_{k-1} = (-1)^k$  for all  $k$ .
- (ii)  $q_k = a_k q_{k-1} + q_{k-2}$  and  $p_k = a_k p_{k-1} + p_{k-2}$
- (iii)  $p_k$  and  $q_k$  are coprime for all  $k$ .
- (iv) We have

$$\frac{p_{2k-3}}{q_{2k-3}} > \frac{p_{2k-1}}{q_{2k-1}} > \frac{p_{2k}}{q_{2k}} > \frac{p_{2k-2}}{q_{2k-2}}.$$

and

$$\left| \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right| = \frac{1}{q_k q_{k+1}}.$$

(v) There exists an  $\alpha \in \mathbb{R}$  such that

$$\frac{p_n}{q_n} \rightarrow \alpha.$$

Further

$$\left| \frac{p_n}{q_n} - \alpha \right| \leq \frac{1}{q_n q_{n+1}}.$$

Naturally we call  $\alpha$  the value of the associated continued fraction.

**Exercise 15.4.** Suppose we have an irrational  $x \in (0, 1]$  and we form a continued fraction

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}$$

is the manner of Section 14. Show that

$$\frac{p_{2k-1}}{q_{2k-1}} > x > \frac{p_{2k}}{q_{2k}}$$

for all  $k$  and deduce that  $x = \alpha$ .

**Theorem 15.5.** Continuing with the ideas and notation of Theorem 15.3,  $p_n/q_n$  is closer to  $\alpha$  than any other fraction with denominator no larger than  $q_n$ . In other words,

$$\left| \frac{p_n}{q_n} - \alpha \right| \leq \left| \frac{p}{q} - \alpha \right|$$

whenever  $p$  and  $q$  are integers with  $1 \leq q \leq q_n$ .

**Theorem 15.6.** If  $\alpha$  is irrational, we can find  $u_n$  and  $v_n$  integers with  $v_n \rightarrow \infty$  such that

$$\left| \frac{u_n}{v_n} - \alpha \right| < \frac{1}{v_n^2}.$$

**Exercise 15.7.** Which earlier result tells us that, if  $\alpha$  is the irrational root of a quadratic, then there exists a  $C$  (depending on  $\alpha$ ) such that

$$\left| \frac{u}{v} - \alpha \right| \geq \frac{C}{v^2}$$

whenever  $u$  and  $v$  are integers with  $v \geq 1$ .

We can treat ‘terminating continued fractions’ and rationals in the same way.

Speaking rather vaguely, we see that the occurrence of large  $a_j$ ’s in a continued fraction expansion gives rise to large  $q_m$ ’s and associated good approximations. It is reasonable to look at the most extreme opposite case.

**Exercise 15.8.** (i) Show that, if we write

$$\sigma = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}$$

then

$$\sigma = \frac{-1 + \sqrt{5}}{2}.$$

(ii) Show that, if we form  $p_n$  and  $q_n$  in the usual way for the continued fraction above, then  $p_n = F_n$ ,  $q_n = F_{n+1}$  where  $F_m$  is the  $m$ th Fibonacci number given by  $F_0 = 0$ ,  $F_1 = 1$  and

$$F_{m+1} = F_m + F_{m-1}.$$

(iii) Show that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}.$$

**Exercise 15.9.** In one of Lewis Carroll’s favourite puzzles an  $8 \times 8$  square is reassembled to form a  $13 \times 5$  rectangle as shown in Figure 1.

What is the connection with Exercise 15.8? Can you design the next puzzle in the sequence?

Hardy and Wright’s *An Introduction to the Theory of Numbers* [4] contains a chapter on approximation by rationals in which they show, among other things, that  $\sigma$  is indeed particularly resistant to being so approximated by rationals. If I was asked to nominate a book to be taken away by some one leaving *professional* mathematics, but wishing to keep up an interest in the subject, this book would be my first choice.

The notion of a continued fraction can be extended in many ways. It is not hard to see how someone trying to imitate our procedure for producing continued fractions might be led to guess that the following formula might

Figure 1: Carroll's puzzle

hold in some sense.

$$\tan x = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \dots}}}}$$

The following theorem makes the guess precise.

**Theorem 15.10.** *If we write*

$$R_n(x) = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{\dots - \frac{x^2}{2n-1}}}},$$

*then  $R_n(x) \rightarrow \tan x$  as  $n \rightarrow \infty$  for all real  $x$  with  $|x| \leq 1$ .*

In order to attack this we start by generalising an earlier result

**Exercise 15.11.** Suppose that  $a_j$  and  $b_j$  [ $j = 0, 1, 2, \dots$ ] are chosen so that we never divide by zero (for example all strictly positive). Show that for appropriate choices of  $p_j$  and  $q_j$ , we have

$$\frac{p_n}{q_n} = a_0 + \frac{b_0}{a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \frac{b_3}{a_4 + \frac{b_4}{\ddots + \frac{b_{n-1}}{a_{n-1} + \frac{b_{n-1}}{a_n}}}}}}}$$

and

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ b_{n-1} & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ b_0 & 0 \end{pmatrix}.$$

Deduce that

$$\begin{aligned} p_n &= a_n p_{n-1} + b_{n-2} p_{n-2} \\ q_n &= a_n q_{n-1} + b_{n-2} q_{n-2}. \end{aligned}$$

It is also useful to recall an earlier result.

**Lemma 12.3.** Let us write

$$S_n(x) = \frac{1}{2^{n_n!}} \int_0^x (x^2 - t^2)^n \cos t \, dt.$$

Then  $S_n(x) = q_n(x) \sin x - p_n(x) \cos x$  where  $p_n$  and  $q_n$  satisfy the recurrence relations

$$\begin{aligned} p_n(x) &= (2n - 1)p_{n-1}(x) - x^2 p_{n-2}(x), \\ q_n(x) &= (2n - 1)q_{n-1}(x) - x^2 q_{n-2}(x) \end{aligned}$$

and  $p_0(x) = 0$ ,  $q_0(x) = 1$ ,  $p_1(x) = x$ ,  $q_1(x) = 1$ .

This result and other interesting topics are discussed in a book [2] which is a model of how a high level recreational mathematics text should be put together.

## 16 Winding numbers

We all know that complex analysis has a lot to say about ‘the number of times a curves goes round a point’. In this final section we make the notion precise.

**Theorem 16.1.** *Let*

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

*If  $g : [0, 1] \rightarrow \mathbb{T}$  is continuous with  $g(0) = e^{i\theta_0}$  then there is a unique continuous function  $\theta : [0, 1] \rightarrow \mathbb{R}$  with  $\theta(0) = \theta_0$ . such that  $g(t) = e^{i\theta(t)}$  for all  $t \in [0, 1]$ .*

**Exercise 16.2.** *Suppose  $\psi, \phi : [0, 1] \rightarrow \mathbb{R}$  are continuous with  $e^{i\psi(t)} = e^{i\phi(t)}$  for all  $t \in [0, 1]$ . Show that there exists an integer  $n$  such that  $\psi(t) = \phi(t) + 2n\pi$  for all  $t \in [0, 1]$ .*

**Corollary 16.3.** *If  $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  is continuous with  $\gamma(0) = |\gamma(0)|e^{i\theta_0}$  then there is a unique continuous function  $\theta : [0, 1] \rightarrow \mathbb{R}$  such that  $\gamma(t) = |\gamma(t)|e^{i\theta(t)}$ .*

**Definition 16.4.** *If  $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  and  $\theta : [0, 1] \rightarrow \mathbb{R}$  are continuous with  $\gamma(t) = |\gamma(t)|e^{i\theta(t)}$  then we define*

$$w(\gamma, 0) = \frac{\theta(1) - \theta(0)}{2\pi}.$$

Exercise 16.2 shows that  $w(\gamma, 0)$  does not depend on the choice of  $\theta$ .

