Introduction to Functional Analysis
Part III, Autumn 2006

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May 9, 2007

Small print This is just a first draft for the course. The content of the course will be what I say, not what these notes say. Experience shows that skeleton notes (at least when I write them) are very error prone so use these notes with care. I should **very much** appreciate being told of any corrections or possible improvements and might even part with a small reward to the first finder of particular errors.

Contents

1 Some notes on prerequisites

Many years ago it was more or less clear what could and what could not be assumed in an introductory functional analysis course. Since then, however, many of the concepts have drifted into courses at lower levels.

I shall therefore assume that you know what is a normed space, and what is a a linear map and that you can do the following exercise.

Exercise 1. *Let* $(X, \| \ \|_X)$ *and* $(Y, \| \ \|_Y)$ *be normed spaces.*

(i) If $T : X \to Y$ *is linear, then* T *is continuous if and only if there exists a constant* K *such that*

$$\|Tx\|_Y \leq K\|x\|_X$$

for all $x \in X$.

(ii) If $T : X \to Y$ *is linear and* $x_0 \in X$, *then* T *is continuous at* x_0 *if and only if there exists a constant* K *such that*

$$\|Tx\|_Y \leq K\|x\|_X$$

for all $x \in X$.

(iii) If we write $\mathcal{L}(X, Y)$ *for the space of continuous linear maps from* X *to* Y *and write*

$$\|T\| = \sup\{\|Tx\|_Y \ : \ \|x\|_X = 1, \ x \in X\}$$

then $(\mathcal{L}(X, Y), \| \ \|)$ *is a normed space.*

I also assume familiarity with the concept of a metric space and a complete metric space. You should be able to do at least parts (i) and (ii) of the following exercise (part (iii) is a little harder).

Exercise 2. *Let* $(X, \| \ \|_X)$ *and* $(Y, \| \ \|_Y)$ *be normed spaces.*

(i) If $(Y, \| \ \|_Y)$ *is complete then* $(\mathcal{L}(X, Y), \| \ \|)$ *is.*

(ii) Consider the set s *of sequences* $x = (x_1, x_2, \dots)$ *in which only finitely many of the* x_j *are non-zero. Explain briefly how* s *may be considered as a vector space. If we write*

$$\|x\| = \sup_j |x_j|$$

show that $(s, \| \ \|)$ is a normed vector space which is not complete.

(iii) If $(X, \| \ \|_X)$ is complete does it follow that $(\mathcal{L}(X,Y), \| \ \|)$ is? Give a proof or a counter-example.

The reader will notice that I have not distinguished between vector spaces over $\mathbb{R}$ and those over $\mathbb{C}$. I shall try to make the distinction when it matters but, if the two cases are treated in the same way, I shall often proceed as above.

Although I shall stick with metric spaces as much as possible, there will be points where we shall need the notions of a topological space, a compact topological space and a Hausdorff topological space. I would be happy, if requested, to give a supplementary lecture introducing these notions. (Even where I use them, no great depth of understanding is required.)

I shall also use, without proof, the famous Stone-Weierstrass theorem.

Theorem 3. *(A) Let X be a compact space and $C(X)$ the space of real valued continuous functions on X. Suppose A is a subalgebra of $C(X)$ (that is a subspace which is algebraically closed under multiplication) and*

(i) $1 \in A$,

(ii) Given any two distinct points x and y in X there is an $f \in A$ with $f(x) \neq f(y)$.

Then A is uniformly dense in $C(X)$.

(B) Let X be a compact space and $C(X)$ the space of complex valued continuous functions on X. Suppose A is a subalgebra of $C(X)$ and

(i) $1 \in A$,

(ii) Given any two distinct points x and y in X there is an $f \in A$ with $f(x) \neq f(y)$.

(iii) If $f \in A$ then $f^* \in A$.

Then A is uniformly dense in $C(X)$.

The proof will not be examinable, but if you have not met it, you may wish to request a supplementary lecture on the topic.

Functional analysis goes hand in hand with measure theory. Towards the end of the course I will need to refer Borel measures on the line. However, I will not use any theorems from measure theory proper and I will make my treatment independent of previous knowledge. Elsewhere I may make a few remarks involving measure theory. These are for interest only and will not be examinable[1]. I intend the course to be fully accessible without measure theory.

[1] In this course, as in other Part III courses you should assume that everything in the lectures and nothing outside them is examinable unless you are explicitly told to the contrary. If you are in any doubt, ask the lecturer.

2 Baire category

If (X, d) is a metric space we say that a set E in X has dense complement[2] if, given $x \in E$ and $\delta > 0$, we can find a $y \notin E$ such that $d(x, y) < \delta$.

Exercise 4. *Consider the space M_n of $n \times n$ complex matrices with an appropriate norm. Show that the set of matrices which do not have n distinct eigenvalues is a closed set with dense complement.*

Theorem 5 (Baire's theorem)**.** *If (X, d) is a complete metric space and E_1, E_2, ... are closed sets with dense complement then $X \neq \bigcup_{j=1}^{\infty} E_j$.*

Exercise 6. *(If you are happy with general topology.) Show that a result along the same lines holds true for compact Hausdorff spaces.*

We call the countable union of closed sets with dense complement a set of first category. The following observations are trivial but useful.

Lemma 7. *(i) The countable union of first category sets is itself of first category.*

(ii) If (X, d) is a complete metric space, then Baire's theorem asserts that X is not of first category.

Exercise 8. *If (X, d) is a complete metric space and X is countable show that there is an $x \in X$ and a $\delta > 0$ such that the ball $B(x, \delta)$ with centre x and radius δ consists of one point.*

The following exercise is a standard application of Baire's theorem.

Exercise 9. *Consider the space $C([0, 1])$ of continuous functions under the uniform norm $\| \ \|$. Let*

$$\begin{aligned} E_m = \{f \in C([0,1]) : & \text{ there exists an } x \in [0,1] \text{ with} \\ & |f(x+h) - f(x)| \leq m|h| \text{ for all } x + h \in [0,1]\}. \end{aligned}$$

(i) Show that E_m is closed in $(C([0, 1], \| \ \|_\infty)$.

(ii) If $f \in C([0, 1])$ and $\epsilon > 0$ explain why we can find an infinitely differentiable function g such that $\|f - g\|_\infty < \epsilon/2$. By considering the function h given by

$$h(x) = g(x) + \tfrac{\epsilon}{2} \sin Nx$$

with N large show that E_m has dense complement.

(iii) Using Baire's theorem show that there exist continuous nowhere differentiable functions.

[2] If the lecturer uses the words 'nowhere dense' correct him for using an old fashioned and confusing terminology

Exercise 10. *(This is quite long and not very central.)*

(i) Consider the space $\mathcal{F}$ of non-empty closed sets in $[0,1]$. Show that if we write
$$d_0(x, E) = \inf_{e \in E} |x - e|$$
when $x \in [0,1]$ and $E \in \mathcal{F}$ and write
$$d(E, F) = \sup_{f \in F} d_0(f, E) + \sup_{e \in E} d_0(e, F)$$
then d is a metric on $\mathcal{F}$.

(ii) Suppose E_n is a Cauchy sequence in $(\mathcal{F}, d)$. By considering
$$E = \{x : \text{there exist } e_n \in E_n \text{ such that } e_n \to x\},$$
or otherwise, show that E_n converges. Thus $(\mathcal{F}, d)$ is complete.

(iii) Show that the set
$$\mathcal{A}_n = \{E \in \mathcal{F} : \text{there exists an } x \in E \text{ with } (x - 1/n, x + 1/n) \cap E = \{x\}\}$$
is closed with dense complement in $(\mathcal{F}, d)$. Deduce that the set of elements of $\mathcal{F}$ with isolated points is of first category. (A set E has an isolated point e if we can find a $\delta > 0$ such that $(e - \delta, e + \delta) \cap E = \{e\}$.)

(iv) Let $I = [r/n, (r+1)/n]$ with $0 \leq r \leq n-1$ and r and n integers. Show that the set
$$\mathcal{B}_{r,n} = \{E \in \mathcal{F} : E \supseteq I\}$$
is closed with dense complement in $(\mathcal{F}, d)$. Deduce that the set of elements of $\mathcal{F}$ containing an open interval is of first category.

(v) Deduce the existence of non-empty closed sets which have no isolated points and contain no intervals.

3 Non-existence of functions of several variables

This course is very much a penny plain rather than tuppence coloured[3]. One exception is the theorem proved in this section.

Theorem 11. *Let λ be irrational We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[1 \leq j \leq 5]$ with the following property. Given any*

[3] And thus suitable for those ‘who want from books plain cooking made still plainer by plain cooks’.

continuous function $f : [0,1]^2 \to \mathbb{R}$ we can find a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x,y) = \sum_{j=1}^{5} g(\phi_j(x) + \lambda\phi_j(y)).$$

The main point of Theorem 11 may be expressed as follows.

Theorem 12. *Any continuous function of two variables can be written in terms of continuous functions of one variable and addition.*

That is, there are no true functions of two variables! (We shall explain why this statement is slightly less shocking than it seems at the end of this section.)

For the moment we merely observe that the result is due in successively more exact forms to Kolmogorov, Arnol'd and a succession of mathematicians ending with Kahane whose proof we use here. It is, of course, much easier to prove a specific result like Theorem 11 than one like Theorem 12.

Our first step is to observe that Theorem 11 follows from the apparently simpler result that follows.

Lemma 13. *Let λ be irrational. We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[1 \le j \le 5]$ with the following property. Given any continuous function $F : [0,1]^2 \to \mathbb{R}$ we can find a function $G : \mathbb{R} \to \mathbb{R}$ such that $\|G\|_\infty \le \|F\|_\infty$ and*

$$\sup_{(x,y)\in[0,1]^2} \left| F(x,y) - \sum_{j=1}^{5} G(\phi_j(x) + \lambda\phi_j(y)) \right| \le \frac{999}{1000} \|F\|_\infty.$$

Next we make the following observation.

Lemma 14. *We can find a sequence of functions $f_n : [0,1]^2 \to \mathbb{R}$ which are uniformly dense in $C([0,1])^2$.*

This enables us to obtain Lemma 13 from a much more specific result.

