Partial Solutions for Questions in Appendix K of A Companion to Analysis

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Introduction

Here is a miscellaneous collection of hints, answers, partial answers and remarks on some of the exercises in the book. I expect that there are many errors both large and small and would appreciate the opportunity to correct them. Please tell me of any errors, unbridgable gaps, misnumberings etc. I welcome suggestions for additions. ALL COMMENTS GRATEFULLY RECEIVED. (If you can, please use LaTeX 2_ε or its relatives for mathematics. If not, please use plain text. My e-mail is **twk@dpmms.cam.ac.uk**. You may safely assume that I am both lazy and stupid so that a message saying 'Presumably you have already realised the mistake in question 33' is less useful than one which says 'I think you have made a mistake in question 33 because not all left objects are right objects. One way round this problem is to quote X's theorem.')

To avoid disappointment note that a number like K15* means that there is no comment. A number marked K15? means that I still need to work on the remarks. Note also that what is given is *at most* a sketch and often very much less.

Please treat the answers as a last resort. You will benefit more from thinking about a problem than from reading a solution. I am inveterate peeker at answers but I strongly advise you to do as I say and not as I do.

It may be easiest to navigate this document by using the table of contents which follow on the next few pages.

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K1

(ii) Every non-empty set of positive integers has a least member. Since x is a rational number, qx is an integer for some strictly positive integer. We can certainly find a k such that $k+1 > x \geq k$ (ultimately by the Axiom of Archimedes). But x is not an integer so $k+1 > x > k$.

Since mx, m and k are integers, $m' = mx - mk$ is. Also

$$m'x = mx^2 - mkx = mN - k(mx)$$

is an integer since mx is. Since $1 > x - k > 0$ we have $m > mx - mk = m' > 0$ and $m > m' \geq 1$ contradicting the definition of m as least strictly positive integer with mx an integer.

K2

Write $c_n = d_n + d_n^{-1}$. Then

★
$$d_n^2 - c_n d_n + 1 = 0$$

$$d_n = \frac{c_n \pm \sqrt{c_n^2 - 4}}{2}.$$

Since the product of the two roots of ★ is 1, we must take the bigger root. Thus

$$d_n = \frac{c_n + \sqrt{c_n^2 - 4}}{2} \to \frac{c + \sqrt{c^2 - 4}}{2}$$

where c is the limit of c_n.

For the second paragraph, just take

$$d_{2n} = (1 + k)/2 \text{ and } d_{2n+1} = 2/(1 + k).$$

In the third paragraph the condition $|d_n| \geq 1$ will not do. Take any d with $|d| = 1$ and $d \neq \pm 1$. Then, if $d_{2n} = d$ and $d_{2n+1} = d^*$, the result fails.

The condition $|d_n| > 1$ will do but we must argue carefully. Look first to see which root of $d^2 - cd + 1 = 0$ lies outside the unit circle.

K3

(Second part of problem.) Choose $a_1 = 2$, $b_1 = 1$, say.

K4

Take $x = 0$, $f(t) = H(t)$, $g(t) = 0$. Take $f(t) = 0$, $g(t) = H(t)$. ($H(t) = 1$ for $t > 0$, $H(t) = 0$ for $t \leq 0$.)

K5

(i), (ii) and (iii) are true and can be proved by, e.g. arguing by cases.

(iv) is false since f is continuous at $1/2$. (f is, however, discontinuous everywhere else.)

(v) is false. Take e.g. $x = 16^{-1} + 17^{-1/2}$, $y = 3^{-1} - 17^{-1/2}$.

K6

$\sum_{n=1}^{N} 2^{-n} H(x-q_n)$ is an increasing sequence in N bounded above by 1.

Observe that if $x \geq y$ then $H(x-q_n) - H(y-q_n) \geq 0$ for all $n \geq 0$. Thus

$$\sum_{n=1}^{N} 2^{-n} H(x-q_n) - \sum_{n=1}^{N} 2^{-n} H(y-q_n)$$

$$= \sum_{n=1}^{N} 2^{-n} \big(H(x-q_n) - H(y-q_n) \big) \geq 0$$

for all N so $f(x) \geq f(y)$.

Observe that, if $x > q_m$, then $H(x-q_n) - H(q_m-q_n) \geq 0$ for all $n \geq 0$ and $H(x-q_m) - H(q_m-q_m) = 1$ so

$$\sum_{n=1}^{N} 2^{-n} H(x-q_n) - \sum_{n=1}^{N} 2^{-n} H(q_m-q_n) \geq 2^{-m}$$

for all $N \geq m$ so $f(x) \geq f(q_m) + 2^{-m}$ for all $x > q_m$. Thus f is not continuous at q_m.

If x is irrational, we can find an $\epsilon > 0$ such that $|q_m - x| > \epsilon$ for all $m \leq M$ and so

$$\left| \sum_{n=1}^{N} 2^{-n} H(x-q_n) - \sum_{n=1}^{N} 2^{-n} H(y-q_n) \right| \leq 2^{-M}$$

whenever $|x-y| < \epsilon$. Thus f is continuous at x.

K7

Observe that $a_{n+1} - a_n$ is decreasing. Thus either

(a) $a_{n+1} - a_n > 0$ for all n and, if a_n is unbounded $a_n \to \infty$, or

(b) There exists an N such that $a_{n+1} - a_n \leq 0$ for all $n \geq N$ and, if a_n is unbounded, $a_n \to -\infty$.

For the reverse inequality, observe that $-a_n$ satisfies the original inequality.

K8

A decreasing sequence bounded below tends to a limit, so, for any fixed $x \in (0,1)$,

$$x_n = f_n(x) \to y, \text{ say.}$$

Since $x > x_n > 0$ we have $1 > y \geq 0$. If $y > 0$ then by continuity

$$x_{n+1} = f(x_n) \to f(y)$$

and so $y = f(y)$. Thus $y = 0$.

If

$$f(x) = \begin{cases} \frac{1}{4} + \frac{x}{2} & \text{for } x > 1/2, \\ \frac{x}{4} & \text{for } x \leq 1/2, \end{cases}$$

then $f_n(x) \to 1/2$ for $1 > x > 1/2$.

We suppose our calculator works in radians. If $a_{n+1} = \sin a_n$ then, whatever a_0 we start from, we have $|a_1| \leq 1$. If $a_1 = -b_1$ and $b_{n+1} = \sin b_n$ then $a_n = -b_n$ for $n \geq 1$. If $0 < a < 1$, then $0 < \sin a < a$.

(If our calculator works in degrees, matters are simpler and less interesting.)

K9

(ii) we need only look at positive integers. If $n \geq 2(k+1)$

$$\begin{aligned}(1+\delta)^n &= \sum_{j=0}^{n} \binom{n}{j} \delta^j \\ &\geq \binom{n}{k+1} \delta^{k+1} \\ &\geq \frac{(n/2)^{k+1}}{(k+1)!} \delta^{k+1} \\ &\geq \frac{n^{k+1}}{2^{k+1}(k+1)!} \delta^{k+1}\end{aligned}$$

so

$$n^{-k}(1+\delta)^n \geq n \frac{\delta^{k+1}}{2^{k+1}(k+1)!} \to \infty$$

as $n \to \infty$.

K10

Observe that

$$\begin{aligned}\frac{x_n}{n} =& \frac{1}{n}\left(n\frac{x_1+x_2+\cdots+x_n}{n} - (n-1)\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)\\ =&\left(\frac{x_1+x_2+\cdots+x_n}{n} - \left(1-\frac{1}{n}\right)\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)\\ &\to 1-(1-0)1=0\end{aligned}$$

as $n\to\infty$.

Either there exists an N such that $m(n)=m(N)$ for all $n\geq N$ and the result is immediate or $m(n)\to\infty$ and

$$\frac{x_{m(n)}}{n}\leq\frac{x_{m(n)}}{m(n)}\to 0$$

as $n\to\infty$.

If $\alpha>1$, then

$$\frac{x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha}{n^\alpha}\leq\frac{x_1+x_2+\cdots+x_n}{n}\left(\frac{x_{m(n)}}{n}\right)^{\alpha-1}\to 1\times 0=0.$$

If $\alpha<1$, then

$$\frac{x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha}{n^\alpha}\geq\frac{x_1+x_2+\cdots+x_n}{n}\left(\frac{x_{m(n)}}{n}\right)^{\alpha-1}\to\infty.$$

K11

(i) True. If $a+b=c$ and $a,\ c\in\mathbb{Q}$ then $b=c-a\in\mathbb{Q}$.

(ii) False. $0\times\sqrt{2}=0$. (However the product of an irrational number with a non-zero rational number is irrational.)

(iii) True. By the axiom of Archimedes we can find a strictly positive integer N with $N^{-1}<\epsilon$ and an integer M such that $mN^{-1}\leq x<(m+1)N^{-1}$. Set $y=mN^{-1}+\sqrt{2}/(2N)$.

(iv) False. Take $x_n=2^{1/2}/N$.

(v) False. If $a^{\sqrt{2}}$ is rational, then, since a is irrational, the statement is false. If $a^{\sqrt{2}}$ is irrational, then, since $(a^{\sqrt{2}})^{\sqrt{2}}=2$ the statement is false.

K12

(i) $|x^n - y^n| = |(x-y)\sum_{j=0}^{n-1} x^j y^{n-1-j}| \leq |x-y| \sum_{j=0}^{n-1} |x|^j |y|^{n-1-j}|$.

(ii) If $P(t) = \sum_{j=0}^{N} a_j t^j$, then

$$|P(x) - P(y)| \leq |x-y| \sum_{j=0}^{N} a_j |j| R^{j-1}.$$

(vi) An increasing sequence bounded above converges.

$$\sum_{k=J+1}^{N} 10^{-k!} \leq 10^{-(J+1)!} \sum_{k=J+1}^{N} 10^{k-(J+1)} \leq \frac{10}{9} \times 10^{-(J+1)!}.$$

K13*

No comments.

K14*

No comments.

K15

Apply the intermediate value theorem.

a_n converges (as in lion hunting) to c say.

Suppose $f'(c) > 0$. We can find an $\epsilon > 0$ such that

$$\left| \frac{f(t) - f(c)}{t - c} - f'(c) \right| < \frac{f'(c)}{2}$$

and so

$$\frac{f(t) - f(c)}{t - c} > \frac{f'(c)}{2}$$

for all t with $|t - c| < \epsilon$. Thus $f(t) > f(c)$ for $c < t < c + \epsilon$ and $f(t) < f(c)$ for $c - \epsilon < t < c$

Choose n such that $2^n < \epsilon$. We have $f(a_n) < f(b_n)$ which is absurd.

K16

Repeat the lion hunting argument for the intermediate value theorem with $f(a_n) \geq g(a_n)$, $f(b_n) \leq g(b_n)$. If $a_n \to c$, say, then, since f is increasing this gives

$$g(b_n) \geq f(b_n) \geq f(c) \geq f(a_n) \geq g(a_n).$$

If $a_n \to c$, say, then since g is continuous $g(a_n) \to g(c)$ and so $f(c) = g(c)$.

First sentence of second paragraph. False. Consider $[a, b] = [-1, 1]$, $g(x) = 0$, $f(x) = x - 1$ for $x \leq 0$, $f(x) = x + 1$ for $x > 0$.

Second sentence false. Consider $[a, b] = [-1, 1]$, $g(x) = -x + 3$, $f(x) = x - 1$ for $x \leq 0$, $g(x) = -x + 3$, $f(x) = x + 1$ for $x > 0$.

Third sentence true. Apply intermediate value theorem to $f - g$.

K17

Let $\epsilon > 0$. Choose $\delta_k > 0$ such that $|f_k(x) - f(c)| < \epsilon$ for all $|x - c| < \delta_k$. Set $\delta = \min_{1 \le k \le N} \delta_k$. We claim that $|g(x) - g(c)| < \epsilon$ for all $|x - c| < \delta$.

To see this, suppose, without loss of generality, that $f_N(c) = g(c)$. Then

$$g(x) \ge f_N(x) > f_N(c) - \epsilon = g(c) - \epsilon$$

and

$$f_j(x) \le f_j(c) + \epsilon \le g(c) + \epsilon$$

for all $1 \le j \le N$ and so

$$g(x) \le g(c) + \epsilon$$

for all $|x - c| < \delta$.

For the example we can take $(a, b) = (-1, 1)$, $c = 0$ and

$$f_n(x) = \begin{cases} 0 & \text{if } x \le 0, \\ nx & \text{if } 0 \le x < 1/n, \\ 1 & \text{if } 1/n \le x. \end{cases}$$

K18

(i) Observe that

$$\{x \in \mathbb{R} \,:\, E \cap (-\infty, x] \text{ is countable}\}$$

is a non-empty bounded set and so has a supremum α say. Observe that $E_n = E \cap (-\infty, \alpha - 1/n]$ is countable so $E \cap (-\infty, \alpha) = \bigcup_{n=1}^{\infty} E_n$ is countable and so $E \cap (-\infty, \alpha]$ is countable. Thus $E \cap (-\infty, \gamma)$ is uncountable if and only if $\gamma > \alpha$. Similarly we can find a β such that $E \cap (\gamma, \infty)$ is uncountable if and only if $\gamma < \beta$. Since E is uncountable, $\beta > \alpha$.

(ii) We have one of four possibilities.

(a) There exist α and β with $\alpha > \beta$ and

$$\{e \in E \,:\, e < \gamma\} \text{ and } \{e \in E \,:\, e > \gamma\}$$

are uncountable if and only if $\alpha < \gamma < \beta$.

(b) There exists an α with

$$\{e \in E \,:\, e < \gamma\} \text{ and } \{e \in E \,:\, e > \gamma\}$$

are uncountable if and only if $\alpha < \gamma$.

(c) There exists a β with

$$\{e \in E \,:\, e < \gamma\} \text{ and } \{e \in E \,:\, e > \gamma\}$$

are uncountable if and only if $\gamma < \beta$.

(d) We have

$$\{e \in E \,:\, e < \gamma\} \text{ and } \{e \in E \,:\, e > \gamma\}$$

uncountable for all γ.

All these possibilities can occur. Look at $\mathbb{Z} \cup [-1, 1]$, $(0, \infty)$, $(-\infty, 0)$ and $\mathbb{R}$.

(iii) Set $E = \{1/n \,:\, n \geq 1,\ n \in \mathbb{Z}\}$

K19

(i) False. Consider $x_j = 1$.

(ii) False. Consider $x_j = -j$.

(iii) True. Follows from proof of Lemma 3.19.

(iv) False. Consider $x_j = (-1)^j$.

K20

(i) Let $\epsilon > 0$. There exists an N such that $a_n \leq \limsup_{r\to\infty} a_r + \epsilon$ for $n \geq N$. There exists an M such that $n(p) \geq N$ for $p \geq M$. Now $\limsup_{r\to\infty} a_r + \epsilon \geq a_{n(p)}$ for $p \geq M$. Since ϵ was arbitrary,

$$\limsup_{r\to\infty} a_r \geq a.$$

(ii) Take $a_n = (1 + (-1)^n)/2$.

Take e.g. $a_{2^n+r} = r/2^n$ for $0 \leq r < 2^n$, $n \geq 0$.

(iii) Take $a_n = -b_n = (-1)^n$.

(True.) Let $\epsilon > 0$. If n is large enough

$$\limsup_{r\to\infty} a_r + \epsilon > a_n \text{ and } \limsup_{r\to\infty} b_r + \epsilon > b_n$$

so

$$\limsup_{r\to\infty} a_r + \limsup_{r\to\infty} b_r + 2\epsilon > a_n + b_n.$$

Thus

$$\limsup_{r\to\infty} a_r + \limsup_{r\to\infty} b_r + 2\epsilon > \limsup_{n\to\infty}(a_n + b_n).$$

But ϵ is arbitrary so

$$\limsup_{r\to\infty} a_r + \limsup_{r\to\infty} b_r \geq \limsup_{n\to\infty}(a_n + b_n).$$

(True) Similarly if $\epsilon > 0$ and n is large enough

$$b_n \geq \liminf_{r\to\infty} b_r - \epsilon$$

and so

$$a_n + b_n \geq a_n + \liminf_{r\to\infty} b_r - \epsilon.$$

Thus

$$\limsup_{n\to\infty}(a_n + b_n) \geq \limsup_{n\to\infty} a_n + \liminf_{r\to\infty} b_r - \epsilon$$

and, since ϵ was arbitrary,

$$\limsup_{n\to\infty}(a_n + b_n) \geq \limsup_{n\to\infty} a_n + \liminf_{r\to\infty} b_r.$$

K21

$$\begin{aligned}\left\|\frac{\mathbf{x}}{\|\mathbf{x}\|^2}-\frac{\mathbf{y}}{\|\mathbf{y}\|^2}\right\|^2 &= \frac{\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2}-2\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}+\frac{\|\mathbf{y}\|^2}{\|\mathbf{y}\|^2}\\ &= \frac{\|\mathbf{y}\|^2-2\mathbf{x}\cdot\mathbf{y}+\|\mathbf{x}\|^2}{\|\mathbf{x}\|^2\|\mathbf{y}\|^2}\\ &= \left(\frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{x}\|\|\mathbf{y}\|}\right)^2.\end{aligned}$$

Thus

$$\left\|\frac{\mathbf{x}}{\|\mathbf{x}\|^2}-\frac{\mathbf{y}}{\|\mathbf{y}\|^2}\right\| = \frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

Ptolomey's result now follows from the triangle inequality for points of the form $\|\mathbf{x}\|^{-2}\mathbf{x}$.

K22

(i), (ii) and (iii) are true. Proof by applying the definitions.

(iv) is false. Take, for example, $n = 1$, $U = \{x : |x| < 1\}$.

K23*

No comments.

K24

(i) If $(x_n, y_n) \in E$ and $\|(x_n, y_n) - (x, y)\| \to 0$, then $x_n \to x$ and $y_n \to y$. Thus $y_n = 1/x_n \to x$ and $x = y$ so $(x, y) \in E$.

$\pi_1(E) = \{x : x > 0\}$ is not closed since $1/n \in \pi_1(E)$ and $1/n \to 0 \notin \pi_1(E)$ as $n \to 0$.

(ii) Let $E = \{(x, 1/x) : x > 0\} \cup \{(1, 0)\}$. Then E is closed (union two closed sets), $\pi_1(E) = \{x : x > 0\}$ is not closed but $\pi_2(E) = \{y : y \geq 0\}$ is.

(iii) If $x_n \in \pi_1(E)$ and $x_n \to x$ we can find y_n such that $(x_n, y_n) \in E$. Since y_n is a bounded sequence there exists a convergent subsequence $y_{n(j)} \to y$ as $j \to \infty$. Since $x_{n(j)} \to x$ the argument of (i) shows that $(x, y) \in E$ and $x \in \pi_1(E)$. (It is worth looking at why this argument fails in (ii).)

K25

(i) If $G \subseteq [-K, K]^{n+m}$, then $E \subseteq [-K, K]^n$.

(ii) Take $n = m = 1$, $E = (0, \infty)$, $f(x) = 1/x$¿ See question K.24.

(iii) Suppose $\|(\mathbf{x}_j, \mathbf{f}(\mathbf{x}_j)) - (\mathbf{x}, \mathbf{y})\| \to 0$. Then $\|\mathbf{x}_j - \mathbf{x}\| \to 0$. Since E is closed, $\mathbf{x} \in E$. Since $\mathbf{f}$ is continuous, $\|\mathbf{f}(\mathbf{x}_j) - \mathbf{f}(\mathbf{x})\| \to 0$. Thus $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $(\mathbf{x}, \mathbf{y}) \in G$.

(iv) Take $n = m = 1$, $E = \mathbb{R}$ and $f(x) = x^{-1}$ for $x \neq 0$, $f(0) = 0$.

(v) Suppose $\mathbf{x}_j \in E$ and $\|\mathbf{x}_j - \mathbf{x}\| \to 0$. Since G is closed and bounded we can find a subsequence $j(k) \to \infty$ such that

$$\|(\mathbf{x}_{j(k)}, \mathbf{f}(\mathbf{x}_{j(k)})) - (\mathbf{x}, \mathbf{y})\| \to 0$$

for some $(\mathbf{x}, \mathbf{y}) \in G$. Since $(\mathbf{x}, \mathbf{y}) \in G$, we have $\mathbf{y} = f(\mathbf{x})$.

Observe that $\mathbf{x} \in E$, so E is closed. If $\mathbf{f}$ was not continuous at $\mathbf{x}$, we could find a $\delta > 0$ and $\mathbf{x}_j \in E$ and $\|\mathbf{x}_j - \mathbf{x}\| \to 0$ such that $\|\mathbf{f}(\mathbf{x}_j) - \mathbf{f}(\mathbf{x})\| > \delta$ and so

$$\|(\mathbf{x}_{j(k)}, \mathbf{f}(\mathbf{x}_{j(k)})) - (\mathbf{x}, \mathbf{y})\| > \delta$$

for every $j(k)$. Our result follows by reductio ad absurdum.

K26

If we write $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$, then f is upper semicontinuous but not continuous.

If f is not bounded we can find $\mathbf{x}_n \in E$ such that $f(\mathbf{x}_n) \geq n$. Extract a convergent subsequence and use upper semicontinuity to obtain a contradiction. To show that the least upper bound is attained, take a sequence for which f approaches its supremum, extract a convergent subsequence and use upper semicontinuity again.

Take $n = 1$, $K = [0, 1]$, $f(x) = -1/x$ for $x \neq 0$, $f(0) = 0$ to obtain an upper semicontinuous function not bounded below.

Take $n = 1$, $K = [0, 1]$, $f(x) = x$ for $x \neq 0$, $f(0) = 1$ to obtain an upper semicontinuous function bounded below which does not attain its infimum.

(As usual it is informative to see how the proof of the true result breaks down when we loosen the hypotheses.)

K27

(i) f is continuous on $[0, a]$ so is bounded and attains its bounds on $[0, a]$. By periodicity f is everywhere bounded and attains its bounds.

(ii) Let $f(x) = (x - [x])^{-1})$ if x is not an integer and $f(x) = 0$ if x is an integer. (Here $[x]$ is the integer part of x.)

(iii) By considering $g(x) - kx$ we see that we may suppose $k = 0$. Given $\epsilon > 0$ there exists an $X > 0$ such that $|g(x+1) - g(x)| < \epsilon$ (and so $|g(x+n) - g(x)| < n\epsilon$ for $x > X$. By hypothesis there exists a K such that $|g(y)| \leq K$ for $0 \leq y \leq X + 1$. Write $n(x)$ for the integer such that $X < x - n(x) \leq X + 1$. If $x > X + 1$ we have

$$\begin{aligned} \left|\frac{g(x)}{x}\right| &\leq \left|\frac{g(x) - g(x - n(x))}{x}\right| + \left|\frac{g(x - n(x))}{x}\right| \\ &\leq \frac{n(x)\epsilon}{x} + \frac{K}{x} \to \epsilon \end{aligned}$$

as $x \to \infty$. Since ϵ was arbitrary, $g(x)/x \to 0$ as $x \to \infty$.

(iv) Any example for (ii) will work here.

(v) False. Consider, for example, $h(x) = kx + x^{1/2}\sin(\pi x/2)$.

K28

No comments.

K29

Observe that x_n can lie in at most two of the three intervals $[a_{n-1}, a_{n-1} + k_{n-1}]$, $[a_{n-1} + k_{n-1}, a_{n-1} + 2k_{n-1}]$and $[a_{n-1} + 2k_{n-1}, b_{n-1}]$ where $k_{n-1} = (b_{n-1} - a_{n-1})/3$.

K30

(i) Lots of different ways. Observe that the map J given by $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$ is continuous since

$$\begin{aligned}|(\mathbf{x}+\mathbf{h}) \cdot (\mathbf{y}+\mathbf{k}) - \mathbf{x} \cdot \mathbf{y}| &\leq |\mathbf{x} \cdot \mathbf{k}| + |\mathbf{h} \cdot \mathbf{y}| + |\mathbf{h} \cdot \mathbf{k}| \\ &\leq \|\mathbf{h}\|\|\mathbf{x}\| + \|\mathbf{y}\|\|\mathbf{k}\| + \|\mathbf{h}\|\|\mathbf{k}\| \to 0\end{aligned}$$

as $\|(\mathbf{h}, \mathbf{k})\| \to 0$. Since α is continuous, the map A given by $\mathbf{x} \mapsto (\mathbf{x}, \alpha\mathbf{x})$ is continuous so, composing the two maps A and J, we see that the given map is continuous.

Last part, continuous real valued function on a closed bounded set attains a maximum.

(ii) Observe that, if $\mathbf{h} \cdot \mathbf{e} = 0$, then $\|\mathbf{e} + \delta\mathbf{h}\|^2 = (1+\delta^2)$ so by (i)

$$(1+\delta^2)\mathbf{e} \cdot (\alpha\mathbf{e}) \geq (\mathbf{e}+\delta\mathbf{h}) \cdot (\alpha(\mathbf{e}+\delta\mathbf{h}))$$

and so

$$\delta(\mathbf{e} \cdot (\alpha\mathbf{h}) + \mathbf{h} \cdot (\alpha\mathbf{e} + \delta(\mathbf{e} \cdot (\alpha\mathbf{e}) - \mathbf{h} \cdot (\alpha\mathbf{h})) \geq 0$$

for all δ so

$$\mathbf{e} \cdot (\alpha\mathbf{h}) + \mathbf{h} \cdot (\alpha\mathbf{e}) = 0.$$

(iv) If we assume the result true for $\mathbb{R}^{n-1}$, then, since U has dimension $n-1$ and $\beta|_U$ is self adjoint, U has an orthonormal basis $\mathbf{e}_2$, $\mathbf{e}_3$, $\ldots$, $\mathbf{e}_n$ of eigenvectors for $\beta|_U$. Taking $\mathbf{e}_1 = \mathbf{e}$, gives the desired result.