**Exercise 16.5.** *(i) If  $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  is continuous and  $\gamma(0) = \gamma(1)$  (that is to say, the path is closed) show that  $w(\gamma, 0)$  is an integer.*

*(ii) Give an example to show that, under the conditions of (i),  $w(\gamma, 0)$  can take any integer value.*

If  $a \in \mathbb{C}$ , it is natural to define the winding number round  $a$  of a curve given by a continuous map

$$\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{a\}$$

to be

$$w(\gamma, a) = w(\gamma - a, 0)$$

but we shall not use this slight extension.

**Lemma 16.6.** *If  $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  then the product  $\gamma_1\gamma_2$  satisfies*

$$w(\gamma_1\gamma_2, 0) = w(\gamma_1, 0) + w(\gamma_2, 0).$$

**Lemma 16.7. [Dog walking lemma]** *If  $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  are continuous,  $\gamma_1(0) = \gamma_1(1)$ ,  $\gamma_2(0) = \gamma_2(1)$  and*

$$|\gamma_2(t)| < |\gamma_1(t)|$$

*for all  $t \in [0, 1]$ , then  $\gamma_1 + \gamma_2$  never takes the value 0 and*

$$w(\gamma_1 + \gamma_2, 0) = w(\gamma_1, 0).$$

Many interesting results in ‘applied complex analysis’ are obtained by ‘deforming contours’. The idea of ‘continuously deforming curves’ can be made precise in a rather clever manner.

**Definition 16.8.** *Suppose that  $\gamma_0, \gamma_1$  are closed paths not passing through 0 (so we have,  $\gamma_j : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$ ). Then we say that  $\gamma_0$  is homotopic to  $\gamma_1$  by closed curves not passing through zero if we can find a continuous function  $\Gamma : [0, 1]^2 \rightarrow \mathbb{C} \setminus \{0\}$  such that*

$$\begin{aligned} \Gamma(s, 0) &= \Gamma(s, 1) && \text{for all } s \in [0, 1], \\ \Gamma(0, t) &= \gamma_0(t) && \text{for all } t \in [0, 1], \\ \Gamma(1, t) &= \gamma_1(t) && \text{for all } t \in [0, 1]. \end{aligned}$$

We often write  $\gamma_s(t) = \Gamma(s, t)$ .

**Exercise 16.9.** *If  $\gamma_1$  and  $\gamma_2$  satisfy the conditions of Definition 16.8, we write  $\gamma_0 \simeq \gamma_1$ . Show that  $\simeq$  is an equivalence relation on closed curves not passing through zero.*

The proof of the next theorem illustrates the utility of Definition 16.8. The proof itself is sometimes referred to as ‘dog walking along a canal’.

**Theorem 16.10.** *If  $\gamma_1$  and  $\gamma_2$  satisfy the conditions of Definition 16.8, then  $w(\gamma_0, 0) = w(\gamma_1, 1)$ .*

As before, let us write

$$\begin{aligned} \bar{D} &= \{z \in \mathbb{C} : |z| \leq 1\} \\ D &= \{z \in \mathbb{C} : |z| < 1\} \\ \partial D &= \{z \in \mathbb{C} : |z| = 1\} \end{aligned}$$

**Corollary 16.11.** *Suppose  $f : \bar{D} \rightarrow \mathbb{C}$  is continuous,  $f(z) \neq 0$  for  $z \in \partial D$ , and we define  $\gamma : [0, 1] \rightarrow \mathbb{C}$  by*

$$\gamma(t) = f(e^{2\pi it})$$

*for all  $t \in [0, 1]$ . If  $w(\gamma, 0) \neq 0$ , then there must exist a  $z \in D$  with  $f(z) = 0$ .*

This gives us another proof of the Fundamental Theorem of Algebra (Theorem 2.8).

**Corollary 16.12.** *If we work in the complex numbers, every polynomial has a root.*

We also obtain a second proof of Brouwer's theorem in two dimensions. (Look at statement (ii) in Lemma 4.7.)

**Corollary 16.13.** *There does not exist a continuous function  $f : \bar{D} \rightarrow \partial D$  with  $f(z) = z$  for all  $z \in \partial D$ .*

The contents of this section show that parts of complex analysis are really just special cases of general 'topological theorem'. On the other hand, other parts (such as Taylor's theorem and Cauchy's theorem itself) depend crucially on the fact that we are dealing with the very restricted class of functions which satisfy the Cauchy-Riemann equations.

In traditional courses on complex analysis, this fact appears, if it appears at all, rather late in the day. Beardon's *Complex Analysis* [3] shows that it is possible to do things differently and is well worth a look<sup>18</sup>.

## References

- [1] A. Baker, *Transcendental number theory*, CUP, 1975.
- [2] K. Ball, *Strange Curves, Counting Rabbits and Other Mathematical Explorations*, Princeton University Press, 2003.
- [3] A. F. Beardon *Complex Analysis*, Wiley, 1979.
- [4] G. H. Hardy and E. M. Wright *An Introduction to the Theory of Numbers* OUP, 1937.
- [5] L. C. Thomas *Games, Theory and Applications*, Dover reprint of a book first published by Wiley in 1984.

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<sup>18</sup>Though, like other good texts, it will not please those 'who want from books, plain cooking made still plainer by plain cooks.'

## 17 Exercise Sheet 1

Students should make sure that they can do all the questions which occur in the main course. The starred questions are not, in general, harder but are less central to the course. I think that students with a strong background in analysis should do the starred questions.

*Exercise 17.1.* (Lemma 1.2.) (i) Consider  $\mathbb{R}^n$ . If we take  $d$  to be ordinary Euclidean distance

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left( \sum_{j=1}^n |x_j - y_j|^2 \right)^{1/2},$$

show that  $(\mathbb{R}^n, d)$  is a metric space.

[Hint: Use Cauchy–Schwartz to obtain the triangle inequality.]

(ii) Consider  $\mathbb{C}$ . If we take  $d(z, w) = |z - w|$ , then  $(\mathbb{C}, d)$  is a metric space.

*Exercise 17.2.* (Exercise 1.3.) (i) Let  $X$  be a non-empty set. If we write

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y, \end{cases}$$

then  $(X, d)$  is a metric space.

(ii) Consider  $X = \{0, 1\}^n$ . If we take

$$d(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^n |x_j - y_j|$$

then  $(X, d)$  is a metric space.

*Exercise 17.3.* (Exercise 1.7.) (i) Show that, if  $x \in X$  and  $r > 0$ , then  $B(x, r)$  is open in the sense of Definition 1.6.

(ii) Show that, if  $x \in X$  and  $r > 0$ , then the set

$$\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$$

is closed.

*Exercise 17.4.* (Exercise 1.14.) Suppose that  $(X, d)$  is a metric space and  $Y$  is a non-empty subset of  $X$ .

(i) Show that, if  $d_Y(a, b) = d(a, b)$  for all  $a, b \in Y$ , then  $(Y, d_Y)$  is a metric space.

(ii) Show that, if  $(X, d)$  is complete and  $Y$  is closed in  $(X, d)$ , then  $(Y, d_Y)$  is complete.

(iii) Show that, if  $(Y, d_Y)$  is complete, then (whether  $(X, d)$  is complete or not)  $Y$  is closed in  $(X, d)$ .

*Exercise 17.5.* (Exercise 2.2.) Prove Theorem 2.1 by using the case  $m = 1$  proved in 1A and the kind of argument we used to prove Theorem 1.15.