Lemma 15. *Let λ be irrational and let the f_n be as in Lemma 14. We can find increasing continuous functions $\phi_j : [0,1] \to \mathbb{R}$ $[1 \le j \le 5]$ with the following property. We can find functions $g_n : \mathbb{R} \to \mathbb{R}$ such that $\|g_n\|_\infty \le \|f_n\|_\infty$ and*

$$\sup_{(x,y)\in[0,1]^2} \left| f_n(x,y) - \sum_{j=1}^{5} g_n(\phi_j(x) + \lambda\phi_j(y)) \right| \le \frac{998}{1000} \|f_n\|_\infty.$$

Now that we have reduced the matter to satisfying a countable set of conditions, we can use a Baire category argument. We need to use the correct metric space.

Lemma 16. *The space Y of continuous functions $\boldsymbol{\phi} : [0,1] \to \mathbb{R}^5$ with norm*

$$\|\boldsymbol{\phi}\|_\infty = \sup_{t\in[0,1]} \|\boldsymbol{\phi}(t)\|$$

is complete. The subset X of Y consisting of those $\boldsymbol{\phi}$ such that each ϕ_j is increasing is a closed subset of Y. Thus if d is the metric on X obtained by restricting the metric on Y derived from $\| \ \|_\infty$ we have (X,d) complete.

Exercise 17. *Prove Lemma 16*

Lemma 18. *Let $f : [0,1]^2 \to \mathbb{R}$ be continuous and let λ be irrational. Consider the set E of $\boldsymbol{\phi} \in X$ such that there exists a $g : \mathbb{R} \to \mathbb{R}$ such that $\|g\|_\infty \leq \|f\|_\infty$*

$$\sup_{(x,y)\in[0,1]^2} \left| f(x,y) - \sum_{j=1}^{5} g(\phi_j(x) + \lambda\phi_j(y)) \right| < \frac{998}{1000}\|f\|_\infty.$$

Then $X \setminus E$ is a closed set with dense complement in (X,d).

(Notice that it is important to take '$<$' rather than '$\leq$' in the displayed formula of Lemma 18.) Lemma 18 is the heart of the proof and once it is proved we can easily retrace our steps and obtain Theorem 11.

By using appropriate notions of information Vitushkin was able to show that we can not replace continuous by continuously differentiable in Theorem 12. Thus Theorem 11 is an 'exotic' rather than a 'central' result.

4 The principle of uniform boundedness

We start with a result which is sometimes useful by itself but which, for us, is merely a stepping stone to Theorem 22.

Lemma 19 (Principle of uniform boundedness)**.** *Suppose that (X,d) is a complete metric space and we have a collection $\mathcal{F}$ of continuous functions $f : X \to \mathbb{R}$ which are pointwise bounded, that is, given any $x \in X$ we can find a $K(x) > 0$ such that*

$$|f(x)| \leq K(x) \text{ for all } f \in \mathcal{F}.$$

Then we can find a ball $B(x_0,\delta)$ and a K such that

$$|f(x)| \leq K \text{ for all } f \in \mathcal{F} \text{ and all } x \in B(x_0,\delta)\ .$$

Exercise 20. *(i) Suppose that (X, d) is a complete metric space and we have a sequence of continuous functions $f_n : X \to \mathbb{R}$ and a function $f : X \to \mathbb{R}$ such that f_n converges pointwise that is*

$$f_n(x) \to f(x) \text{ for all } f \in \mathcal{F}.$$

Then we can find a ball $B(x_0, \delta)$ and a K such that

$$|f_n(x)| \leq K \text{ for all } n \text{ and all } x \in B(x_0, \delta)\ .$$

(ii) (This is elementary but acts as a hint for (iii).) Suppose $y \in [0, 1]$. Show that we can find a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ such that $1 \geq f_n(x) \geq 0$ for all x and n, f_n converges pointwise to 0 everywhere, f_n converges uniformly on $[0, 1] \setminus (y - \delta, y + \delta)$ and fails to converge uniformly on $[0, 1] \cap (y - \delta, y + \delta)$ for all $\delta > 0$.

(iii) State with reasons whether the following statement is true or false. Under the conditions of (i) we can obtain the stronger conclusion that we can find a ball $B(x_0, \delta)$ such that

$$f_n(x) \to f(x) \text{ uniformly on } B(x_0, \delta).$$

Exercise 21. *Suppose that (X, d) is a complete metric space and Y is a subset of X which is of first category in X. Suppose further that we have a collection $\mathcal{F}$ of continuous functions $f : X \to \mathbb{R}$ which are pointwise bounded on $X \setminus Y$, that is, given any $x \notin Y$, we can find a $K(x) > 0$ such that*

$$|f(x)| \leq K(x) \text{ for all } f \in \mathcal{F}.$$

Show that we can find a ball $B(x_0, \delta)$ and a K such that

$$|f(x)| \leq K \text{ for all } f \in \mathcal{F} \text{ and all } x \in B(x_0, \delta)\ .$$

We now use the principle of uniform boundedness to prove the Banach-Steinhaus theorem[4].

Theorem 22. (Banach-Steinhaus theorem) *Let $(U, \|\ \|_U)$ and $(V, \|\ \|_V)$ be normed spaces and suppose $\|\ \|_U$ is complete. If we have a collection $\mathcal{F}$ of continuous linear maps from U to V which are pointwise bounded then we can find a K such that $\|T\| \leq K$ for all $T \in \mathcal{F}$.*

Here is a typical use of the Banach-Steinhaus theorem.

[4] You should be warned that a lot of people, including the present writer, tend to confuse the names of these two theorems. My research supervisor took the simpler course of referring to all the theorems of functional analysis as 'Banach's theorem'.

Theorem 23. *There exists a continuous 2π periodic function $f : \mathbb{R} \to \mathbb{R}$ whose Fourier series fails to converge at a given point.*

The next exercise contains results that most of you will have already met.

Exercise 24. *(i) Show that the set l^∞ of bounded sequences over $\mathbb{F}$ (with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$)*

$$\mathbf{a} = (a_1,\ a_2,\ \dots)$$

can be made into a vector space in a natural manner. Show that $\|\mathbf{a}\|_\infty = \sup_{j\geq 1} |a_j|$ defines a complete norm on l^∞.

(ii) Show that s, the set of convergent sequences and s_0 the set of sequences convergent to 0 are both closed subspaces of $(l^\infty, \|\ \|_\infty)$.

(iii) Show that the set l^1 of sequences

$$\mathbf{a} = (a_1,\ a_2,\ \dots) \text{ such that } \sum_{j=1}^{\infty} |a_j| \text{ converges}$$

can be made into vector space in a natural manner. Show that $\|\mathbf{a}\|_1 = \sum_{j=1}^\infty |a_j|$ defines a complete norm on l^1.

(iv) Show that, if $\mathbf{a} \in l^1$, then

$$T_{\mathbf{a}}(\mathbf{b}) = \sum_{j=1}^{\infty} a_j b_j$$

defines a continuous linear map from l^∞ to $\mathbb{F}$ and that $\|T_{\mathbf{a}}\| = \|\mathbf{a}\|_1$.

Here is another use of the Banach-Steinhaus theorem.

Lemma 25. *Let $a_{ij} \in \mathbb{R}$ $[i, j \geq 1]$. We say that the a_{ij} constitute a summation method if whenever $c_j \to c$ we have $\sum_{j=1}^\infty a_{ij}c_j$ convergent for each i and*

$$\sum_{j=1}^{\infty} a_{ij}c_j \to c$$

as $i \to \infty$.

The following conditions are necessary and sufficient for the a_{ij} to constitute a summation method:-

(i) There exists a K such that

$$\sum_{j=1}^{\infty} |a_{ij}| \leq K \text{ for all } i.$$

(ii) $\displaystyle\sum_{j=1}^{\infty} a_{ij} \to 1$ *as* $i \to \infty$.

(iii) $a_{ij} \to 0$ *as* $i \to \infty$ *for each* j.

Exercise 26. *Cesàro's summation method takes a sequence* $c_0,\ c_1,\ c_2, \dots$ *and replaces it with a new sequence whose nth term*

$$b_n = \frac{c_1 + c_2 + \dots + c_n}{n}$$

is the average of the first n terms of the old sequence.

(i) By rewriting the statement above along the lines of Lemma 25 show that if the old sequence converges to c so does the new one.

(ii) Examine what happens when $c_j = (-1)^j$. *Examine what happens if* $c_j = (-1)^k$ *when* $2^k \le j < 2^{k+1}$.

(iii) Show that, in the notation of Lemma 25, taking $a_{n,2n} = 1$, $a_{n,m} = 0$, *otherwise, gives a summation method. Show that taking* $a_{n,2n+1} = 1$, $a_{n,m} = 0$, *otherwise, also gives a summation method. Show that the two methods disagree when presented with the sequence* $1,\ -1,\ 1,\ -1,\ \dots$.

Another important consequence of the Baire category theorem is the open mapping theorem. (Recall that a complete normed space is called a Banach space.)

Theorem 27 (Open mapping theorem)**.** *Let E and F be Banach spaces and* $T : E \to F$ *be a continuous linear surjection. Then T is an open map (that is to say, if U is open in E we have TU open in F.)*

This has an immediate corollary.

Theorem 28 (Inverse mapping theorem)**.** *Let E and F be Banach spaces and let* $T : E \to F$ *be a continuous linear bijection. Then* T^{-1} *is continuous.*

The next exercise is simple, and if you can not do it this reveals a gap in your knowledge (which can be remedied by asking the lecturer) rather than in intelligence.

Exercise 29. *Let* (X, d) *and* (Y, ρ) *be metric spaces with associated topologies* τ *and* σ. *Then the product topology induced on* $X \times Y$ *by* τ *and* σ *is the same as the topology given by the metric*

$$\triangle((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2).$$

The inverse mapping theorem has the following useful consequence.

Theorem 30 (Closed graph theorem)**.** *Let E and F be Banach spaces and let* $T : E \to F$ *be linear. Then T is continuous if and only the graph*

$$\{(x, Tx) : x \in E\}$$

is closed in $E \times F$ *with the product topology.*

5 Zorn's lemma and Tychonov's theorem

Let A be a non-empty set and, for each $\alpha \in A$, let X_α be a non-empty set. Is $\prod_{\alpha \in A} X_\alpha$ non-empty (or, equivalently, does there exist a function $f : A \to \bigcup_{\alpha \in A} X_\alpha$ with $f(\alpha) \in X_\alpha$)? It is known that the standard axioms of set theory do not suffice to answer this question in general. (In particular cases they do suffice. If $X_\alpha = A$ for all $\alpha \in A$ then $f(\alpha) = \alpha$ will do.) Specifically, if there exists any model for standard set theory, then there exist models for set theory obeying the standard axioms in which the answer to our question is always yes (such systems are said to obey the axiom of choice) and there exist models in which the answer is sometimes no.