K31

(i) Observe that

$$|g(\mathbf{u}) - g(\mathbf{v})| \leq \|\mathbf{uv}\|$$

since, as simple consequence of the triangle inequality,

$$|\|\mathbf{a}\| - \|\mathbf{b}\|| \leq \|\mathbf{a} - \mathbf{b}\|.$$

Choose any $\mathbf{z} \in E$. Set $R = \|\mathbf{z} - \mathbf{y}\|$. The continuous function $\mathbf{g}$ attains a minimum on the closed bounded set $\bar{B}(\mathbf{y}, R) \cap E$ at $\mathbf{x}_0$, say. If $\mathbf{x} \in E$ then either $\mathbf{x} \in \bar{B}(\mathbf{y}, R)$ and $g(\mathbf{x}) \geq g(\mathbf{x}_0)$ automatically, or $\mathbf{x} \notin \bar{B}(\mathbf{y}, R)$ and

$$g(\mathbf{x}) > R = g(\mathbf{z}) \geq g(\mathbf{x}_0).$$

(ii) Let $\mathbf{u} = \mathbf{x}_0 - \mathbf{y}$. Use the definition of $\mathbf{x}_0$ to show that

$$\mathbf{u} \cdot \mathbf{u} \geq (\mathbf{u} + \delta\mathbf{h}) \cdot (\mathbf{u} + \delta\mathbf{h})$$

whenever $\mathbf{h} \in E$. By considering what happens when δ is small, deduce that $\mathbf{u} \cdot \mathbf{h} = 0$.

If $\|\mathbf{x}_1 - \mathbf{y}\| = \|\mathbf{x}_0 - \mathbf{y}\|$ and $\mathbf{x}_1 \in E$, then, setting $\mathbf{h} = \mathbf{x}_1 - \mathbf{x}_0$, we get $\mathbf{h} \cdot \mathbf{h} = 0$.

(iii) Since $\alpha \neq 0$ we can find a $\mathbf{y} \notin E$. Set $\mathbf{u} = \mathbf{y} - \mathbf{x}_0$ and $\mathbf{b} = (\|\mathbf{u}\|)^{-1}\mathbf{u}$. Observe that

$$(\mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b}) \cdot \mathbf{b} = 0$$

so, since E has dimension $n - 1$, we have

$$\mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b} \in E$$

and

$$\alpha(\mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b}) = 0$$

so

$$\alpha\mathbf{x} = \mathbf{b} \cdot \mathbf{x}\alpha\mathbf{b}.$$

Set $\mathbf{a} = (\alpha\mathbf{b})\mathbf{b}$.

K32

(i) Argument as in K31.

(ii) Suppose $\mathbf{x} \in E$ and $\mathbf{x} \cdot \mathbf{y} > 0$. Then, if δ is small and positive, we have

$$\delta\mathbf{x} = (1-\delta)\mathbf{0} + \delta\mathbf{x} \in E$$

but the same kind of argument as in K30 and K31 shows that $\|\mathbf{y}\| > \|\mathbf{y} - \delta\mathbf{x}\|$.

(iii) Translation.

K33

Write B for the closed unit ball. If $\|\mathbf{z}\| < 1$ then

$$\mathbf{z} = (1 - \|\mathbf{z}\|)\mathbf{z} + \|\mathbf{z}\|\mathbf{0}$$

so $\mathbf{z}$ is not an extreme point.

On the other hand, if $\|\mathbf{x}\| = 1$ and

$$\mathbf{z} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$$

with $0 < \lambda < 1$ and $\mathbf{x}$, $\mathbf{y} \in B$, then, by the triangle inequality,

$$\|\mathbf{z}\| \leq \|\lambda\mathbf{x}\| + \|(1 - \lambda)\mathbf{y}\|$$

with equality only if $\lambda\mathbf{x}$, $(1 - \lambda)\mathbf{y}$ and $\mathbf{z}$ are collinear. But

$$\|\lambda\mathbf{x}\| + \|(1 - \lambda)\mathbf{y}\| \leq \lambda + (1 - \lambda) = 1$$

and $\|\mathbf{z}\| = 1$. Thus $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$ and $\lambda\mathbf{x}$, $(1-\lambda)\mathbf{y}$ and $\mathbf{z}$ are collinear. Inspection now shows that $\mathbf{x} = \mathbf{y} = \mathbf{z}$.

A simpler pair of arguments gives the extreme points of the cube.

(ii) Since K is closed and bounded the continuous function $\mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{a}$ attains a maximum on K.

Suppose $\mathbf{u}_0$ is an extreme point of K'. If $\mathbf{v}$, $\mathbf{w} \in K$, $1 > \lambda > 0$ and

$$\mathbf{x}_0 + \mathbf{u}_0 = \lambda\mathbf{v} + (1 - \lambda)\mathbf{w}$$

then

$$\begin{aligned} \mathbf{x}_0 \cdot \mathbf{a} &= (\mathbf{x}_0 + \mathbf{u}_0) \cdot \mathbf{a} \\ &= \lambda\mathbf{v} \cdot \mathbf{a} + (1 - \lambda)\mathbf{w}) \cdot \mathbf{a} \\ &\geq \lambda\mathbf{x}_0 \cdot \mathbf{a} + (1 - \lambda)\mathbf{x}_0 \cdot \mathbf{a} \\ &= \mathbf{x}_0 \cdot \mathbf{a}. \end{aligned}$$

The inequality in the last set of equations must thus be replaced by equality and so $\mathbf{v}$, $\mathbf{w} \in K'$. Since $\mathbf{u}_0$ is an extreme point of K', $\mathbf{v} = \mathbf{w} = \mathbf{u}_0$.

(iv) Use a similar argument to (ii) (or (ii) itself) to show that extreme points of

$$\{\mathbf{x} \in K \,:\, T(\mathbf{x}) = T(\mathbf{x}_0)\},$$

are extreme points of K.

K34*

No comments.

K35*

No comments

K36

(i) If neither $\mathcal{K}_1$ nor $\mathcal{K}_2$ have the finite intersection property, we can find $K_1,\ K_2,\ \ldots\ K_N,\ K_{N+1},\ \ldots,\ K_{N+M} \in \mathcal{K}$ such that

$$\bigcap_{j=1}^{N} K_j \cap [a,c] = \emptyset \text{ and } \bigcap_{j=N+1}^{M} K_j \cap [c,b] = \emptyset$$

and so

$$\bigcap_{j=1}^{N+M} K_j \cap [a,b] = \emptyset.$$

(ii) We find $[a_n, b_n]$ such that

$$\mathcal{L}_n = \{K \cap [a_n, b_n] \,:\, K \in \mathcal{K}\}$$

has the finite intersection property

$$a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_2 \leq b_1 \leq b_0 = b$$

and $b_n - a_n = 2^{-n}(b-a)$. We have $a_n,\ b_n \to \alpha$ for some $\alpha \in [a,b]$.

We claim that $\alpha \in K$ for each $K \in \mathcal{K}$. For, if not, we can find $K_0 \in \mathcal{K}$ such that $\alpha \notin K_0$. Since K_0 is closed, we can find a $\delta > 0$ such that $(\alpha-\delta, \alpha+\delta) \cap K_0 = \emptyset$. If N is sufficiently large, $[a_N, b_N] \subseteq (\alpha-\delta, \alpha+\delta)$ so $[a_N, b_N] \cap K_0 = \emptyset$ contradicting the finite intersection property of $\mathcal{L}_N$.

Thus, by reductio ad absurdum, $\alpha \in \bigcap_{K \in \mathcal{K}} K$.

K37*

No comments

K38

Observe that, if $[a_j, b_j] \in \mathcal{K}$, then

$$\bigcap_{j=1}^{N} [a_j, b_j] = [\max_{1 \le j \le N} a_j, \min_{1 \le j \le N} b_j] \in \mathcal{K}$$

so, in particular, $\mathcal{K}$ has the finite intersection property.

If $c \in \bigcap_{K \in \mathcal{K}} K$ then $c \in [a, b]$, that is to say $a \le c \le b$ whenever $a \in E$ and $b \ge e$ for all $e \in E$. Thus c is a (and thus the) greatest lower bound for E.

K39

(i) $f_1'(t) = 1 - \cos t \geq 0$ for all t. Thus f_1 is everywhere increasing. But $f_1(0) = 0$ so $f_1(t) \geq 0$ for $t \geq 0$ so $t \geq \sin t$ for $t \geq 0$.

(ii) $f_2'(t) = f_1(t) \geq 0$ for $t \geq 0$ and $f_2(0) = 0$.

(iv) We have

$$\sum_{j=0}^{2N} \frac{(-1)^j t^{2j+1}}{(2j+1)!} \geq \sin t \geq \sum_{j=0}^{2N+1} \frac{(-1)^j t^{2j+1}}{(2j+1)!}$$

for $t \geq 0$ and, using the fact that $\sin(-t) = -\sin t$,

$$\sum_{j=0}^{2N} \frac{(-1)^j t^{2j+1}}{(2j+1)!} \leq \sin t \leq \sum_{j=0}^{2N+1} \frac{(-1)^j t^{2j+1}}{(2j+1)!}$$

for $t \leq 0$.

(v) Thus

$$\left| \sin t - \sum_{j=0}^{N} \frac{(-1)^j t^{2j+1}}{(2j+1)!} \right| \leq \frac{|t|^{N+1}}{(2N+3)!} \to 0$$

as $N \to \infty$.

K40

(i) Observe that g' is increasing since g'' is positive. If $g'(x_1) > 0$ then $g'(t) \geq g'(x_1) > 0$ for all $t \in [x_1, x_2]$ so $0 = g(x_2) > g(x_1) = 0$ which is absurd. Thus $g'(x_1) \leq 0$ and $g'(x_2) \geq 0$. By the intermediate value theorem there exist a $c \in [x_1, x_2]$ such that $g'(c) = 0$. Since g' is increasing $g'(t) \leq 0$ and g is decreasing on $[x_1, c]$ whilst $g'(t) \geq 0$ and g is increasing on $[c, x_2]$. Thus $g(t) \leq g(x_1) = 0$ on $[x_1, c]$ and $g(t) \leq g(x_2) = 0$. We use a similar argument to get the result on $[c, x_2]$.

(ii) Look at $g(t) = f(t) - A - Bt$ with A and B chosen to make the hypotheses of (i) hold.

(iii) We may assume that $1 > \lambda_{n+1} > 0$. The key algebraic manipulation in a proof by induction runs as follows.

$$\begin{aligned}
f\left(\sum_{j=1}^{n+1} \lambda_j x_j\right) &= f\left(\lambda_{n+1}x_{n+1} + (1-\lambda_{n+1})\sum_{j=1}^{n} \frac{\lambda_j}{\sum_{k=1}^{n}\lambda_k} x_j\right) \\
&\leq \lambda_{n+1} f(x_{n+1}) + (1-\lambda_{n+1}) f\left(\sum_{j=1}^{n} \frac{\lambda_j}{\sum_{k=1}^{n}\lambda_k} x_j\right) \\
&\leq \lambda_{n+1} f(x_{n+1}) + (1-\lambda_{n+1}) \sum_{j=1}^{n} \frac{\lambda_j}{\sum_{k=1}^{n}\lambda_k} f(x_j) \\
&= \sum_{j=1}^{n+1} \lambda_j f(x_j)
\end{aligned}$$

since

$$\sum_{j=1}^{n} \frac{\lambda_j}{\sum_{k=1}^{n}\lambda_k} = 1 \text{ and } \sum_{k=1}^{n} \lambda_k = (1-\lambda_{n+1}).$$

(iv) Apply Jensen as suggested and take exponentials.

K41

The area of the inscribed quadrilateral is

$$\alpha = \sum_{j=1}^{4} a^2 \sin\theta_j \cos\theta_j = \tfrac{1}{2}\sum_{j=1}^{4} a^2 \sin 2\theta_j.$$

with a the radius of the circle. Also $\sum_{j=1}^{4} \theta_j = \pi$.

Now $\sin'' t = -\sin t \leq 0$ for $t \in [0,\pi]$, so $-\sin$ is convex on $[0,\pi]$ and Jensen's inequality gives

$$\alpha = 2a^2 \sum_{j=1}^{4} \tfrac{1}{4}\sin 2\theta_j \leq 2a^2 \sin\left(\sum_{j=1}^{4} \tfrac{1}{4}(2\theta_j)\right) = 2a^2 \sin(\pi/2) = 2a^2.$$

The area is attained when the θ_j are all equal (and with a little thought, observing that $\sin'' t < 0$ on $(0,\pi)$) only then.

Consider a circumscribing n-gon $A_1A_2\ldots A_n$. If A_jA_{j+1} touches the circle at X_j let ϕ_{2j-1} be the angle $\angle A_{j-1}OX_j$ and ϕ_{2j} be the angle $\angle X_jOA_j$. Then $\sum_{r=1} 2n\phi_r = 2\pi$ and the area of the circumscribed polygon is

$$\beta = 2^{-1}a^2 \sum_{r=1}^{2n} \tan\phi_r \geq 2^{-1}a^2 2n \tan\left(\sum_{r=1}^{2n} \phi_r/2n\right) = na^2\tan(\pi/n)$$

(since tan is convex on $(0,\pi/2)$). Equality is attained for a regular polygon.

K42

Since g is continuous on $[0, 1]$, it attains a maximum M say. Let $E = \{x \in [0, 1] : f(x) = M\}$. Since E is non-empty it has a supremum γ. Since f is continuous, $g(\gamma) = M$. If $\gamma = 0$ or $\gamma = 1$ then $M = 0$. If not then we can find a k with $\gamma - k, \gamma + k \in (0, 1)$ and

$$M = g(\gamma) = \tfrac{1}{2}(g(\gamma - k) + g(\gamma + k)) \geq M$$

with equality if and only if $g(\gamma - k) = g(\gamma + k) = M$. Thus $\gamma + k \in E$ contradicting the definition of γ. We have shown that $M = 0$ so $g(t) \leq 0$ for all t. Similarly $g(t) \geq 0$ for all t so $g = 0$.

Suppose that we drop the end conditions. By replacing $g(t)$ by $G(t) = g(t) - A - Bt$ with $G(0) = G(1) = 0$, we can show that g is linear.

[In higher dimensions the condition becomes 'the average of g over the surface of any sufficiently small sphere equals the value at the centre' and this turns out to be equivalent to saying that g satisfies Laplace's equation $\nabla^2 g = 0$. Observe that in one dimension this reduces to saying that $g'' = 0$ so g is linear as we saw above. However in higher dimensions the solutions of Laplace's equation are much more diverse.]

K43

Observe that

$$\left|\frac{f(h)-f(0)}{h}\right| = |h\sin h^{-4}| \le |h| \to 0$$

as $h \to 0$.

If $x \neq 0$, then $f'(x) = 2x\sin x^{-4} - 4x^{-2}\cos x^{-4}$. Thus we have $f\big(((2n+1)\pi)^{-1/4}\big) \to \infty$ as $n \to \infty$.

K44

Since g is continuous on $[a, b]$, it attains a maximum at some $c \in [a, b]$. Since $g'(c) = 0$, we have $c \in (a, b)$ and $f'(c) = k$.

For the final paragraph, note that $f''(a) \leq 0 \leq f''(b)$.

K45

(i) The statement 'f must jump by at least 1 at z' is not well defined. If the reader feels that there should be some way of making this statement well defined in a useful way she should consider $f(x) = 2^{-1}\sin(1/x)$ for $x \neq 0$, $f(0) = 0$.

However f can not be continuous at z. Observe that

$$\max(|f(z) - f(x_n)|, |f(z) - f(y_n|) \\ \geq \tfrac{1}{2}(|f(z) - f(x_n)| + |f(z) - f(y_n|) \geq \tfrac{1}{2}.$$

Thus we can find $z_n \to z$ with $|f(z) - f(Z_n)| \geq 1/2$.

(ii) The behaviour of $f'(z_{n(j)})$ does not control the value of $f'(z)$. Proposed result is false, see K43.

K46*

No comments.

K47

Since g' is constant, the conclusions of intermediate value theorem hold trivially for g'.

K48

Observe that

$$\begin{aligned}\cos n\theta &= \Re(\cos\theta + i\sin\theta)^n\\ &= \sum_{0\le 2r\le n}\binom{n}{2r}(-1)^r\cos^{n-2r}\theta\sin^{2r}\theta\\ &= \sum_{0\le 2r\le n}\binom{n}{2r}(-1)^r\cos^{n-2r}(1-\cos^2\theta)^r\end{aligned}$$

a real polynomial in $\cos\theta$ of degree at most n. (Part (ii) shows that the degree is exactly n.)

$T_0(t) = 1$, $T_1(t) = t$, $T_2(t) = 2t^2 - 1$, $T_3(t) = 4t^3 - 3t$.

(a) Observe that

$$\cos n\theta\cos\theta = (\cos(n+1)\theta + \cos(n-1)\theta)/2.$$

Thus $tT_n(t) = (T_{n+1}(t) + T_{n-1}(t))/2$ for all $t \in [-1, 1]$.

(b) But a polynomial of degree at most $n+1$ which vanishes at $n+2$ points is identically zero, so $tT_n(t) = (T_{n+1}(t) + T_{n-1}(t))/2$ for all t.

(c) Induction using (i).

(d) $|T_n(\cos\theta)| = |\cos n\theta| \le 1$.

(e) T_n vanishes at $\cos((r + 1/2)\pi/n)$ with $0 \le r \le n - 1$.

For the last part make the change of variables $x = \cos\theta$ with range $0 \le \theta \le \pi$.

If $m = 0$ replace $\pi/2$ by π.

K49

Two polynomials with the same roots and the same leading coefficient are equal.

Use the fact that $|T_n(t)| \leq 1$ for $T \in [-1, 1]$.

Choose α and β so that $-\alpha + \beta = -1$, $\alpha + \beta = 1$. If P is the interpolating polynomial of degree n for f on $[a, b]$, then taking g and Q such that

$$g(t) = f(\alpha t + \beta),\ Q(t) = P(\alpha t + \beta),$$

we see that Q is the interpolating polynomial of degree n for g on $[-1, 1]$. If $|f^n(x)| \leq A$ on $[a, b]$ then $|g^n(x)| \leq A\alpha^n = A((b-a)/2)^n$ on $[-1, 1]$ so

$$|f(t) - P(t)| \leq \frac{(b-a)^n A}{2^{2n-1}}$$

on $[a, b]$.

K50

(ii) $P(t) = \sum_{j=0}^{n} \frac{f^j(a)}{j!} t^j$.

(vii) Observe that
$$\frac{f(t)}{g(t)} = \frac{f^{(n+1)}(\theta_t)}{g^{(n+1)}(\phi_t)}$$
with θ_t, $\phi_t \to a$ as $t \to a$ and use continuity.

K51

If we set

$$P(t) = \sum_{j=0}^{n} A_j t^j (1-t)^{n+1} + \sum_{j=0}^{n} B_j t^{n+1} (1-t)^j,$$

then the resulting system of equations for A_j is triangular with non-vanishing diagonal entries and thus soluble. The same holds for B_j.

If $y \in (0,1)$, set

$$F(x) = f(x) - P(x) - E(y)\frac{x^{n+1}(1-x)^{n+1}}{y^{n+1}(1-y)^{n+1}}.$$

F has vanishing r-th derivative at 0 and 1 for $0 \leq r \leq n$ and vanishes at y. By Rolle's theorem F' vanishes at least twice in $(0,1)$, F'' vanishes at least three times in $(0,1)$, F''' vanishes at least four times in $(0,1)$, $\ldots$, $F^{(n+1)}$ vanishes at least $n+2$ times in $(0,1)$, $F^{(n+2)}$ vanishes at least $n+1$ times in $(0,1)$, $F^{(n+3)}$ vanishes at least n times in $(0,1)$, $\ldots$, F^{2n+2} vanishes at least once (at ξ say) and this gives the result.

K52

(i) $g(b) - g(a) = g'(c)(b-a) \neq 0$ for some $c \in (a,b)$.

(ii) $A = (f(b) - f(a))/(g(b) - g(a))$.

(iv) Observe that
$$\frac{f(t) - f(a)}{g(t) - g(a)} = \frac{f'(c_t)}{g'(c_t)}$$
with $c_t \to a$ as $t \to a$.

K53

(i) We have

$$f(b) = f(a) + f'(a)h + f''(c)\frac{h^2}{2!}$$

for some $c \in (a, b)$.

(ii) Thus

$$f'(a)h = f(b) - f(a) - f''(c)\frac{h^2}{2!}$$

and so

$$|f'(a)||h| \leq 2M_0 + 2^{-1}M_2|h|^2$$

for all a and all h. Thus

$$|f'(t)| \leq \frac{2M_0}{h} + \frac{M_2}{2h}$$

for all $h > 0$ and, choosing $h = 2(M_0M_1)^{1/2}$, we have the required result.

(iii) Take $f(t) = t$ to see that (a) and (b) are false.

(c) is true by the method of (ii), provided that $L \geq 2(M_0M_1)^{1/2}$.

(d) True. $g(t) = \sin t$.

(e) True. Scaling (d), $G(t) = M_0 \sin(M_0^{-1/2}M_2^{1/2}t)$ will do.

(f) Fails scaling. If G as in (e) then

$$\frac{G'(0)}{(M_0M_2)^{1/4}} = (M_0M_1)^{1/4}$$

can be made as large as we wish.

K54

(i) True. Choose a $\delta > 0$ such that $|f(t) - f(s)| < 1$ whenever $|t - s| < 2\delta$ and an integer $N > 2 + \delta^{-1}$. If $x \in (0,1)$ we can find $M \leq N$ and $x_1,\ x_2,\ \ldots,\ x_M \in (0,1)$ such that $x_1 = x$, $x_M = 1/2$, $|x_{j+1} - x_j| \leq \delta$ for $1 \leq j \leq M-1$. Since $|f(x_{j+1}) - f(x_j)| \leq 1$ we have $|f(x) - f(1/2)| \leq N$ and $|f(x)| \leq N + |f(1/2)|$.

(ii) False. If $f(t) = t$ then f does not attain its bounds.

(iii) False. Set $f(t) = \sin(1/t)$.

K55

(i) True. By the intermediate value theorem, we see that $f(x) \to l$ for some l as $|x| \to \infty$. Given $\epsilon > 0$, we can find a K such that $|f(x) - l| < \epsilon/2$ for all $|x| \geq K$. Since a continuous function on a closed bounded set is uniformly continuous, we can find a δ with $1 > \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all x, $y \in [-K-2, K+2]$ with $|x-y| < \delta$. By going through cases, we see that $|f(x) - f(y)| < \epsilon$ for all x, $y \in \mathbb{R}$ with $|x - y| < \delta$.

(ii) False. Take $f(t) = \sin t$.

(iii) False. Take $f(t) = \sin t^2$.

(iv) False. Take $f(t) = t$.

(v) False. Take $f(z) = \exp(i|z|)$.

(vi) True. Observe that $||f(z)| - |f(w)|| \leq |f(z) - f(w)|$.

(vii) True. Let $\epsilon > 0$. Choose $\delta > 0$ such that $|x - y| < \delta$ implies $||f(x)| - |f(y)|| < \epsilon/2$.

Now suppose $y > x$ and $|x - y| < \delta$. If $f(x)$ and $f(y)$ have the same sign, then automatically, $|f(x) - f(y)| < \epsilon/2$. If they have opposite sign, then by the intermediate value theorem, we can find u with $y > u > x$ and $f(u) = 0$. Since $|x - u|$, $|y - u| < \delta$ we have

$$|f(y) - f(x)| \leq |f(y) - f(u)| + |f(u) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

(vii) False. Take $f(x) = g(x) = x$. If $\delta > 0$ then, taking $y = 8\delta^{-1}$ and $x = y + \delta/2$, we have $|x - y| < \delta$ but $|x^2 - y^2| = (x + y)(x - y) > 1$ so $x \mapsto x^2$ is not uniformly continuous.

(viii) True. Let $\epsilon > 0$. We can find a $\delta > 0$ such that $|u - w| < \delta$ implies $|f(u) - f(w)| < \epsilon$ and an $\eta > 0$ such that $|x - y| < \eta$ implies $|g(x) - g(y)| < \delta$. Thus $|x - y| < \eta$ implies $|g(f(x)) - g(f(y))| < \epsilon$.

K56

Observe that

$$\|(\mathbf{x}+\mathbf{u})-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{u}\|$$

so that

$$f(\mathbf{x}+\mathbf{u}) \leq f(\mathbf{x})+\|\mathbf{u}\|.$$

Thus

$$\|f(\mathbf{a})-f(\mathbf{b})\| \leq \|\mathbf{a}-\mathbf{b}\|$$

and f is uniformly continuous.

If E is closed, then we can find $\mathbf{y}_n \in E$ such that $\|\mathbf{x}-\mathbf{y}_n\| \leq f(\mathbf{x})+1/n$. The $\mathbf{y}_n$ belong to the closed bounded set $E \cap \bar{B}(\mathbf{x}, 2)$ so we can find $n(j) \to \infty$ and a $\mathbf{y}$ with $\|\mathbf{y}_{n(j)}-\mathbf{y}\| \to 0$. Since E is closed, $\mathbf{y} \in E$. We observe that

$$\begin{aligned} f(\mathbf{x})+1/n(j) &\geq \|\mathbf{x}-\mathbf{y}_n\| \\ &\geq \|\mathbf{x}-\mathbf{y}\|-\|\mathbf{y}-\mathbf{y}_n\| \\ &\to \|\mathbf{x}-\mathbf{y}\| \geq f(\mathbf{x}) \end{aligned}$$

as $j \to \infty$. Thus $\|\mathbf{x}-\mathbf{y}\|=f(\mathbf{x})$.

Conversely, if the condition holds, suppose $\mathbf{y}_n \in E$ and $\|\mathbf{y}_n-\mathbf{x}\| \to 0$. We have $f(\mathbf{x})=0$ so there exists a $\mathbf{y} \in E$ with $\|\mathbf{x}-\mathbf{y}\|=f(\mathbf{x})=0$. We have $\mathbf{x}=\mathbf{y} \in E$ so E is closed.

K57

(i) Given $\epsilon > 0$ we can find a $\delta > 0$ such that, if $u, v \in \mathbb{Q}$, then $|u - v| < \delta$ implies $|f(u) - f(v)| < \epsilon$. We can now find an N such that $|x_n - x| < \delta/2$ for $n \geq N$. If $n, m \geq N$ then $|x_n - x_m| < \delta$ so $|f(x_n) - f(x_m)| < \epsilon$. Thus the sequence $f(x_n)$ is Cauchy and converges.

(ii) Using the notation of (i), we can find an M such that $|x_m - x| < \delta/2$ and $|y_m - x| < \delta/2$ for $m \geq M$. Thus $|x_m - y_m| < \delta$ and $|f(x_m) - f(y_m)| < \epsilon$ for $m \geq M$. Since ϵ was arbitrary, $f(x_m)$ and $f(y_m)$ tend to the same limit.

(iii) Observe that (ultimately by the Axiom of Archimedes) any $x \in \mathbb{R}$ is the limit of $x_n \in \mathbb{Q}$.

(iv) Take $x_n = x$.