*Exercise 17.6.* (Exercise 2.7.) Use Theorem 2.6 to obtain the theorem from 1A which states that, if  $f : [c, d] \rightarrow \mathbb{R}$  is continuous, then  $f$  is bounded and attains its bounds. (A couple of sentences should suffice.)

*Exercise 17.7.* Suppose that  $d$  is the usual metric on  $\mathbb{R}^m$   $X$  is a compact set in  $\mathbb{R}^m$  and  $f : X \rightarrow X$  is a continuous *distance increasing* map. In other words

$$d(f(x), f(y)) \geq d(x, y)$$

for all  $x, y \in X$ . The object of this question is to show that  $f$  must be a surjection.

(i) Explain why  $f(X)$  is closed.

(ii) If  $x \in X$ , consider the sequence  $x_n$  defined by  $x_0 = x$  and  $x_{n+1} = f(x_n)$ . By using compactness and the distance increasing property show that  $x \in f(X)$ . Thus  $f$  is surjective.

(iii) Let  $X = \{x \in \mathbb{R} : x \geq 0\}$ . Find a continuous function  $f : X \rightarrow X$  such that  $|f(x) - f(y)| \geq 2|x - y|$  for all  $x, y \in X$  but  $f(X) \neq X$ . Why does this not contradict (ii)?

(iv) We work in  $\mathbb{C}$ . Let  $\alpha$  be irrational and let  $\omega = \exp(2\pi\alpha i)$ . If

$$X = \{\omega^n : n \geq 1\},$$

find a continuous function  $f : X \rightarrow X$  such that  $|f(w) - f(z)| = |w - z|$  for all  $w, z \in X$  but  $f(X) \neq X$ . Why does this not contradict (ii)?

*Exercise 17.8.* (Exercise 2.9.) We work in the complex numbers.

(i) Use induction on  $n$  to show that, if  $P$  is a polynomial of degree  $n$  and  $a \in \mathbb{C}$ , then there exists a polynomial  $Q$  of degree  $n - 1$  and an  $r \in \mathbb{C}$  such that

$$P(z) = (z - a)Q(z) + r.$$

(ii) Deduce that, if  $P$  is a polynomial of degree  $n$  with root  $a$ , then there exists a polynomial  $Q$  of degree  $n - 1$  such that

$$P(z) = (z - a)Q(z).$$

(iii) Use induction and the fundamental theorem of algebra to show that every polynomial  $P$  of degree  $n$  can be written in the form

$$P(z) = A \prod_{j=1}^n (z - a_j)$$

with  $A, a_1, a_2, \dots, a_n \in \mathbb{C}$  and  $A \neq 0$ .

(iv) Show that, if  $P$  is a polynomial of degree at most  $n$  which vanishes at  $n + 1$  points, then  $P$  is the zero polynomial.

*Exercise 17.9.* (Exercise 3.2) (You may prefer to do question 17.11 first.)

(i) Show that  $\text{Int } E$  is open. Show also that, if  $V$  is open and  $V \subseteq E$ , then  $V \subseteq \text{Int } E$ .

(ii) Show that  $\text{Cl } E$  is closed. Show also that, if  $F$  is closed and  $F \supseteq E$ , then  $F \supseteq \text{Cl } E$ .

(Thus  $\text{Int } E$  is the largest open set contained in  $E$  and  $\text{Cl } E$  is the smallest closed set containing  $E$ .)

(iii) Show that  $\partial E$  is closed.

(iv) Suppose that we work in  $\mathbb{R}^m$  with the usual metric. Show that, if  $E$  is bounded, then so is  $\text{Cl } E$ .

*Exercise 17.10.* (Exercise 3.5.) Let  $\Omega$  be a bounded open subset of  $\mathbb{C}$ . Suppose that

$$f : \text{Cl } \Omega \rightarrow \mathbb{C}$$

is continuous and  $f$  is analytic on  $\Omega$ . Recall that the real and imaginary parts of  $f$  satisfy Laplace's equation on  $\text{Int } \Omega$ . Show that  $|f|$  attains its maximum on  $\partial\Omega$ .

[Hint: Why can we assume that  $\Re f(z_0) = |f(z_0)|$  at any particular point  $z_0$ ?]

*Exercise 17.11.* (Exercise 4.4.) Let  $(X, d)$  be a metric space. Show that  $E$  is open if and only if  $X \setminus E$  is closed.

*Exercise 17.12.* (Exercise 4.5.) Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces and let  $f : X \rightarrow Y$  be a function. Prove the following statements

(i)  $f$  is continuous if and only if  $f^{-1}(U)$  is open whenever  $U$  is.

(ii)  $f$  is continuous if and only if  $f^{-1}(E)$  is closed whenever  $E$  is.

*Exercise 17.13.* (Exercise 4.9.) Prove that the following four statements are equivalent.

(i) If  $f : [0, 1] \rightarrow [0, 1]$  is continuous, then we can find a  $w \in [0, 1]$  such that  $f(w) = w$ .

(ii) There does not exist a continuous function  $g : [0, 1] \rightarrow \{0, 1\}$  with  $g(0) = 0$  and  $g(1) = 1$ . (In topology courses we say that  $[0, 1]$  is connected.)

(iii) If  $A$  and  $B$  are closed subsets of  $[0, 1]$  with  $0 \in A$ ,  $1 \in B$  and  $A \cup B = [0, 1]$  then  $A \cap B \neq \emptyset$ .

(iv) If  $h : [0, 1] \rightarrow \mathbb{R}$  is continuous and  $h(0) \leq c \leq h(1)$ , then we can find a  $y \in [0, 1]$  such that  $h(y) = c$ .

*Exercise 17.14.* (Exercise 4.10.) Suppose that we colour the points  $r/n$  red or blue [ $r = 0, 1, \dots, n$ ] with 0 red and 1 blue. Show that there are a pair

of neighbouring points  $u/n$ ,  $(u + 1)/n$  of different colours. Use this result to prove statement (iii) of Exercise 17.13

## 18 Exercise Sheet 2

Students should make sure that they can do all the questions which occur in the main course. The starred questions are not, in general, harder but are less central to the course. I think that students with a strong background in analysis should do the starred questions.

*Exercise 18.1.* (i) Give an example of a continuous bijection  $f : \mathbb{R} \rightarrow \mathbb{R}$  with no fixed points.

(ii) Give an example of a closed convex set  $E$  in  $\mathbb{R}^2$  with  $E \neq \mathbb{R}^2$  and a continuous bijection  $f : E \rightarrow E$  with no fixed points.

(iii) Give an example of a bijection  $f : [0, 1] \rightarrow [0, 1]$  of the closed interval into itself with no fixed points.

(iv) If  $A$  is an infinite countable set show that there exists a bijection  $f : A \rightarrow A$  with no fixed points. What can you say if  $A$  is finite?

(v) Give an example of a bijection  $f : \bar{D} \rightarrow \bar{D}$  of the closed disc into itself with no fixed points.

[I think (v) is harder. One way is to write  $\bar{D} = A \cup B$  where  $A = \bar{D} \cap \mathbb{Q}^2$  and  $B = \bar{D} \setminus A$ . After doing Exercise 18.2 you may think of another way.]

*Exercise 18.2.* Let  $f : D \rightarrow D$  be a continuous map from the *open* disc

$$D = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < 1\}$$

into itself. Must it have a fixed point? Does your answer change if  $f$  is bijective? Give reasons.