Most mathematicians are happy to add the axiom of choice to the standard axioms and this is what we shall do. Note that if we prove something using the standard axioms and the axiom of choice then we will be unable to find a counter-example using only the standard axioms. Note also that, when dealing with specific systems we may be able to prove the result for that system without using the axiom of choice.

The axiom of choice is not very easy to use in the form that we have stated it and it is usually more convenient to use an equivalent formulation called Zorn's lemma.

Definition 31. *Suppose X is a non-empty set. We say that $\succeq$ is a partial order on X, that is to say, that $\succeq$ is a relation on X with*

(i) $x \succeq y$, $y \succeq z$ implies $x \succeq z$,

(ii) $x \succeq y$ and $y \succeq x$ implies $x = y$,

(iii) $x \succeq x$

for all x, y, z.

We say that a subset C of X is a chain if, for every x, $y \in C$ at least one of the statements $x \succeq y$, $y \succeq x$ is true.

If Y is a non-empty subset of X we say that $z \in X$ is an upper bound for Y if $z \succeq y$ for all $y \in Y$.

We say that m is a maximal element for $(X, \succeq)$ if $x \succeq m$ implies $x = m$.

You must be able to do the following exercise.

Exercise 32. *(i) Give an example of a partially ordered set which is not a chain.*

(ii) Give an example of a partially ordered set and a chain C such that (a) the chain has an upper bound lying in C, (b) the chain has an upper bound but no upper bound within C, (c) the chain has no upper bound.

(iii) If a chain C has an upper bound lying in C, show that it is unique. Give an example to show that, even in this case C may have infinitely many upper bounds (not lying in C).

(iv) Give examples of partially ordered sets which have (a) no maximal elements, (b) exactly one maximal element, (b) infinitely many maximal elements.

(v) how should a minimal element be defined? Give examples of partially ordered sets which have (a) no maximal or minimal elements, (b) exactly one maximal element and no minimal element, (c) infinitely many maximal elements and infinitely many minimal elements.

Axiom 33 (Zorn's lemma)**.** *Let $(X, \succeq)$ be a partially ordered set. If every chain in X has an upper bound then X contains a maximal element.*

Zorn's lemma is associated with a proof routine which we illustrate in Lemmas 34 and 36

Lemma 34. *Zorn's lemma implies the axiom of choice.*

The converse result is less important to us but we prove it for completeness.

Lemma 35. *The axiom of choice implies Zorn's lemma.*

Proof. (Since the proof we use is non-standard, I give it in detail.) Let X be a non-empty set with a partial order $\succeq$ having no maximal elements. We show that the assumption that every chain has a upper bound leads to a contradiction.

We write $x \succ y$ if $x \succeq y$ and $x \neq y$. If C is a chain we write

$$C_x = \{c \in C \,:\, x \succ c\}.$$

Observe that, if C is a chain in X, we can find an $x \in X$ such that $x \succ c$ for all $c \in C$. (By assumption, C has an upper bound, x', say. Since X has no maximal elements, we can find an $x \in X$ such that $x \succ x'$.) We shall take $\emptyset$ to be a well ordered chain.

We shall look at well ordered chains, that is to say, chains for which every non-empty subset has a minimum. (Formally, if $S \subseteq C$ is non-empty we can find an $s_0 \in C$ such that $s \succeq s_0$ for all $s \in S$. We write $\min C = s_0$.) By the previous paragraph

$$A_C = \{x \,:\, x \succ c \text{ for all } c \in C\} \neq \emptyset.$$

Thus, if we write $\mathcal{W}$ for the set of all well ordered chains, the axiom of choice tells us that there is a function $\kappa : \mathcal{W} \to X$ such that $\kappa(C) \succ c$ for all $c \in C$.

We now consider 'special chains' defined to be well ordered chains C such that

$$\kappa(C_x) = x \text{ for all } x \in C.$$

(Note that 'well ordering' is an important general idea, but 'special chains' are an ad hoc notion for this particular proof. Note also that if C is a special chain and $x \in C$ then C_x is a special chain.)

The key point is that, if K and L are special chains, then either $K = L$ or $K = L_x$ for some $x \in L$ or $L = K_x$ for some $x \in K$.

Subproof If $K = L$, we are done. If not, at least one of $K \setminus L$ and $L \setminus K$ is non-empty. Suppose, without loss of generality, that $K \setminus L \neq \emptyset$. Since K is well ordered, $x = \min K \setminus L$ exists. We observe that $K_x \subseteq L$. If $K_x = L$, we are done.

We show that the remaining possibility $K_x \neq L$ leads to contradiction. In this case, $L \setminus K_x \neq \emptyset$ so $y = \min L \setminus K_x$ exists.

If $L_y = K_x$ then

$$y = \kappa(L_y) = \kappa(K_x) = x$$

so $x = y \in L \cap K$ contradicting the statement that $x \in K \setminus L$.

If $L_y \neq K_x$ let z be the least member of $K_x \setminus L_y$. Observe that, since $K_x \subseteq L$ and so

$$w \in L_y,\ z' \in K_x,\ w \succ z' \Rightarrow z' \in L,\ y \succ w \succ z' \Rightarrow z' \in L_y$$

whence

$$z' \notin L_y,\ z' \in K_x,\ w \succ z' \Rightarrow w \notin L_y.$$

Thus $K_z = L_y$ and

$$y = \kappa(L_y) = \kappa(K_z) = z$$

so $y = z \in K \cap L$ contradicting the definition of y.

End subproof

We now take S to be the union of all special chains. Using the key observation, it is routine to see that:

(i) S is a chain. (If $a, b \in S$, then $a \in L$ and $b \in K$ for some special chains. By our key observation, either $L \supseteq K$ of $K \supseteq L$. Without loss of generality, $K \supseteq L$ so $a, b \in K$ and $a \succeq b$ or $b \succeq a$.)

(ii) If $a \in S$, then S_a is a special chain. (We must have $a \in K$ for some special chain K. Since $K \subseteq S$, we have $K_a \subseteq S_a$. On the other hand, if $b \in S_a$ then $b \in L$ for some special chain L and each of the three possible relationships given in our key observation imply $b \in K_a$. Thus $S_a \subseteq K_a$, so $S_a = K_a$ and S_a is a special chain.)

(iii) S is well ordered. (If E is a non empty subset of S, pick an $x \in E$. If $S_x \cap E = \emptyset$, then x is a minimum for E. If not, then $S_x \cap E$ is a non-empty subset of the special, so well ordered, chain S_x, so $\min S_x \cap E$ exists and is a minimum for E.)

(iv) S is a special chain. (If $x \in S$, we can find a special chain K such that $x \in K$. Let $y = \kappa(K)$. Then $L = K \cup \{y\}$ is a special chain. As in (ii), $S_y = L_y$, so $S_x = L_x$ and $\kappa(S_x) = \kappa(L_x) = x$.)

We can now swiftly obtain a contradiction. Since S is well ordered $\kappa(S)$ exists and does not lie in S. But S is special, so $S \cup \kappa(S)$ is, so $S \cup \kappa(S) \subseteq S$, so $\kappa(S)$ lies in S. The required result follows by reductio ad absurdum[5]. $\square$

Lemma 36 (Hamel basis theorem)**.** *(i) Every vector space has a basis.*

(ii) If U is an infinite dimensional normed space over $\mathbb{F}$ then we can find a linear map $T : U \to \mathbb{F}$.

(iii) If U is an infinite dimensional normed space over $\mathbb{F}$ then we can find a discontinuous linear map $T : U \to \mathbb{F}$.

[Note that we do not claim that T in (ii) is continuous.]

Exercise 37. *(i) Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the equation*

$$f(x + y) = f(x) + f(y)$$

for all $x,\ y \in \mathbb{R}$, then there exists a c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

(ii) Show that there exists a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x + y) = f(x) + f(y)$$

for all $x,\ y \in \mathbb{R}$.
[Hint. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$.]

The rest of this section is devoted to a proof of Tychonov's theorem. We recall a definition.

Added 1 (Definition)**.** *Let (X_α, τ_α) be a topological space for each $\alpha \in A$. The weak (or Tychonov) topology τ on $\prod_{\alpha \in A} X_\alpha$ is the collection of sets U such that if $\mathbf{u} \in U$ we can find $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ and $O_{\alpha_j} \in \tau_{\alpha_j}$ such that $\mathbf{x} \in U$ whenever $x_{\alpha_j} \in O_{\alpha_j}$ for $1 \leq j \leq n$.*

Added 2 (Exercise)**.** *We retain the notation of Definition Added 1 and write $\pi_\alpha \mathbf{x} = x_\alpha$.*

(i) Show that τ is indeed a topology and that, with this topology, the maps $\pi_\alpha : X \to X_\alpha$ are continuous.

(ii) Show that τ is the weakest topology for which the π_α are continuous. [Thus, if σ is a topology for which the π_α are continuous, we have $\sigma \supseteq \tau$.]

[5] To the best of my knowledge, this particular proof is due to Jonathon Letwin (*American Mathematical Monthly*, Volume 98, **1991**, pp. 353–4). If you know about transfinite induction, there are more direct proofs.

(iii) Show that if all the τ_α are Hausdorff so is τ.

(iv) Suppose that $A = [0,1]$ and $X_t = \mathbb{R}$ and τ_t is the usual Euclidean topology on X_t. Explain how an $f \in \prod_{t\in[0,1]} X_t$ can be identified in a natural manner with a function $f : [0,1] \to \mathbb{R}$. With this identification show that the sequence of functions $f_n \to f$ pointwise if and only if, given any $U \in \tau$ with $f \in U$ we can find an N such that $f_n \in U$ for all $n \geq N$.

Theorem 38 (Tychonov)**.** *The product of compact spaces is itself compact.*

We follow the presentation in [1]. (The method of proof is due to Bourbaki.)

The following result should be familiar to almost all of my readers.

Lemma 39 (Finite intersection property)**.** *(i) A topological space is compact if and only if whenever a non-empty collection of closed sets $\mathcal{F}$ has the property that $\bigcap_{j=1}^n F_j \neq \emptyset$, for any F_1, F_2, $\dots$, $F_n \in \mathcal{F}$ it follows that $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$.*

(ii) A topological space is compact if and only if whenever a non-empty collection of sets $\mathcal{A}$ has the property that $\bigcap_{j=1}^n A_j \neq \emptyset$ for any A_1, A_2, $\dots$, $A_n \in \mathcal{A}$ it follows that $\bigcap_{A\in\mathcal{A}} \bar{A} \neq \emptyset$.