(v) We use the notation of (i). If $|x-y| < \delta/3$ we can find $x_n, y_n \in \mathbb{Q}$ with $x_n \to x$, $y_n \to y$ and $|x_n - x| < \delta/3$, $|y_n - y| < \delta/3$. Then $|x_n - y_n| < \delta$ for all n so $|f(x_n) - f(y_n)| < \epsilon$ and $|F(x) - F(y)| \leq \epsilon$.

K58

(i) If $|z| < \min(R, S)$ then $\sum_{n=0}^N a_n z^n$ and $\sum_{n=0}^N b_n z^n$ tends to limit and so

$$\sum_{n=0}^N (a_n + b_n) z^n = \sum_{n=0}^N a_n z^n + \sum_{n=0}^N b_n z^n$$

tends to limit as $N \to \infty$.

Suppose $R < S$ If $R < |z| < S$ then $\sum_{n=0}^\infty a_n z^n$ converges and $\sum_{n=0}^\infty b_n z^n$ diverges so $\sum_{n=0}^\infty (a_n + b_n) z^n$ diverges. Thus the sum has radius of convergence R.

(ii) The argument of the first paragraph of (i) still applies.

(iii) We deal with the case $\infty > T \geq R > 0$. Let $a_n = R^{-n} + T^{-n}$ and $b_n = -R^{-n}$.

(iv) The radius of convergence is R unless $\lambda = 0$ in which case it is infinite.

(v) If $|z| > 1$ then, provided n is sufficiently large, $|z|^{n^2} \geq (2R)^n$ so $\sum_{n=0}^\infty a_n z^n$ diverges. If $|z| < 1$ then, provided n is sufficiently large, $|z|^{n^2} \leq (R/2)^n$ so $\sum_{n=0}^\infty a_n z^n$ converges. The radius of convergence is 1.

If $R = 0$, then any radius of convergence ρ with $0 \leq \rho \leq 1$ is possible. For $\rho = 0$ take $a_n = n^{n^n}$. For $0 < \rho < 1$ take $a_n = \rho^{-n^2}$. For $\rho = 1$ take $a_n = \rho_n^{-n^2}$ where ρ_n tends to 1 very slowly from below.

Similar ideas for $R = \infty$.

(vi) Let $|c_n| = \max(|a_n|, |b_n|)$. Since $\sum_{j=0}^n |c_n z^n| \geq \sum_{j=0}^n |a_n z^n|$ radius of convergence R' at most R. Thus $R' \leq \min(R, S)$. But $|c_n| \leq |a_n| + |b_n|$ so $R' \geq \min(R, S)$. Thus $R' = \min(R, S)$.

If $|c_n| = \min(|a_n|, |b_n|)$ then similar arguments show that the radius of convergence $R'' \geq \max(R, S)$. But every value $A \geq \max(R, S)$ is possible for R''. (If $A < \infty$ and R, $S > 0$ can take eg. $a_{2n} = R^{-2n}$, $b_{2n} = A^{-2n}$, $a_{2n+1} = A^{-2n-1}$, $b_{2n} = S^{-2n-1}$.

K59

Let $\epsilon > 0$. We can find an N such that $|h(n) - \gamma| < \epsilon$ for $n \geq N$.

Observe that if $n \geq N$ then $h(n) < \gamma + \epsilon$ and so

$$a_n > 2^{\left(-(\gamma+\epsilon)+(1-(\gamma+\epsilon)\right)n} = 2^{1-2(\gamma+\epsilon)n}.$$

Thus if $|z| > 2^{2(\gamma+\epsilon)-1}$ we have $|a_n z^n| \to \infty$. Thus the radius of convergence R satisfies $R \leq 2^{2(\gamma+\epsilon)-1}$. Since ϵ was arbitrary $R \leq 2^{2\gamma-1}$. A similar argument shows that, if $|z| < 2^{2(\gamma-\epsilon)-1}$ $|a_n z^n| \to 0$, and thus $R \geq 2^{2\gamma-1}$ so $R = 2^{2\gamma-1}$.

Suppose $1 \geq \gamma \geq 0$. Define E inductively by taking $a_{n+1} = a_n/2$ if $a_n/2 \geq 2^{1-2\gamma}$, $a_{n+1} = 2a_n$ otherwise. Then $h(n) \to \gamma$ and $|a_n z^n| \geq 1$ whenever $|z| = 2^{2\gamma-1}$ and n is large. Thus we have divergence everywhere on the circle of convergence.

Suppose $1 \geq \gamma \geq 0$. Define E inductively by taking $a_{n+1} = 2a_n$ if $2a_n \leq 2^{(1-2\gamma)n}n^{-2}$, $a_{n+1} = a_n/2$ otherwise. Then $|a_n z^n| \leq n^{-2}$ whenever $|z| = 2^{2\gamma-1}$ and n is large. Thus we have convergence everywhere on the circle of convergence. When n is large we have

$$a_n \leq 4 \times 2^{(1-2\gamma)n}n^{-2}$$

so

$$2^{n-2h(n)n} \leq 2^{2+(1-2\gamma)n}n^{-2}$$

whence

$$(\log 2)(n - 2h(n)n) \leq (\log 2)(2 + (1-2\gamma)n - 2\log n$$

and dividing by $n \log 2$ and allowing $n \to \infty$ we see that

$$1 - 2\limsup_{n\to\infty} h(n) = 1 - 2\gamma$$

so $\limsup_{n\to\infty} h(n) = \gamma$. A similar argument (but there are alternative ways of proceeding) shows that $\liminf_{n\to\infty} h(n) = \gamma$ so $h(n) \to \gamma$.

K60

If $|a_n|^{1/n}$ is unbounded, the radius of convergence is zero.

We deal with the case $(\limsup_{n\to\infty} |a_n|^{1/n})^{-1} = R$ with $0 < R < \infty$. The other cases are similar.

Suppose $R > \epsilon > 0$. Then we can find an N such that $|a_n|^{1/n} \leq (R - \epsilon/2)^{-1}$ for all $n \geq N$. Thus if $|z| < R - \epsilon$ we have

$$|a_n z^n| < \left(\frac{R-\epsilon}{R-\epsilon/2}\right)^n$$

for $n \geq N$ and $\sum_{n=0}^{\infty} |a_n z^n|$ converges by comparison with a convergent geometric sum.

We can also find $n(j) \to \infty$ such that $|a_n|^{1/n(j)} \geq (R+\epsilon/2)^{-1}$. Thus, if $|z| > R + \epsilon$ we have

$$|a_n z^{n(j)}| > \left(\frac{R+\epsilon}{R+\epsilon/2}\right)^{n(j)} \to \infty.$$

Since ϵ was arbitrary, R is the radius of convergence as required.

K61

Write $\alpha = \limsup_{n\to\infty} a_{n+1}/a_n$. If $\epsilon > 0$, we can find an N with $a_{n+1}/a_n \leq \alpha + \epsilon$ for $n \geq N$. Thus, by induction, $a_n \leq A(\alpha+\epsilon)^n$ for $n \geq N$ where $A = a_N(\alpha+\epsilon)^{-N}$. Thus, if $n \geq N$,

$$a_n^{1/n} \leq A^{1/n}(\alpha+\epsilon) \to \alpha + \epsilon$$

as $n \to \infty$. Since ϵ was arbitrary

$$\limsup_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} a_{n+1}/a_n.$$

Remaining inequalities similar or simpler.

All 8 possibilities can arise. Here are d.

(1) Set $a_n = 1$.

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \liminf_{n\to\infty} a_n^{1/n} = \limsup_{n\to\infty} a_n^{1/n} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

(2) Set $a_1 = 1$, $a_{2^n+1}/a_{2^n} = 2$, $a_{r+1}/a_r = 1$ otherwise.

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \liminf_{n\to\infty} a_n^{1/n} = \limsup_{n\to\infty} a_n^{1/n} = 1 < 2 = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

(3) Set $a_1 = 1$, $a_{2^n+1}/a_{2^n} = 2$, $a_{2^n+2}/a_{2^n+1} = 2^{-1}$ for $n \geq 2$ $a_{r+1}/a_r = 1$ otherwise.

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 1/2 < \liminf_{n\to\infty} a_n^{1/n} = \limsup_{n\to\infty} a_n^{1/n} = 1 < 2 = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

(4) Set $a_1 = 1$, $a_{r+1}/a_r = 2$ if $2^{2^{2n}} \leq r < 2^{2^{2n+1}}$, $a_{r+1}/a_r = 1$ otherwise

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \liminf_{n\to\infty} a_n^{1/n} = 1 < 2 = \limsup_{n\to\infty} a_n^{1/n} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

(5) Set $a_1 = 1$, $a_{r+1}/a_r = 2$ if $2^{2^{2n}} + 1 \leq r < 2^{2^{2n+1}}$, $a_{r+1}/a_r = 4$ if $r = 2^{2^{2n}}$ $a_{r+1}/a_r = 1$ otherwise

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \liminf_{n\to\infty} a_n^{1/n} = 1 < 2 = \limsup_{n\to\infty} a_n^{1/n} < 4 = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

(6) Set $a_1 = 1$, $a_{r+1}/a_r = 2$ if $2^{2^{2n}} + 2 \leq r < 2^{2^{2n+1}}$, $a_{r+1}/a_r = 4$ if $r = 2^{2^{2n}}$, $a_{r+1}/a_r = 1/2$ if $r = 2^{2^{2n}} + 1$ $a_{r+1}/a_r = 1$ otherwise

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 2^{-1} < \liminf_{n\to\infty} a_n^{1/n} = 1 < 2 = \limsup_{n\to\infty} a_n^{1/n} < 4 = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Other two obtained similarly.

Note lim sup formula will always give radius of convergence. Ratio may not.

K62

(iii) Could write $b_n = c_n + d_n$ with $d_n = 0$ for n large and $|b_n| \leq \epsilon$. Thus $\limsup_{n\to\infty} |\mathcal{C}_n| \leq \epsilon$.

(v) If $b_j = (-1)^j$ limit is 0.

(viii) Pick $N(j)$, M_j, s_j and r_j inductively so that $N(0) = 0$, $M(0) = 1$, $s_0 = 1/2$. we pick r_{j+1} so that $0 < s_j < r_{j+1} < 1$ and $1 - r_{j+1}^{M(j)+1} < 2^{-j}$. Pick an $N(j+1) > M(j)$ such that $r_{j+1}^{N(j+1)} < 2^{-j}$. Pick s_{j+1} so that $0 < r_{j+1} < s_{j+1} < 1$ and $s_{j+1}^{N(j+1)} > 1 - 2^{-j-1}$. Pick $M(j+1) > N(j+1)$ so that $s_{j+1}^{M(j+1)} < 2^{-j-1}$ Set $b_n = 1$ if $N(j) \leq n \leq M(j)$, $b_n = 0$ otherwise.

$$\begin{aligned}
\mathcal{A}_{r(j+1)} &\leq \sum_{n=0}^{M(j)} (1 - r_{j+1}) r_{j+1}^n + \sum_{n=N(j+1)}^{\infty} (1 - r_{j+1}) r_{j+1}^n \\
&= (1 - r_{j+1}^{M(j)+1}) + r_{j+1}^{N(j)} < 2^{-j} + 2^{-j} = 2^{-j+1}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_{s(j+1)} &\geq \sum_{n=N(j+1)}^{M(j+1)} (1 - s_{j+1}) s_{j+1}^n \\
&= s_{j+1}^{N(j+1)} - s_{j+1}^{M(j+1)+1} > 1 - 2^{-j-1} - 2^{-j-1} = 1 - 2^{-j}.
\end{aligned}$$

Thus $\mathcal{A}_r$ does not converge.

K63

(ii) $\mathbb{C}$ can be identified with $\mathbb{R}^2$.

(iii) (a) If $z \neq 1$, $\sum_{n=0}^{N} z^n = (1 - z^{N+1})/(1 - z)$ tends to a limit if and only if $|z| < 1$. The limit is $1/(1-z)$. If $z = 1$ we have divergence.

(b) Observe that, if $z \neq 1$,

$$\begin{aligned}
\mathcal{C}_n &= (n+1)^{-1} \sum_{k=0}^{n} \sum_{j=0}^{k} z^j \\
&= (n+1)^{-1} \sum_{k=0}^{n} \frac{1 - z^{k+1}}{1 - z} \\
&= \frac{1}{n+1} \left((n+1) + z \frac{1 - z^{n+1}}{1 - z} \right) \frac{1}{1 - z} \\
&= \frac{1}{1 - z} + \frac{z}{n+1} \frac{1 - z^{n+1}}{1 - z}.
\end{aligned}$$

Thus we have convergence if and only if $|z| \leq 1$, $z \neq 1$ and the limit is $1/(1 - z)$.

(c) $\mathcal{A}_r = (1 - rz)^{-1}$. Abel sum exists if and only if $|z| \leq 1$, $z \neq 1$ and is then $1/(1 - z)$.

K64

(v) If $\lambda_k \to \infty$ then, using part (i), $\mathcal{B}_{\lambda_k} \to b$. Since this is true for every such sequence, $\mathcal{B}_\lambda \to b$ as $\lambda \to \infty$.

(vi) We seek to Borel sum z^j. Observe that, if $z \neq 1$

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}(1 - z e^{-\lambda(1-z)}).$$

We have convergence if and only if $\Re(1 - z) < 1$ and the Borel sum is then $1/(1 - z)$.

If $z = 1$,

$$\mathcal{B}_\lambda = \sum_{n=0}^{\infty}(n+1)\frac{\lambda^n}{n!}e^{-\lambda} = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda} + \lambda \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda} = 1 - \lambda \to \infty.$$

K65

(i) True. Since the sequence is bounded, there exists a b and a strictly increasing sequence $n(j)$ such that $b_{n(j)} \to b$. Set $u_{jn(j)} = 1$, $u_{jk} = 0$ otherwise.

(ii) True. We can extract subsequences with different limits and use (i).

(iii) False. Let $b_j = 1$ for $2^n - n \leq j \leq 2^n - 1$, $b_j = -1$ for $2^n + 1 \leq j \leq 2^n + n$ $[n \geq 4]$, $b_j = 0$ otherwise Observe that, if $u_{nk} = 1$ for $2^n - N \leq k \leq 2^n$, $u_{nk} = -1$ for $2^n + 1 \leq k \leq 2^n + N$, then $U \in G$ but $\sum_{k=0}^{\infty} u_{jk} b_k = 2N$ for j sufficiently large. But N is arbitrary.

(iv) True. Define $N(r)$ and $j(r)$ inductively as follows. Set $N(0) = j(0) = 1$. Choose $N(r+1) > N(r)$ such that $\sum_{k=N(r+1)}^{\infty} |u_{j(r)k}| < 2^{-r}$ and $j(r+1) > j(r)$ such that $\sum_{k=0}^{N(r+1)} |u_{j(r+1)k}| < 2^{-r-1}$. Set $b_k = (-1)^{r+1}$ for $N(r) \leq k < N(r+1)$. Then

$$
\begin{aligned}
(-1)^{r+1} & \sum_{k=0}^{\infty} u_{j(r)k} b_k \\
&= (-1)^{r+1} \left(\sum_{k=0}^{N(r)} u_{j(r)k} b_k + \sum_{k=N(r)+1}^{N(r+1)} u_{j(r)k} b_k + \sum_{k=N(r+1)+1}^{\infty} u_{j(r)k} b_k \right) \\
&\leq \sum_{k=0}^{\infty} u_{j(r)k} - 2 \sum_{k=0}^{N(r)} |u_{j(r)k}| - 2 \sum_{k=N(r+1)+1}^{\infty} |u_{j(r)k}| \\
&\leq \sum_{k=0}^{\infty} u_{j(r)k} - 2^{-r+3} \to 1.
\end{aligned}
$$

as $r \to \infty$.

(v) False. Try $b_j = 0$.

K66

(i) Try $b_k = 1$ for $k = N$, $b_k = 0$ otherwise. Try $b_k = 1$ for all k.

K67

(i) implies (ii) since absolute convergence implies convergence.

For the converse observe that, writing $\mathbf{x}_n = (x_{n1}, x_{n2}, \dots, x_{nm})$, we have

$$\|\mathbf{x}_j\| \leq |x_{j1}| + |x_{j2}| + \cdots + |x_{jm}|.$$

Thus, if $\sum_{j=1}^{\infty} \|\mathbf{x}_j\|$ diverges, we must be able to find a k with $1 \leq k \leq m$ such that $\sum_{j=1}^{\infty} |x_{jk}|$ diverges. Choose ϵ_j so that $\epsilon_j x_{jk} \geq 0$. Then

$$\left\| \sum_{j=1}^{N} \epsilon_j \mathbf{x}_j \right\| \geq \left| \sum_{j=1}^{N} \epsilon_j x_{jk} \right| = \sum_{j=1}^{N} |x_{jk}| \to \infty.$$

as $N \to \infty$.

K68

If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\sum_{n\in S_1, n\leq N} a_n \leq \sum_{n\leq N} a_n \leq \sum_{n=1}^{\infty} a_n$$

and, since an increasing sequence bounded above converges, $\sum_{n\in S_1, n\leq N} a_n$ tends to a limit as $N \to \infty$.

If $\sum_{n\in S_j, n\leq N} a_n$ tends to a limit so does

$$\sum_{n\leq N} a_n = \sum_{n\in S_2, n\leq N} a_n + \sum_{n\in S_2, n\leq N} a_n.$$

Second paragraph does not depend on positivity a_j so 'if' remains true. However, taking $a_{2n-1} = -a_{2n} = 1/n$, S_1 evens and S_2 odds then we see 'only if' false.

Let $S_1 = \{n : a_n^{1/2}/n \leq a_n\}$ and $S_2 = \{n : a_n^{1/2}/n > a_n\}$. $\sum_{n\in S_1} a_n^{1/2}/n$ converges by comparison with $\sum_{n=1}^{\infty} a_n$. If $n \in S_2$ then

$$a_n = (a_n^{1/2})^2 < (n^{-1})^2 = n^{-2}$$

so, since $\sum_{n=1}^{\infty} n^{-2}$ converges, so does $\sum_{n\in S_2} a_n^{1/2}/n$. Hence the result.

By Cauchy Schwarz,

$$\left(\sum_{n=1}^{N} \frac{a_n^{1/2}}{n}\right)^2 \leq \sum_{n=1}^{N} a_n \sum_{n=1}^{N} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} \frac{1}{n^2}$$

so, since an increasing sequence bounded above converges, we are done.

Observe that $c_n \geq (a_n + b_n)/2$ and use comparison.

K69

(i) Diverges
$$\sum_{n=1}^{2N}\frac{1+(-1)^n}{n}=\sum_{n=1}^{N}\frac{1}{n}\to\infty$$
as $N\to\infty$.

(ii) Converges by comparison since $p_n^{-2}\leq n^{-2}$.

(iii) Diverges by comparison. We have $n^{1/n}\to 1$ (e.g. by taking logarithms) so $n^{-1/n}\geq 1/2$ for n large and so $n^{-1-1/n}>2^{-1}n^{-1}$ for n large.

(iv) If $\sum_{n=1}^{\infty}a_n$ converges, then taking $N(j)=j$ we have that $\sum_{n=1}^{N(j)}a_n$ tends to a limit.

If $\sum_{n=1}^{N(j)}a_n$ tends to a limit A, then for any M we can find a j with $N(j)>M$ so that
$$\sum_{n=1}^{M}a_n\leq\sum_{n=1}^{N(j)}a_n\leq A$$
so, since an increasing sequence bounded above converges, we are done.

(v) If $\int_1^\infty f(t)\,dt$ converges then
$$\sum_{n=1}^{N}a_n\leq\sum_{n=1}^{N}\int_n^{n+1}f(t)\,dt=\int_1^{N+1}f(t)\,dt\leq\int_0^\infty f(t)\,dt$$
since an increasing sequence bounded above converges $\sum_{n=1}^{\infty}a_n$ converges.

For the converse observe that if $\sum_{n=1}^{\infty}a_n$ converges then, if $X\leq N$,
$$\int_1^X f(t)\,dt\leq\int_1^N f(t)\,dt=\sum_{n=1}^{N-1}\int_n^{n+1}f(t)\,dt\leq K\sum_{n=1}^{N-1}a_n\leq\sum_{n=1}^{\infty}a_n.$$
Now use something like Lemma 9.4.

(vi) $a_n=(-1)^n$.

(vii) Let $\sum_{n=1}^{MN}\mathbf{a}_n$ tend to $\mathbf{b}$. Given $\epsilon>0$ we can find an N_0 such that $\|\sum_{n=1}^{MN}\mathbf{a}_n-\mathbf{b}\|\leq\epsilon/2$ for $N\geq N_0$. We can also find an N_1 such that $\|\mathbf{a}_n\|\leq\epsilon/2M$ for $n\geq N_1$. Let $N_2=(M+1)(N_1+N_0)$. If $m\geq N_2$ we can write $m=NM+u$ with $N\geq N_0$, $0\leq u\leq M-1$ and $MN+r\geq N_1$ for all $r\geq 0$. We then have
$$\|\sum_{n=1}^{m}\mathbf{a}_n-\mathbf{b}\|\leq\|\sum_{n=1}^{MN}\mathbf{a}_n-\mathbf{b}\|+\sum_{n=MN+1}^{MN+u}\|\mathbf{a}_n\|<\frac{\epsilon}{2}+\frac{u\epsilon}{2M}\leq\epsilon$$
as desired.

(viii) Set $N(j) = 2^{2j}$, $a_n = -j^{-1}$ if $2^{2j} \leq n < 2 \times 2^{2j}$, $a_n = j^{-1}$ if $2 \times 2^{2j} \leq n < 3 \times 2^{2j}$, $a_n = 0$ otherwise.

K70

(i) Observe that $(a_nb_n)^{1/2} \leq (a_n + b_n)/2$ and use comparison.

(ii) Take $b_n = a_{n+1}$ in (i).

(iii) Observe that $a_{n+1} \leq (a_na_{n+1})^{1/2}$ and use comparison.

(iv) $a_{2n} = 1$, $a_{2n+1} = n^{-4}$.

K71

(i) True. $a_n^4 \to 0$ so there exists an N with $|a_n| \leq 1$ for $n \geq N$. Thus $a_n^4 \geq |a_N^5|$ for $n \geq N$ and the result follows by comparison and the fact that absolute converge implies convergence.

(ii) False. Take $a_n = (-1)n^{-1/6}$ and use the alternating series test.

(iii) False. Take $a_{2^k} = 2^{-k}$, $a_r = 0$ otherwise.

(iv) False. Same counterexample as (iii).

(v) True. Given $\epsilon > 0$ we can find an N such that $\sum_{j=N}^{M} a_j < \epsilon/2$ for $M \geq N$. Thus, if $n \geq 2N + 2$

$$na_n \leq 2(n - N - 1)a_n \leq 2\sum_{j=N}^{M} a_j < \epsilon.$$

(vi) False. Take $a_n = (n \log n)^{-1}$ for $n \geq 3$ and use the integral comparison test.

(vii) True. Abel's test.

(viii) True. By Cauchy–Schwarz,

$$\left(\sum_{n=1}^{N} |a_n| n^{-3/4}\right)^2 \leq \sum_{n=1}^{N} |a_n|^2 \sum_{n=1}^{N} n^{-3/2} \leq \sum_{n=1}^{\infty} |a_n|^2 \sum_{n=1}^{\infty} n^{-3/2}.$$

(ix) False. Take $a_n = (-1)^n (\log n)^{-1}$ for $n \geq 3$. Use alternating series test and integral test.

(x) True. We must have $a_n \to 0$ so there exists a K such that $|a_n| \leq K$. Since $|n^{-5/4} a_n| \leq K n^{-5/4}$ the comparison test tells us that $\sum_{n=1}^{\infty} n^{-5/4} a_n$ is absolutely convergent and so convergent.

K72

$a_n^k \to 0$ so there exists an N with $a_n \leq 1$ for $n \geq N$. Thus $a_n^k \geq |a_n^{k+1}|$ for $n \geq N$ and the result follows by comparison.

Set $f(n) = \log(n+1)n$ and observe that, if $k \geq 2$

$$\sum_{j=1}^{3N} a_j^k \geq \sum_{j=1}^{N} (2^k - 2)(\log(n+1))^{-k}) \to \infty.$$

(Use comparison and the fact that $n \geq (\log n)^k$ for n sufficiently large.)

K73

(i) Write

$$\mathbf{S}_n = \sum_{j=1}^{n} \mu_j^{-1}\mathbf{b}_j.$$

Then, by partial summation (or use Exercise 5.17),

$$\begin{aligned}\sum_{j=1}^{n} \mathbf{b}_j &= \sum_{j=1}^{n} \mu_j(\mathbf{S}_j - \mathbf{S}_{j-1}) \\ &= \sum_{j=1}^{n-1} (\mu j - \mu_{j+1})\mathbf{S}_j + \mu_n \mathbf{S}_n\end{aligned}$$

so that

$$\begin{aligned}\left\|\sum_{j=1}^{n} \mathbf{b}_j\right\| &\leq \sum_{j=1}^{n-1} (\mu_{j+1} - \mu_j)\|\mathbf{S}_j\| + \mu_n\|\mathbf{S}_n\| \\ &\leq \sum_{j=1}^{n-1} (\mu_{j+1} - \mu_j)K + \mu_n K = (2\mu_n - \mu_1)K \leq 2k\mu_n\end{aligned}$$

whence the result.

(ii) If $\epsilon > 0$ then we can find an N such that $\| \sum_{j=N}^{n} \mu_j^{-1}\mathbf{b}_j\| \leq \epsilon$ for $n \geq N$. Thus, by (i),

$$\begin{aligned}\mu_n^{-1}\| \sum_{j=1}^{n} \mathbf{b}_j\| &\leq \mu_n^{-1} \sum_{j=1}^{N-1} \|\mathbf{b}_j\| + \mu_n^{-1}\| \sum_{j=N}^{n} \mathbf{b}_j\| \\ &\leq \mu_n^{-1} \sum_{j=1}^{N-1} \|\mathbf{b}_j\| 2\epsilon \to 2\epsilon\end{aligned}$$

as $n \to \infty$. Kronecker's lemma follows.

(iii) ((a) fails) Let $b_j = j^4$, $\mu_j = j^6$ for $j \neq 2^r$, $b_{2^r} = 1$ and $\mu_{2^r} = 2^{-2r}$.

((b) fails) Let $b_j = j^{-2}$, $\mu_j = 1$.

((c) fails) Let $b_j = j^4$, $\mu_j = j^2$.