*Exercise 18.3.* (Exercise 4.12.) Suppose that  $A = (a_{ij})$  is a  $3 \times 3$  matrix such that  $a_{ij} \geq 0$  for all  $1 \leq i, j \leq 3$  and  $\sum_{i=1}^3 a_{ij} = 1$  for all  $1 \leq j \leq 3$ . Let

$$T = \{\mathbf{x} \in \mathbb{R}^3 : x_j \geq 0 \text{ for all } j \text{ and } x_1 + x_2 + x_3 = 1\}.$$

By considering the effect of  $A$  on  $T$  show that  $A$  has an eigenvector lying in  $T$  with eigenvalue 1.

*Exercise 18.4.* Suppose that  $A = (a_{ij})$  is a  $3 \times 3$  matrix with positive entries and all columns non-zero vectors. Show that  $A$  has an eigenvector with positive entries.

*Exercise 18.5.\*\** (A brain teaser) An English nursery school teacher is on an exchange visit to Hungary. As part of the visit, the teacher conducts a

kindergarten class (of course this is done in English and no interpreter is needed) and tells the children to take  $10 \times 10$  grid of points and colour all its vertices red, blue, green or yellow. They must do this in such a way that all the points along the bottom are red, all the the ones on the right are blue, all the ones along the top are green and all the ones on the left are yellow. (The children politely point out that this will lead to conflict at the four corners and are told to choose one of the two possible colours in such a way that the corners are all of different colours.)

Little Moricz remarks that at least one of the  $9 \times 9$  small squares will require at least three colours. However, his playmate Malika paints her points so that no small square uses all four colours.

Explain little Moricz's remark and give a picture that Malika could have drawn.

*Exercise 18.6.\*\** (Only for the enthusiast.) Let  $\mathbf{a}_1, \mathbf{a}_2$  and  $\mathbf{a}_3$  be unit vectors making angles of  $\pm 2\pi/3$  with each other. We take  $T$  to be the closed triangle with vertices  $\mathbf{a}_1, \mathbf{a}_2$  and  $\mathbf{a}_3$  and sides  $I, J$  and  $K$  (just as in Lemma 4.7).

Let  $A, B$  and  $C$  be *open* subsets of  $\mathbb{R}^2$  with  $A \cup B \cup C \supseteq T$  and  $A \supseteq I, B \supseteq J, C \supseteq K$ . Show that  $A \cap B \cap C \cap T \neq \emptyset$ .

Here is one way to go about things. We assume the sides of  $T$  are of unit length.

(i) Show that there is a  $10^{-2} > \delta > 0$  such that if  $d(\mathbf{x}, I) < 4\delta$ , then  $\mathbf{x} \in A$  with similar results for  $B$  and  $C$ .

(ii) Suppose  $I'$  is the closed line segment in  $I$  consisting of all those points of  $I$  at least  $\delta$  from a vertex and  $J', K'$  are similarly defined. Let  $A' = A \setminus (J' \cup K')$  and let  $B', C'$  be similarly defined. Explain why  $A \subset A', A'$  is open  $A' \cap J' = A' \cap K' = \emptyset$ , and  $A' \cup B' \cup C' \supseteq T$ .

Conclude that it suffices to prove the result when

$$A \cap J' = A \cap K' = B \cap I' = B \cap K' = C \cap I' = C \cap J' = \emptyset.$$

(iii) Now consider

$$F = \{\mathbf{x} \in T : d(\mathbf{x}, A^c) \geq \max(d(\mathbf{x}, B^c), d(\mathbf{x}, C^c))\}$$

and similar sets  $G$  and  $H$ .

*Exercise 18.7.\** As usual  $\bar{D}$  is the closed unit disc in  $\mathbb{R}^2$  and  $\partial D$  its boundary. Let us write

$$\Delta = \{(\mathbf{x}, \mathbf{y}) \in \bar{D}^2 : \mathbf{x} \neq \mathbf{y}\}$$

and consider  $\Delta$  as a subset of  $\mathbb{R}^4$  with the usual metric. We define  $F : \Delta \rightarrow \partial D$  as follows.

Given  $(\mathbf{x}, \mathbf{y}) \in \Gamma$ , take the line from  $\mathbf{x}$  to  $\mathbf{y}$  and extend it (in the  $\mathbf{x}$  to  $\mathbf{y}$  direction) until it first hits the boundary at  $\mathbf{z}$ . We write  $F(\mathbf{x}, \mathbf{y}) = \mathbf{z}$ .

The lecture claimed that it was obvious that  $F$  was continuous. Suppose, if possible, that  $g : \bar{D} \rightarrow \bar{D}$  is a continuous map with  $g(\mathbf{x}) \neq \mathbf{x}$  for all  $\mathbf{x} \in \bar{D}$ . Explain why it follows from the lecturers claim that the map

$$\mathbf{x} \mapsto F(\mathbf{x}, g(\mathbf{x}))$$

is a continuous map.

The claimed result is obvious (in some sense) and you may take it as obvious in the exam. However, if we can not prove the obvious it ceases to be obvious. This, starred, question outlines one method of proof.

(i) Show that given any  $\epsilon > 0$  and any  $K > 0$  we can find a  $\delta > 0$  such that if  $f$  is a linear function (that is to say  $f(t) = at + b$ ) with the property that

$$|f(t_1)|, |f(t_2)| < \delta$$

for some  $t_1 \in [1/2, 2]$  and some  $t_2 \in [-2, -1/2]$  then  $|f(t)| < \epsilon$  for all  $|t| \leq K$ .

(ii) Suppose that  $(x_1, y), (x_2, y) \in \bar{D}$ ,  $y \geq 0$  and  $x_1 \neq x_2$ . By using (i), or otherwise, show that, given any  $\epsilon > 0$ , we can find an  $\eta > 0$  such that, whenever

$$\|(x'_1, y'_1) - (x_1, y)\|, \|(x'_2, y'_2) - (x_2, y)\| < \eta$$

and  $(x'_1, y'_1), (x'_2, y'_2) \in \bar{D}$ , we have  $(x'_1, y'_1) \neq (x'_2, y'_2)$  and

$$F((x'_1, y'_1), (x'_2, y'_2)) = (u, v) \text{ with } |v - y| < \epsilon.$$

(iii) Hence show that  $F : \Delta \rightarrow \bar{D}$  is continuous.

*Exercise 18.8.\** (Exercise 6.1.) Consider two rival firms  $A$  and  $B$  engaged in an advertising war. So long as the war continues, the additional costs of advertising mean that the larger firm  $A$  loses 3 million pounds a year and the smaller firm  $B$  loses 1 million pounds a year. If they can agree to cease hostilities then  $A$  will make 8 million a year and  $B$  will make 1 million a year. According to Nash, how much should  $A$  pay  $B$  per year to achieve this end.

*Exercise 18.9.\** We use the notation of Theorem 5.1. Suppose that  $a_{ij} = -b_{ij}$ , that is to say that Albert's gain is Bertha's loss. Show that if  $(p^*, q^*) \in E$ ,  $(p^{**}, q^{**}) \in E$  and

$$\begin{aligned} B(p^*, q^*) &\geq B(p^*, q) \text{ for all } (p^*, q) \in E, \\ A(p^*, q^*) &\geq A(p, q^*) \text{ for all } (p, q^*) \in E, \\ B(p^{**}, q^{**}) &\geq B(p^{**}, q) \text{ for all } (p^{**}, q) \in E, \\ A(p^{**}, q^{**}) &\geq A(p, q^{**}) \text{ for all } (p, q^{**}) \in E, \end{aligned}$$

then  $A(p^*, q^*) = A(p^{**}, q^{**})$ .

If you have done the 1B Optimisation course, explain why, in the kind of competitive zero-sum game considered there, the solution will always be a Nash equilibrium point.