Definition 40. *A system $\mathcal{F}$ of subsets of a given set S is said to be of finite character if*

(i) whenever every finite subset of a set $A \subseteq S$ belongs to $\mathcal{F}$ it follows that $A \in \mathcal{F}$ and

(ii) whenever $A \in \mathcal{F}$ every finite subset of A belongs to $\mathcal{F}$.

Lemma 41 (Tukey's lemma)**.** *If a system $\mathcal{F}$ of subsets of a given set S has finite character and $F \in \mathcal{F}$ then $\mathcal{F}$ has a maximal (with respect to inclusion) element containing F.*

We now prove Tychonov's theorem.

Added 3 (Exercise)**.** *If (X,τ) is a Hausdorff space and $\mathcal{G}$ is a maximal collection of sets with the finite intersection property explain why $\bigcap_{G\in\mathcal{G}} \bar{G}$ consists of one point.*

If you are interested, examine how the second (but not the first) appeal to the axiom of choice may be avoided in our proof of Tychonov's theorem if all our spaces are Hausdorff.

The reason why Tychonov's theorem demands the axiom of choice is made clear by the final result of this section.

Lemma 42. *Tychonov's theorem implies the axiom of choice.*

6 The Hahn-Banach theorem

A good example of the use of Zorn's lemma occurs when we ask if given a Banach space $(U, \| \ \|)$ (over $\mathbb{C}$, say) there exist any non-trivial continuous linear maps $T : U \to \mathbb{C}$. For any space that we can think of, the answer is obviously yes, but to show that the result is always yes we need Zorn's lemma[6]. Our proof uses the theorem of Hahn-Banach.

One form of this theorem is the following.

Theorem 43. (Hahn–Banach) *Let U be a real vector space. Suppose $p : U \to \mathbb{R}$ is such that*

$$p(u+v) \leq p(u) + p(v) \text{ and } p(au) = ap(u)$$

for all $u,\ v \in U$ and all real and positive a.

If E is a subspace of U and there exists a linear map $T : E \to \mathbb{R}$ with $Tx \leq p(x)$ for all $x \in E$ then there exists a linear map $\tilde{T} : U \to \mathbb{R}$ with $Tx \leq p(x)$ for all $x \in U$ and $\tilde{T}(x) = Tx$ for all $x \in E$.

[Note that we do not assume that the vector space U is normed but we do assume that the vector space is real.]

Added 4 (Exercise)**.** *We say that a function $f : \mathbb{R} \to \mathbb{R}$ is convex if*

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x,\ y \in \mathbb{R}$. Using the ideas of the proof of the Hahn–Banach theorem but not the result itself show that a convex function f is continuous and that given any $a \in \mathbb{R}$ we can find a c such that

$$f(x) \geq f(a) + c(x-a)$$

for all $x \in \mathbb{R}$. Give an example to show that f need not be differentiable.

Added 5 (Exercise)**.** *Let X be a real vector space and $p,\ q : X \to \mathbb{R}$ be functions such that $p(\lambda x) = \lambda p(x)$, $q(\lambda x) = \lambda q(x)$ for all $\lambda \in \mathbb{R}$ with $\lambda \geq 0$ and all $x \in X$, whilst*

$$p(x+y) \leq p(x) + p(y),\ \ q(x) + q(y) \leq q(x+y)$$

for all $x,\ y \in X$.

[6] In fact the statement is marginally weaker than Zorn's lemma but you need to be logician either to know or care about this.

(i) Suppose that Y is a subspace of X and $S : Y \to \mathbb{R}$ a linear function such that

$$S(y) \leq p(x+y) - q(x)$$

for all $x \in X$, $y \in Y$. Show that

$$S(y') - p(x' + y' - z) + q(x') \leq -S(y) + p(x + y + z) - q(x)$$

for all x, x', $z \in X$ and y, $y' \in Y$.

(ii) Suppose that Y_0 is a subspace of X and $T_0 : Y \to \mathbb{R}$ a linear function such that

$$T_0(y) \leq p(x+y) - q(x)$$

for all $x \in X$, $y \in Y_0$. Show that there exists a linear function $T_0 : X \to \mathbb{R}$ such that

$$T(y) \leq p(x+y) - q(x)$$

for all x, $y \in X$ and $Tu = Tu_0$ for all $u \in Y_0$. Show that

$$q(x) \leq T(x) \leq p(x)$$

for all $x \in X$.

(iii) Suppose $p(x) \geq q(x)$ for all $x \in X$. Show that there exists a linear function (possibly the zero function) $U : X \to \mathbb{R}$ such that

$$q(x) \leq U(x) \leq p(x)$$

for all $x \in X$.

(iv) Let $X = \mathbb{R}^2$, $\mathbf{n}$ be a unit vector and $p(\mathbf{x}) = |\mathbf{n}.\mathbf{x}|$ (the absolute value of the usual inner product) and $q(\mathbf{x}) = -|\mathbf{n}.\mathbf{x}|$. Show that p and q obey the conditions of the introductory paragraph and part (iii). What can you say about U?

We have the following important corollary to Theorem 43.

Theorem 44. *Let $(U, \| \ \|)$ be a real normed vector space. If E is a subspace of U and there exists a continuous linear map $T : E \to \mathbb{R}$, then there exists a continuous linear map $\tilde{T} : U \to \mathbb{R}$ with $\|\tilde{T}\| = \|T\|$.*

The next result is famous as 'the result that Banach did not prove'.

Theorem 45. *Let $(U, \| \ \|)$ be a complex normed vector space. If E is a subspace of U and there exists a continuous linear map $T : E \to \mathbb{C}$ then there exists a continuous linear map $\tilde{T} : U \to \mathbb{C}$ with $\|\tilde{T}\| = \|T\|$.*

We can now answer the question posed in the first sentence of this section.

Lemma 46. *If $(U, \| \ \|)$ is normed space over the field $\mathbb{F}$ of real or complex numbers and $a \in U$ with $a \neq 0$, then we can find a continuous linear map $T : U \to \mathbb{F}$ with $Ta \neq 0$.*

The importance of this lemma becomes more obvious if we state it in reverse. If $Ta = 0$ for all continuous linear maps then $a = 0$.

Here are a couple of results proved by Banach using his theorem.

Theorem 47 (Generalised limits)**.** *Consider the vector space l^∞ of bounded real sequences. There exists a linear map $L : l^\infty \to \mathbb{R}$ such that*

(i) If $x_n \geq 0$ for all n then $L\mathbf{x} \geq 0$.

(ii) $L((x_1, x_2, x_3, \dots)) = L((x_0, x_1, x_2, \dots))$.

(iii) $L((1, 1, 1, \dots)) = 1$.

The theorem is illustrated by the following lemma.

Lemma 48. *Let L be as in Theorem 47. Then*

$$\limsup_{n\to\infty} x_n \geq L(\mathbf{x}) \geq \liminf_{n\to\infty} x_n.$$

In particular, if $x_n \to x$ then $L(\mathbf{x}) = x$.

Exercise 49. *(i) Show that, even though the sequence $x_n = (-1)^n$ has no limit, $L(\mathbf{x})$ is uniquely defined.*

(ii) Find, with reasons, a sequence $\mathbf{x} \in l^\infty$ for which $L(\mathbf{x})$ is not uniquely defined.

Banach used the same idea to prove the following odd result.

Lemma 50. *Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the unit circle and let $B(\mathbb{T})$ be the vector space of real valued bounded functions. Then we can find a linear map $I : B(\mathbb{T}) \to \mathbb{R}$ obeying the following conditions.*

(i) $I(1) = 1$.

(ii) $If \geq 0$ if f is positive.

(iii) If $f \in B(\mathbb{T})$, $a \in \mathbb{T}$ and we write $f_a(x) = f(x - a)$ then $If_a = If$.

Exercise 51. *Show that if I is as in Lemma 50 and f is Riemann integrable then*

$$If = \int_{\mathbb{T}} f(t)\, dt.$$

However, Lemma 50 is put in context by the following.

Lemma 52. *Let G be the group freely generated by two generators and $B(G)$ be the vector space of real valued bounded functions on G. If $f \in B(G)$ let us write $f_c(x) = f(xc^{-1})$ for all $x,\ c \in G$.*

There exists a function $f \in B(G)$ and $c_1,\ c_2,\ c_3$ such that $f(x) \geq 0$ for all $x \in G$ and

$$f(x) + f_{c_1}(x) - f_{c_2}(x) - f_{c_3}(x) \leq -1$$

for all $x \in G$.

Exercise 53. *If G is as in Lemma 52 then there is no linear map $I : B(G) \to \mathbb{R}$ obeying the following conditions.*

(i) $I(1) = 1$.

(ii) $If \geq 0$ if f is positive.

(iii) $If_c = If$ for all $c \in G$.

It can be shown that there is a finitely additive, congruence respecting integral for $\mathbb{R}$ and $\mathbb{R}^2$ but not $\mathbb{R}^n$ for $n \geq 3$.

7 Banach algebras

Many of the objects studied in analysis turn out to be Banach algebras.

Definition 54. *An algebra $(B, +, ., \times)$ is a vector space $(B, +, ., \mathbb{C})$ equipped with a multiplication $\times$ such that*

(i) $x \times (y \times z) = (x \times y) \times z$,

(ii) $(x + y) \times z = x \times z + y \times z$ and $z \times (x + y) = z \times x + z \times y$,

(iii) $(\lambda x) \times y = x \times (\lambda y) = \lambda(x \times y)$ for all $x,\ y,\ z \in B$.

[We shall write $x \times y = xy$.]

Note that there is no assumption that multiplication is commutative. In principle, we could talk about real Banach algebras (in which $\mathbb{C}$ is replaced by $\mathbb{R}$) but, though some elementary results carry over, our treatment will only cover complex Banach algebras.

Definition 55. *A Banach algebra $(B, +, ., \times, \| \ \|)$ is an algebra $(B, +, ., \times, \mathbb{C})$ such that $(B, +, ., \mathbb{C}, \| \ \|)$ is a Banach space and such that the map $(x, y) \mapsto xy$ is continuous.*

Note that the two definitions above are not to be memorised; so far as this course is concerned the following definition is all that is required.