K74

Consider the set A_n of integers between 10^n and $10^{n+1}-1$ inclusive who's decimal expansion does not contain the integer 9. We observe that $\{0\}\cup\bigcup_{j=0}^{n} A_j$ contains $(9/10)^{n+1}\times 10^{n+1} = 9^{n+1}$ elements. Thus A_n contains at most 9^{n+1} elements and

$$\sum_{j\in A_n} j^{-1} \leq 10^{-n}\times 9^{n+1}.$$

The partial sums of the given series thus can not exceed $\sum_{n=0}^{\infty} 10^{-n}\times 9^{n+1} = 90$ and the series converges.

K75

(i) Observe that $1/x \geq 1/(n+1)$ for $n \leq x \leq n+1$ and integrate.

$$T_n - T_{n+1} = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x}\,dx.$$

Observe that $T_1 = 1$ and

$$T_n \geq \sum_{r=1}^{n-1} \left(\frac{1}{r} - \int_r^{r+1} \frac{1}{x}\,dx \right).$$

Decreasing sequence bounded below tends to a limit.

(v) Given l we can choose integers $p_n, q_n \geq 1$ such that $p_{n+1} > 2p_n$, $q_{n+1} > 2q_n$, p_n/q_n is strictly decreasing and $\log 2 + (1/2)\log(p_n/q_n) \to l$. At the n-th stage, if the ratio of positive terms to negative exceeds p_n/q_n use p_n positive terms followed by $q_n + 1$ terms if the ratio is less than p_n/q_n, use $p_n + 1$ positive terms followed by q_n negative terms, otherwise usep_n positive terms followed by q_n negative terms. Switch from the n-th stage to the $n+1$-th stage when the size of each unused terms is less than $(p_{n+1} + q_{n+1} + 1)^{-1}2^{-n}$ and the ratio of positive terms to negative differs from p_n/q_n by at most 2^{-n}.

K76

(ii) Observe that, if $\sum_{j=0}^{\infty} b_j$ converges, we can find an $N(\epsilon)$ such that $|\sum_{j=M)}^{N} b_j \leq \epsilon|$ for $N \leq M \leq N(\epsilon)$ so

$$\left|\sum_{j=M}^{\infty} b_j x^j\right| \leq \epsilon x^M \leq \epsilon$$

for $M \geq N(\epsilon)$.

(iii) If $M \geq N(\epsilon)$, then

$$\begin{aligned}\left|\sum_{j=0}^{\infty} b_j x^j - \sum_{j=0}^{\infty} b_j\right| &\leq \sum_{j=0}^{M-1} |b_j x^j - b_j| + \left|\sum_{j=M}^{\infty} b_j\right| + \left|\sum_{j=M}^{\infty} b_j x^j\right| \\ &\leq \sum_{j=0}^{M-1} |b_j x^j - b_j| + 2\epsilon \to 2\epsilon\end{aligned}$$

as $x \to 1-$. Since ϵ is arbitrary the required result follows.

(iv) Observe that, if $0 < x < 1$, then $\sum_{j=0}^{\infty} a_j x^j$, $\sum_{j=0}^{\infty} b_j x^j$ and $\sum_{j=0}^{\infty} c_j x^j$ are absolutely convergent so (using Exercise 5.38) we may multiply them to get

$$\sum_{j=0}^{\infty} a_j x^j \sum_{j=0}^{\infty} b_j x^j = \sum_{j=0}^{\infty} c_j x^j.$$

Now let $x \to 1-$.

K77

(a) Since $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}x^n y^m$ converges for $|x| = |y| = \delta$, we have $|c_{n,m}|\delta^{n+m} = |c_{n,m}x^n y^m| \to 0$ for n, $m \to \infty$ so there exists a K with $|c_{n,m}|\delta^{n+m} < K$ for all n, $m \geq 0$. Set $\rho = \delta^{-1}$.

(b) (i) $\sum_{n\leq N,\ m\leq N} |x^n y^m| = \sum_{n\leq N} |x|^n \sum_{m\leq N} |y|^m$.

E is the square $(-1, 1) \times (-1, 1)$.

(ii) $\sum\sum_{n+m\leq N} \binom{n+m}{n}|x^n y^m| = \sum_{r=0}^{N}(|x| + |y|)^r$

E is the square $\{(x, y) : |x| + |y| < 1\}$.

(iii) $\sum\sum_{n+m\leq N} \binom{n+m}{n}|x^{2n}y^{2m}| = \sum_{r=0}^{N}(|x|^2 + |y|^2)^r$.

E is the circle $\{(x, y) : x^2 + y^2 < 1\}$.

(iv) $\sum_{n,\, m\leq N} |x^n y^m/(n!m!)| = \sum_{n\leq N} |x|^n/n! \sum_{m\leq N} |y|^m/m!$.

$E = \mathbb{R}^2$.

(v) $\sum_{n\, m\leq N} |x^n y^m n!m!| = \sum_{n\leq N} |x|^n n! \sum_{m\leq N} |y|^m m!$.

$E = \{0\}$.

(vi) $\sum_{n\leq N} |x^n y^n| = \sum_{n\leq N} |xy|^n$.

$E = \{(x, y) : |xy| < 1\}$ a set bounded by branches of hyperbola.

(c) Observe that $|c_{nm}x^n y^m| \leq |c_{nm}x_0^n y_0^m|$ when $|x| \leq |x_0|$, $|y| \leq |y_0|$.

K78

(i) Choose $N_j(\epsilon)$ so that $|a_j - a_{j,n}| < \epsilon/M$ for $n \geq N_j(\epsilon)$ and set $N(\epsilon, M) = \max_{1\leq j\leq M} N_j(\epsilon)$.

Observe that
$$\sum_{j=1}^{\infty} a_j \geq \sum_{j=1}^{M} a_j \geq \sum_{j=1}^{M} a_{j,n}$$
for all M.

(ii) If $\sum_{j=1}^{\infty} a_j$ converges with value A, then given $\epsilon > 0$ we can find an M such that $\sum_{j=1}^{M} a_j > A - \epsilon$. By (i) we can find $N(\epsilon, M)$ such that
$$\sum_{j=1}^{\infty} a_{j,n} \geq \sum_{j=1}^{M} a_j - \epsilon$$
and so
$$\sum_{j=1}^{\infty} a_j \geq \sum_{j=1}^{\infty} a_{j,n} \geq A - 2\epsilon$$
for all $n \geq N(\epsilon, M)$. Thus $\sum_{j=1}^{\infty} a_{j,n} \to A$.

If $\sum_{j=1}^{\infty} a_j$ fails to converge, then given any $K > 0$, we can find an M such that $\sum_{j=1}^{M} a_j > 2K$. But we can find an N such that $\sum_{j=1}^{M} a_{j,n} \geq \sum_{j=1}^{M} a_j - K$ for $n \geq N$ and so $\sum_{j=1}^{\infty} a_{j,n} \geq K$ for $n \geq N$. Thus $\sum_{j=1}^{\infty} a_{j,n}$ can not converge.

(iii) Yes we can drop the condition. Consider $b_{j,n} = a_{j,n} - a_{j,1}$ instead.

K79*

No comments

K80

It is possible to do this by fiddling with logarithms but I prefer to copy the proof of the radius of convergence. Show that if $\sum_{n=0}^{\infty} b_n e^{nz}$ converges then $\sum_{n=0} b_n e^{nw}$ converges absolutely when $\Re w < \Re z$.

Next observe that the sum $\sum_{n=0}^{\infty} b_n e^{nz}$ converges everywhere (set $X = -\infty$) or nowhere (set $X = \infty$) or we can find z_1 and z_2 such that $\sum_{n=0}^{\infty} b_n e^{nz_1}$ converges and $\sum_{n=0}^{\infty} b_n e^{nz_2}$ diverges. Explain why this means that

$$X = \sup\{\Re z : \sum_{n=0}^{\infty} b_n e^{nz} \text{ converges}\}$$

exists and has the desired properties.

For $X = -\infty$ can take $b_n = 0$, for $X = \infty$ can take $b_n = e^{n^2}$, for $X = R$ with R finite can take $b_n = e^{-Rn}$.

$\sum_{n=0}^{\infty} 2^n e^{nz}/(n+1)^2$ converges when $z = -\log 2$. Also $2^n e^{nx}/(n+1)^2 \to \infty$ when x is real and $x > -\log 2$ so $X = -\log 2$.

By the first paragraph there exists a $Y \geq 0$ such that $\sum_{n=0}^{\infty} c_n e^{inz}$ converges for $\Im z < Y$ and diverges for$\Im z > Y$. By taking complex conjugates we see that $\sum_{n=0}^{\infty} c_n e^{-inz}$ converges for $\Im z > -Y$ and diverges for $\Im z < -Y$.

Since $\sum_{n=0}^{\infty} s_n + t_n$ diverges if exactly one of $\sum_{n=0}^{\infty} s_n$ and $\sum_{n=0}^{\infty} t_n$ converges and converges if both converge, $\sum_{n=0}^{\infty} c_n(e^{inz} + e^{-inz})$ and thus $\sum_{n=0}^{\infty} c_n \cos nz$ converges if $\Im z < Y$ and diverges if $\Im z > Y$.

K81

(i) By integration by parts,

$$\begin{aligned} I_{n+1} &= [-\sin^n x \cos x]_0^{\pi/2} + n\int_0^{\pi/2} \sin^{n-1} x \cos^2 x\, dx \\ &= n\int_0^{\pi/2} \sin^{n-1} x(1-\sin^2 x)\, dx \\ &= nI_{n-1} - nI_{n+1}. \end{aligned}$$

(ii) Observe that $\sin^{n-1} x \leq \sin^n x \leq \sin^{n+1} x$ for $x \in [0, \pi/2]$. Integrating gives $I_{n+1} \leq I_n \leq I_{n-1}$ so

$$\frac{n}{n+1} = \frac{I_{n+1}}{I_{n-1}} \leq \frac{I_n}{I_{n-1}} \leq 1.$$

so $I_n/I_{n-1} \to 1$ as $n \to \infty$.

(iii) $I_0 = \pi/2$, $I_1 = 1$

$I_{2n} = \dfrac{(2n-1)(2n-3)\dots 1}{2n(2n-2)\dots 2}\dfrac{\pi}{2}$, and$I_{2n+1} = \dfrac{2n(2n-2)\dots 2}{(2n+1)(2n-1)\dots 1}$.

Thus

$$\prod_{k=1}^{n} \frac{4k^2}{4k^2-1}\frac{2}{\pi} = \frac{I_{2n+1}}{I_{2n}} \to 1$$

and the Wallis formula follows.

K82

(i) Power series or set $f(x) = \exp x - x$ and observe $f(0) = 0$, $f'(x) \geq 0$ for all $x \geq 0$.

(iii) Observe that

$$\begin{aligned}\left|\prod_{j=1}^{n}(1+a_j) - \prod_{j=1}^{m}(1+a_j)\right| &= \left|\prod_{j=1}^{n}(1+a_j)\right|\left|\prod_{j=n+1}^{m}(1+a_j) - 1\right| \\ &\leq \prod_{j=1}^{n}(1+|a_j|)\left(\prod_{j=n+1}^{m}(1+|a_j|) - 1\right).\end{aligned}$$

Now use the inequality $(1+|a_j|) \leq \exp|a_j|$.

(iv) Use the general principle of convergence and (iv).

(v) We must have $a_n \to 0$. If $|a_n| < 1/2$ then

$$|b_n| = \left|\frac{a_n}{1+a_n}\right| \leq 2|a_n|.$$

Thus $\sum_{n=1}^{\infty} b_n$ converges absolutely.

$$1 = \prod_{j=1}^{n}(1+a_j)\prod_{j=1}^{n}(1+b_j) \to \prod_{j=1}^{\infty}(1+a_j)\prod_{j=1}^{\infty}(1+b_j)$$

so $\prod_{j=1}^{\infty}(1+a_j)\prod_{j=1}^{\infty}(1+b_j) = 1$.

(vi) Observe that

$$\prod_{j=1}^{N}(1+a_j/R_j) = \prod_{j=1}^{N}\frac{R_{j-1}}{R_j} = \frac{R_0}{R_N} \to 0$$

as $N \to \infty$ and use (v).

K83

(i) Left to reader.

(ii) Let $a_n = n^{-\alpha}$, $a_{n,N} = n^{-\alpha}$ if all the prime factors of N lie among the first N primes, $a_{n,N} = 0$ otherwise. Observe that

$$0 \leq a_{n,N} \leq a_{n,N+1} \leq a_n,$$

that $a_{n,N} \to a_n$ as $N \to \infty$ and that $\sum_{n=1}^{\infty} a_n$ converges.

(iii) If

$$\prod_{p \in P,\ p \leq N} \frac{1}{1 - p^{-1}} \leq K$$

for all N then

$$\prod_{p \in P,\ p \leq N} \frac{1}{1 - p^{-\alpha}} \leq K$$

for all N and all $\alpha > 1$ so

$$\prod_{p \in P} \frac{1}{1 - p^{-\alpha}} \leq K$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \leq K$$

for all $\alpha > 1$. Thus

$$\sum_{n=1}^{N} \frac{1}{n^{\alpha}} \leq K$$

for all $\alpha > 1$ and all N so

$$\sum_{n=1}^{N} \frac{1}{n} \leq K$$

for all $\sum_{n=1}^{\infty} \frac{1}{n}$ which is absurd.

(iv) Let E be a set of positive integers. Write

$$E_n = \{r \in E \,:\, 2^n \leq r < 2^{n+1}\}.$$

If E_n has $M(n)$ elements and $M(n) \leq A2^{\beta n}$ then

$$\sum_{r \in E, r \leq 2^N} \frac{1}{r} = \sum_{j=0}^{N-1} \sum_{r \in E_j} \frac{1}{r} \leq \sum_{j=0}^{N-1} 2^{-j} M(j) \leq A \sum_{j=0}^{N-1} 2^{n(\beta-1)} = \frac{A}{1 - 2^{\beta-1}}$$

for all N.

K84

Suppose that $P_n > 1$ for all $n \geq N$. Then $P_m = P_N \prod_{j=N}^{m}(1 - a_n)$. If $\sum_{j=1}^{\infty} a_n$ diverges then (see K82) $P_m \to 0$ which is impossible by the definition of N. Thus $\sum_{j=1}^{\infty} a_n$ converges and P_m tends to a limit. A similar argument applies if $P_n \leq 1$ for all $n \geq N$.

Thus we need only consider the case $P_n - 1$ changes sign (or is zero) infinitely often. But $a_n \to 0$ and if (A) $P_{N-1} - 1 \leq 0 \leq P_N - 1$ or $P_{N-1} - 1 \geq 0 \geq P_N - 1$ and (B) $0 \leq a_n \leq \epsilon$ for $n \geq N - 1$ then we have $1 + \epsilon \geq P_n \geq 1 - \epsilon$ for $n \geq N$. Thus $P_n \to 1$.

If $\sum_{j=1}^{\infty} a_n$ diverges we can say nothing about l. (If $l \geq 1$ consider $a_1 = l - 1$, $a_n = 0$ for $n \geq 2$. If $l \geq 1$ consider $a_1 = 1$, $a_2 = 1 - l/2$, $a_n = 0$ for $n \geq 3$.)

K85

Let $a_n = z^n$, $a_{n,N} = z^n$ if $0 \leq n \leq 2^{N+1} - 1$ and $a_{n,N} = 0$, otherwise. Observe that $0 \leq |a_{n,N} \leq |a_n|$, and that $a_{n,N} \to a_n$ as $N \to \infty$. By dominated convergence (Lemma 5.25)

$$\prod_{j=1}^{N}(1 + z^{2^j}) = \sum_{n=0}^{\infty} a_{n,N} \to \sum_{n=0}^{\infty} a_n = (1-z)^{-1}.$$

K86

Observe that

$$2\cot 2\phi = \frac{\cos^2\phi - \sin^2\phi}{\cos\phi\sin\phi} = \cot\phi + \tan\phi$$

and use induction.

Observe that

$$\frac{1}{2^n}\cot\frac{\theta}{2^n} = \frac{1}{\theta}\frac{2^{-n}\theta}{\sin 2^{-n}\theta}\cos 2^{-n}\theta \to \frac{1}{\theta}$$

as $n \to \infty$.

K87

Very similar to K86. To obtain the formula for u_n, observe that $2\cos^2\phi = 1 + \cos 2\phi$ and $\cos(\pi/4) = 2^{-1/2}$.

K88

The square of a rational is rational.

(ii) Observe, by induction or Leibniz's rule, that $f^{(k)}$ is the sum of powers of x with integral coefficients plus terms of the form

$$M\frac{x^r(1-x)^s}{(\min(r,s))!}$$

with M an integer and r, $s \geq 1$. Thus $f^{(k)}(0)$ and $f^{(k)}(1)$ are always integers. Since $b^n\pi^{2n-r}$ is an integer, $G(0)+G(1)$ is always an integer.

$$0 < \pi a^n f(x)\sin \pi x \leq \frac{a^n}{n!} \to 0$$

as $n \to \infty$.

K89

(a)(i) Given $c \in C$ we can find $b \in B$ with $g(b) = c$ and $a \in A$ with $f(a) = b$.

(ii) $g \circ f(a) = g \circ f(a')$ implies $f(a) = f(a')$ and so $a = a'$.

(iii) $f(a) = f(a')$ implies $g \circ f(a) = g \circ f(a')$ and so $a = a'$.

(iv) Given $c \in C$, we can find $a \in A$ with $g \circ f(a) = b$. Observe that $f(a) \in B$ and $g\big(f(a)\big) = c$

(b) Let $A = \{a\}$, $B = \{a, b\}$, $C = \{a\}$ with $a \neq b$ $f(a) = a$, $g(a) = g(b) = a$.

K90

If f satisfies the conditions of the first sentence of the second paragraph, we can write $f = g \circ J$ where $J(x) = -x$ and g satisfies the conditions of the first paragraph. Thus

$$f'(x) = J'(x)g'(J(x)) = -g'(J(x)) < 0$$

for all $x \in E$.

If $ad - bc = 0$ then f is constant and $f' = 0$.

K91

(i) Suppose that f is continuous. If f is not strictly monotonic we can find $x_1 < x_2 < x_2$ so that $f(x_1)$, $f(x_2)$ and $f(x_3)$ are not a strictly increasing or a strictly decreasing sequence. Without loss of generality (or by considering $f(1-x)$ and $1-f(x)$ if necessary), we may suppose $f(x_3) \geq f(x_1) \geq f(x_2)$. If either of the inequalities are equalities then f is not injective. Thus we may suppose $f(x_3) > f(x_1) > f(x_2)$. But by the intermediate value theorem we can find $c \in (x_2, x_3)$ so that $f(c) = f(x_1)$ and f is not injective.

Suppose f is strictly monotonic. Without loss of generality we suppose f increasing. We shall show that f is continuous at $t \in (0,1)$ (the cases $t = 0$ and $t = 1$ are handled similarly). Observe that $f(1) > f(t) > f(0)$. Let $\epsilon > 0$. Set $\epsilon' = \min(1 - f(t), f(t), \epsilon)/2$. We can find s_1 and s_2 with $f(s_1) = f(t) - \epsilon'$ $f(s_2) = f(t) + \epsilon'$. If $\nabla/ta = \min(t - s_1, t + s_2)$, then $|s - t| > \delta$ implies $s_2 > s > s_1$ so $f(s_2) > f(s) > f(s_1)$ and $|f(s) - f(t)| \leq \epsilon' < \epsilon$.

(ii) The same proof as in (i) shows that, if g is continuous and injective, it must be strictly monotonic. The example of $g(t) = t/2$ for $t \leq 1/2$, $g(t) = t/2 + 1/2$ shows that the converse is false.

(iii) The same proof as in (i) shows that, if g is strictly monotonic and surjective, it must be continuous. The example $g(t) = \sin \pi t$ shows that the converse is false.

K92

Write $f(x) = (\log x)/x$. By looking at f' we see that f is strictly increasing from $-\infty$ to e^{-1} on the interval $(0, e]$, has a maximum value of e^{-1} at e and is strictly decreasing from e^{-1} to 0 on the interval $(0, \infty)$. The function f has a unique zero at 1,

We have $x^y = y^x$ if and only if $f(x) = f(y)$. The set

$$\{(x, y) : x^y = y^x, \ x, \, y > 0\}$$

is thus the union of the straight line $\{(x, x) : x > 0\}$ and a curve with reflection symmetry in that line, having asymptotes $x = 1$ and $y = 1$ and passing through (e, e).

If $f(x) = f(y)$ and $x < y$, then $x \leq e$ so we need only examine the cases $m \leq 2$ to see that the integer solutions of $n^m = m^n$ are $n = m$, $(m, n) = (2, 4)$ and $(m, n) = (4, 2)$.

Since $f(\pi) < f(e)$ we have $e^\pi > \pi^e$.

K93

Observe that

$$xy = \tfrac{1}{4}\big((x+y)^2 - (x-y)^2.$$

K94

(i) Choose γ with $\beta > \gamma > 1$. If we set $f(x) = x^\alpha \gamma^x$, then (recalling that $\gamma^x = \exp(x \log \gamma)$)

$$f'(x) = \alpha x^{\alpha-1}\gamma^x + x^\alpha(\log\gamma)\gamma^x = \alpha x^{\alpha-1}\gamma^x(\alpha + \gamma x) > 0$$

for x sufficiently large ($x \geq x_0$, say). Thus

$$x^\alpha \beta^x = f(x)(\beta/\gamma)^x \geq f(x_0)(\beta/\gamma)^x \to \infty.$$

(ii) Set

$$f_4(x) = \exp\left(\frac{1}{x}\right),\ f_3(x) = \left(\frac{1}{x}\right)^{1/2},\ f_2(x) = \exp\left\{\left(\log\frac{1}{x}\right)^{1/2}\right\}$$

and

$$f_1(x) = \left(\log\frac{1}{x}\right)^3.$$

Set $y = 1/x$ so $y \to \infty$ as $x \to 0+$.

Observe that $\log f_4(x)/\log f_3(x) = 2y/(\log y) \to \infty$ as $x \to \infty$.

Observe that $\log f_3(x)/\log f_2(x) = (\log y)^{1/2}/2 \to \infty$ as $x \to \infty$.

Observe that $\log f_2(x)/\log f_1(x) = (\log y)^{1/2}/(3\log\log y) \to \infty$ as $x \to \infty$.

K95

(ii) If Q has a root $\alpha \neq 0$ of order $n \geq 1$ then differentiating the given equation $n-1$ times and setting $z = \alpha$ we get $-P(\alpha)Q^{(n)}(\alpha) = 0$ so $P(\alpha) = 0$ and P and Q have a common factor. Thus $Q(z) = Az^N$ for some $N \geq 1$, $A \neq 0$ and it is easy to check that this too is impossible.

(iv) If the degree of P is no smaller than that of Q then

$$\frac{P(x)}{Q(x)} \log x \to \infty.$$

If the degree of P is smaller than that of Q then

$$\frac{P(x)}{Q(x)} \log x \to 0.$$

If the degree of P is no smaller than that of Q then

$$\frac{P(x)}{Q(x)} \log x \to \infty.$$

K96

Set $f(x) = x - \log(1+x)$. Then $f(0) = 0$ and $f'(x) = 1-(1+x)^{-1} \geq 0$ for $0 < x$ and $f'(x) \leq 0$ for $x > 0$. so $f(x) \geq 0$ and $-x \geq \log(1-x)$ for $0 \leq x \leq 1$.

Set $g(x) = \log(1+x) - x + x^2$. Then $g(0) = 0$ and

$$g'(x) = (1+x)^{-1} - 1 + 2x = (x + 2x^2)(1+x)^{-1} \geq 0$$

for $x > 0$ and $g'(x) \leq 0$ for $-1/2 \leq x \leq 0$ so $g(x) \geq 0$ and $\log(1+x) \geq x - x^2$ for $x \geq -1/2$.

Since

$$\log \prod_{r=1}^{kn} \left(1 - \frac{r}{n^2}\right) = \sum_{r=1}^{kn} \log\left(1 - \frac{r}{n^2}\right)$$

we have

$$-\sum_{r=1}^{kn} \frac{r}{n^2} \geq \log \prod_{r=1}^{kn} \left(1 - \frac{r}{n^2}\right) \geq -\sum_{r=1}^{kn} \frac{r}{n^2} - \sum_{r=1}^{kn} \frac{r^2}{n^4}.$$

But

$$\sum_{r=1}^{kn} \frac{r}{n^2} = \frac{kn(kn+1)}{2n^2} = \frac{k}{2}\left(k + \frac{1}{n}\right) \to \frac{k^2}{2}$$

and

$$\sum_{r=1}^{kn} \frac{r^2}{n^4} \leq \frac{(kn)^3}{n^4} = \frac{k}{n} \to 0$$

as $n \to \infty$ so

$$\log \prod_{r=1}^{kn} \left(1 - \frac{r}{n^2}\right) \to -\frac{k^2}{2}$$

and the result follows on applying exp.

K97

Suppose that $x \geq 1$. Then $x_n \geq 1$ and

$$\begin{aligned} 2^n(x_n - 1) - 2^{n+1}(x_{n+1} - 1) &= 2^n(x_{n+1}^2 - 2x_{n+1} + 1) \\ &= 2^n(x_{n+1} - 1)^2 \geq 0. \end{aligned}$$

Thus the sequence $2^n(x_n - 1)$ is decreasing bounded below by 0 and so tends to a limit. A similar argument applies if $1 > x > 0$.

Since

$$\frac{2^n(x_n - 1)}{2^n(1 - 1/x_n)} = x_n \to 1$$

$2^n(x_n - 1)$ and $2^n(1 - 1/x_n)$ tend to the same limit.

(i) If $x = 1$, then $x_n = 1$ so $\log 1 = 0$.

(ii) If $x_n = 1 + \delta_n$, $y = 1 + \eta_n$ then

$$\left(\frac{(1 + \delta_n)(1 + \eta_n)}{1 + \delta_n + \eta_n}\right)^{2^n} = \left(1 + \frac{\delta_n \eta_n}{1 + \delta_n + \eta_n}\right)^{2^n}.$$

Since $2^n\delta_n$ tends to a limit we can find an A such that $|\delta_n \leq A2^{-n}$. Similarly we can find a B such that $|\eta_n \leq B2^{-n}$. Thus we can find a C such that

$$\left|\frac{\delta_n \eta_n}{1 + \delta_n + \eta_n}\right| \leq C2^{-2n}.$$

It follows that

$$\begin{aligned} \left|\left(\frac{(1 + \delta_n)(1 + \eta_n)}{1 + \delta_n + \eta_n}\right)^{2^n} - 1\right| &\leq \sum_{r=1}^{2^n} \binom{2^n}{r} C^r 2^{-2nr} \\ &\leq \sum_{r=1}^{2^n} 2^{nr} C^r 2^{-2nr} \\ &=\leq \sum_{r=1}^{2^n} 2^{nr} C^r 2^{-nr} \\ &\leq \frac{C2^{-n}}{1 - C2^{-n}} \to 0 \end{aligned}$$

as $n \to \infty$.