*Exercise 18.10.* (Extension of Exercise 6.3.) Suppose that  $E$  is a bounded closed convex set in  $\mathbb{R}^n$ , that  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear map and that  $\mathbf{b} \in \mathbb{R}^m$ . Show that

$$\alpha(E) = \{\mathbf{b} + \alpha\mathbf{x} : \mathbf{x} \in E\}$$

is a bounded closed convex set in  $\mathbb{R}^m$ .

*Exercise 18.11.* (Exercise 7.1. It is important that you should have thought carefully about this example at least once in your life. It explains why we can not build real variable analysis on the basis of some kind of Taylor's theorem.)

Let  $E : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$E(t) = \begin{cases} \exp(-1/t^2) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

(i)  $E$  is infinitely differentiable, except, possibly, at 0, with

$$E^{(n)}(t) = P_n(1/t)E(t)$$

for all  $t \neq 0$  for some polynomial  $P_n$ .

(ii)  $E$  is infinitely differentiable everywhere with

$$E^{(n)}(0) = 0.$$

(iii) We have

$$E(t) \neq \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!} t^n$$

for all  $t \neq 0$ .

*Exercise 18.12.* (Exercise 7.4.) Show that the following define metrics on the space  $C([0, 1])$  of continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ .

(i)  $d_1(f, g) = \|f - g\|_1 = \int_0^1 |f(t) - g(t)| dt.$

(ii)  $d_2(f, g) = \|f - g\|_2 = \left( \int_0^1 |f(t) - g(t)|^2 dt \right)^{1/2}.$

(iii)  $d_3(f, g) = \|f - g\|_\infty = \sup_{t \in [0, 1]} |f(t) - g(t)|.$

Show that

$$\|f - g\|_\infty \geq \|f - g\|_1 \geq \|f - g\|_2.$$

[Hint: You may find the Cauchy–Schwartz inequality for integrals useful in at least two places.]

Let

$$f_n(t) = \begin{cases} (1 - nt) & \text{for } 0 \leq t \leq 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

Compute  $\|f_n\|_\infty/\|f_n\|_1$  and  $\|f_n\|_1/\|f_n\|_2$ . Comment.

*Exercise 18.13.* Consider the continuous functions on  $[0, 1]$  with the uniform norm. Show that the unit ball

$$\{f \in C([0, 1]) : \|f\|_\infty \leq 1\}$$

is a closed bounded subset of the complete space  $C([0, 1])$  but is not compact.

*Exercise 18.14.* (i) (Exercise 7.9) Suppose we only deal with a random variable  $X$  which takes a finite number of values. State and prove the algebraic inequality corresponding to Chebychev’s inequality without explicitly mentioning probability. Do you think there is any advantage in doing this?

(ii) Suppose that  $X$  is a real valued random variable taking a finite number of values and  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing function with  $\phi(0) = 0$ . Show that

$$\Pr(|X| \leq a) \leq \frac{\mathbb{E}\phi(|X|)}{\phi(a)}$$

for all  $a > 0$

*Exercise 18.15.* Use the ideas of Theorem 7.10 to show that, if  $f : [0, 1]^2 \rightarrow \mathbb{R}$  is continuous, then, given  $\epsilon > 0$ , we can find a polynomial  $P$  in two variables such that

$$|f(x, y) - P(x, y)| < \epsilon$$

for all  $x, y \in [0, 1]$ .

*Exercise 18.16.\** (Not very much to do with the course but a nice question.) Suppose  $f : [0, 1]^2 \rightarrow \mathbb{R}$  is such that the map  $x \mapsto f(x, y)$  is continuous for each fixed  $y$  and the map  $y \mapsto f(x, y)$  is continuous for each fixed  $x$ . By means of a proof or counterexample establish whether  $f$  is necessarily continuous.

## 19 Exercise Sheet 3

Students should make sure that they can do all the questions which occur in the main course. The starred questions are not, in general, harder but are less central to the course. I think that students with a strong background in analysis should do the starred questions.

*Exercise 19.1.* Suppose that  $(X, d)$  is a compact metric space,  $(Y, \rho)$  is a metric space and  $f : X \rightarrow Y$  is continuous. Show that, given any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that, whenever  $u, v \in X$  and  $d(u, v) < \delta$ , it follows that  $\rho(f(u), f(v)) < \epsilon$ . (In other words, continuity implies uniform continuity.)

[Hint: Suppose not. Then we can find an  $\epsilon > 0$  and  $u_n, v_n \in X$  with  $d(u_n, v_n) < 1/n$  but  $\rho(f(u_n), f(v_n)) \geq \epsilon$ .]

*Exercise 19.2.\** If  $n, r \in \mathbb{Z}$  and  $n \geq 1$ , let us define  $\Delta_{n,r} : [-1, 1] \rightarrow \mathbb{R}$  by

$$\Delta_{n,r}(x) = \max(0, 1 - n|x - rn^{-1}|).$$

Sketch  $\Delta_{n,r}$ .

Now suppose  $f : [-1, 1] \rightarrow \mathbb{R}$ . Show that

$$f_n(x) = \sum_{m=0}^n f(m/n) \Delta_{n,m}(x)$$

is a piecewise linear function with  $f_n(r/n) = f(r/n)$ .

Show that, if  $f$  is continuous,  $\|f_n - f\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ .

*Exercise 19.3.\** (i) Let  $\eta > 0$ . Define

$$g_n(x) = (1 + \eta - x^2/4)^n$$

and

$$G_n(x) = \left( \int_{-2}^2 g_n(t) dt \right)^{-1} g_n(x).$$

Let

$$H_n(t) = \int_{-2}^t G_n(x) dx, \quad K_n(t) = 2H_n(t) - 1, \quad \text{and} \quad F_n(t) = \int_{-2}^t K_n(x) dx.$$

If  $\eta$  is small, sketch the graphs of  $G_n$ ,  $H_n$ ,  $K_n$  and  $F_n$  on the interval  $[-2, 2]$  when  $n$  is very large.

(ii) Show that, given  $\epsilon > 0$  we can find a polynomial  $P$  such that

$$|P(x) - |x|| < \epsilon.$$

for  $x \in [-2, 2]$ .

*Exercise 19.4.\** (i) Sketch the function  $F$  given by

$$F(x) = |x - 1| - |x| + |x + 1|.$$

(ii) If  $\Delta_{n,r}$  is defined as in Exercise 19.2, find  $\alpha_j$ ,  $A_j$ , and  $k$  so that

$$\Delta_{n,r}(x) = \sum_{j=1}^k A_j |x - \alpha_j|.$$

(iii) Combine the ideas of Exercises 19.2 and 19.3 with part (ii) to obtain another proof of Weierstrass approximation theorem. (This idea is due to Lebesgue.)

*Exercise 19.5.\** In Theorem 8.4 we say that, if  $f : [a, b] \rightarrow \mathbb{R}$  is a continuous function there exists a polynomial  $P$  of degree at most  $n$  such that  $\|P - f\|_\infty \leq \|Q - f\|_\infty$  for all polynomials  $Q$  degree  $n$  or less. We claim that we can find  $a \leq a_0 \leq a_1 \leq \dots \leq a_n \leq b$  such that, writing  $\sigma = \|f - P\|_\infty$  we have either

$$f(a_j) - P(a_j) = (-1)^j \sigma \text{ for all } 0 \leq j \leq n$$

or

$$f(a_j) - P(a_j) = (-1)^j \sigma \text{ for all } 0 \leq j \leq n.$$

Our proof will be by reductio ad absurdum.

We assume without loss of generality that  $[a, b] = 0$  and  $\sigma = 1$ .