Definition 56. *A Banach algebra $(B, \| \ \|)$ is a Banach space equipped with a continuous multiplication which makes it an algebra.*

As usual there is a little amount of playing about with the definition.

Lemma 57. *The following statements about* $(B, \| \ \|)$ *a Banach space equipped with a multiplication are equivalent.*

(i) Multiplication is left and right continuous (that is, the map $x \mapsto xy$ *is continuous for all* y *and the map* $y \mapsto xy$ *is continuous for all* x*).*

(ii) There exists a K *such that* $\|xy\| \leq K\|x\|\|y\|$ *for all* x *and* y*.*

(iii) $(B, \| \ \|)$ *is a Banach algebra.*

Lemma 58. *If* $(B, \| \ \|)$ *is a Banach algebra we can find a norm* $\| \ \|_B$ *on* B *which is equivalent to* $\| \ \|$ *(that is, there exists a* $C > 0$ *such that* $C^{-1}\|x\| \leq \|x\|_B \leq C\|x\|$*) such that*

$$\|xy\|_B \leq \|x\|_B\|y\|_B$$

for all $x, \ y \in B$*.*

Unless specifically indicated otherwise you may assume both in the rest of the notes and in the literature generally that the norm on a Banach algebra has been chosen to satisfy

$$\|xy\| \leq \|x\|\|y\|$$

for all x and y.

Definition 59. *We say that a Banach algebra* B *has a unit* e *if* $xe = ex = x$ *for all* $x \in B$*.*

The following remarks form part of the course but are left as an exercise.

Exercise 60. *(i) If a Banach algebra has a unit that unit is unique.*

(ii) If $(B, \| \ \|)$ *is a Banach algebra with unit* e *we can find a norm* $\| \ \|_B$ *on* B *which is equivalent to* $\| \ \|$ *such that*

$$\|xy\|_B \leq \|x\|_B\|y\|_B$$

for all $x, \ y \in B$ *and*

$$\|e\|_B = 1.$$

Unless specifically indicated otherwise you may assume both in the rest of the notes and in the literature generally that the norm on a Banach algebra with unit e has been chosen to satisfy $\|e\| = 1$ and $\|xy\|_B \leq \|x\|_B\|y\|_B$ for all x and y.

Example 61. *(i) A Banach space $(B, \| \ \|)$ becomes a commutative Banach algebra if we define $xy = 0$ for all $x,\ y \in B$. If B is non-trivial the resulting algebra has no unit.*

(ii) Consider the Banach space l^1 of sequences $\mathbf{a} = (a_0, a_1, \dots)$. If we define $\mathbf{a} * \mathbf{b} = \mathbf{c}$ with

$$c_r = \sum_{k+j=r,\, k\geq 0, j\geq 0} a_j b_k$$

then $*$ is a well defined multiplication and l^1 is a Banach algebra with this multiplication. As a Banach algebra, l^1 is commutative with a unit.

(iii) Consider the Banach space l^1 of sequences $\mathbf{a} = (a_1, a_2, \dots)$. If we define $\mathbf{a} * \mathbf{b} = \mathbf{c}$ with

$$c_r = \sum_{k+j=r,\, k\geq 1, j\geq 1} a_j b_k$$

then $*$ is a well defined multiplication and l^1 is a Banach algebra with this multiplication. As a Banach algebra, l^1 is commutative but has no unit.

(iv) Consider the Banach space l^1 of two sided sequences $\mathbf{a} = (\dots, a_{-2}, a_{-1}, a_0, a_1, \dots)$. If we define $\mathbf{a} * \mathbf{b} = \mathbf{c}$ with

$$c_r = \sum_{k+j=r} a_j b_k$$

then $*$ is a well defined multiplication and l^1 is a Banach algebra with this multiplication. As a Banach algebra, l^1 is commutative and has a unit.

Exercise 62. *If you know measure theory you ought to work through this exercise. We work in L^1 the space of Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{C}$.*

(i) Use Fubini's theorem to show that, if $f,\ g \in L^1$, then

$$f * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\, dx$$

is well defined almost everywhere and that $f * g \in L^1$ with

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

(ii) Use Fubini's theorem to show that, if $f,\ g \in L^1$ then

$$\widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$$

for all $\lambda \in \mathbb{R}$.

(iii) If $e_a(t) = e^{-iat}$ compute $\widehat{e_a f}$ for $f \in L^1$. Show that if $e \in L^1$ is a unit then $\hat{e} = 1$.

(iv) Show that if $f \in L^1$ then $\sup_{\lambda \in \mathbb{R}} |\hat{f}(\lambda)| \leq \|f\|_1$. Show that if f is once continuously differentiable with $f,\ f' \in L^1$ and $f(t),\ f'(t) \to 0$ as $|t| \to \infty$ then $\hat{f}(\lambda) \to 0$ as $|\lambda| \to \infty$. Use a density argument to show that $\hat{g}(\lambda) \to 0$ as $|\lambda| \to \infty$ whenever $g \in L^1$ (this is the Lebesgue-Riemann lemma).

(v) Use (iii) and (iv) to show that $(L^1, *)$ has no unit.

Lemma 63. *If B is a Banach algebra without unit we can find $\tilde{B}$ a Banach algebra with a unit e such that*

(i) B is a sub Banach algebra of $\tilde{B}$,

(ii) B is closed in $\tilde{B}$,

(iii) $\tilde{B} = \operatorname{span}(B, e)$ in the algebraic sense.

Exercise 64. *(i) Suppose we apply the construction of Lemma 63 to a Banach algebra B with unit u. Is u a unit of the extended algebra $\tilde{B}$? Does $\tilde{B}$ have a unit?*

(ii) (Needs measure theory.) Can you find a natural identification for the unit of $\widetilde{L^1}$ where L^1 is the Banach algebra of Exercise 62.

Thus any Banach algebra B without a unit can be studied by 'adjoining a unit and then removing it'. This is our excuse for only studying Banach algebras with a unit.

The following result is easy but fundamental.

Lemma 65. *Let B be a Banach algebra with unit e.*

(i) If $\|e - a\| < 1$ then a is invertible (that is has a multiplicative inverse).

(ii) If E is the set of invertible elements in B then E is open.

Lemma 65 (i) can be improved in a useful way.

Theorem 66. *(i) If B is a Banach algebra and $b \in B$ then, writing $\rho(b) = \inf_n \|b^n\|^{1/n}$ we have*

$$\|b^n\|^{1/n} \to \rho(b)$$

as $n \to \infty$.

(ii) If B is a Banach algebra with unit e and $\rho(e - a) < 1$ then a is invertible.

We call $\rho(a)$ the spectral radius of a.

Exercise 67. *Consider the space M_n of $n \times n$ matrices over $\mathbb{C}$ with the operator norm.*

(i) Show that M_n is a Banach algebra with unit. For which values of n is it commutative?

(ii) Give an example of an $A \in M_2$ with $A \neq 0$ but $\rho(A) = 0$.

(iii) If A is diagonalisable show that

$$\rho(A) = \max\{|\lambda| : \lambda \textit{ an eigenvalue of } A\}.$$

(iv) (Harder and not essential.) Show that the formula of (iii) holds in general.

8 Maximal ideals

We now embark on a line of reasoning which will eventually lead to a characterisation of a large class of commutative Banach algebras.

Initially we continue to deal with Banach algebras which are not necessarily commutative. The generality is more apparent than real as the next exercise reveals.

Exercise 68. *Let B be a Banach algebra with unit e. Let A be the closed Banach algebra generated by e and some $a \in B$. (Formally, A is the smallest closed sub Banach algebra containing e and a.) Then A is commutative.*

Definition 69. *Let B be a Banach space with unit e. If $x \in B$ the resolvent $R(x)$ of x is defined by*

$$R(x) = \{\lambda \in \mathbb{C} : x - \lambda e \textit{ is invertible}\}.$$

Lemma 70. *We use the notation of Definition 69.*
(i) $\mathbb{C} \setminus R(x)$ is bounded.
(ii) $R(x)$ is open.
(iii) If $\mu \in R(x)$ we can find a $\delta > 0$ and $a_0,\ a_1,\ \ldots \in B$ such that $\sum_{j=0}^{\infty} a_j z^j$ converges for all $|z| < \delta$ and

$$(x - \lambda e)^{-1} = \sum_{j=0}^{\infty} a_j (\lambda - \mu)^j$$

for $\lambda \in \mathbb{C}$ and $|\lambda - \mu| < \delta$.
(iv) $R(x) \neq \mathbb{C}$.

Lemma 70 gives us our first substantial result on the nature of commutative Banach algebras.

Theorem 71 (Gelfand-Mazur)**.** *Any Banach algebra which is also a field is isomorphic as a Banach algebra to $\mathbb{C}$.*

9 Analytic functions

In order to extract more information on the resolvent we take a detour through a little (easy) integration theory and complex variable theory.

Theorem 72. *Let U be a Banach space, $[a,b]$ a closed bounded interval in $\mathbb{R}$. Then we can define an integral $\int_a^b F(t)\,dt$ for every $F:[a,b]\to U$ a continuous function having the following properties (here $F,\ G:[a,b]\to U$ are continuous and $\lambda,\ \mu\in\mathbb{C}$).*

(i) $\displaystyle\int_a^b \lambda F(t)+\mu G(t)\,dt = \lambda\int_a^b F(t)\,dt + \mu\int_a^b G(t)\,dt.$

(ii) If $a<c<b$ then

$$\int_a^b F(t)\,dt = \int_a^c F(t)\,dt + \int_c^b F(t)\,dt.$$

(iii) $\displaystyle\left\|\int_a^b F(t)\,dt\right\| \le \int_a^b \|F(t)\|\,dt.$

(iv) If $T:U\to\mathbb{C}$ is a continuous linear functional

$$\int_a^b T(F(t))dt = T\int_a^b F(t)\,dt.$$

Using the integral just defined we can define contour integrals as we did in the complex variable course.

Definition 73. *If $\gamma:[a,b]\to\mathbb{C}$ is continuously differentiable with $\gamma(a)=\gamma(b)$ and $F:[a,b]\to U$ a continuous function we define*

$$\int_\gamma F(z)\,dz = \int_a^b F(\gamma(t))\gamma'(t)\,dt.$$

(We shall talk about the 'closed contour' γ.)

We can now introduce the notion of an analytic Banach algebra valued function.