By the mean value theorem

$$(b^{2^{-n}} - a^{2^{-n}}) = 2^{-n}c^{2^{-n}-1}(b - a)$$

for some $c \in (a, b)$ so

$$2^n((1 + \delta_n)(1 + \eta_n) - (1 + \delta_n + \eta_n) \to 0$$

that is to say

$$2^n((xy)^{2^{-n}} - (x_n + y_n - 1)) \to 0$$

so writing $z = xy$

$$2^n(z_n - 1) - 2^n(x_n - 1) - 2^n(y_n - 1) \to 0$$

and

$$\log xy = \log x + \log y.$$

Result (ii) shows that we need only prove (iii), (iv), (v) and (vi) at the point $x = 1$. To do this observe that a simple version of Taylor's theorem

$$|f(1+\delta) - f(1) - f'(1)\delta| \leq \sup_{|\eta|\leq|\delta|} |f''(1+\eta)||\delta^2|$$

applied to $f(x) = x^{-2^n}$ yields

$$|(1+\delta)^{-2^n} - 1 - 2^{-n}\delta| \leq 2^{-n} \times 4 \times |\delta|^2$$

for all $|\delta| \leq 10^{-1}$ and $n \geq 3$. (We can do better but we do not need to.) Thus if $|\delta| \leq 10^{-1}$ and we set $x = 1 + \delta$ we obtain

$$|2^n(x_n - 1) - \delta| \leq 4|\delta|^2$$

for $n \geq 3$ so that

$$|\log(1+\delta) - \delta| \leq 4|\delta|^2.$$

Thus log is continuous and differentiable at 1 with derivative 1.

Since $\log(x+t) = \log x + \log(1+t/x)$ the chain rule gives $\log' x = 1/x$ and since $\log' x > 0$ we see that log is strictly increasing.

(vii) Since $\log 2 > \log 1 = 0$ we have $\log 2^n = n \log 2 \to \infty$ so $\log x \to \infty$ as $x \to \infty$ and $\log y = -\log y^{-1} \to -\infty$ as $y \to 0+$. Since log is continuous and strictly monotonic, part (vii) follows.

K98

If

$$\left(\frac{f(x+\eta)}{f(x)}\right)^{1/\eta} \to l$$

then, taking logarithms,

$$\frac{\log f(x+\eta) - \log f(x)}{\eta} \to \log l$$

so $\log f$ is differentiable and, by the chain rule, $f = \exp \log f$ is differentiable at x. Since

$$\log l = \frac{d}{dx} \log f(x) = \frac{f'(x)}{f(x)}$$

we have

$$l = \exp \frac{f'(x)}{f(x)}.$$

K99

If
$$f(x) = \frac{\gamma x^{\beta-1}}{x^\beta - a^\beta} - \frac{1}{x-a}$$
tends to a limit as $x \to a$ then $(x-a)f(x) \to 0$. Since
$$\frac{x^\beta - a^\beta}{x-a} \to \beta a^{\beta-1}$$
as $x \to a$ this gives
$$\frac{\gamma}{\beta} - 1 = 0$$
so $\beta = \gamma$.

If $\beta = \gamma$ then $f(x) = F(x)/G(x)$ with
$$F(x) = \beta x^\beta - \beta a x^{\beta-1} - x^\beta + a^\beta$$
$$G(x) = x^{\beta+1} - a^\beta x - a x^\beta + a^{\beta+1}$$

We find $F(a) = F'(a) = 0$, $G(a) = G'(a) = 0$ and use L'Hôpital's rule to give
$$f(x) \to \frac{F''(a)}{G''(a)} = \frac{\beta - 1}{2a}.$$

K100

We only consider x^α when x is real and we define it as $\exp(\alpha \log x)$ where log is the real logarithm.

K101

(iii) Set $f(x) = \log g(x)$ and apply (i) to get $g(x) = x^b$ for some real b. Easy to see that this is a solution.

(iv) Observe that, if $x > 0$, $g(x) = g(x^{1/2})^2 > 0$. Thus, by (iii) $g(x) = b^x$ for all $x > 0$ and some $b > 0$.

Now $g(-1)^2 = g(1) = 1$ so $g(-1) = \pm 1$ and either $g(x) = b^{|x|}$ for all x or $g(x) = \operatorname{sgn}(x) b^{|x|}$ for all x. (Recall $\operatorname{sgn} x = 1$ if $x > 0$, $\operatorname{sgn} x = -1$ if $x < 0$.) It is easy to see that these are solutions.

K102

(i) We know that

$$\chi(t/2)^2 = \chi t$$

but the equation $(\exp is)^2 = \exp iu$, where $u \in \mathbb{R}$, has two real solutions in s with $-\pi < s \leq \pi$ (the solutions being $s/2$ and one of $s/2 \pm \pi$). As a specific example, if $\chi(t) = \exp 2it$, then $\theta(3\pi/4) = -\pi/2$ but $\theta(3\pi/8) = 3\pi/4$.

However since χ is continuous we can find a $\delta > 0$ such that $|\chi(t) - 1| \leq 100^{-1}$ for $|t| \leq \delta$. If $|t| \leq \delta$ then $|\theta(t)| \leq \pi/4$ and $|\theta(t/2)| \leq \pi/4$ so $\theta(t/2) = \theta(t)/\pi$ (since $|\theta(t/2) \pm \pi| > \pi/4$).

(iii) The same ideas show that the continuous homomorphisms are precisely the maps $\lambda \mapsto \lambda^n$ for some integer n.

K103

(i) $f(0) = 2f(0)$ so $f(0) = 0$.

(ii) Set $F(2^k p_j^{-1}) = 2^k j$ whenever j and k are integers with $j \geq 1$ and $F(t) = 0$ otherwise. (By the uniqueness of factorisation F, is well defined.)

Since $F(p_j^{-1}) \to \infty$ and $p_j^{-1} \to 0$ as $j \to \infty$, F is not continuous at 0.

(iii)
$$f(t) = 2^n f(t2^{-n}) = t\frac{f(t2^{-n}) - f(0)}{t2^{-n}} \to f'(0)t$$
as $n \to \infty$ so $f(t) = f'(0)t$ for all $t \neq 0$ and so for all t.

(iv) Set $F(2^k p_j^{-1}) = 2^k p_j^{-1}$ whenever j and k are integers with $j \geq 1$ and $F(t) = 0$ otherwise. Since $|F(t) - F(0)| = |f(t)| \leq |t|$, we have F continuous at 0. Since
$$\frac{F(p_j^{-1}) - F(0)}{p_j^{-1}} = 1 \to 1 \text{ but } \frac{F(2^{1/2}j^{-1}) - F(0)}{2^{1/2}j^{-1}} \to 0$$
as $j \to \infty$, F is not differentiable at 0.

(v) Observe that $u(t) = u(t/2)^2 \geq 0$ for all t. If $u(x) \leq 0$ then $u(2^{-n}x) = u(x)^{2^{-n}} \to 1$ so by continuity $u(0) = 1$. Thus by continuity there exists a $\delta > 0$ such that $u(t) > 0$ for $|t| < \delta$. Thus $u(t) = u(2^{-n}t)^{2^n} > 0$ for $|t| < 2^n\delta$ and so u is everywhere strictly positive. Now set $f(t) = \log u(t)$ and apply part (iii).

K104

Consider the case $\mathbf{x} \neq \mathbf{0}$. Observe that

$$\frac{1}{(1+\delta)^{1/2}} = 1 - \frac{\delta}{2} + \epsilon(\delta)|\delta|$$

with $\epsilon(\delta) \to 0$ as $\delta \to 0$. (Use differentiation or Taylor series.) Observe also that $|\mathbf{x} \cdot \mathbf{h}| \leq \|\mathbf{x}\|\|\mathbf{h}\|$. Thus

$$\begin{aligned}
\mathbf{f}(\mathbf{x}+\mathbf{h}) &= \frac{\mathbf{x}+\mathbf{h}}{(\|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{h} + \|\mathbf{h}\|^2)^{1/2}} \\
&= \frac{\mathbf{x}+\mathbf{h}}{\|\mathbf{x}\|\big(1 + (2\mathbf{x}\cdot\mathbf{h} + \|\mathbf{h}\|^2)/\|\mathbf{x}\|^2\big)^{1/2}} \\
&= \frac{\mathbf{x}+\mathbf{h}}{\|\mathbf{x}\|}\left(1 - \frac{2\mathbf{x}\cdot\mathbf{h} + \|\mathbf{h}\|^2}{2\|\mathbf{x}\|^2}\right) + \epsilon_1(\mathbf{h})\|\mathbf{h}\| \\
&= \mathbf{f}(\mathbf{x}) - \frac{\mathbf{h}}{\|\mathbf{x}\|} - \frac{(\mathbf{x}\cdot\mathbf{h})\mathbf{x}}{\|\mathbf{x}\|^3} + \epsilon_2(\mathbf{h})\|\mathbf{h}\|
\end{aligned}$$

where $\|\epsilon_j(\mathbf{h})\| \to 0$ as $\|\mathbf{h}\| \to 0$. Thus $\mathbf{f}$ is differentiable at $\mathbf{x}$ and

$$D\mathbf{f}(\mathbf{x})\mathbf{h} = \frac{\mathbf{h}}{\|\mathbf{x}\|} - \frac{(\mathbf{x}\cdot\mathbf{h})\mathbf{x}}{\|\mathbf{x}\|^3}.$$

$$(D\mathbf{f}(\mathbf{x})\mathbf{h}) \cdot \mathbf{x} = \frac{\mathbf{h}\cdot\mathbf{x}}{\|\mathbf{x}\|} - \mathbf{x}\cdot\mathbf{h}\frac{\|\mathbf{x}\|^2}{\|\mathbf{x}\|^3} = 0.$$

Since

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{0})\| = 1 \nrightarrow 0$$

as $\|\mathbf{x}\| \to 0$, $\mathbf{f}$ is not even continuous at $\mathbf{0}$

Observe that $\mathbf{f}$ is constant along radii.

K105

Without loss of generality take $z_0 = 0$.

(i)$\Rightarrow$(ii) If f is complex differentiable at 0, then writing $f'(0) = A + Bi$ with A and B real, and taking h and k real

$$f(h+ik) = f(0) + (A+Bi)(h+ik) + \epsilon(h+ik)|h+ik|$$

with $\epsilon(h+ik) \to 0$ as $|h+ik| \to 0$. Taking real and imaginary parts,

$$\begin{pmatrix} u(h,k) \\ v(h,k) \end{pmatrix} = \begin{pmatrix} u(0.0) \\ v(0,0) \end{pmatrix} + \begin{pmatrix} Ah - Bk \\ Ak + Bh \end{pmatrix} + \begin{pmatrix} \epsilon_1(h,k) \\ \epsilon_2(h,k) \end{pmatrix} (h^2+k^2)^{1/2}$$

with

$$\left\| \begin{pmatrix} \epsilon_1(h,k) \\ \epsilon_2(h,k) \end{pmatrix} \right\| \to 0$$

as $(h^2+k^2)^{1/2} \to 0$. Thus F is differentiable at $\mathbf{0}$ with Jacobian matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

If $A = B = 0$ take $\lambda = 0$ and $\theta = 0$. Otherwise take $\lambda = (A^2+B^2)^{1/2}$ and θ such that $\cos\theta = A$, $\sin\theta = B$.

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) Just a matter of correct interpretation.

(iv)$\Rightarrow$(i) Carefully reverse the first proof.

K106

(i) Observe that

$$\begin{aligned}g(x+h) &= f(x+h, c-x-h)\\ &= f(x, c-x)f_{,1}(x, c-x)h + f_{,2}(x, c-x)(-h) + \epsilon(h,h)(h^2+h^2)^{1/2}\\ &= g(x) + \big(f_{,1}(x, c-x) - f_{,2}(x, c-x)\big) + 2^{1/2}\epsilon(h,h)|h|\end{aligned}$$

where $\epsilon(h,k) \to 0$ as $(h^2+k^2)^{1/2} \to 0$. Thus g is differentiable with derivative

$$g'(x) = f_{,1}(x, c-x) - f_{,2}(x, c-x).$$

(ii) Observe that $g = f \circ \mathbf{u}$ where

$$\mathbf{u}(x) = \begin{pmatrix} x \\ c-x \end{pmatrix}.$$

Thus $\mathbf{u}$ has Jacobian matrix

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

By chain the rule g is differentiable with 1×1 Jacobian matrix

$$\big(f_{,1}(x, c-x) \quad f_{,2}(x, x-c)\big)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (f_{,1}(x, c-x) - f_{,2}(x, c-x)).$$

Define g as stated. If $f_{,1} = f_{,2}$, then $g' = 0$ so, by the constant value theorem, g is constant. Thus $f(x, x-c) = f(y, y-c)$ for all x, y, $c \in \mathbb{R}$. Writing $h(x+y) = f(0, x+y)$, we have the required result.

K107

(i) Differentiating with respect to λ and applying the chain rule, we have

$$\alpha\lambda^{\alpha-1}f(\mathbf{x}) = \sum_{j=1}^{m} x_j f_{,j}(\lambda\mathbf{x}).$$

Taking $\lambda = 1$ gives the required result.

(ii) We observe that v is differentiable with

$$v'(\lambda) = \sum_{j=1}^{m} x_j f_{,j}(\lambda\mathbf{x}) = \lambda^{-1}\sum_{j=1}^{m} \lambda x_j f_{,j}(\lambda\mathbf{x}) = \alpha\lambda^{-1}v(\lambda).$$

Thus

$$\frac{d}{d\lambda}(\lambda^{-\alpha}v(\lambda)) = 0$$

and by the constant value theorem

$$\lambda^{-\alpha}v(\lambda) = v(1)$$

and $f(\lambda\mathbf{x}) = \lambda^{\alpha}f(\mathbf{x})$.

K108

This is really just a question for thinking about.

(i) We seek to maximise the (square root of)

$$f(\theta) = \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\|^2 = (a\cos\theta + b\sin\theta)^2(c\sin\theta + b\cos\theta)^2$$

and this we can do, in principle, by examining the points with $f'(\theta) = 0$.

(ii) We seek to maximise the (square root of)

$$\sum_{r=1}^{p} \left(\sum_{s=1}^{m} a_{rs}x_s \right)^2$$

subject to the constraint

$$\sum_{s=1}^{m} x_s^2 = 1$$

which can, in principle, be handled using Lagrange multipliers.

However, this involves solving an $m \times m$ set of linear equations involving a parameter λ leading to polynomial of degree m in λ. The m roots must then be found and each one inspected. This neither sounds nor is a very practical idea even when m is quite small.

K109

If $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_m$ is the standard basis (or any orthonormal basis) and (a_{ij}) is the associated matrix of α, then

$$\begin{aligned} a_{ij} &= \sum_{r=1}^{m} a_{rj}\langle \mathbf{e}_r, \mathbf{e}_i\rangle = \left\langle \sum_{r=1}^{n} a_{rj}\mathbf{e}_r, \mathbf{e}_i \right\rangle \\ &= \langle \alpha\mathbf{e}_j, \mathbf{e}_i\rangle = \langle \mathbf{e}_j, \alpha\mathbf{e}_i\rangle = \langle \alpha\mathbf{e}_i, \mathbf{e}_j\rangle = a_{ji} \end{aligned}$$

Conversely if $a_{ij} = a_{ji}$ for all i, j, then essentially the same calculation shows that

$$\langle \alpha\mathbf{e}_j, \mathbf{e}_i\rangle = \langle \alpha\mathbf{e}_i, \mathbf{e}_j\rangle$$

and so by linearity

$$\begin{aligned} &\left\langle \alpha\left(\sum_{r=1}^{n} x_r\mathbf{e}_r\right), \sum_{s=1}^{m} y_s\mathbf{e}_s \right\rangle \\ &\qquad = \sum_{r=1}^{m}\sum_{s=1}^{m} x_r y_s \langle \alpha\mathbf{e}_r, \mathbf{e}_s\rangle \\ &\qquad = \sum_{r=1}^{m}\sum_{s=1}^{m} x_r y_s \langle \mathbf{e}_r, \alpha\mathbf{e}_s\rangle \\ &\qquad = \left\langle \sum_{r=1}^{n} x_r\mathbf{e}_r, \alpha\left(\sum_{s=1}^{m} y_s\mathbf{e}_s\right) \right\rangle. \end{aligned}$$

Let $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{e}_m$ be an orthonormal basis of eigenvectors with eigenvalues λ_1, λ_2, … λ_m with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$. Then

$$\left\| \alpha\left(\sum_{j=1}^{m} x_j\mathbf{u}_j\right) \right\|^2 = \left\| \sum_{j=1}^{m} x_j\lambda_j\mathbf{u}_j \right\|^2 = \sum_{j=1}^{m} x_j^2\lambda_j^2 \geq \lambda_1^2 \sum_{j=1}^{m} x_j^2$$

with equality when $x_2 = x_3 = \cdots = x_m = 0$ (and possibly for other values). This proves †.

The matrix A given has eigenvalue 0 only. However $\|A\| = 1$.

K110

(ii) Let $\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j$. Then

$$\mathbf{y}_k = \sum_{j=1}^{n} \frac{\lambda_j^k x_j}{\left(\sum_{r=1}^{n} (\lambda_r^k x_r)^2\right)^{1/2}} \to \mathbf{e}_1$$

unless $x_1 = 0$.

If the largest eigenvalue is negative, then (in general) $\|\mathbf{x}_k\| \to |\lambda_1|$ and $\|(-1)^k \mathbf{y}_k - \mathbf{u}_1\| \to 0$ where $\mathbf{u}_1 = \pm\mathbf{e}_1$.

(iii) (c) The number of operations for each operation is bounded by An^2 for some A ($A = 6$ will certainly do),

Suppose $|\lambda_1| > \mu \geq |\lambda_j|$ for $j \geq 2$. Then the $\mathbf{e}_1$ component of $\alpha^k \mathbf{x}$ will grow at a geometric rate λ_j/μ relative to the other components.

(v) Look at (iii). If the two largest eigenvalues are very close the same kind of thing will occur.

(iv) Suppose $\mathbf{e}_1$ the eigenvector with largest eigenvalue known. Use the iteration

$$\mathbf{x} \mapsto (\alpha(\mathbf{x} - \mathbf{x} \cdot \mathbf{e}_1)$$

or something similar. However, the accuracy to which we know $\mathbf{e}_1$ will limit the accuracy to which we can find $\mathbf{e}_2$ and matters will get rapidly worse if try to find the third largest eigenvalue and so on.

(vii) No problem in real symmetric case because eigenvectors orthogonal.

K111

(ii) $\langle \mathbf{x}, \alpha^*\alpha\mathbf{x}\rangle = \langle \alpha\mathbf{x}, \alpha\mathbf{x}\rangle = \|\alpha\mathbf{x}\|^2$.

Thus

$$\|\alpha^*\alpha\|\|\mathbf{x}\|^2 = \|\mathbf{x}\|\|\alpha^*\alpha\mathbf{x}\| \geq |\langle \mathbf{x}, \alpha^*\alpha\mathbf{x}\rangle| = \|\alpha\mathbf{x}\|^2 \geq \|\alpha\|^2\|\|\mathbf{x}\|^2$$

for all $\mathbf{x}$ and so $|\alpha^*\alpha\| \geq \|\alpha\|^2$.

(iii) $\|\alpha^*\|\|\alpha\| \geq \|\alpha^*\alpha\| \geq \|\alpha\|^2$ and so either $\alpha = 0$ (in which case the result is trivial) or $\|\alpha\| \neq 0$ and $\|\alpha^*\| \geq \|\alpha\|$. But $\alpha^{**} = \alpha$ so $\|\alpha\| = \|\alpha^{**}\| \geq \|\alpha\|$. Thus $\|\alpha\| = \|\alpha^*\|$ and $\|\alpha\|^2 = \|\alpha^*\|\|\alpha\| \geq \|\alpha^*\alpha\| \geq \|\alpha\|^2$ so $\|\alpha^*\alpha\| = \|\alpha\|^2$.

Since $(\alpha^*\alpha)^* = \alpha^*\alpha^{**} = \alpha^*\alpha$ we have $\alpha^*\alpha$ symmetric. If $\alpha\mathbf{e} = \lambda\mathbf{e}$ with $\mathbf{e} \neq \mathbf{0}$ then

$$\lambda\|\mathbf{e}\|^2 = \langle \mathbf{e}, \lambda\mathbf{e}\rangle = \langle \mathbf{e}, \alpha^*\alpha\mathbf{e}\rangle == \langle \alpha\mathbf{e}, \alpha\mathbf{e}\rangle = \|\alpha\mathbf{e}\|^2.$$

Thus $\lambda \geq 0$.

(vi) Multiplication of A with A^* takes of the order of m^3 operations (without tricks) and dominates.

(vii) We have $A^*A = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$.

A has eigenvalues 1 and 2. A^*A has eigenvalues $4 \pm 5^{1/2}$. $\|A\| = (4 + 5^{1/2})^{1/2}$.

K112

Follow the proof of Rolle's theorem. If $\|\mathbf{c}\| < 1$ and f has a maximum at $\mathbf{c}$ then $f_{,1}(\mathbf{c}) = f_{,2}(\mathbf{c}) = 0$ so $Df(\mathbf{c})$ is the zero map.

$g(x, y) = x^2$ will do. Observe that $\cos^2\theta - a\cos\theta - b\sin\theta$ is not identically zero (consider $\theta = \pi/2$ and other values of θ).

We can not hope to have a 'Rolle's theorem proof' in higher dimensions.

K113

Since $(x, y) \mapsto x - y$, $(x, y) \mapsto xy$, $x \mapsto x^{-1}$ and $x \mapsto f(x)$ are all continuous we see that F which can be obtained by composition of such functions is continuous except perhaps at points (x, x).

If F is continuous at (x, x) then $F(x+h, x)$ must tend to the limit $F(x, x)$ as $h \to 0$ so f is differentiable and $F(x, x) = f'(x)$. Since $F(x+h, x+h) \to F(x, x)$ as $h \to 0$, f' must be continuous.

Conversely if f' exists and is continuous then, setting $F(x, x) = f'(x)$, we saw above that F is continuous at all points (x, y) with $x \neq y$. If $h \neq k$ the mean value theorem gives

$$\begin{aligned} F(x+h, x+k) - F(x, x) &= \frac{f(x+h) - f(x+k)}{x - y} - f'(x) \\ &= f'\big(x + (\theta_{h,k}h + (1 - \theta_{h,k})k)\big) - f'(x) \end{aligned}$$

for some $\theta_{h,k}$ with $0 < \theta_{h,k} < 1$. But it is also true that, if we set $\theta_{h,h} = 1/2$,

$$\begin{aligned} F(x+h, x+h) - F(x, x) &= f'(x+h) - f'(x) \\ &= f'\big(x + (\theta_{h,h}h + (1 - \theta_{h,h})h)\big) - f'(x). \end{aligned}$$

Thus

$$F(x+h, x+k) - F(x, x) = f'\big(x + (\theta_{h,k}h + (1 - \theta_{h,k})k)\big) - f'(x) \to 0$$

as $(h^2 + k^2)^{1/2} \to 0$.

Suppose now that f is twice continuously differentiable. Much as before, the chain rule shows that F is differentiable at all points (x, y) with $x \neq y$. If f is twice continuously differentiable then the local form of Taylor's theorem shows us that

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \epsilon(h)h^2$$

with $\epsilon(h) \to 0$ as $h \to 0$. Thus, if $h \neq k$,

$$\begin{aligned} &F(x+h, x+k) \\ &= \frac{(f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \epsilon(h)h^2) - (f(x) + f'(x)k + \frac{1}{2}f''(x)k^2 + \epsilon(k)k^2}{h - k} \\ &= f'(x) + \frac{f''(x)}{2}(h + k) + \frac{\epsilon(h)h^2 + \epsilon(k)k^2}{h - k} \\ &= F(x, x) + \frac{f''(x)}{2}(h + k) + \epsilon'(h, k)(h^2 + k^2)^{1/2} \end{aligned}$$

where $\epsilon'(h, k) \to 0$ as $(h^2 + k^2)^{1/2} \to 0$. The formula can be extended to the case $h = k$ by applying the appropriate Taylor theorem to f'. Thus F is differentiable everywhere.

K114

If $k \neq 0$, then we can find an $X > 0$ such that $|f'(t) - k| < |k|/4$ and so $|f'(t)| > 3|k|/4$ for $t \geq X$. Thus, using the mean value inequality,

$$\begin{aligned}
\left|\frac{f(x)}{x}\right| &= \left|\frac{f(x) - F(X)}{x - X}\frac{x - X}{x} + \frac{F(X)}{x}\right| \\
&\geq \left|\frac{f(x) - f(X)}{x - X}\right|\left|\frac{x - X}{x}\right| - \left|\frac{f(X)}{x}\right| \\
&\geq \frac{3|k|}{4}\left|\frac{x - X}{x}\right| - \left|\frac{f(X)}{x}\right| \\
&\geq \frac{|k|}{2} - \frac{|k|}{4} = \frac{|k|}{4}
\end{aligned}$$

whenever $x \geq \max(3X, (4|f(X)|)^{-1})$.

$g(x) = x^{-2}\sin x^4$ will do.

K115

The local form of Taylor's theorem

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \epsilon(h)h^2$$

applied to log (or other arguments) show us that

$$\log(1-h) = -h - \frac{h^2}{2} + \epsilon(h)h^2$$

with $\epsilon(h) \to 0$ as $h \to 0$.

Thus since $q_n \to \infty$ we can find an N such that $-2^{-1}q_n^{-1} \geq \log(1-q_n^{-1})$ and $0 \geq \log(1-q_n^{-1}) + q_n^{-1} \geq -2q_n^{-2} \geq -2n^{-2}$ for all $n \geq N$, so the results follow from the comparison test.

K116

If $a = b = 0$ maximum c.

If $a = 0$ maximum c if $b \leq 0$, $10b + c$ if $b \geq 0$

If $a > 0$ then maximum c if $-b/2a \geq 5$, $100a + 10b + c$ if $-b/2a \leq 5$.

If $a < 0$ then maximum c if $-b/2a \leq 0$, $c - b^2/(4a)$ if $0 \leq -b/2a \leq 10$, $100a + 10b + c$ if $10 \leq -b/2a$.

K117

(i) Take $x_1 = 1$, $x_j = 0$ otherwise.