(i) Write  $g = f - P$ . Explain why we can find an integer  $N \geq 1$  such that if  $1 \leq r \leq N$  at least one of the following statements must be true

$$g(x) \geq 1/2 \text{ for all } x \in [(r-1)/N, r/N],$$

or

$$g(x) \leq -1/2 \text{ for all } x \in [(r-1)/N, r/N],$$

or

$$|g(x)| \leq 3/4 \text{ for all } x \in [(r-1)/N, r/N].$$

(ii) Using the result of (i) show that, if our putative theorem is false, we can find an integer  $q \leq n$  and integers

$$0 = u(1) < v(1) < u(2) < v(2) < \dots < u(q) < v(q) = N$$

and  $w \in \{0, 1\}$  such that

$$\begin{aligned} (-1)^{w+j} g(x) &> -1 \text{ for all } x \in [u(j)/N, v(j)/N] \\ |g(x)| &< 1 \text{ for all } x \in [v(j)/N, u(j+1)/N]. \end{aligned}$$

Without loss of generality, we take  $w = 0$ .

(iii) Explain why we can find an  $\eta > 0$  with

$$\begin{aligned} (-1)^j g(x) &> -1 + \eta \text{ for all } x \in [u(j)/N, v(j)/N] \\ |g(x)| &< 1 - \eta \text{ for all } x \in [v(j)/N, u(j+1)/N], \end{aligned}$$

for all  $j$ . We may take  $\eta < 1/8$  and will do so.

(iv) Explain how to find a polynomial  $R$  of degree  $n$  or less with  $\|R\|_\infty = 1$  such that

$$(-1)^j R(x) > 0 \text{ for all } x \in [u(j)/N, v(j)/N]$$

and  $j = 1, 2, \dots, q$ .

(v) Show that

$$|g(x) - (\eta/2)R(x)| < 1$$

for all  $x \in [0, 1]$  and deduce that  $\|g - (\eta/2)R\|_\infty < 1$ . Hence obtain a contradiction.

*Exercise 19.6.* (See Theorem 7.3 for definitions if necessary.) Compute the Chebychev polynomials  $T_n$  of the first kind and the Chebychev polynomials  $U_{n-1}$  of the second kind for  $n = 1, 2, \dots, 5$ .

*Exercise 19.7.* (i) Use the Gram–Schmidt method (see Lemma 9.3) to compute the Legendre polynomials  $p_n$  for  $n = 0, 1, 2, 3, 4$ . You need not compute the normalising factor for  $p_4$  explicitly.

(ii) Explain why

$$\frac{d^m}{dx^m}(1-x)^n(1+x)^n$$

vanishes at when  $x = 1$  or  $x = -1$  whenever  $m < n$ .

Suppose that

$$P_n(x) = \frac{d^n}{dx^n}(1-x^2)^n.$$

Use integration by parts to show that

$$\int_{-1}^1 P_n(x)P_m(x) dx = 0$$

for  $m \neq n$ . Conclude that the  $P_n$  are scalar multiple of the Legendre polynomials  $p_n$ .

(iii) Compute  $P_n$  for  $n = 0, 1, 2, 3, 4$  and check that these verify the last sentence of (ii).

*Exercise 19.8.* (Exercises 10.2 and 10.4.) Let  $E$  and  $F$  be a non-empty compact sets in  $\mathbb{R}^m$ .

(i) Suppose that  $\mathbf{a} \in \mathbb{R}^m$ . Give an example to show that there may be distinct points  $\mathbf{e}_1, \mathbf{e}_2 \in E$  with

$$d(\mathbf{a}, E) = d(\mathbf{a}, \mathbf{e}_1) = d(\mathbf{a}, \mathbf{e}_2).$$

(ii) Show that, if  $E$  is convex, there is only one point  $\mathbf{e} \in E$  with

$$d(\mathbf{a}, E) = d(\mathbf{a}, \mathbf{e}).$$

(iii) Give an example to show that there may be distinct points  $\mathbf{e}_1, \mathbf{e}_2 \in E$  and  $\mathbf{f}_1, \mathbf{f}_2 \in F$  such that

$$\inf_{\mathbf{y} \in E} d(\mathbf{y}, F) = \|\mathbf{e}_1 - \mathbf{f}_1\| = \|\mathbf{e}_2 - \mathbf{f}_2\|.$$

(iv) Is it true that, if  $E$  and  $F$  convex there is only one pair of points  $\mathbf{e} \in E$  and  $\mathbf{f} \in F$  such that

$$\inf_{\mathbf{y} \in E} d(\mathbf{y}, F) = \|\mathbf{e} - \mathbf{f}\|?$$

*Exercise 19.9.\** Recall that we say that a function  $f; [-1, 1] \rightarrow \mathbb{R}$  is *even* if  $f(x) = f(-x)$  for all  $x$  and *odd* if  $f(x) = -f(-x)$  for all  $x$ .

Explain why we know, without calculation, that the Chebychev polynomials  $T_n$  are even when  $n$  is even and odd when  $n$  is odd. What can you say about the Chebychev polynomials  $U_n$  of the second kind?

Explain why we know, without calculation, that the Legendre polynomials  $p_n$  are even when  $n$  is even and odd when  $n$  is odd.

*Exercise 19.10.* (Exercise 10.5.) Suppose that we consider the space  $\mathcal{K}$  of non-empty compact sets in  $\mathbb{R}^m$  and we set

$$\tau(E, F) = \inf_{\mathbf{y} \in E} d(\mathbf{y}, F).$$

Demonstrate, by proofs and counterexamples, which of the conditions for a metric  $\tau$  does and does not satisfy.

Show that, if  $E, F \in \mathcal{K}$ , then  $\tau(E, F) = 0$  if and only if  $E \cap F \neq \emptyset$ .

*Exercise 19.11.* (Exercises 10.6 and 10.7.) If  $E$  and  $F$  are non-empty compact sets in  $\mathbb{R}^m$ , let us write

$$\sigma(E, F) = \sup_{\mathbf{y} \in E} d(\mathbf{y}, F).$$

Show that there exists an  $\mathbf{e} \in E$  such that  $d(\mathbf{e}, F) = \sigma(E, F)$ .

Consider the space  $\mathcal{K}$  of non-empty compact sets in  $\mathbb{R}^m$ . Demonstrate, by proofs and counterexamples, which of the conditions for a metric (apart from the triangle inequality)  $\sigma$  does and does not satisfy.

Show that  $\sigma(E, F) = 0$  if and only if  $E \subseteq F$ .

*Exercise 19.12.* (Exercise 11.8.) (i) If  $w \neq 0$  show that we can find a power series  $\sum_{j=0}^{\infty} a_j(z-w)^j$  with radius of convergence  $|w|$  such that

$$z^{-1} = \sum_{j=0}^{\infty} a_j(z-w)^j$$

for all  $|z| < |w|$ .

(ii) Let

$$\Omega = \{z : |z| < 1\} \setminus \{z : |\Im z| \leq 10^{-2}, \Re z \geq -10^{-2}\}.$$

Show that  $\Omega$  is open and bounded and  $f(z) = 1/z$  defines a bounded analytic function on  $\Omega$  but we can not find  $z_0$  and  $b_j$  such that

$$z^{-1} = \sum_{j=0}^{\infty} b_j(z-z_0)^j$$

for all  $z \in \Omega$ .

*Exercise 19.13.\** (Note that some starred questions are very easy and this gives an example.) Show by quoting the appropriate theorems that, if  $\Omega$  is an open set in  $\mathbb{C}$ , then  $f : \Omega \rightarrow \mathbb{C}$  is analytic if and only if whenever  $K$  is a compact subset of  $\Omega$  and  $\epsilon > 0$  we can find a polynomial  $P$  such that

$$|P(z) - f(z)| < \epsilon$$

for all  $z \in K$ .