Definition 74. *Let B be a Banach algebra and Ω a simply connected*[7] *open set in $\mathbb{C}$. A function $f:\Omega\to B$ is said to be analytic on Ω if there exists an $f':\Omega\to B$ such that, for all $z\in\Omega$*

$$\left\|\frac{f(z+h)-f(z)}{h} - f'(z)\right\| \to 0$$

as $h\to 0$ through values of h such that $z+h\in\Omega$.

[7] Informally 'with no holes'.

Theorem 75. *Let B be a Banach algebra, Ω an open simply connected set in $\mathbb{C}$, and γ a closed contour in Ω. Then, if f is analytic on Ω,*

$$\int_\gamma f(z)\,dz = 0.$$

We can follow a first undergraduate complex variable course to prove the following results.

Lemma 76. *Let B be a Banach algebra with a unit e, Ω an open set in $\mathbb{C}$ containing a disc $D(z_0, R)$, $f : \Omega \to B$ an analytic function and γ a contour describing a circle centre z_0 radius $0 < r < R$. If $|z_0 - z| < r$ then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}\,dw.$$

Lemma 77. *Let B be a Banach algebra with a unit e, Ω an open set in $\mathbb{C}$ containing a disc $D(z_0, R)$ and f an analytic function on Ω. There exist unique $a_0, a_1, a_2, \ldots \in B$ such that $\sum_{j=0}^{\infty} a_r(z - z_0)^r$ converges and*

$$f(z) = \sum_{j=0}^{\infty} a_r(z - z_0)^r$$

for all $|z - z_0| < R$.

Theorem 78. *If B is a Banach algebra with unit*

$$\sup\{|\lambda| \,:\, \lambda \notin R(x)\} = \rho(x).$$

Added 6 (Exercise)**.** *It may help the reader's understanding to prove the corresponding classical result.*

We work in $\mathbb{C}$. Suppose that $a_j \in \mathbb{C}$ and that $\limsup_{n\to\infty} |a_n|^{1/n} < \infty$. Write

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

and

$$D(0, r) = \{z \in\in \mathbb{C} \,:\, |z| < r\}.$$

(i) Show that there exists an analytic function $f : D(0, r) \to \mathbb{C}$ such that $f^{(j)}(0) = a_j$ for all j.

(ii) Show that, if $\delta > 0$ there does not exist an analytic function $g : D(0, r + \delta) \to \mathbb{C}$ such that $g^{(j)}(0) = a_j$ for all j.

10 Maximal ideals

One way of exploiting the Gelfand-Mazur theorem is to introduce the notion of maximal ideals. (From now on all our Banach algebras will be commutative.)

Lemma 79. *Every proper ideal in a commutative algebra with unit is contained in a maximal ideal.*

(Recall that an ideal I in a commutative algebra B is a vector subspace of B such that if $a \in B$ and $b \in I$ then $ab \in I$. An ideal J is maximal if $J \neq B$ but whenever an ideal K satisfies $J \subseteq K \subseteq B$ either $K = J$ or $K = B$.)

Added 7 (Lemma)**.** *If I is a closed ideal in a Banach algebra then B/I is a Banach algebra under the norm*

$$\|x + I\|_{B/I} = \inf\{\|y\| \,:\, y - x \in I\}$$

If $I \neq B$, then

$$\|e + I\|_{B/I} = 1.$$

Added 8 (Exercise)**.** *The following remark is useful in proving results like the above.*

Show that a normed space $(V, \| \ \|)$ is complete if and only if $\sum_{j=1}^{\infty} x_j$ converges whenever $\sum_{j=1}^{\infty} \|x_j\|$ does.

Lemma 80. *Every maximal ideal M in a commutative Banach algebra with unit is closed.*

Lemma 81. *If M is a maximal ideal in a commutative Banach algebra with unit then the quotient B/M is isomorphic to $\mathbb{C}$ as a Banach algebra.*

The notion of a maximal ideal is closely linked to that of a multiplicative linear functional.

Definition 82. *A multiplicative linear functional on a Banach algebra is a non-trivial (i.e. not the zero map) linear map $\chi : B \to \mathbb{C}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x,\ y \in B$.*

Lemma 83. *If B is a commutative Banach algebra with identity and χ is a multiplicative linear functional then the following results hold.*

(i) $\ker \chi$ *is a maximal ideal.*

(ii) The map $x + \ker \chi \mapsto \chi(x)$ is an algebraic isomorphism of $B/\ker \chi$ with $\mathbb{C}$.

(iii) χ is continuous and $\|\chi\| = 1$.

Theorem 84. *If B is a commutative Banach algebra with identity then the mapping $\chi \mapsto \ker \chi$ is a bijection between the set of multiplicative linear functionals on B and its maximal ideals.*

We now have the following useful corollary.

Lemma 85. *If B is a commutative Banach algebra with identity then an element $x \in B$ is invertible if and only $\chi(x) \neq 0$ for all multiplicative linear functionals χ.*

The Banach algebra proof Theorem 87 was the first result to convince classical analysts of the utility of these ideas. The lemma that precedes it places the result in context.

Lemma 86. *If $f \in C(\mathbb{T})$ has an absolutely convergent Fourier series (that is to say, $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$) then*

$$f(t) = \sum_{-\infty}^{\infty} \hat{f}(n) \exp(int).$$

Theorem 87 (Wiener's theorem)**.** *Suppose $f \in C(\mathbb{T})$ has an absolutely convergent Fourier series. Then, if $f(t) \neq 0$ for all $t \in \mathbb{T}$, $1/f$ also has an absolutely convergent Fourier series.*

Exercise 88. *Let B be any Banach space. Make it into a Banach algebra by defining $xy = 0$ for all $x,\ y \in B$. Now add an identity in the usual manner. Identify all the multiplicative linear functionals.*

11 The Gelfand representation

Throughout this section B will be a commutative Banach algebra with a unit e and $\mathcal{M}$ will be the space of maximal ideals. If $x \in B$ and $M \in \mathcal{M}$ we know by Theorem 84 that there is a unique multiplicative linear functional χ_M with kernel M so we may write $M(x) = \chi_M(x)$. We give the space $\mathcal{M}$ the weak star topology, that is to say, the smallest topology containing sets of the form

$$\{M \in \mathcal{M} \,:\, |M(x) - M_0(x)| < \epsilon\}$$

with $M_0 \in \mathcal{M}$ and $x \in B$.

Lemma 89. *Under the weak topology $\mathcal{M}$ is a compact Hausdorff space.*

If $x \in B$ and $M \in \mathcal{M}$ we now write $\hat{x}(M) = M(x)$.

Lemma 90. *Let B be a commutative Banach algebra with unit. The mapping $x \mapsto \hat{x}$ is an algebraic homomorphism of B into $C(\mathcal{M})$. As linear map from $(B, \| \, \|)$ to $C(\mathcal{M}, \| \, \|_\infty)$ it is continuous with operator norm exactly* 1.

We know that the homomorphism $x \mapsto \hat{x}$ need not be injective

Exercise 91. *Justify this statement by considering the Banach algebra of Exercise 88.*

The following simple observation is the key to the question of when we have isomorphism.

Lemma 92. *Suppose x is an element of a commutative Banach algebra with unit. Then the complement of the resolvent $R(x)$ is the range of $\hat{x}$.*

That is to say,

$$\{\hat{x}(M) \, : \, M \in \mathcal{M}\} = \{\lambda \in \mathbb{C} \, : \, (x - \lambda e) \text{ is not invertible}\}.$$

There are two immediate corollaries.

Lemma 93. *If x is an element of a commutative Banach algebra with unit, then $\|\hat{x}\|_\infty = \rho(x)$.*

Lemma 94. *If x is an element of a commutative Banach algebra with unit, then $\rho(x) = 0$ if and only if x is contained in every maximal ideal.*

We make the following definitions.

Definition 95. *If B is a commutative Banach algebra with unit we define the radical of B to be the set of all elements contained in every maximal ideal.*

Thus $x \in \text{radical}(B)$ if and only if $\rho(x) = 0$.

Definition 96. *We say that a commutative Banach algebra with unit is semi-simple if and only if its radical consists of* 0 *alone.*

Theorem 97. *Let B be a commutative Banach algebra with unit. The mapping $x \mapsto \hat{x}$ is injective if and only if B is semi-simple.*

Exercise 98. *Consider the Banach algebra X of continuous linear maps $T : l^\infty \to l^\infty$. Let S be the map given by*

$$S(a_1, a_2, \dots) = (0, c_1 a_1, c_2 a_2, \dots),$$

with the sequence c_j bounded. Explain why the closed Banach subalgebra generated by I and S is a commutative Banach algebra. Show that with an appropriate choice of c_j we can have $S^n \neq 0$ for all n but $\rho(S) = 0$.

Theorem 99. *Let B be a commutative Banach algebra with unit. If there exists a $K > 0$ such that $\|x\|^2 \leq K\|x^2\|$ for all $x \in B$, then ρ is a norm equivalent to the original norm on B.*

12 Finding the Gelfand representation

Suppose we are given a commutative Banach algebra B and we wish to find its Gelfand representation. It is not enough to find its maximal ideals (or, equivalently its multiplicative linear functionals). We must also find the correct topology on the space of maximal ideals. The following simple remarks resolve the problem in all the cases that we shall consider.

Exercise 100. *Write out the proof that if (X,τ) and (Y,σ) are topological spaces with (X,τ) compact and (Y,σ) Hausdorff then, if $f:(X,\tau)\to(Y,\sigma)$ is a continuous bijection, f is a homeomorphism.*

Lemma 101. *Suppose τ is a compact topology on the space $\mathcal{M}$ of maximal ideals of commutative Banach space B with identity. If the maps $\hat{x}:(\mathcal{M},\tau)\to\mathbb{C}$ are continuous for each $x\in B$ then τ is the weak star topology on $\mathcal{M}$.*

Our first identification was adumbrated in our proof of Wiener's theorem.

Example 102. *Consider the space $A(\mathbb{T})$ of continuous functions $f:\mathbb{T}\to\mathbb{C}$ with absolutely convergent Fourier series (that is to say, $\sum_{-\infty}^{\infty}|\tilde{f}(n)|<\infty$ where $\tilde{f}(n)$ is the nth Fourier coefficient). If we set*

$$\|f\|_A=\sum_{-\infty}^{\infty}|\tilde{f}(n)|,$$

then $(A(\mathbb{T}),\|\ \|_A)$ is a commutative Banach algebra with unit 1 under pointwise multiplication. $(A(\mathbb{T}),\|\ \|_A)$ has maximal ideal space (identified with) $\mathbb{T}$ under its usual topology. We have $\hat{f}(t)=f(t)$.