(ii) To see that B is positive definite if A is, take

$$x_1 = -(a_{11}^{-1}a_{12}x_2 + a_{11}^{-1}a_{13}x_3 + \cdots + a_{11}^{-1}a_{1n}x_n).$$

K118

(i) One way of seeing that the λ_j are continuous is to solve the characteristic equation. Now use the intermediate value theorem.

(iii) $B(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ will do.

K119

The reduction to Figure K2 can not be rigorous unless we state clearly what paths are allowed. However if we have a path which is not symmetric under reflection in the two axes of symmetry joining mid points of opposite sides then by considering the reflected path we can reconstruct a shorter symmetric path.

[A more convincing route involves the observation that the shortest path system for three points M, N, P is three paths MZ, NZ, PZ making angle $2\pi/3$ to each other provided the largest angle of the triangle MNP is less than $2\pi/3$ and consists of the two shortest sides otherwise. (See Courant and Robbins *What is Mathematics?*.)]

One we have the diagram we see that the total path length is

$$f(\theta) = 2a\sec\theta + (a - a\tan\theta)$$

where a is the length of the side of the square and θ is a base angle of the isosceles triangles.

$$f'(\theta) = \frac{1 - 2\sin\theta}{\cos^2\theta}$$

so f increases on $[0, \pi/6]$ and decreases on $[\pi/6, \pi/4]$ so the best angle is $\theta/6$ (check that this gives an arrangement consistent with the statement in square brackets). IN PARTICULAR $\theta = \pi/4$ is not best.

There are two such road patterns the other just being a rotation through $\pi/4$

K120

The time taken is

$$f(x) = \frac{h-x}{u} + \frac{(1+x^2)^{1/2}}{v}$$

for $0 \le x \le h$. We observe that

$$f'(x) = -\frac{1}{u} + \frac{x}{v((1+x^2)^{1/2}}$$

so f' is negative and f is decreasing for $0 \le x \le \alpha$ with $0 \le x \le (1 - v^2/u^2)^{-1/2}$ and f' is positive and f is increasing for $x \ge (1-v^2/u^2)^{-1/2}$. Thus if $(1 - v^2/u^2)^{-1/2} < h$ (i.e. $(1 - v^2/u^2)h^2 > 1$) he should take $x = (1 - v^2/u^2)^{-1/2}$. However, if $(1 - v^2/u^2)h^2 < 1$ his best course if $0 \le x \le h$ is to take $x = h$. Since choosing x outside this range is clearly worse he should take $x = h$.

If h is large his initial run is almost directly towards the tree and his greater land speed makes this desirable. If h is small his run is almost perpendicular to the direction of the tree and does him very little good.

K121

We wish to find the maxima and minima of $f(x,y) = y^2 - \frac{1}{3}x^3 + ax$ in the region $x^2 + y^2 \leq 1$. We first examine the region $x^2 + y^2 < 1$.

The stationary points are given by

$$0 = f_{,1}(x,y) = -x^2 + a,\ 0 = f_{,2}(x,y) = 2y.$$

Thus, if $a < 0$ or $a \geq 1$, there are no interior stationary points and both the global maximum and minimum must occur on the boundary.

If $0 < a < 1$, then there are two interior stationary points at $x = \pm a^{1/2}$, $y = 0$. The Hessian is

$$\begin{pmatrix} -2x & 0 \\ 0 & 2 \end{pmatrix}$$

so $(a^{1/2}, 0)$ is a saddle and $(-a^{1/2}, 0)$ a minimum. Thus the global maximum must occur on the boundary and the global minimum may occur at $(-a^{1/2}, 0)$ with value $-2a^{3/2}/3$ or may occur on the boundary.

If $a = 0$ then there is one stationary point at $(0,0)$, the Hessian is singular there but inspection of the equation $f(x,y) = y^2 - \frac{1}{3}x^3$ shows that we have a saddle and both the global maximum and minimum must occur on the boundary.

Now we look at the boundary $x^2 + y^2 = 1$. Here

$$f(x,y) = g(x) = 1 - x^2 - \tfrac{1}{3}x^3 + ax$$

with $x \in [-1, 1]$. Notice that g is a cubic so we can draw sketches to help ourselves.

Now $g'(x) = -2x - x^2 + a$ and g' has roots at $-1 \pm (1 + a^2)$. If $|a| \geq 1$ then g is decreasing on $(-1, 1)$ so has maximum at $x = -1$ (which is thus the global maximum for f at $(-1, 0)$) and a minimum at $x = 1$ 9which is thus the global minimum for f at $(0, 1)$).

If $|a| < 1$, g increases for x running from -1 to $-1 \pm (1 + a^2)$ (which is thus a global maximum and f has its global maximum at $x = -1 + (1 + a^2)$ $y = \left(1 - (1 + (1 + a^2))^2\right)^{1/2}$) and decreases as x runs from from $-1 \pm (1 + a^2)$ to 1. Thus g has minima at $x = -1$ and 1 and (by evaluating g at these points) a global minimum at 1 if $a \leq -1/3$. Since $a \leq -1/3$ we know that this gives a global minimum for f at $(0, 1)$. If $a \geq -1/3$, g has a global minimum at -1 and if $0 \geq a \geq -1/3$ we know that it gives a global minimum for f at $(0, 1)$.

If $1 > a > 0$ we observe that we know that the global minimum occurs when $y = 0$ and by looking at $h(x) = f(x, 0) = -\frac{1}{3}x^3 + ax$ we see that it occurs at $(-a^{1/2}, 0)$.

K122

The composition of differentiable functions is differentiable and the product of differentiable functions (with values in $\mathbb{R}$) are differentiable so since the maps $(x, y) \mapsto r$ and $(x, y) \mapsto \theta$ are differentiable except at $\mathbf{0}$, f is. (To see that $(x, y) \mapsto r$ is differentiable observe that $r = (x^2 + y^2)^{1/2}$ and use basic theorems again. To see that $(x, y) \mapsto \theta$ is differentiable on

$$S = \{(x, y) \,:\, |y| > 9x\}$$

observe that on S, $\theta = \tan^{-1} x/y$. To extend to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ either use rotational symmetry or cover $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ with rotated copies of S on which similar formulae hold.

Since

$$\frac{f(r\cos\theta, r\sin\theta) - f(0,0)}{r} \to g(\theta)$$

as $r \to 0+$, f has a directional derivative at 0 if and only if $g(\theta) = -g(-\theta)$.

If f is differentiable at $\mathbf{0}$ then

$$\frac{f(r\cos\theta, r\sin\theta) - f(0,0)}{r} \to \cos\theta f_{,1}(0,0) + \sin\theta f_{,2}(0,0)$$

so

$$g(\theta) = g(0)\cos\theta + g(\pi/2)\sin\theta$$

ie

$$g(\theta) = A\sin\theta + B\cos\theta$$

for some constants A and B so

$$f(x, y) = Ax + By.$$

The necessary condition is sufficient by inspection.

K123

(i) Observe that.

$$f(\alpha t, \beta t) = t^2(\beta - a\alpha^2 t)(\beta - b\alpha^2 t)$$

which has a strict local minimum at $t = 0$.

However, if $a > c > b$, then

$$f(t, ct^2) = (c - a)(c - b)t^4$$

which has a strict local maximum at $t = 0$.

By looking at the behaviour of f along curves $(x, y) = (\alpha t, \beta t^2)$ the function f has no minima.

(ii) In part (i), f has Hessian

$$\begin{pmatrix} f_{,11}(\mathbf{0}) & f_{,12}(\mathbf{0}) \\ f_{,21}(\mathbf{0}) & f_{,22}(\mathbf{0}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

which is singular.

K124

(i) The second sentence holds by Exercise 8.22 which says that if f is Riemann integrable so is $|f|$.

(ii) First sentence false. Take $f = -g = F$.

Second sentence false. Take

$$\begin{aligned} f(x) &= F(x), \quad g(x) = 0 && \text{for } x < 1/2 \\ f(x) &= 0, \quad g(x) = F(x) && \text{for } x \geq 1/2. \end{aligned}$$

(iii) By considering functions of the form

$$\begin{aligned} f(x) &= F(x), \quad g(x) = 0 && \text{for } x < 1/2 \\ f(x) &= 0, \quad g(x) = F(x) && \text{for } x \geq 1/2. \end{aligned}$$

and of the form

$$\begin{aligned} f(x) &= F(x), \quad g(x) = 0 && \text{for } x < 1/2 \\ f(x) &= 0, \quad g(x) = 1 && \text{for } x \geq 1/2. \end{aligned}$$

and functions like $f(1-x)$ and $g(x)$ we see that the product of two functions that are not Riemann integrable may or may not be Riemann integrable and that the product of one that is Riemann integrable with one that is not may or may not be Riemann integrable.

K125

(i) Consider the dissection $\mathcal{D}_N = \{r/N : 0 \le r \le N\}$ where $N \ge M^2$. At most $2M^2$ intervals $[(r-1)/N, R/N]$ contain points of the form p/q with p and q coprime and $1 \le q \le M$. Thus

$$0 = s(f, \mathcal{D}_N) \le S(f, \mathcal{D}_N) \le 2M^2 N^{-1} + M^{-1} \to M^{-1}$$

as $N \to \infty$. Since M is arbitrary f is Riemann integrable with $\int_0^1 f(t)\,dt = 0$.

(ii) f is continuous at irrational points x. Since there are only finitely many points of the form p/q with p and q coprime and $1 \le q \le M$ and x is not one of them we can find a $\delta > 0$ such that $(x-\delta, x+\delta) \cap [0,1]$ contains no such points. Thus $|f(x) - f(y)| = f(y) \le M^{-1}$ for all $|x - y| < \delta$.

f is discontinuous at rational points x. Choose x_n irrational such that $x_n \to x$. Since $0 = f(x_n) \nrightarrow f(x)$, we are done.

(iii) Nowhere differentiable. By (ii) need only look at irrational points x. If q is a prime we know that there exists a p such that $(p-1)/q \le x < p/q$ so

$$\frac{(f(p/q) - f(x)}{p/q - x} \ge \frac{1/p}{1/p} = 1$$

but choosing x_n irrational such that $x_n \to x$ we have

$$\frac{(f(x_n) - f(x)}{x_n - x} = 0 \to 0.$$

K126

Observe that, if $0 \leq u < v \leq 1$, then $u^n - v^n \leq n(y-x)$ (by the mean value theorem or algebra). Write $f_n(x) = f(x^n)$. If $\mathcal{D}$ is a dissection write

$$\mathcal{D}_n = \{y^{1/n} \, : \, y \in \mathcal{D}\}.$$

By the observation of the first sentence

$$S(f_n, \mathcal{D}_n) - s(f_n, \mathcal{D}_n 0 \leq n\big(S(f, \mathcal{D}) - s(f, \mathcal{D})\big)$$

so, if f is Riemann integrable so is f_n.

(ii) and so (i) is true

Suppose that $|f(x)| \leq K$. Given $\epsilon > 0$, we can find a $1 > \delta > 0$ such that $|f(x) - f(0)| \leq \epsilon/2$ for $0 \leq x \leq \delta$. Since $\delta^{1/n} \to 1$ as $n \to \infty$ we can find an N such that $1 - \delta^{1/n} \leq (K + |k|)^{-1}\epsilon/2$ for $n \geq N$.

If $N \geq n$ then $|f_n(x) - k| \leq \epsilon/2$ for $x \in [0, \delta^{1/n}]$ and $|f_n(x)| \leq K$ for $x \in [\delta^{1/n}, 1]$ so

$$\left| \int_0^{\delta^{1/n}} f(x)\, dx - k(1 - \delta^{1/n} \right| \leq \epsilon/2 \text{ and } \left| \int_{\delta^{1/n}}^1 f(x)\, dx - k\delta^{1/n} \right| \leq \epsilon/2$$

so that

$$\left| \int_0^1 f(x)\, dx - k \right| \leq \epsilon.$$

(iii) is false. Take $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$.

K127

Without loss of generality, suppose that $f\big((a+b)\big) \geq 0$. Observe that, since $f(a) = 0$ and $|f'(t)| \leq K$ it follows that, by the mean value inequality, $|f(s)| \leq K(s-a)$ for $a \leq s \leq (a+b)/2$ with equality when $s = (b+a)/2$ if and only if $f(s) = K(s-a)$ for all $a \leq s \leq (a+b)/2$. Similarly $|f(s)| \leq K(b-s)$ for $b \geq s \geq (a+b)/2$ with equality when $s = (b+a)/2$ if and only if $f(s) = K(b-s)$ for all $a \leq s \leq (a+b)/2$. Unless $K = 0$ (in which case the result is trivial) the conditions for equality give f non-differentiable at $(b+a)/2$. Thus if we write

$$K(b-a)/2 - |f\big((b+a)/2\big) = 4\eta(b-a)$$

we have $4\eta > 0$ and, by continuity, we can find a $\delta > 0$ such that

$$\begin{aligned}
|f(s)| &\leq (K-\eta)(s-a) \text{ for } (b+a)/2 \geq s \geq (b+a)/2 - \eta\\
|f(s)| &\leq (K-\eta)(b-s) \text{ for } (b+a)/2 \leq s \leq (b+a)/2 + \eta
\end{aligned}$$

Thus

$$\begin{aligned}
\int_a^b |f(s)| &\leq \int_a^{(a+b)/2-\eta} + \int_{(a+b)/2-\eta}^{(b+a)/2} + \int_{(a+b)/2}^{(a+b)/2+\eta} + \int_{(a+b)/2+\eta}^{b} |f(s)|\,ds\\
&\leq K\big((b-a)/2-\eta\big)^2 + (K-\eta)(\big(b-a)/2\big)^2 - \big((b-a)/2-\eta\big)^2\\
&< K(b-a)^2/4
\end{aligned}$$

as required.

To see that this is best possible define

$$\begin{aligned}
f(x) &= K(x-a) && \text{for } a \leq x \leq (a+b)/2 - \epsilon,\\
f(x) &= C - \tau(x-(a+b)/2)^2 && \text{for } (a+b)/2 - \epsilon < x \leq (a+b)/2,
\end{aligned}$$

and $f((a+b)/2-t) = f((a+b)/2+t)$ where C is chosen to make f continuous and τ is chosen to make f differentiable at $(a+b)/2-\epsilon$ (thus we take $\tau = K/(2\epsilon)$).

K128

(v) Observe that once we know that f is Riemann integrable we know that $\frac{b-a}{n}\sum_{j=0}^{n-1} f(a+j(b-a)/n)$ converges but the fact that $\frac{b-a}{n}\sum_{j=0}^{n-1} f(a+j(b-a)/n)$ converges does not imply that f is Riemann integrable.

K129

For (A) note that, if F_j and G_j are positive bounded Riemann integrable functions with $F_1 - G_1 = F_2 - G_2$ then $F_1 + G_2 = F_2 + G_1$ and

$$\begin{aligned}\int_a^b F_1(t)\,dt + \int_a^b G_2(t)\,dt &= \int_a^b F_1(t) + G_2(t)\,dt\\ &= \int_a^b F_2(t) + G_1(t)\,dt\\ &= \int_a^b F_2(t)\,dt + \int_a^b G_1(t)\,dt\end{aligned}$$

and so

$$\int_a^b F_1(t)\,dt - \int_a^b G_1(t)\,dt = \int_a^b F_2(t)\,dt - \int_a^b G_2(t)\,dt.$$

The problems with (C) begin (I think) when we try to prove that if f and g are Riemann integrable so is $f+g$. We also get problems (which, I think have the same cause) when we try to show that anything which is Riemann integrable under the old definition are Riemann integrable under the new. One way forward is to show that anything which is (A) integrable is (C) integrable. To do this we can first prove that a bounded positive function f is Riemann integrable if and only if given any $\epsilon > 0$ we can find a dissection

$$\mathcal{D} = \{x_0,\, x_1,\, \dots, x_N\}$$

with $a = x_0 < x_1 < \cdots < x_N = b$ and a subset Θ of $\{r\ :< 1 \leq r \leq N\}$ such that

$$|f(x) - f(y)| \leq \epsilon \text{ for } x,\, y \in [x_{r-1}, x_r] \text{ when } r \in \Theta.$$

and

$$\sum_{r \notin \Theta} (x_r - x_{r-1}) \leq \epsilon.$$

K130

(iii) Everything works well until we try to define the upper and lower integrals but the set of $s(\mathcal{D}, f)$ has no supremum in $\mathbb{Q}$ and $S(\mathcal{D}, f)$ has no infimum.

K131

(i) Since F is continuous on a closed bounded interval the infimum and supremum are attained at α and β say. Now use the intermediate value theorem.

(ii) Observe that

$$\sup_{s\in[a,b]} f(s)w(t) \geq f(t)w(t)$$

for all $t \in [a,b]$ so

$$\sup_{s\in[a,b]} f(s) \int_a^b w(t)\,dt \geq \int_a^b f(t)w(t)\,dt$$

Using a similar result for the infimum we have

$$\sup_{s\in[a,b]} f(s) \int_a^b w(t)\,dt \geq \int_a^b f(t)w(t)\,dt \geq \inf_{s\in[a,b]} f(s) \int_a^b w(t)\,dt.$$

Now use part (i).

(iii) Take $a = -1$, $b = 1$, $F(t) = w(t) = t$.

If w everywhere negative, consider $-w$.

K132

The graph of f splits the rectangle $[a,b]\times[f(a),f(b)]$ into two parts whose areas correspond to the given integrals.

Consider a dissection of $[a,b]$

$$\mathcal{D}=\{x_0,\,x_1,\,\ldots,x_N\}$$

with $a=x_0<x_1<\cdots<x_N=b$. This gives rise to a dissection of $[f(a),f(b)]$

$$\mathcal{D}'=\{f(x_0),\,f(x_1),\,\ldots,\,f(x_N)\}$$

with $f(a)=F(x_0)<f(x_1)<\cdots<f(x_N)=f(b)$.

We have

$$S(\mathcal{D},f)+s(\mathcal{D}',g)=bf(b)-af(a)$$

so, taking an infimum over all $\mathcal{D}$, we get

$$I^*(f)+I_*(g)\geq bf(b)-af(a)$$

Similarly

$$s(\mathcal{D},f)+S(\mathcal{D}',g)=bf(b)-af(a)$$

so

$$I_*(f)+I^*(g)\leq bf(b)-af(a).$$

But $I^*(f)=I_*(f)=\int_a^b f(t)\,dt$ and $I^*(g)=I_*(g)=\int_{f(a)}^{f(b)} g(s)\,ds$ so

$$\int_a^b f(t)\,dt+\int_{f(a)}^{f(b)} g(s)\,ds=bf(b)-af(a).$$

(ii) Without loss of generality, assume $Y\geq F(X)$. Then

$$\begin{aligned}\int_0^X f(x)\,dx+\int_0^Y g(s)\,ds&=\int_0^X f(x)\,dx+\int_0^{f(X)} g(s)\,ds+\int_{f(X)}^Y g(s)\,ds\\&\geq XF(X)+(Y-F(X))X=YX\end{aligned}$$

with equality if and only if $Y=f(X)$.

(iii) Take $f(x)=x^{p^{-1}-1}$, $g(y)=y^{q^{-1}-1}$. Equality if and only if $X^{1/q}=Y^{1/p}$.

K133

Since f is continuous at c, we can find a $\delta > 0$ such that $|f(c) - f(x)| < \kappa/4$ when $|x - c| < \delta$. Thus if $[\alpha, \beta] \subseteq (c - \delta, c + \delta)$

$$I^*_{[\alpha,\beta]}(f) - I_{*[\alpha,\beta]}(f) \leq \kappa(\beta - \alpha)/2.$$

K134

Suppose $\gamma \in [a,b]$. Then given $1 > \epsilon > 0$ we can find a $\delta > 0$ such that
$$|f(s) - f(\gamma)|,\ |f(g(s)) - f(g(\gamma))|,\ |g'(s) - g'(\gamma)| < \epsilon$$
for all $s \in [a,b]$ with $|s - \gamma| < \delta$.

Now suppose $[\alpha, \beta] \subseteq (\alpha - \delta, \alpha + \delta)$. Without loss generality we suppose $g(\beta) \geq g(\alpha)$. Then
$$\begin{aligned}
\left|\int_{g(\alpha)}^{g(\beta)} f(s)\,ds - \big(g(\beta)\ - g(\alpha)\big) f(g(\gamma))\right| &\leq \int_{g(\alpha)}^{g(\beta)} |f(s) - f(g(\gamma))|\,ds \\
&\leq \big(g(\beta) - g(\alpha)\big) \sup_{s \in [g(\alpha), g(\beta)]} |f(s) - f(g(\gamma))| \\
&\leq \big(g(\beta) - g(\alpha)\big) \sup_{t \in [\alpha, \beta]} |f(g(s)) - f(g(\gamma))| \\
&\leq \big(g(\beta) - g(\alpha)\big)\epsilon \\
&\leq \sup_{t \in [\alpha,\beta]} |g'(t)|(\beta - \alpha) \\
&\leq (|g'(\gamma)| + \epsilon)(\beta - \alpha)\epsilon
\end{aligned}$$

We also have, by a simpler argument
$$\left|\int_{\alpha}^{\beta} f(g(t))\,dt - \big(g(\beta) - g(\alpha)\big) f(g(\gamma))\right| \leq (\beta - \alpha)\epsilon.$$
Thus
$$\begin{aligned}
\left|\int_{g(\alpha)}^{g(\beta)} f(s)\,ds - \int_{\alpha}^{\beta} f(g(t))\,dt\right| &\leq (1 + |g'(\gamma)| + \epsilon)(\beta - \alpha)\epsilon \\
&\leq (2 + |g'(\gamma)|)\epsilon(\beta - \alpha).
\end{aligned}$$
We can make ϵ as small as we want.

K135*

No comments.

K136

The formula

$$\sup_{x\in[x_{j-1},x_j]} f'(t)(x_j - x_{j-1}) \geq f(x_j) - f(x_{j-1}) \geq \inf_{x\in[x_{j-1},x_j]} f'(t)(x_j - x_{j-1})$$

is a version of the mean value inequality.

Consider a dissection of $[a, b]$

$$\mathcal{D} = \{x_0,\, x_1,\, \ldots, x_N\}$$

with $a = x_0 < x_1 < \cdots < x_N = b$. By summing the formula of the first paragraph we get

$$S(\mathcal{D}, f') \geq f(b) - f(a) \geq s(\mathcal{D}, f')$$

so

$$I^*(f') \geq f(b) - f(a) \geq I_*(f')$$

and the result follows.

K137*

No comments.

K138

Let $N \geq 4$. Take a dissection $\mathcal{D}_N$ consisting of the points -1, 0, 1, $1 - N^{-3}$ and $rN^{-1} \pm N^{-3}$ with $1 \leq r \leq N - 1$. Then

$$0 = s(\mathcal{D}_N, f) \leq S(\mathcal{D}_N, f) \leq N^{-1} + N \times 2N^{-3} = N^{-1} + 2N^{-2} \to 0$$

as $N \to \infty$. Thus f is Riemann integrable and $\int_{-1}^{1} f(t)\, dt = 0$.

Similarly $F(t) = 0$ for all t. Thus $F'(0)$ exists with value $0 = f(0)$. Since $f(1/n) = 1 \nrightarrow 0$, f is not continuous at 0.

K139

(i)

$$\sum_{r=1}^{n} r(r-1) = \sum_{r=1}^{n} \frac{(r+1)r(r-1) - r(r-1)(r-2)}{3} = \frac{(n+1)n(n-1)}{3}.$$

$$\sum_{r=1}^{n} r = \sum_{r=1}^{n} \frac{(r+1)r - r(r-1)}{2} = \frac{(n+1)n}{2}$$

$$\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} r(r-1) + \sum_{r=1}^{n} r = \frac{n(n+1)(2n+1)}{6}.$$

(ii)

$$\sum_{r=1}^{n} n^3 = \frac{n^2(n+1)^2}{4}.$$

(iii) If $f(x) = x^m$

$$S(\mathcal{D}, f) = \sum_{r=1}^{n} n^{-1}(\frac{r}{n})^m = \frac{1}{m+1} + \frac{P_m(n)}{n^{m+1}} \to \frac{1}{m+1}$$

and

$$s(\mathcal{D}, f) = \sum_{r=0}^{n-1} n^{-1}(\frac{r}{n})^m = \frac{1}{m+1}(1 - n^{-1})^{m+1} + \frac{P_m(n-1)}{n^{m+1}} \to \frac{1}{m+1}$$

so $I^* f = I_* f = (m+1)^{-1}$.

(iv) If $f(x) = x^m$ and $F(x) = x^{m+1}/(m+1)$ then $F' = f$ and

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

K140

(v) We have $|g(t)-g(-1)(t+1)| \leq K(t-1)$ so

$$\left|\int_{-1}^{1} g(t)\,dt - g(-1)\right| \leq K\int_{-1}^{1}(t-1)\,dt = 2K.$$

Scaling, we get

$$\left|\int_{a-(r-1)h}^{a-rh} f(t)\,dt - f(a-(r-1)h)\right| \leq Kh^2$$

if $|f'(t)| \leq K$ for $t \in [a,b]$ and $Nh = b-a$. Thus, summing,

$$\left|\int_{a}^{b} f(t)\,dt - S_h(f)\right| \leq K(b-a)h.$$

Taking $a=0$, $b=\pi$, $Nh=\pi$ and $F(t)=\sin^2(nt)$, we see that the result cannot be substantially improved.

K141

(ii) Let $s = yt$.

(viii) We have

$$\frac{1}{1-t} - (1 + t + \cdots + t^n) = \frac{t^{n+1}}{1-t}.$$

So, integrating both sides from 0 to x, we have

$$-\log(1-x) - \left(x + \frac{x^2}{2} + \cdots + \frac{x^{n+1}}{n+1}\right) = \int_0^x \frac{t^{n+1}}{1-t}\, dt \to 0$$

as $n \to \infty$ for all $|x| < 1$. Thus

$$\log(1-x) = -\sum_{r=1}^{\infty} \frac{x^{r+1}}{r+1}$$

for $|x| < 1$. [The result also holds if $x = -1$ but the argument above does not prove this without extra work.]

(ix) By (viii)

$$\begin{aligned} \log\left(\frac{1+x}{1-x}\right) &= \log(1+x) - \log(1-x) \\ &= \sum_{r=1}^{\infty} \frac{(-1)^r x^{r+1}}{r+1} + \sum_{r=1}^{\infty} \frac{x^{r+1}}{r+1} \\ &= 2\sum_{r=1}^{\infty} \frac{x^{2r+1}}{2r+1} \end{aligned}$$

for $|x| < 1$.