*Exercise 19.14.* (i) (Exercise 12.5.) Show that a real number  $\alpha$  is algebraic if and only if it is a zero of a polynomial with rational coefficients.

(ii) Is it true that if all the roots of a polynomial are algebraic, then the polynomial must have rational coefficients. Give a proof or counterexample

*Exercise 19.15.* (i) Show that  $\cosh 1$  is irrational.

(ii) (Traditional and only requiring 1A knowledge.) Show that  $\sqrt{3} + \sqrt{5}$  is irrational.

*Exercise 19.16.* (Exercise 12.9.) By considering

$$\sum_{n=0}^{\infty} \frac{b_n}{10^{n!}}$$

with  $b_j \in \{1, 2\}$ , give another proof that the set of transcendental numbers is uncountable.

*Exercise 19.17.* (i) We work in  $\mathbb{C}$ . Show that there exists a sequence of polynomials  $P_n$  such that

$$P_n(z) \rightarrow \begin{cases} 1 & \text{if } |z| \leq 1 \text{ and } \Re z \geq 0 \\ 0 & \text{if } |z| \leq 1 \text{ and } \Re z < 0 \end{cases}$$

as  $n \rightarrow \infty$ .

[Hint: Recall that, if  $\Omega_1$  and  $\Omega_2$  are disjoint open sets and  $f(z) = 0$  for  $z \in \Omega_1$  and  $f(z) = 1$  for  $z \in \Omega_2$ , then  $f$  is analytic on  $\Omega_1 \cup \Omega_2$ .]

(ii) Show that there exists a sequence of polynomials  $Q_n$  such that

$$P_n(z) \rightarrow \begin{cases} 1 & \text{if } \Re z \geq 0 \\ 0 & \text{if } \Re z < 0 \end{cases}$$

as  $n \rightarrow \infty$ .

## 20 Exercise Sheet 4

Students should make sure that they can do all the questions which occur in the main course. The starred questions are not, in general, harder but are less central to the course. I think that students with a strong background in analysis should do the starred questions.

*Exercise 20.1.* (Exercise 13.12.) (i) Show that we can find continuous functions  $f_n : [0, 1] \rightarrow \mathbb{R}$  such that  $f_n(x) \rightarrow 0$  for each  $x \in [0, 1]$  but

$$\sup_{t \in [0, 1]} f_n(t) \rightarrow \infty$$

as  $n \rightarrow \infty$ .

[Hint: Witch's hat.]

(ii) Let  $N$  be a strictly positive integer. Show that we can find continuous functions  $g_n : [0, 1] \rightarrow \mathbb{R}$  such that  $g_n(x) \rightarrow 0$  for each  $x \in [0, 1]$  but

$$\sup_{t \in [r/N, (r+1)/N]} g_n(t) \rightarrow \infty$$

for each  $0 \leq r \leq N - 1$  as  $n \rightarrow \infty$ .

(iii) Show that we can find continuous functions  $h_n : [0, 1] \rightarrow \mathbb{R}$  such that  $h_n(x) \rightarrow 0$  for each  $x \in [0, 1]$  but whenever  $0 \leq a < b \leq 1$  we can find a  $\kappa(a, b) > 0$  and  $N_0(a, b)$  such that

$$\sup_{t \in [a, b]} h_n(t) \geq \kappa(a, b)$$

for  $n \geq N_0(a, b)$ .

*Exercise 20.2.* The following exercise is traditional. It outlines a beautiful proof due to Banach showing the existence of continuous nowhere differentiable functions. We work on the space continuous functions  $C([0, 1])$  with the uniform norm. We consider the set  $E_m$  consisting of all those  $f \in C([0, 1])$  such that there exists an  $x_m \in [0, 1]$  with the property

$$|f(x) - f(y)| \leq m|x - y|$$

for all  $y \in [0, 1]$ .

(i) Show that if  $f \in C([0, 1])$  and  $f$  is differentiable at some point  $x \in [0, 1]$  then there exists an  $m \geq 1$  such that  $f \in E_m$ . Conclude that every  $g \in C([0, 1]) \setminus \bigcup_{m=1}^{\infty} E_m$  is nowhere differentiable.

(ii) Suppose that  $f_n \in E_m$  and  $\|f_n - f\|_{\infty} \rightarrow 0$ . By definition there exists an  $x \in [0, 1]$  with the property

$$|f_n(x_n) - f(y)| \leq m|x_n - y|$$

for all  $y \in [0, 1]$ . Explain why we can find  $x \in [0, 1]$  and  $n(r) \rightarrow \infty$  such that  $x_{n(r)} \rightarrow x$  as  $r \rightarrow \infty$ . Hence, or otherwise, show that  $f \in E_m$ . Thus  $E_m$  is closed.

(iii) If  $f \in E_m$  and  $\epsilon > 0$ , show that we can find a piecewise linear function  $g \in C([0, 1])$  such that  $\|f - g\| < \epsilon/2$ . (See Exercise 19.2 if necessary.) Show that there is a  $K$  such that

$$|g(x) - g(y)| \leq K|x - y|$$

for all  $x, y \in [0, 1]$ .

(iv) Let  $g$  be the function obtained in (iii). By considering functions of the form

$$h(x) = g(x) + \frac{\epsilon}{4} \sin Nx$$

with  $N$  very large, or otherwise, show that that we can find an  $h \in C([0, 1])$  such that  $\|g - h\|_{\infty} < \epsilon/2$  but  $h \notin E_m$ . Conclude that  $E_m$  is nowhere dense.

(v) Deduce that the set of nowhere differentiable functions is of second category in  $C([0, 1])$ . Why does this imply the existence of continuous nowhere differentiable functions?

*Exercise 20.3.\** (i) If  $0 < a < b$ , show that there is a  $K > 0$  such that

$$\bigcup_{n=1}^{\infty} [na, nb] \supseteq [K, \infty).$$

(ii) Suppose that  $f : (0, \infty) \rightarrow \mathbb{R}$  is a continuous function such that  $f(nx) \rightarrow 0$  as  $n \rightarrow \infty$  for each fixed  $x$ . Show that  $f(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

*Exercise 20.4.* (Exercise 14.1.) Carry out the process outlined in Section 14 for producing continued fractions by applying it  $100/37$ . Carry out the process for the rational of your choice.

*Exercise 20.5.* (i) (Exercise 14.3.) Show that  $\sqrt{2}$  has continued fraction expansion

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

Deduce that  $\sqrt{2}$  is irrational.

(ii) Find the continued fraction expansion for  $\sqrt{3}$ .

(iii) Find the value of the continued fraction

$$1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \dots}}}}$$

with the 1s and 2s continuing to alternate.

*Exercise 20.6.* What rational with denominator less than 10 best approximates  $71/49$  and why?

*Exercise 20.7.\** Take your electronic calculator out of your old school satchel (or use the expensive piece of equipment on which you play DVDs) and find the first few terms of the continued fraction for  $\pi$  (or, more strictly for the rational number that your calculator gives when you ask it for  $\pi$ .) Compute first few associated convergents (what we have called  $3 + p_n/q_n$ ).

Verify that  $355/113$  is an extremely good approximation for  $\pi$  and explain why this is so. Apparently the approximation was first discovered by the astronomer Tsu Ch'ung-Chih in the fifth century A.D.

The entries  $a_n$  in the continued fraction expansion for  $\pi$  look, so far as anyone knows, just like those you would expect from a random real number (in a sense made precise in Corollary 14.6).