Example 103. *The 'transform' nature of the Gelfand transform is clearer if we seek the maximal ideal space and transform associated with the Banach algebra $l^1(\mathbb{Z})$ with standard norm and addition and multiplication given by convolution (that is $\mathbf{a}*\mathbf{b}=\mathbf{c}$ where $c_m=\sum_{r=-\infty}^{\infty}a_{m-r}b_r$).*

Here is a variation on the theme.

Lemma 104. *Let $D=\{z\in\mathbb{C}:|z|<1\}$ and $\bar{D}=\{z\in\mathbb{C}:|z|\leq 1\}$. Consider $A(D)$ the set of continuous functions $f:\bar{D}\to\mathbb{C}$ such that f is analytic in D. If $f_1,\ f_2,\ \dots,\ f_n\in A(D)$ are such that $\sum_{j=1}^{n}|f_j(z)|>0$ for all $z\in\mathbb{C}$ (that is to say that the f_j do not vanish simultaneously) show that we can find $g_1,\ g_2,\ \dots,\ g_n\in A(D)$ such that $\sum_{j=1}^{n}f_j(z)g_j(z)=1$ for all $z\in\bar{D}$.*

We deviate slightly from our main theme to introduce some interesting algebras.

Added 9 (Lemma)**.** *If K is a compact set in $\mathbb{C}$, then the following are closed subalgebras of $C(K)$.*

(i) $A(K)$ the collection of $f \in C(K)$ such that F is analytic on $\operatorname{Int} K$.

(ii) $R(K)$ the collection of $f \in C(K)$ which are uniform limits of rational functions with poles outside K.

(iii) $P(K)$ the collection of $f \in C(K)$ which are uniform limits of polynomials.

We have $C(K) \supseteq A(K) \supseteq R(K) \supseteq P(K)$.

Added 10 (Exercise)**.** *(i) If K is the closed unit disc, then $P(K) = R(K) = A(K) \neq C(K)$.*

(ii) If $K = \{z : |z| = 1\}$, then $P(K) \neq R(K) = A(K) = C(K)$.

Added 11 (The Swiss Cheese)**.** *There exists a K with $R(K) \neq A(K)$.*

Added 12 (Exercise)**.** *There exists a K with $P(K) \neq R(K) \neq A(K) \neq C(K)$.*

The next example is a key one in understanding the kind of problem we face.

Example 105. *Consider the sub Banach algebra $A_+(\mathbb{T})$ of $A(\mathbb{T})$ consisting of elements f of $A(\mathbb{T})$ with $\tilde{f}(n) = 0$ for $n < 0$. Show that $A_+(\mathbb{T})$ has maximal ideal space (identified with) $\mathbb{D}$ the closed unit disc. We have $\hat{f}(z) = \sum_{n=0}^{\infty} \tilde{f}(n) z^n$.*

Exercise 106. *Consider the sub Banach algebra $A_-(\mathbb{T})$ of $A(\mathbb{T})$ consisting of elements f of $A(\mathbb{T})$ with $\tilde{f}(n) = 0$ for $n > 0$. Find the maximal ideal space and associated Gelfand transform.*

Exercise 107. *Consider the space $B(\mathbb{T})$ of continuous functions $f : \mathbb{T} \to \mathbb{C}$ with $\sum_{-\infty}^{\infty} |n\tilde{f}(n)| < \infty$. Show that if we set*

$$\|f\|_B = \sum_{-\infty}^{\infty} (|n| + 1)|\tilde{f}(n)|$$

then $(B(\mathbb{T}), \|\ \|_B)$ is a commutative Banach algebra with unit 1 under pointwise multiplication. Find the maximal ideal space and associated Gelfand transform.

Exercise 108. *Consider the sub Banach algebra $B_+(\mathbb{T})$ of $B(\mathbb{T})$ consisting of elements f of $B(\mathbb{T})$ with $\tilde{f}(n) = 0$ for $n < 0$. Find the maximal ideal space and associated Gelfand transform.*

Our next example is fundamental.

Example 109. *Let (X, τ) be a compact Hausdorff space. The space $C(X)$ of continuous functions $f : X \to \mathbb{C}$ with the uniform norm is a commutative Banach algebra with unit 1 under pointwise operations. $C(X)$ has maximal ideal space (identified with) X under its usual topology. We have $\hat{f}(t) = f(t)$.*

One way of expressing many of our results is in terms of function algebras.

Definition 110. *Let (X, τ) be a compact Hausdorff space. If we consider $C(X)$ as a Banach algebra in the usual way then any subalgebra A with a norm which makes it a Banach algebra is called a function algebra.*

Lemma 111. *With the notation of Definition 110, if A is a function algebra containing 1 with norm $\| \ \|$, then $\|f\| \geq \|f\|_\infty$ for all $f \in A$.*

Lemma 112. *We use the notation of Definition 110.*

(i) If A separates points (that is, given $x, y \in X$ with $x \neq y$, we can find an $f \in A$ such that $f(x) \neq f(y)$), if $f \in A$ implies $f^* \in A$ and if $f(t) \neq 0$ for all $t \in X$ and $f \in A$ implies $1/f \in A$, then A has maximal ideal space (identified with) X under its usual topology. We have $\hat{f}(t) = f(t)$.

(ii) If A satisfies (i) and, in addition, there exists a K such that $\|f\|^2 \leq K\|f^2\|$, then $A = C(X)$ and there exists a κ such that

$$\kappa\|f\|_\infty \geq \|f\| \geq \|f\|_\infty$$

for all $f \in A$ (so the norms $\| \ \|$ and $\| \ \|_\infty$ are Lipschitz equivalent)

Exercise 113. *Show that the space B of continuous functions $f : [0, 1] \cup [2, 3] \to \mathbb{C}$ such that $f(2 + t) = f(t)$ for $t \in [0, 1]$ equipped with the uniform norm is a function algebra. Find the maximal ideal space and associated Gelfand transform.*

Exercise 114. *Show that the space $C^1([0, 1])$ of once continuously differentiable functions equipped with norm*

$$\|f\| = \|f\|_\infty + \|f'\|_\infty$$

is a function algebra. Find the maximal ideal space and associated Gelfand transform.

Our final set of results on Gelfand transforms is introduced by the following fairly simple exercise.

Added 13 (Exercise)**.** *(i) If B is a (not necessarily commutative) Banach algebra with unit then* $\exp a$ *can be defined in a natural manner using power series. Show that if a and b commute then*

$$\exp(a+b) = \exp(a)\exp(b).$$

(ii) Show that this result may be false if a and b do not commute.

[Hint: Suppose that a and d do not commute. Consider the expression $\exp(\eta a + \eta b) - \exp(\eta a)\exp(\eta b)$ *when η is very small.]*

Added 14 (Definition)**.** *If B is a commutative Banach algebra with a unit we say that B has an involution if there exists a map $B \to B$ given by $f \mapsto f^*$ with the following properties.*

(i) $f^{**} = f$.

(ii) $(\lambda f)^* = \bar{\lambda} f^*$ *where* $\lambda \in \mathbb{C}$ *and* $\bar{\lambda}$ *is the complex conjugate of* λ.

(iii) $(f+g)^* = f^* + g^*$.

(iv) $(fg)^* = f^* g^*$.

(v) $\|ff^*\| = \|f\|^2$

Added 15 (Theorem)**.** *If B is a commutative Banach algebra with a unit and B has an involution $f \mapsto f^*$ then the Gelfand transform $B \to C(\mathcal{M})$ is a norm preserving isomorphism with* $(f^*)\hat{} = (f\hat{})^-$.

Added 16 (Corollary)**.** *Suppose that V is a finite dimensional complex inner product space and $\alpha : V \to V$ is a linear map which is normal (that is to say $\alpha\alpha^* = \alpha^*\alpha$ where α^* denotes the adjoint map). Then we can find a set π_1, π_2, $\dots$, π_m of commuting orthogonal projections and λ_1, λ_2, $\dots$, $\lambda_m \in \mathbb{C}$ such that*

$$\alpha = \sum_{j=1}^{m} \lambda_j \pi_j.$$

The result and proof extend without problems to normal compact operators on Hilbert space. The general spectral theorem for normal operators in Hilbert space can also be obtained in the same way but requires some discussion of integration theory.

Added 17 (Exercise)**.** *Consider the Banach algebra of $n \times n$ complex matrices. Let A be an $n \times n$ diagonal matrix. Write down a typical matrix in the closed subalgebra generated by A, A^* and I. What are the multiplicative linear functionals.*

13 Three more uses of Hahn-Banach

The following exercise provides background for our first discussion but is not examinable. For the moment $C([a,b])$ will be the set of real valued continuous functions.

Exercise 115. *We say that a function $G : [a,b] \to \mathbb{R}$ is of bounded variation if there exists a K such that whenever we have a dissection*

$$\mathcal{D} = \{x_0,\ x_1,\ x_2,\ \ldots,\ x_n\}$$

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$ we have

$$\sum_{j=1}^{n} |G(x_j) - G(x_{j-1})| \leq K.$$

We write

$$\|G\|_{BV} = \sup_{\mathcal{D}} \sum_{j=1}^{n} |G(x_j) - G(x_{j-1})|$$

where the supremum is taken over all possible dissections.

Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Let us write

$$S(\mathcal{D}, f, G) = \sum_{j=1}^{n} f(x_j)\big(G(x_j) - G(x_{j-1})\big).$$

If $\mathcal{D} = \{x_0,\ x_1,\ x_2,\ \ldots,\ x_n\}$ and $\mathcal{D}' = \{x'_0,\ x'_1,\ x'_2,\ \ldots,\ x'_{n'}\}$ are such that $|f(t) - f(s)| < \epsilon$ for all $t,\ s \in [x_{j-1}, x_j]$ $[1 \leq j \leq n]$ and for all $t,\ s \in [x'_{j-1}, x'_j]$ $[1 \leq j \leq n']$ show by considering $\mathcal{D} \cup \mathcal{D}'$, or otherwise that

$$|S(\mathcal{D}, f, G) - S(\mathcal{D}', f, G)| \leq 2K\epsilon.$$

Hence, or otherwise, show that there exists a unique $I(f,G)$ such that, given any $\epsilon > 0$ we can find a $\delta > 0$ such that, given any

$$\mathcal{D} = \{x_0,\ x_1,\ x_2,\ \ldots,\ x_n\}$$

with $|x_{j-1} - x_j| < \delta$ $[1 \leq j \leq n]$ we have

$$|S(\mathcal{D}, f, G) - I(f,G)| < \epsilon.$$

We write

$$I(f,G) = \int_a^b f(t)\, dG(t).$$

(i) Let $[a,b]=[0,1]$. Find elementary expressions for $\int_a^b f(t)\,dG(t)$ in the three cases when $G(t)=t$, when $G(t)=-t$ and when $G(t)=0$ for $t<1/2$, $G(t)=1$ for $t\geq 1/2$.