If $y > 0$ and we seek an x with

$$y = \frac{1+x}{1-x}$$

we obtain

$$x = \frac{y-1}{y+1}$$

so $|x| < 1$ and substitution in the formula of (ix) gives the desired result.

K142

Power series can be multiplied term by term within their radius of convergence. We observe that

$$\sum_{r=1}^{n-1} \frac{(-1)^r}{r} \times \frac{(-1)^{n-r}}{n-r} = \frac{(-1)^n}{n} \sum_{r=1}^{n-1} \left(\frac{1}{r} + \frac{1}{n-r} \right) = \frac{(-1)^n 2S_{n-1}}{n}.$$

The result is valid for $|x| < 1$.

K143

(i) Write
$$\lambda = \frac{x_3 - x_2}{x_3 - x_1}$$
and observe that $1 > \lambda > 0$, so
$$\lambda f(x_1) + (1-\lambda)f(x_3) \geq f(\lambda x_3 + (1-\lambda)x_1)$$
so that
$$(x_3 - x_2)f(x_1) + (x_2 - x_1)f(x_3) \geq (x_3 - x_1)f(x_2)$$
and the result follows on rearrangement.

Observe, either similarly, or as a consequence that, if $y_1 < y_2 < y_3$, then
$$\frac{f(y_3) - f(y_1)}{y_3 - y_1} \geq \frac{f(y_2) - f(y_1)}{y_2 - y_1} \text{ and } \frac{f(y_3) - f(y_2)}{y_3 - y_2} \geq \frac{f(y_3) - f(y_1)}{y_3 - y_1}.$$

(ii) By (i), if $h > 0$, then
$$\frac{f(c+h) - f(c)}{h} \geq \frac{f(c) - f(d)}{c - d}$$
where d is fixed with $c > d > a$. Since a non-empty set bounded below has an infimum, $\sigma_f(c+)$ exists.

By definition, given $\epsilon > 0$ we can find a $\delta > 0$ such that
$$\sigma_f(c+) + \epsilon \geq \frac{f(c+\delta) - f(c)}{\delta}.$$
If $h > \delta > 0$ then (looking at the last paragraph of our discussion of (i))
$$\sigma_f(c+) + \epsilon \geq \frac{f(c+\delta) - f(c)}{\delta} \geq \frac{f(c+h) - f(c)}{h} \geq \sigma_f(c+)$$
so
$$\frac{f(c+h) - f(c)}{h} \to \sigma_f(c+)$$
as $h \to 0+$.

(iii) We can find a $\delta > 0$ such that, if $\delta > h > 0$, then
$$\sigma_f(c+) + 1 \geq \frac{f(c+h) - f(c)}{h} \geq \sigma_f(c+)$$
and so
$$|f(c+h) - f(c)| \leq (|\sigma_f(c+)| + 1)|h| \to 0$$
as $h \to 0+$. Similarly $f(c+h) - f(c) \to 0$ as $h \to 0-$ so we are done.

(iv) We have $f(x) - f(c) \geq B(x - c)$ for all x whenever $\sigma_f(c+) \geq B \geq \sigma_f(c-)$.

(v) Observe that
$$\alpha g(t) \geq t g(t) \geq \beta g(t)$$

and integrate to obtain $c \in [\alpha, \beta]$.

We know that there exists a B such that $f(x) - f(c) \geq B(x-c)$ and so
$$(f(t) - f(c))g(t) \geq B(t-c)g(t).$$
Now integrate.

(vi) We use the notation of Exercise K40 (iii). Let $c = \lambda_j x_j$. We know that there exists a B such that $f(x) - f(c) \geq B(x-c)$ and so
$$f(x_j) - f(c) \geq B(x_j - c)$$
whence
$$\sum_{j=1}^{n} \lambda_j f(x_j) - f\left(\sum_{j=1}^{n} \lambda_j x_j\right) = \sum_{j=1}^{n} \lambda_j (f(x_j) - f(c)) \geq \sum_{j=1}^{n} \lambda_j (x_j - c) = 0.$$

Similarly we know that there exists a B such that $f(x) - f(\mathbb{E}X) \geq B(x - \mathbb{E}X)$ and so
$$f(X) - f(\mathbb{E}X) \geq B(X - \mathbb{E}X).$$
Taking expectations gives the result.

(vii) Pick a', b' so that $a < a' < c < b' < b$ Observe that,
$$E = \{(x,y) \in \mathbb{R}^2 \,:\, y \geq f(x),\ x \in [a', b']\}$$
is a closed convex set. Applying Exercise 32 (iii) with $\mathbf{y} = (c, f(c) - n^{-1}0$ we see that there exist a_n, b_n and c_n such that
$$b_n x + a_n y \leq c_n \text{ whenever } (x,y) \in E$$
and $b_n c + a_n(f(c) - n^{-1}) > c_n$. By considering what happens when y is very large, we see that $a_n \leq 0$ and the conditions given are incompatible if $a_n = 0$. Thus $a_n < 0$ and, setting $B_n = b_n/a_n$,$C_n = c_n/a_n$, we obtain
$$B_n x + y \geq C_n \text{ whenever } (x,y) \in E$$
and $B_n c + (f(c) - n^{-1}) < C_n$.

In particular we have
$$B_n x + f(x) > B_n c + (f(c) - n^{-1})$$
for all $x \in [a', b']$. Thus
$$f(x) - f(c) > B_n(x-c) - n^{-1}$$
for all $x \in [a', b']$. If $\delta > 0$ is such that $[c-\delta, c+\delta] \subseteq [a', b']$ then
$$f(c+\delta) - f(c) > B_n\delta - n^{-1} \text{ and } f(c) - f(c+\delta) - f(c) < B_n\delta + n^{-1}.$$
Thus B_n is bounded and we can find a convergent subsequence $B_{n(j)} \to B$. We have
$$f(x) - f(c) \geq B(x-c)$$
for all $x \in [a', b']$.

To extend the result to (a,b), observe that the result just proved shows that that if $a < a+n^{-1} < c < b-n^{-1} < b$ we can find a B_n such that
$$f(x)-f(c) \geq B_n(x-c)$$
for all $x \in [a+n^{-1}, b-n^{-1}]$. A subsequence argument now gives the existence of a B with
$$f(x)-f(c) \geq B(x-c)$$
for all $x \in (a,b)$.

K144

(ii) Suppose that $\gamma < \beta$. If $0 < h, k < (\beta - \gamma)/2$ then

$$\frac{f(\beta) - f(\beta - h)}{h} \geq \frac{f(\beta - h) - f(\gamma + h}{\beta - \gamma - h - k} \geq \frac{f(\gamma + k) - f(\gamma)}{k}$$

so

$$\sigma_f(\beta-) \geq \sigma_f(\gamma+).$$

Thus

$$\begin{aligned}
&\sum_{j=1}^{N} \big(\sigma_f(c_j+) - \sigma_f(c_j-)\big) \\
&\qquad = \sigma_f(c_N+) - \sum_{j=1}^{N} \big(\sigma_f(\sigma_f(c_j-) - \sigma_f(c_{j-1}+))\big) - \sigma_f(c_1-) \\
&\qquad \leq \sigma_f(c_N+) - \sigma_f(c_1-) \\
&\qquad \leq \sigma_f(\alpha+) - \sigma_f(\beta-).
\end{aligned}$$

In particular the set

$$E_n = \{c \in (a, b) \,:\, \sigma_f(c-) - \sigma_f(c+)\}$$

contains at most $[(\sigma_f(\alpha+) - \sigma_f(\beta-))/N]$ points (with $[x]$ meaning the integer part of x). Thus

$$E = \{a,\, b\} \cup \bigcup_{j=1}^{\infty} E_j$$

is countable. If $c \notin E$, then $\sigma_f(c-) = \sigma_f(c+) = \sigma_f(c)$, say, and

$$\frac{f(c + h) - f(c)}{h} \to \sigma_f(c)$$

as $h \to 0$.

(iv) Enumerate the rational points in $(-1, 1)$ as $q_1,\, q_2,\, \ldots$. If we set

$$f_n(x) = n^{-2}|x - q_n|$$

then f_n is convex and

$$0 \leq f_n(x) \leq 2n^{-2}$$

Thus, using the comparison test, $\sum_{n=1}^{\infty} f_n(x)$ converges everywhere to $f(x)$, say.

Noting that, by convexity, $f_m(x+h)+f_m(x-h)-2f_m(x)\geq 0$ for all m when $x, x-h, x+h \in (-1,1)$, we have

$$\begin{aligned}
\frac{f(q_n+h)-f(q_n)}{h}-\frac{f(q_n)-f(q_n-h)}{h} &= \frac{f(q_n+h)+f(q_n-h)-2f(q_n)}{h}\\
&\geq \frac{f_n(q_n+h)+f(q_n-h)-2f(q_n)}{h}\\
&= 2n^{-2} \nrightarrow 0
\end{aligned}$$

as $h \to 0+$, so f is not differentiable at q_n

K145

(i) Given $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for $|x - c| < \delta$. If $[a_n, b_n] \subseteq (c - \delta, c + \delta)$, then

$$\begin{aligned}\left|\frac{1}{|I_n|}\int_{I_n} f(t)\,dt - f(c)\right| &= \left|\frac{1}{|I_n|}\int_{I_n} f(t) - f(c)\,dt\right|\\ &\leq \frac{1}{|I_n|}\int_{I_n} |f(t) - f(c)|\,dt\\ &\leq \frac{1}{|I_n|}\int_{I_n} \epsilon\,dt = \epsilon\end{aligned}$$

(ii) The following is one generalisation (interpreting the notation appropriately). Let $B(\mathbf{c}, r)$ be the ball centre $\mathbf{c}$ then (provided the integrals exist)

$$\frac{1}{\operatorname{Vol} B(\mathbf{c}, r)}\int_{B(\mathbf{c},r)} f(\mathbf{t})\,dV(\mathbf{t}) \to f(\mathbf{c})$$

as $r \to 0+$ whenever f is continuous at $\mathbf{c}$.

K146*

No comments.

K147

(i) The range of integration is not fixed.

(ii) If x is fixed the fundamental theorem of the calculus shows that $F_{,2}$ exists and
$$F_{,2}(x,y) = g(x,y)$$
so $F_{,2}$ is continuous.

Theorem 8.57 tells us that, if x is fixed we may differentiate under the integral to obtain
$$F_{,1}(x,y) = \int_0^y g_{,1}(x,y)\,dx.$$

If (x_0,y_0) is fixed Take $R \geq 2+|y_0|+|x_0|$. Since $g_{,1}$ is continuous on $K=[-R,R]\times[-R,R]$ it is uniformly continuous and bounded. Thus
$$\begin{aligned}
&|F_{,1}(x,y) - F_{,1}(x_0,y_0)| \\
&\quad \leq |F_{,1}(x,y) - F_{,1}(x_0,y)| + |F_{,1}(x_0,y) - F_{,1}(x_0,y_0)| \\
&\quad \leq |y| \sup_{|t|\leq|y|} |g_{,1}(x,t) - g_{,1}(x_0,t)| + |y-y_0| \sup_{|t|\leq \max(|y|,|y_0|)} |g_{,1}(x_0,t)| \\
&\quad \leq R \sup_{(s,t),(s',t)\in K, |s-s'|\leq|x-x_0|} |g_{,1}(s,t) - g_{,1}(s',t)| \\
&\qquad\quad + |y-y_0| \sup_{(s,t),(s',t)\in K} |g_{,1}(x_0,t)| \to 0
\end{aligned}$$
as $\|(x,y)-(x_0,y_0)\| \to 0$. Thus $F_{,2}$ is continuous and F is once differentiable.

(iii) The chain rule shows that G is differentiable and
$$G'(x) = F_1(x,x) + F_2(x,x) = \int_0^x g_{,1}(x,t)\,dt + g(x,x).$$

(iv) We have $H(x) = F(x,h(x))$ so the chain rule gives
$$H'(x) = F_1(x,h(x)) + F_2(x,h(x))h'(x) = \int_0^{h(x)} g_{,1}(x,t)\,dt + h'(x)g(x,h(x)).$$

K148

Observe that differentiating under the integral, using the symmetry of partial derivatives and the fundamental theorem of the calculus

$$\begin{aligned}
E'(t) &= \int_0^1 \frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \left(\frac{\partial u}{\partial x}(x,t)\right)^2\right)\,dx \\
&= 2\int_0^1 u_{,2}(x,t)u_{,22}(x,t) + u_{,1}(x,t)u_{,12}(x,t)\,dx \\
&= 2\int_0^1 -u_{,2}(x,t)u_{,11}(x,t) + u_{,1}(x,t)u_{,21}(x,t)\,dx \\
&= \int_0^1 \frac{d}{dx}\left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \left(\frac{\partial u}{\partial x}(x,t)\right)^2\right)\,dx \\
&= \left[\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \left(\frac{\partial u}{\partial x}(x,t)\right)^2\right]_0^1 = 0
\end{aligned}$$

so by the constant value theorem (and thus the mean value theorem) E is constant.

K149

H_δ is a smooth function with

$$\begin{aligned} H_\delta(t) &= 0 \qquad &\text{for } t \leq 0 \\ H_\delta(t) &= c_\delta \qquad &\text{for } t \geq 2\delta \end{aligned}$$

and $0 \leq H_\delta(t) \leq c_\delta$ for all t, where c_δ is a strictly positive number.

Let $L_\delta(t) = c_\delta^{-1}(H_\delta(t) - H_\delta(t + 1 - 2\delta)$. If $0 < \delta < 1/4$ then

$$\begin{aligned} L_\delta(t) &= 0 \qquad &\text{for } t \notin [0, 1] \\ L_\delta(t) &= 1 \qquad &\text{for } t \in [2\delta, 1 - 2\delta] \end{aligned}$$

and $0 \leq L_\delta(t) \leq 1$ for all t.

Now let $k_n(x) = (-1)^{[2nx]} L_{\epsilon/20}(2nx - 2[nx])$ and follow Exercise 8.66.

K150

We seek to minimise

$$\int_a^b F(x, y(x), y'(x))\,dx \text{ with } F(u, v, w) = g(u, v)(1 + w^2)^{1/2}.$$

The Euler-Lagrange equation gives

$$\begin{aligned}
0 &= F_{,2}(x, y(x), y'(x)) - \frac{d}{dx} F_{,3}(x, y(x), y'(x)) \\
&= g_{,2}(x, y)(1 + y'^2)^{1/2} - \frac{d}{dx}\frac{g(x, y)y'}{(1 + y'^2)^{1/2}} \\
&= g_{,2}(x, y)(1 + y'^2)^{1/2} - \frac{g_{,1}(x, y)y' + g_{,2}(x, y)y'^2 + g(x, y)y''}{(1 + y'^2)^{1/2}} + \frac{g(x, y)y'^2 y''}{(1 + y'^2)^{3/2}}.
\end{aligned}$$

Multiplying through by $1 + y'2$ gives the required result.

We seek to minimise

$$2\pi \int_a^b x\,ds.$$

We take $g(x, y) = x$ and obtain

$$-y' - \frac{xy''}{1 + y'^2} = 0$$

or

$$\frac{d}{dx}\left(\frac{xy'}{(1 + y'^2)^{1/2}}\right) = 0$$

(observe that $g(x, y)$ has no direct dependence on y and recall Exercise 8.63). Thus

$$\frac{xy'}{(1 + y'^2)^{1/2}} = c$$

whence

$$y' = \frac{c}{(x^2 - c^2)^{1/2}}$$

and

$$y + a = c \cosh^{-1} \frac{x}{c}$$

for appropriate constants a and c.

K151

Precisely as in the standard case, we consider

$$G(\eta) = \int_a^b f(x, y(x) + \eta g(x), y'(x) + \eta g'(x), y''(x) + \eta g''(x))\,dx$$

and require

$$0 = G'(0) = \int_a^b f_{,2}(x, y, y', y'') + g'(x) f_{,3}(x, y, y', y'') + g''(x) f_{,4}(x, y, y', y'')\,dx$$

for all well behaved g with $g(a) = g'(a) = 0$, $g(b) = g'(b) = 0$.

Integrating by parts once and twice gives

$$0 = \int_a^b g(x) \left(f_{,2}(x, y(x), y'(x), y''(x)) - \frac{d}{dx} f_{,3}(x, y(x), y'(x), y''(x)) + \frac{d^2}{dx^2} f_{,4}(x, y(x), y'(x), y''(x)) \right) dx$$

for all g as above, so we get an Euler-Lagrange type equation

$$0 = f_{,2}(x, y(x), y'(x), y''(x)) - \frac{d}{dx} f_{,3}(x, y(x), y'(x), y''(x)) + \frac{d^2}{dx^2} f_{,4}(x, y(x), y'(x), y''(x)).$$

In the case given this becomes

$$0 = -24 + \frac{d^2}{dx^2} y''(x)$$

that is to say

$$y''''(x) = 24$$

so $y = A + Bx + Cx^2 + Dx^3 + x^4$. The boundary conditions give us $A = B = 0$, $0 = C + D + 1$, $4 = 2C + 3D + 4$ so $C = -3$, $D = 2$ and

$$y(x) = -3x^2 + 2x^3 + x^4.$$

If we consider $y(x) + \epsilon x^2(1-x)^2 \sin Nx$, we can make small changes in y increasing I so we can not have a maximum.

K152*

No comments.

K153

(i) $g(1/2) = 1$, $g(x) = 0$ otherwise.

(ii) Could take

$$g(x) = \begin{cases} 1 - m2^{n+3}|x - r2^{-n}| & \text{if } |x - r2^{-n}| < m^{-1}2^{-n-3},\ 0 \leq r \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

K154

Second sentence done exactly as the case $h = 1$ (Lemma 9.6).

If $\int_1^\infty f(t)\,dt$ converges then, since

$$h\sum_{n=0}^{N-1} f(nh) \geq \int_0^{Nh} f(t)\,dt \geq h\sum_{n=1}^{N-1} f(nh),$$

we have (allowing $N \to \infty$)

$$h\sum_{n=0}^{\infty} f(nh) \geq \int_0^{\infty} f(t)\,dt \geq h\sum_{n=1}^{\infty} f(nh),$$

and so

$$\left| h\sum_{n=1}^{\infty} f(nh) - \int_0^{\infty} f(t)\,dt \right| \leq hf(0) \to 0$$

as $h \to 0+$.

Without loss of generality, suppose $1 > \epsilon > 0$. Choose $K > 8N\epsilon^{-1}$ and set

$$g(x) = \begin{cases} 1 - K|x - rN^{-1}| & \text{if } |x - rN^{-1}| < K,\ 0 \leq r \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We could set

$$G(x) = \begin{cases} 2^{-k/2}(1 - 2^{4k})|x - r2^{-k}| & \text{if } |x - r2^{-k}| < 2^{-4k},\ 2^k \leq r \leq 2^{k+1} - 1,\ k \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

K155

(i) Observe that

$$f(x) \leq f(n)+\int_n^x f'(t)\,dt \leq f(n)+\int_n^x |f'(t)|\,dt \leq f(n)+\int_n^{n+1} |f'(t)|\,dt$$

for $n \leq x \leq n+1$ and so

$$\int_n^{n+1} f(x)\,dx \leq f(n) + \int_n^{n+1} |f'(t)|\,dt.$$

Similarly

$$\int_n^{n+1} f(x)\,dx \geq f(n) - \int_n^{n+1} |f'(t)|\,dt.$$

It follows that,

$$\left|\sum_{r=n}^{m} f(r)\right| \leq \left|\int_n^{m+1} f(x)\,dx\right| + \int_n^{m+1} |f'(x)|\,dx$$

and, if $\int_1^\infty f(x)\,dx$ converges, $\sum_{r=0}^n f(r)$ is a Cauchy sequence and converges by the general principle of convergence.

Conversely, if $\sum_{r=0}^\infty f(r)$ converges, a similar argument shows that $\int_1^n f(x)\,dx$ tends to a limit. Further, if $\sum_{r=0}^\infty f(r)$ converges we have $f(n) \to 0$ and the arguments above show that

$$\int_n^{n+1} |f(x)|\,dx \leq |f(n)| + \int_n^{n+1} |f'(x)|\,dx \to 0$$

as $n \to \infty$ and so $\int_1^\infty f(x)\,dx$ converges.

(ii) If f is decreasing, positive and has a continuous derivative

$$\int_1^X |f'(x)|\,dx = -\int_1^X f'(x)\,dx = f(1) - f(X) \leq f(1).$$

(iii) False. We consider a continuously differentiable function g such that

$$g(x) = \begin{cases} (2n)^{-1}2^{-2n} & \text{if } 2^{2n} + 2^{-1} \leq x \leq 2^{2n+1} - 2^{-1},\ n \geq 0, \\ -(2n+1)^{-1}2^{-2n-1} & \text{if } 2^{2n+1} + 2^{-1} \leq x \leq 2^{2n+2} - 2^{-1},\ n \geq 0, \end{cases}$$

and, in addition, g is decreasing on $[2^{2n+1}-2^{-1}, 2^{2n+1}+2^{-1}]$, increasing on $[2^{2n} - 2^{-1}, 2^{2n} + 2^{-1}]$ and $g(2^m) = 0$ for all m.

K156

$$\int_{n-1/2}^{n+1/2} \log x\,dx - \log n = \int_{n}^{n+1/2} (\log x - \log n)\,dx + \int_{n-1/2}^{n} (\log x - \log n)\,dx$$
$$= \int_{n}^{n+1/2} \log(x/n)\,dx + \int_{n-1/2}^{n} \log(x/n)\,dx$$
$$= \int_{0}^{1/2} \left(\log\left(1+\frac{t}{n}\right) + \log\left(1-\frac{t}{n}\right)\right)\,dt$$

making substitutions like $x = t+n$ and $x = n-t$.

But by the mean value inequality,

$$\left|\log\left(1+\frac{t}{n}\right) + \log\left(1-\frac{t}{n}\right)\right| \leq 2^{-1} \sup_{t\in(0,1/2)} \left|\frac{1}{t+n} - \frac{1}{n-t}\right|$$
$$= 2^{-1} \sup_{t\in(0,1/2)} \frac{2t}{n^2-t^2} \leq \frac{4}{3n^2}.$$

Thus by the comparison test

$$\sum_{n=1}^{\infty} \left(\log\left(1+\frac{t}{n}\right) + \log\left(1-\frac{t}{n}\right)\right)$$

is absolutely convergent and so convergent to c, say. Thus

$$\int_{1/2}^{N+1/2} \log x\,dx - \log N! \to c.$$

But, integrating by parts,

$$\int_{1/2}^{N+1/2} \log x\,dx = [x\log x]_{1/2}^{N+1/2} - \int_{1/2}^{N+1/2} dx$$
$$= (N+\tfrac{1}{2})\log(N+\tfrac{1}{2}) - \tfrac{1}{2})\log\tfrac{1}{2} - (N+1).$$

Thus

$$(N+\tfrac{1}{2})\log(N+\tfrac{1}{2}) - N - \log N! \to c'$$

for some constant C and the result follows on taking exponentials (and noting that exp is continuous).

K157

With the notation of the previous question
$$\frac{2n!}{(2n+1/2)^{(2n+1/2)}}e^{-2n}\bigg/\left(\frac{n!}{(n+1/2)^{(n+1/2)}}e^{-n}\right)^2 \to C^{-1}$$
where C is the constant of K156. Thus
$$\binom{2n}{n}\frac{(1+\frac{1}{4n})^{2n}}{(1+\frac{1}{2n})^{2n}}2^{2n}n^{1/2} \to C'$$
and
$$4^n n^{1/2}\binom{2n}{n} \to C''$$
as $n \to \infty$ for appropriate constants C' and C''. In particular, we can find constants K_1 and K_2 with $K_1 > K_2 > 0$ such that
$$K_1 4^{-n}n^{-1/2} \geq \binom{2n}{n} \geq 4^{-n}n^{-1/2}.$$
Thus
$$\left|n^\alpha\binom{2n}{n}z^n\right| \to 0 \text{ if } |z| < 1/4 \text{ and } \left|n^\alpha\binom{2n}{n}z^n\right| \to 0 \text{ if } |z| > 1/4$$
so $\sum_{n=0}^\infty n^\alpha\binom{2n}{n}z^n$ has radius of convergence $1/4$.

If $\alpha < -1/2$ we have $|n^\alpha\binom{2n}{n}z^n| \leq K_1 n^{\alpha-1/2}$ for all $|z| = 1/4$ so, by the comparison test, $\sum_{n=0}^\infty n^\alpha\binom{2n}{n}z^n$ converges absolutely and so converges everywhere on the circle of convergence.

If $\alpha \geq 1/2$ we have $|n^\alpha\binom{2n}{n}z^n| \geq K_2$ for all $|z| = 1/4$ so $\sum_{n=0}^\infty n^\alpha\binom{2n}{n}z^n$ diverges everywhere on the circle of convergence.

If $\alpha \geq -1/2$ then $n^\alpha\binom{2n}{n} \geq K_2 n^{\alpha-1/2}$ and the comparison test tells us that $\sum_{n=0}^\infty n^\alpha\binom{2n}{n}z^n$ diverges when $z = 1/4$.

The only remaining case is $|z| = 1/4$, $z \neq 1/4$ and $1/2 > \alpha \geq -1/2$. Let us write $w = 4z$ and $u_n = 4^{-n}n^\alpha\binom{2n}{n}$. We know that $u_n \to 0$ so, if we can show that $u_n > u_{n+1}$ for all sufficiently large n, Abel's test will tell us that $\sum_{n=0}^\infty u_n w^n$. But
$$\frac{u_{n+1}}{u_n} = \frac{(1+\frac{1}{2n})}{(1+\frac{1}{n})^{1-\alpha}} < 1$$
so the conditions of Abel's test are satisfied. (Observe that $(1+x)^\beta \geq 1+\beta x$ for $\beta \geq 0$ and $x \geq 0$.) If $1/2 > \alpha \geq -1/2$ then $\sum_{n=0}^\infty n^\alpha\binom{2n}{n}z^n$ converges everywhere on the circle of convergence except the point $z = 1/4$.

K158

(i) Observe that

$$\int_0^{g(X)} f(s)\,ds = \int_0^X f(x)g'(x)\,dx$$

and that $g(X) \to \infty$ as $X \to \infty$ and, if $g(h(Y)) \to \infty$ as $Y \to \infty$, then $h(Y) \to \infty$ as $Y \to \infty$.

Only one limit so no problem of interchange.

(ii) $(\sin x)/x \to 1$ as $x \to 0$ (by considering differentiation, if you wish).