I would be inclined to say that this was precisely what one should expect if there was not a beautiful expansion (using a generalisation of the kind of

continued fraction discussed in the course) found by Lord Brouncker in 1654.

$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \dots}}}}$$

Huygens refused to believe the result until Brouncker showed that it gave the correct value of  $\pi$  to 10 decimal places. Since no proof is given here, the reader may wish to carry out a similar check. However, the continued fraction is very slowly convergent so, if you are using a hand calculator, you will probably be satisfied with one decimal place ... 'men were men' in those pre-calculator days.

*Exercise 20.8.* (Exercise 15.4.) Suppose we have an irrational  $x \in (0, 1]$  and we form a continued fraction

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

is the manner of Section 14. Show that (using the notation of Section 15)

$$\frac{p_{2k-1}}{q_{2k-1}} > x > \frac{p_{2k}}{q_{2k}}$$

for all  $k$  and deduce that  $x = \alpha$ .

*Exercise 20.9.* (Exercise 15.7.) Which earlier result tells us that, if  $\alpha$  is the irrational root of a quadratic, then there exists a  $C$  (depending on  $\alpha$ ) such that

$$\left| \frac{u}{v} - \alpha \right| \geq \frac{C}{v^2}$$

whenever  $u$  and  $v$  are integers with  $v \geq 1$ .

*Exercise 20.10.* (Exercise 15.8.) (i) Show that, if we write

$$\sigma = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}$$

then

$$\sigma = \frac{-1 + \sqrt{5}}{2}.$$

(ii) Show that, if we form  $p_n$  and  $q_n$  in the usual way for the continued fraction above, then  $p_n = F_n$ ,  $q_n = F_{n+1}$  where  $F_m$  is the  $m$ th Fibonacci number given by  $F_0 = 0$ ,  $F_1 = 1$  and

$$F_{m+1} = F_m + F_{m-1}.$$

(iii) Show that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}.$$

*Exercise 20.11.\** (Exercise 15.9.) In one of Lewis Carroll's favourite puzzles an  $8 \times 8$  square is reassembled to form a  $13 \times 5$  rectangle as shown in Figure 2.

Figure 2: Carroll's puzzle

What is the connection with Exercise 20.11? Can you design the next puzzle in the sequence?

*Exercise 20.12.\** The Fibonacci sequence has many interesting aspects. (It is, so far as I know the only series with its own Fanzine — *The Fibonacci Quarterly*.)

(i) Find the general solution of the difference equation

$$u_{n+1} = u_n + u_{n-1}.$$

The Fibonacci series is the particular solution  $F_n = u_n$  with  $u_0 = 0$ ,  $u_1 = 1$ . Write  $F_n$  in the appropriate form.

(ii) Show, by using (i) or otherwise that if  $n \geq 1$ ,  $F_n$  is the closest integer to

$$\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n.$$

We call

$$\tau = \frac{1 + \sqrt{5}}{2}$$

the golden ratio.

Show that  $F_n = C(\tau^n + (-1)^{n+1}\tau^{-n})$  for some suitable  $C$ .

(iii) Prove the two identities

$$\begin{aligned} F_{2n+1} &= F_n^2 + F_{n+1}^2 \\ F_{2n} &= F_n(F_{n-1} + F_{n+1}) \end{aligned}$$

both by induction (it may be worth trying to prove them simultaneously) and by using the result of (i).

(iv) Let  $x_n = F_{n+1}/F_n$ . Use (iii) to express  $x_{2n}$  as a rational function of  $x_n$ .

(v) Suppose now we take  $y_k = x_{2^k}$ . Use (iv) to write  $y_{k+1}$  as a rational function of  $y_k$ . Show that  $y_k$  converges very rapidly to  $\tau$ . Can you link this with the Newton–Raphson method for finding a root of a particular function?

*Exercise 20.13.* (Exercise 15.11.) Suppose that  $a_j$  and  $b_j$  [ $j = 0, 1, 2, \dots$ ] are chosen so that we never divide by zero (for example all strictly positive). Show that for appropriate choices of  $p_j$  and  $q_j$ , we have

$$\frac{p_n}{q_n} = a_0 + \frac{b_0}{a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \frac{b_3}{a_4 + \frac{b_4}{\ddots + \frac{b_{n-1}}{a_{n-1} + \frac{b_n}{a_n}}}}}}}$$

and

$$\begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ b_{n-1} & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ b_0 & 0 \end{pmatrix}.$$

Deduce that

$$\begin{aligned} p_n &= a_n p_{n-1} + b_{n-1} p_{n-2} \\ q_n &= a_n q_{n-1} + b_{n-1} q_{n-2} \end{aligned}$$

*Exercise 20.14.\** Observe that the proof of Theorem 15.10 continues to work in the more general case when  $x = z \in \mathbb{C}$  and  $|z| \leq 1$ .

Hence obtain an expansion for  $\tanh y$  with  $y$  real. Deduce that

$$\frac{e+1}{e-1} = 2 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \frac{1}{18 + \dots}}}}$$

Why does this give another proof that  $e$  is irrational?

*Exercise 20.15.* (i) Suppose that  $x$  has the continued fraction expansion

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

Show that there is a sequence of fractions  $p_n/q_n$  such that

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{a_n q_n^2}.$$

(ii) If the coefficients are unbounded explain why this tells us that  $x$  is not the root of a quadratic equation with integer coefficients. What does the result of Exercise 20.14 tell you about  $e$ .

*Exercise 20.16.* (Exercise 16.2.) Suppose  $\psi, \phi : [0, 1] \rightarrow \mathbb{R}$  are continuous with  $e^{i\psi(t)} = e^{i\phi(t)}$  for all  $t \in [0, 1]$ . Show that there exists an integer  $n$  such that  $\psi(t) = \phi(t) + 2n\pi$  for all  $t \in [0, 1]$ .

*Exercise 20.17.* (Exercise 16.5) (i) If  $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  is continuous and  $\gamma(0) = \gamma(1)$  (that is to say, the path is *closed*) show that  $w(\gamma, 0)$  is an integer.

(ii) Give example to show that, under the condition of (i),  $w(\gamma, 0)$  can take any integer value.

*Exercise 20.18.* (i) (Exercise 16.9.) If  $\gamma_1$  and  $\gamma_2$  satisfy the conditions of Definition 16.8 we write  $\gamma_0 \simeq \gamma_1$ . Show that  $\simeq$  is an equivalence relation on closed curves not passing through zero.

(ii) Show that there are infinitely many equivalence classes.

*Exercise 20.19.* Let  $p(z) = z^2 - 4z + 3$  and let  $\gamma : [0, 1] \rightarrow \mathbb{C}$  be given by  $\gamma(t) = p(2e^{2\pi it})$ . Show that closed path associated with  $\gamma$  does not pass through 0.

Compute  $w(\gamma, 0)$

(i) Non-rigorously direct from the definition by obtaining enough information about  $\gamma$ , (You could write the real and imaginary parts of  $\gamma(t)$  in terms of  $\cos t$  and  $\sin t$  and find where and how  $\gamma$  crosses the real axis.)

(ii) by factoring, and

(iii) by the dog walking lemma.

*Exercise 20.20.\** Suppose that  $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$  is a continuously differentiable function with  $\gamma(0) = \gamma(1)$ .

If we define  $r : [0, 1] \rightarrow \mathbb{C}$  by

$$r(t) = \exp \left( \int_0^t \frac{\gamma'(s)}{\gamma(s)} ds \right)$$

compute the derivative of  $r(t)/\gamma(t)$  and deduce that

$$w(\gamma, 0) = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(s)}{\gamma(s)} ds.$$

Use this result and the residue theorem to compute  $w(\gamma, 0)$  in Exercise 20.19.