(ii) Show that the map $T:(C([a,b]),\|\ \|_\infty)\to\mathbb{R}$ given by

$$Tf=\int_a^b f(t)\,dG(t)$$

is linear and continuous with $\|T\|\leq\|G\|_{BV}$.

(In order to obtain the more satisfactory result $\|T\|=\|G\|_{BV}$ we must put an extra condition on G such as left continuity.)

Added 18 (Exercise)**.** *(i) Suppose $G:\mathbb{R}\to\mathbb{R}$ is of bounded variation in every interval $[a,b]$. Show that if we write*

$$G_+(t)=G(a)+\sup\left\{\sum_{j=1}^m G(b_j)-G(a_j)\ :\ a\leq a_1\leq b_1\leq a_2\leq b_2\leq\cdots\leq a_m\leq b_m\leq t,\ m\geq 1\right\}$$

and

$$G_-(t)=\sup\{\sum_{j=1}^m G(a_j)-G(b_j)\ :\ a\leq a_1\leq b_1\leq a_2\leq b_2\leq\cdots\leq a_m\leq b_m\leq t,\ m\geq 1\}$$

then G_+, $G_-:[a,b]\to\mathbb{R}$ are increasing functions with

$$G(t)=G_+(t)-G_-(t)$$

and $\|G\|_{BV}=(G_+(b)-G_+(a))+(G_-(b)-G_-(a))$.

(ii) By first considering increasing functions, or otherwise, show that if G is a function of bounded variation in every interval then the left and right limits

$$G(t+)=\lim_{h\to 0,h>0}G(t+h),G(t-)=\lim_{h\to 0,h>0}G(t-h)$$

exist everywhere.

Theorem 116. *If $T:C([a,b])\to\mathbb{R}$ is a continuous linear function then we can find a left continuous function $G:mathbbR\to\mathbb{R}$ of bounded variation in every interval such that*

$$Tf=\int_a^b f(t)\,dG(t)$$

for all $f\in C([a,b])$.

If you know a little measure theory you can restate the theorem in more modern language.

Theorem 117. (The Riesz representation theorem.) *The dual of $C([a,b])$ is the space of Borel measures on $[a,b]$.*

The method used can easily be extended to all compact spaces.

Our second result is more abstract. We require Aloaoglu's theorem.

Theorem 118. *The unit ball of the dual of a normed space X is compact in the weak star topology.*

Our proof of the Riesz representation theorem used the Hahn-Banach theorem as a convenience. Our proof of the next result uses it as basic ingredient.

Theorem 119. *Every Banach space is isometrically isomorphic to some subspace of $C(K)$ for some compact space K.*

(In my opinion this result looks more interesting than it is.)

Our third result requires us to recast the Hahn Banach theorem in a geometric form.

Lemma 120. *If V is a real normed space and E is a convex subset of V containing $B(\mathbf{0},\epsilon)$ for some $\epsilon > 0$, then, given any $\mathbf{x} \notin E$ we can find a continuous linear map $T : V \to \mathbb{R}$ such that $T\mathbf{x} = 1 \geq T\mathbf{e}$ for all $\mathbf{e} \in E$.*

Theorem 121. *If V is a real normed space and F is a closed convex subset of V, then, given any $\mathbf{x} \notin F$ we can find a continuous linear map $T : V \to \mathbb{R}$ and a real α such that $T\mathbf{x} > \alpha > T\mathbf{k}$ for all $\mathbf{k} \in F$.*

Definition 122. *Let V be a real or complex vector space. If K is a non-empty subset of V we say that $E \subseteq K$ is an* extreme set *of K if, whenever $u,\ v \in K$, $1 > \lambda > 0$ and $\lambda u + (1-\lambda)v \in E$, it follows that $u,\ v \in E$. If $\{e\}$ is an extreme set we call e an extreme point.*

Exercise 123. *Define an extreme point directly.*

Exercise 124. *We work in $\mathbb{R}^2$. Find the extreme points, if any, of the following sets and prove your statements.*

(i) $E_1 = \{\mathbf{x} : \|\mathbf{x}\| < 1\}$.

(ii) $E_2 = \{\mathbf{x} : \|\mathbf{x}\| \leq 1\}$.

(iii) $E_3 = \{(x,0) : x \in \mathbb{R}\}$.

(iv) $E_4 = \{(x,y) : |x|, |y| \leq 1\}$.

Theorem 125. (Krein-Milman). *A non-empty compact convex subset K of a normed vector space has at least one extreme point.*

Theorem 126. *A non-empty compact convex subset K of a normed vector space is the closed convex hull of its extreme points (that is, is the smallest closed convex set containing its extreme points).*

Our hypotheses in our version of the Krein-Milman theorem are so strong as to make the conclusion practically useless. However the hypotheses can be much weakened as is indicated by the following version.

Theorem 127. (Krein-Milman). *Let E be the dual space of a normed vector space. A non-empty convex subset K which is compact in the weak star topology has at least one extreme point.*

Theorem 128. *Let E be the dual space of a normed vector space. A non-empty convex subset K which is compact in the weak star topology is the weak star closed convex hull of its extreme points.*

The results follow at once from a new version of Lemma 120

Added 19 (Lemma)**.** *If V is a real normed space and E is a convex subset of the dual space V' containing an open neighbourhood of $\mathbf{0}$, then, given any $\mathbf{x} \notin E$ we can find a continuous linear map $T : V \to \mathbb{R}$ such that $T\mathbf{x} = 1 \geq T\mathbf{e}$ for all $\mathbf{e} \in E$.*

Lemma 129. *The extreme points of the closed unit ball of the dual of $C([0,1])$ are the delta masses δ_a and $-\delta_a$ with $a \in [0,1]$.*

14 The Rivlin-Shapiro formula

In this section we give an elegant use of extreme points due to Rivlin and Shapiro.

Lemma 130. Carathéodory *We work in $\mathbb{R}^n$. Suppose that $\mathbf{x} \in \mathbb{R}^n$ and we are given a finite set of points $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_N$ and positive real numbers λ_1, λ_2, …, λ_N such that*

$$\sum_{j=1}^{N} \lambda_j = 1, \ \sum_{j=1}^{N} \lambda_j \mathbf{e}_j = \mathbf{x}.$$

Then **after renumbering** *the $\mathbf{e}_j$ we can find positive real numbers λ'_1, λ'_2, …, λ'_m with $m \leq n+1$ such that*

$$\sum_{j=1}^{m} \lambda'_j = 1, \ \sum_{j=1}^{m} \lambda'_j \mathbf{e}_j = \mathbf{x}.$$

Added 20 (Exercise). *Show by means of an example that we can not necessarilly take $m = n$ in Carathéodory's lemma.*

Lemma 131. *Consider $\mathcal{P}_n$, the subspace of $C([-1,1])$ consisting of real polynomials of degree n or less. If $S : \mathcal{P}_n \to \mathbb{R}$ is linear then we can find an $N \leq n+2$ and distinct points $x_1,\ x_2,\ \ldots,\ x_N \in [-1,1]$ and non-zero real numbers $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_N$ such that*

$$\sum_{j=1}^{N} |\lambda_j| = 1,\ \ \|S\| \sum_{j=1}^{N} \lambda_j P(x_j) = SP$$

for all $P \in \mathcal{P}_n$.

Lemma 132. *We continue with the hypotheses and notation of Lemma 131 There exists a $P_* \in \mathcal{P}_n$ such that*

$$P_*(x_j) = \|P_*\|_\infty \operatorname{sgn} \lambda_j$$

for all j with $0 \leq j \leq N$. Further, if $P \in \mathcal{P}_n$ satisfies

$$P(x_j) = \|P\|_\infty \operatorname{sgn} \lambda_j$$

then $\|P\|_\infty \|S\| = SP$.

The following results are of considerable interest in view of Lemma 132.

Lemma 133. *We have $\cos n\theta = T_n(\cos\theta)$ where T_n is a real polynomial of degree n. Further*

(i) $|T_n(x)| \leq 1$ for all $x \in [-1,1]$.

(ii) There exist $n+1$ distinct points $x_1,\ x_2,\ \ldots,\ x_{n+1} \in [-1,1]$ such that $|T_n(x_j)| = 1$ for all $1 \leq j \leq n+1$.

Lemma 134. *Suppose that P is a real polynomial of degree n or less such that*

(i) $|P(x)| \leq 1$ for all $x \in [-1,1]$ and

(ii) there exist $n+1$ distinct points $x_1,\ x_2,\ \ldots,\ x_{n+1} \in [-1,1]$ such that $|P(x_j)| = 1$ for all $1 \leq j \leq n+1$.

Then $P = \pm T_n$.

Note that Lemma 134 tells us that there is no real polynomial of degree n or less which takes its extreme absolute value on $[-1,1]$ at $n+2$ points. (Thus we can replace the condition $N \leq n+2$ in Lemma 131 by $N \leq n+1$.)

Theorem 135. *If P is a real polynomial of degree at most n and $t \notin [-1,1]$, then*

$$|P(t)| \leq \sup_{|x|\leq 1} |P(x)||T_n(t)|.$$

Exercise 136. *If P is a real polynomial of degree at most n and $t \notin [-1,1]$, then then*

$$|P^{(r)}(t)| \leq |T^{(r)}(t)| \sup_{|x|\leq 1} |P(x)|.$$

Exercise 137. *(This exercise is part of the course.) (i) Show that if $n \geq 1$ the coefficient of t^n in $T_n(t)$ is 2^{n-1}.*

(ii) Show that if $n \geq 1$ and P is a real polynomial of degree n or less with $|P(t)| \leq 1$ then the coefficient of t^n in $P(t)$ has absolute value at most 2^{n-1}.

(iii) Find, with proof, a polynomial P of degree at most $n-1$ which minimises

$$\sup_{t\in[-1,1]} |t^n - P(t)|.$$

Show that P is unique. (Tchebychev introduced his polynomials T_n in this context.)

References

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