Observe that

$$0 \le (-1)^{n+2}\frac{\sin(x+(n+1)\pi)}{x+(n+1)\pi} \le (-1)^{n+1}\frac{\sin(x+n\pi)}{x+n\pi}$$

so

$$0 \le (-1)^{n+2}\int_0^{\pi} \frac{\sin(x+(n+1)\pi)}{x+(n+1)\pi}\,dx \le (-1)^{n+1}\int_0^{\pi} \frac{\sin(x+n\pi)}{x+n\pi}\,dx$$

and, if we write

$$v_n = (-1)^{n+1}\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx,$$

the v_N are a decreasing sequence with

$$0 \le v_n \le \int_{n\pi}^{(n+1)\pi} \frac{1}{x}\,dx \le (n\pi)^{-1} \to 0$$

as $n \to \infty$. Thus, by the alternating series test,

$$\int_0^{n\pi} \frac{\sin x}{x}\,dx = \sum_{r=0}^{n-1}(-1)^r v_r$$

converges to a limit L with $v_0 \ge L \ge v_0 - v_1 > 0$.

Since

$$\int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right|\,dx \le (n\pi)^{-1} \to 0,$$

we have

$$\int_0^X \frac{\sin x}{x}\,dx \to L$$

as $X \to \infty$.

(iii) Change of variable $s = xt$ with $t > 0$ gives $I(t) = I(1) = L$. Since $(\sin -x)/x = -(\sin x)/x$, $I(-t) = -I(t)$.

(vi) By convexity $\sin x \leq 2x/\pi$ for $0 \leq x \leq \pi/2$, so $g(x) = g(\pi - x) = 2x/\pi^2$ for $0 \leq x \leq \pi/2$ will do. Observe that

$$\begin{aligned}\int_{\pi}^{n\pi} \left|\frac{\sin x}{x}\right| dx &= \sum_{r=1}^{n-1} \int_{r\pi}^{(r+1)\pi} \left|\frac{\sin x}{x}\right| dx \\ &\geq \sum_{r=1}^{n-1} \int_{r\pi}^{(r+1)\pi} \frac{x - r\pi}{r} dx \\ &= \int_0^{\pi} g(x)\, dx \sum_{r=1}^{n-1} \frac{1}{r} \to \infty\end{aligned}$$

as $n \to \infty$.

(v) If $\alpha > 0$, then the arguments above show that $\int_1^\infty \frac{\sin x}{x^\alpha} dx$ converges. If $\alpha \leq 0$, then

$$\left|\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x^\alpha} dx\right| \geq \int_{n\pi}^{(n+1)\pi} |\sin x|\, dx \nrightarrow 0$$

so $\int_1^\infty \frac{\sin x}{x^\alpha} dx$ does not converge.

However, if $0 \leq x \leq 1$ we have $x \geq \sin x \geq 2x/\pi$ so

$$x^{1-\alpha} \geq \frac{\sin x}{x^\alpha} \geq 2x^{1-\alpha}/\pi$$

and comparison shows that

$$\int_\epsilon^1 \frac{\sin x}{x^\alpha} dx$$

converges as $\epsilon \to 0+$ if and only if $\alpha < 2$.

Thus $\int_0^\infty \frac{\sin x}{x^\alpha} dx$ converges if and only if $2 > \alpha > 0$.

K159

(i) Observe that

$$1+2\sum_{r=1}^{n}\cos rx=\sum_{r=-n}^{n}e^{irx}=e^{-inx}\frac{1-e^{i(2n+1)x}}{1-e^{ix}}=\frac{\sin((n+\frac{1}{2})x)}{x}$$

for $|x|\leq\pi$.

(ii) Change of variable $t=\lambda x$.

(iii) Observe that the integrals

$$\int_{r\pi/(n+1/2)}^{(r+1)\pi/(n+1/2)}\frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\frac{1}{2}x}\,dx$$

alternate in sign and decease in absolute value as r increases from 1 to n and as r decreases from $-!$ to $-n$. Observe further that if $\epsilon\leq\alpha\leq\beta\leq\pi$ and $\alpha-\beta\leq\pi/(n+1/2)$ then

$$\left|\int_{\alpha}^{\beta}\frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\frac{1}{2}x}\,dx\right|=\left|\int_{-\beta}^{-alpha}\frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\frac{1}{2}x}\,dx\right|\leq\frac{\pi}{(n+frac12)|\sin\epsilon|}\to 0$$

as $n\to\infty$.

(iv) Write

$$\frac{2}{x}-\frac{1}{\sin\frac{1}{2}x}=\frac{2(\sin\frac{1}{2}x-\frac{1}{2}x)}{x\sin\frac{1}{2}x}$$

and use L'Hôpital or Taylor expansions.

(v) Given any $\eta>0$ we can find an $\epsilon>0$ with $\eta>\epsilon$ such that

$$\left|\frac{2}{x}-\frac{1}{\sin\frac{1}{2}x}\right|<\eta$$

for $|x|<\epsilon$ and so

$$\left|\int_{-\epsilon}^{\epsilon}\frac{\sin((n+\frac{1}{2})x)}{\sin\frac{x}{2}}\,dx-\frac{2\sin((n+\frac{1}{2})x)}{x}\,dx\right|\leq 2\epsilon\eta\leq 2\eta^2.$$

Thus, allowing $n\to\infty$ and using (ii) and (iii) we have

$$\left|2\pi-2\int_{-\infty}^{\infty}\frac{\sin x}{x},dx\right|\leq 2\eta^2.$$

Since η was arbitrary

$$\int_{-\infty}^{\infty}\frac{\sin x}{x},dx=\pi$$

and we are done.

K160

(i) and (ii) (Here ‘permits’ means ‘permits but does not imply’.)

$f(n) = O(g(n))$, permits $f(n) = o(g(n))$, permits $f(n) = \Omega(g(n))$ and permits $f(n) \sim g(n)$.

$f(n) = o(g(n))$, implies $f(n) = O(g(n))$, forbids $f(n) = \Omega(g(n))$ and forbids $f(n) \sim g(n)$.

$f(n) = \Omega(g(n))$ permits $f(n) = O(g(n))$, permits $f(n) \sim g(n)$ and forbids $f(n) = o(g(n))$.

$f(n) \sim g(n)$ implies $f(n) = O(g(n))$, implies $f(n) = \Omega(g(n))$ and forbids $f(n) = o(g(n))$.

Suitable examples will be found amongst the pairs $f(n) = g(n) = 1$; $f(n) = n$, $g(n) = 1$; $f(n) = 1$, $g(n) = n$.

(iii) $f(n) = o(1)$ means $f(n) \to 0$ as $n \to 0$. $f(n) = O(1)$ means f is bounded.

(iv) Could take $f(n) = 1 + n(1 + (-1)^n)$, $g(n) = 1 + n(1 + (-1)^{n+1})$.

K161

(i) True. If $0 \leq f_j(n) \leq A_j g_j(n)$ with $A_j \geq 0$ then
$$0 \leq f_1(n) + f_2 \leq (A_1 + A_2)(g_1(n) + g_2(n)).$$

(ii) False. Take $f_1(n) = f_2(n) = g_1(n) = -g_2(n)$.

(iii) True. If $0 \leq f_j(n) \leq A_j g_j(n)$ with $A_j \geq 0$ then
$$0 \leq f_1(n) + f_2 \leq (A_1 + A_2)\max(g_1(n), g_2(n)).$$

(iv) True. $n! \leq n^n \leq (2^n)^n = 2^{n^2}$.

(v) True. Use L'Hôpital or (I think better) observe that
$$\cos x - 1 + x^2/2 = O(x^4),\ \sin x = x + \alpha(x)x^3,$$
with $|\alpha(x)| < 1$ when $|x|$ is small so
$$(\sin x)^2 = x^2 + 2\alpha(x)x^4 + \alpha(x)^2 x^5$$
and
$$(\sin x)^2 - x^2 = O(x^4).$$

(vi) False. Use L'Hôpital or (I think better) observe that
$$\cos x - 1 + x^2/2 - x^4/4! = O(x^6),\ \sin x = x - x^3/6 + \beta(x)x^5,$$
with $|\gamma(x)| < 1$ when $|x|$ is small so
$$\sin^2 x = x^2 - x^4/3 + \gamma_1(x)x^6,\ \text{and}\ \sin^4 x = x^4 + \gamma_2(x)x^6$$
where $|\gamma_1(x)|$, $|\gamma_2(x)| \leq 10$ for $|x|$ small (we can do better but we do not need to) so
$$\cos x - 1 + \sin^2 x/2 - \sin^4 x/4! + x^4/6 = O(x^6).$$

K162

(i) If $-1 < \alpha \leq 0$, then x^α is decreasing so

$$\sum_{r=1}^{n} r^\alpha \geq \sum_{r=1}^{n} \int_r^{r+1} x^\alpha \, dx = \int_1^{n+1} x^\alpha \, dx \geq \sum_{r=2}^{n+1} r^\alpha.$$

Since

$$\int_1^n x^\alpha \, dx = \frac{n^{\alpha+1} - 1}{\alpha + 1},$$

this gives

$$\left| \frac{(\alpha+1) \sum_{r=1}^{n} r^\alpha}{n^{\alpha+1}} \right| \leq \frac{1 + 2n^\alpha}{n^{\alpha+1}} \to 0$$

as $n \to \infty$.

If $\alpha \geq 0$, then x^α is increasing so

$$\sum_{r=1}^{n} r^\alpha \leq \sum_{r=1}^{n} \int_r^{r+1} x^\alpha \, dx = \int_1^{n+1} x^\alpha \, dx \leq \sum_{r=2}^{n+1} r^\alpha$$

and much the same argument works.

(ii) If $\alpha = -1$, then much the same argument gives

$$\sum_{r=1}^{n} \frac{1}{r} \sim \log n.$$

(iv) No such modification, since $\sum_{r=n}^{\infty} r^{-1}$ diverges.

K163

Argument similar to that in K156. We have $|g''(x)| \leq Ax^{-\lambda}$ for $x \geq R$ say. If $|t| \leq \frac{1}{2}$ and $n \geq R+1$, then applying the mean value theorem twice gives

$$\begin{aligned}
&|(g(t+n+\tfrac{1}{2}) - g(n+\tfrac{1}{2})) + (g(-t+n+\tfrac{1}{2}) - g(n+\tfrac{1}{2}))| \\
&\qquad = |g'(s+n+\tfrac{1}{2}) - g'(-s+n+\tfrac{1}{2})| \\
&\qquad = |2sg'(u+n+\tfrac{1}{2})| \leq A(n-\tfrac{1}{2})^{-\lambda}
\end{aligned}$$

for some s and u with $\frac{1}{2} > s > 0$ and $|u| < s$. Thus we can find a B such that

$$|(g(t+n+\tfrac{1}{2}) - g(n+\tfrac{1}{2})) + (g(-t+n+\tfrac{1}{2}) - g(n+\tfrac{1}{2}))| \leq Bn^{-\lambda}$$

for $n \geq R$ and $|t| \leq \frac{1}{2}$.

We have

$$\begin{aligned}
&\left|\int_n^{n+1} (g(x) - g(n+\tfrac{1}{2})\,dx\right| \\
&\quad = \left|\int_0^{\frac{1}{2}} (g(t+n+\tfrac{1}{2}) - g(n+\tfrac{1}{2})) + (g(-t+n+\tfrac{1}{2}) - g(n+\tfrac{1}{2}))\,dt\right| \\
&\quad \leq \int_0^{\frac{1}{2}} |(g(t+n+\tfrac{1}{2}) - g(n+\tfrac{1}{2})) + (g(-t+n+\tfrac{1}{2}) - g(n+\tfrac{1}{2}))|\,dt \\
&\quad \leq Bn^{-\lambda}
\end{aligned}$$

so

$$\int_n^{n+1} (g(x) - g(n+\tfrac{1}{2})dx = O(n^{-\lambda}).$$

We have

$$\left|\int_n^{n+1} g(x)\,dx - g(n+\tfrac{1}{2})\right| \leq Bn^{-\lambda}$$

for $n \geq R$ so adding and using K162 or direct argument

$$\left|\int_1^{n+1} g(x)\,dx - \sum_{r=1}^{n} g(r)\right| \leq C + \frac{Bn^{-\lambda+1}}{-\lambda+1}$$

for some constant C. Dividing through by $\int_1^{n+1} g(x)\,dx$ gives the required result.

K164

Since $\sin x > 2x/\pi$ for $0 \leq x \leq \pi/2$, we have

$$0 \leq -\log \sin x \leq \log(\pi/2) - \log x \leq \log(\pi/2) + x^{-1/2}$$

when x is small and positive, so, by comparison, $\int_0^{\pi/2} \log(\sin x)\, dx$ converges. (Similar arguments apply to all the integrals of this question.)

Using the change of variables formula with $t = 2x$ and symmetry we have

$$\int_0^{\pi/2} \log(\sin 2x)\, dx = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt = \int_0^{\pi/2} \log(\sin x)\, dx.$$

But, using symmetry again,

$$\begin{aligned}\int_0^{\pi/2} \log(\sin 2x)\, dx &= \int_0^{\pi/2} \log(2 \sin x \cos x)\, dx \\ &= \int_0^{\pi/2} \log 2\, dx + \int_0^{\pi/2} \log(\sin x)\, dx + \int_0^{\pi/2} \log(\cos x)\, dx \\ &= \frac{\pi}{2}\log 2 + 2\int_0^{\pi/2} \log(\sin x)\, dx.\end{aligned}$$

Combining our formulae gives the result.

K165

Suppose that $\int_0^a f(x)\,dx$ converges. Since f is decreasing

$$\int_0^a f(x)\,dx \geq \frac{a}{N}\sum_{r=1}^{N} f\left(\frac{ar}{N}\right) \geq \int_{a/N}^a f(x)\,dx \to \int_0^a f(x)\,dx$$

so

$$\sum_{r=1}^{N} f\left(\frac{ar}{N}\right) \to \int_0^a f(x)\,dx.$$

Suppose that $\int_0^a f(x)\,dx$ diverges. Then given any K we can find an $\epsilon > 0$ such that

$$\int_\epsilon^a f(x)\,dx \geq K + 1.$$

However, since f is continuous

$$\sum_{\epsilon N \leq r \leq N} f\left(\frac{ar}{N}\right) \to \int_\epsilon^a f(x)\,dx$$

so we can find an N_0 such that

$$\sum_{\epsilon N \leq r \leq N} f\left(\frac{ar}{N}\right) \to \int_\epsilon^a f(x)\,dx - 1$$

and so

$$\sum_{r=1}^{N} f\left(\frac{ar}{N}\right) \geq K$$

for all $N \geq N_0$. Thus

$$\sum_{r=1}^{N} f\left(\frac{ar}{N}\right) \to \infty$$

as $N \to \infty$.

(ii) Consider $a = 1$,

$$f(x)\begin{cases} 2^n(1 - 2^{4n}|x - 2^{-n}|) & \text{if } |x - 2^{-n}| \leq 2^{-4n}, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Observe that

$$(1 - z)\sum_{r=0}^{N-1} z^n = 1 - z^N = \prod_{r=0}^{N-1}(z - \omega^r)$$

so, dividing by $z - 1$,

$$\sum_{r=0}^{N-1} z^n = 1 - z^N = \prod_{r=1}^{N-1}(z - \omega^r)$$

for $z \neq 1$. Since a polynomial of degree $N-1$ which vanishes at N points vanishes identically we have

$$\sum_{r=0}^{N-1} z^n = 1 - z^N = \prod_{r=1}^{N-1} (z - \omega^r)$$

for all z.

In particular, setting $z = 1$, we have

$$N = \prod_{r=1}^{N-1} (1 - \omega^r) = \tau^{1+2+\cdots+(N-1)} \prod_{r=1}^{N-1} (\tau^{-r} - \tau^r)$$

where $\tau = \pi/N$. Thus

$$N = \tau^{N(N-1)/2} (-2i)^{N-1} \prod_{r=1}^{N-1} \sin \frac{r\pi}{N} = 2^{N-1} \prod_{r=1}^{N-1} \sin \frac{r\pi}{N}.$$

(iv) Observe that

$$\begin{aligned}
\frac{\pi}{2N} \sum_{r=1}^{N} \log\left(\sin\left(\frac{\pi r}{N}\right)\right) &= \frac{\pi}{2N} \log\left(\sum_{r=1}^{N-1} \sin\left(\frac{\pi r}{N}\right)\right) \\
&= \frac{\pi}{2N} \log \frac{N}{2^{N-1}} \\
&= -\pi \frac{N-1}{2N} \log 2 + \pi \frac{\log N}{2N} \to -\frac{\pi}{2} \log 2.
\end{aligned}$$

K166

Observe that

$$E_k = \{x \,:\, f(x) = k\}$$

is the union of a finite set of disjoint intervals (of the form $[a, b)$) of total length $(9/10)^{k-1}/10$. Thus (writing $\mathbb{I}_A$ for the indicator function of A)

$$f_X = \sum_{1\leq k\leq X} k\mathbb{I}_{E_k} + X\left(1 - \sum_{1\leq k\leq X} \mathbb{I}_{E_k}\right)$$

is Riemann integrable with

$$\int_0^1 f_X(x)\,dx = 10^{-1}\sum_{1\leq k\leq X} k(9/10)^{k-1} + X\left(1 - 9^{-1}\sum_{1\leq k\leq X}(9/10)^{k-1}\right).$$

Now writing $[X]$ for the integer part of X,

$$0 \leq X\left(1 - 10^{-1}\sum_{1\leq k\leq X}(9/10)^{k-1}\right) = X(9/10)^{[X]} \leq ([X]{+}1)(9/10)^{[X]} \to 0$$

and using the Taylor expansion

$$(1-t)^{-2} = \sum_{j=0}^{\infty}(j+1)t^j$$

(Newton's binomial expansion) we have

$$\int_0^1 f_X(x)\,dx \to 10^{-1}(1 - 9/10)^{-2} = 10.$$

It makes no difference if we replace f by g with $g(x) = f(x)$ except at points where $f(x) = \infty$. However we have to work a little harder to show that g_X is Riemann integrable. One way is the observe that

$$g_X(x) - f_X(x) = 0$$

for all $x \in \bigcup_{k=1}^{M} E_k$ so, given any $\epsilon > 0$, we know that $g_X(x) - f_X(x) = 0$ on the union of a finite set of disjoint intervals of total length $1 - \epsilon/2$ and so

$$If_X - X\epsilon = I_*f_X - X\epsilon \leq I_*g_X \leq I^*g_X \leq\leq I^*f_X + X\epsilon = If_X + X\epsilon.$$

Since ϵ is arbitrary g_X is integrable and $Ig_X = If_X$.

K167

(ii) Observe that if $\eta > 0$ is fixed, we can find an $R_0(\eta)$ such that
$$\int_a^b f_{R,S}(x)\,dx \geq \int a^{b-\eta} f(x)\,dx$$
for all R, $S \geq R_0(\eta)$, and if R and S are fixed, we can find an $\eta_0(R,S) > 0$ such that
$$\int a^{b-\eta} f(x)\,dx \geq \int_a^b f_{R,S}(x)\,dx$$
for all $0 < \eta < \eta_0(R,S)$.

(iii) If (A) is true, the argument of (i) shows that, if we write $f_+(x) = \max(f(x),0)$ and $f_-(x) = \min(f(x),0)$, then (A) remains true for both f_+ and f_-. There is this no loss of generality in assuming that f is positive. The argument of (ii) now applies.

(iv) The substitution $t = x^{-1}$ suggests
$$f(t) = \frac{\sin t^{-1}}{t}.$$

K168

(Note that, since $(x,t) \mapsto e^{-tx}$ is continuous on $[0,X] \times [a,b]$, the apparent singularity at $x = 0$ must be a trick of the notation.)

$$\begin{aligned}\int_0^X \frac{e^{-ax} - e^{-bx}}{x}\,dx &= \int_0^X \int_a^b e^{-tx}\,dt\,dx\\ &= \int_a^b \int_0^X e^{-tx}\,dx\,dt\\ &= \int_a^b \frac{1 - e^{-tX}}{t}\,dt.\end{aligned}$$

Now

$$0 \leq \frac{e^{-tX}}{t} \leq \frac{e^{-aX}}{t}$$

for $t \in [a,b]$ so

$$0 \leq \int_a^b \frac{e^{-tX}}{t}\,dt \leq e^{-aX} \log\left(\frac{b}{a}\right) \to 0$$

as $X \to 0$ so

$$\int_0^X \frac{e^{-ax} - e^{-bx}}{x}\,dx = \int_a^b \frac{1}{t}\,dt - \int_a^b \frac{e^{-tX}}{t}\,dt \to \log\frac{b}{a}$$

as $x \to \infty$.

K169

By the fundamental theorem of the calculus
$$\frac{d}{dt}\int_a^t \left(\int_c^d f(u,v)\,dv\right)\,du = \int_c^d f(t,v)\,dv.$$
By differentiating under the integral sign and using the fundamental theorem of the calculus
$$\frac{d}{dt}\int_c^d \left(\int_a^t f(u,v)\,du\right)\,dv = \int_c^d \frac{\partial}{\partial t}\left(\int_a^t f(u,v)\,du\right)\,dv = \int_c^d f(t,v)\,dv.$$
Thus
$$\frac{d}{dt}\left(\int_a^t \left(\int_c^d f(u,v)\,dv\right)\,du - \int_c^d \left(\int_a^t f(u,v)\,du\right)\,dv\right) = 0,$$
and by the constant value theorem
$$\begin{aligned}\int_a^t \left(\int_c^d f(u,v)\ \ dv\right)\,du - \int_c^d \left(\int_a^t f(u,v)\,du\right)\,dv\\ = \int_a^a \left(\int_c^d f(u,v)\,dv\right)\,du - \int_c^d \left(\int_a^a f(u,v)\,du\right)\,dv = 0.\end{aligned}$$

It is weaker because we say nothing about the existence or value of $\int_R f(\mathbf{x})\,dA$.

K170

(i) We have
$$\int_0^1 f(x,y)\,dx = 0$$
for all x and
$$\int_0^1 f(x,y)\,dy = \begin{cases} 2 & \text{if } 2^{-1} < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$
so
$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0 \neq 1 = \int_0^1 \int_0^1 f(x,y)\,dy\,dx.$$

(ii) We have
$$\int_0^1 g(x,y)\,dx = 0$$
for all x and
$$\int_0^1 g(x,y)\,dy = \begin{cases} 2u(2y-1) & \text{if } 2^{-1} < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$
so
$$\int_0^1 \int_0^1 g(x,y)\,dx\,dy = 0 \neq 1 = \int_0^1 \int_0^1 g(x,y)\,dy\,dx.$$
Theorem 9.20 demands continuity everywhere. (The underlying problem is that g is not bounded.)

(iii) We have
$$\begin{aligned}\int_0^1 h(x,y)\,dx &= \frac{1}{2}\int_{y^2}^{1+y^2} \frac{y(w-2y^2)}{w^3}\,dw \\ &= \frac{1}{2}\left[-\frac{y}{w} + \frac{y^3}{w^2}\right]_{y^2}^{1+y^2} \\ &= \frac{-y}{2(1+y^2)^2}\end{aligned}$$
so that
$$\int_0^1 \int_0^1 h(x,y)\,dx\,dy = \left[\frac{1}{4(1+y^2)}\right]_0^1 = -1/8.$$
Observe that $h(y,x) = -h(x,y)$.

K171

(ii) Observe that, if $R \geq 1$, then

$$\bigstar \qquad \left|\int_0^\infty f(x,y)\,dx - \int_0^R f(x,y)\,dx\right| \leq \int_R^\infty \frac{A}{(1+x^2)(1+y^2)}\,dx$$
$$= A\frac{\frac{\pi}{2} - \tan^{-1} R}{1+y^2}$$
$$\leq A(\tfrac{\pi}{2} - \tan^{-1} R)$$

Thus given $\epsilon > 0$ we can find an R such that

$$\left|\int_0^\infty f(x,y)\,dx - \int_0^R f(x,y)\,dx\right| \leq \epsilon/4$$

for all y. Since $\int_0^R f(x,y)\,dx$ is a continuous function of y it follows that given any y_0 we can find a $\delta > 0$ such that $|y_0 - y| < \delta$ implies

$$\left|\int_0^R f(x,y)\,dx - \int_0^R f(x,y_0)\,dx\right| \leq \epsilon/2$$

and so

$$\left|\int_0^\infty f(x,y)\,dx - \int_0^\infty f(x,y_0)\,dx\right| < \epsilon.$$

If follows that $\int_0^\infty f(x,y)\,dx$ is a continuous function of y. (This can be done better using uniform convergence.)

Since $\int_0^\infty f(x,y)\,dx$ is continuous in y and

$$\left|\int_0^\infty f(x,y)\,dx\right| \leq \frac{A\pi/2}{1+y^2},$$

comparison shows that $\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy$ exists. A similar but simpler argument shows that $\int_0^\infty \int_0^R f(x,y)\,dx$ exists Equation $\bigstar$ shows that

$$\left|\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy - \int_0^\infty \int_0^R f(x,y)\,dx\,dy\right| \leq (\tfrac{\pi}{2} - \tan^{-1} R)\frac{\pi A}{2}.$$

We also know that

$$\left|\int_0^\infty \int_0^R f(x,y)\,dx\,dy - \int_0^R \int_0^R f(x,y)\,dx\,dy\right| \leq (\tfrac{\pi}{2} - \tan^{-1} R)\frac{\pi A}{2}$$

so that

$$\left|\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy - \int_0^R \int_0^R f(x,y)\,dx\,dy\right| \leq (\tfrac{\pi}{2} - \tan^{-1} R)\pi A.$$

Similarly

$$\left|\int_0^\infty \int_0^\infty f(x,y)\,dy\,dx - \int_0^R \int_0^R f(x,y)\,dy\,dx\right| \leq (\tfrac{\pi}{2} - \tan^{-1} R)\pi A,$$

so, using Theorem 9.20 (Fubini on products of intervals),

$$\left| \int_0^\infty \int_0^\infty f(x,y)\,dy\,dx - \int_0^\infty \int_0^\infty f(x,y)\,dx\,dy \right| \leq 2(\tfrac{\pi}{2} - \tan^{-1} R)\pi A.$$

Allowing $R \to \infty$ gives the result.

K172

Just use the rectangles $[0,1] \times [0, N^{-1}]$ and $[0,1] \times [N^{-1}, 1]$.

$F_2(0)$ is not defined because $x \mapsto f(x,0)$ is not Riemann integrable.

K173*

No comments

K174

(iv) Observe that U is open, $[-2,2] \supseteq U$ and U does not have Riemann length so $[-3,3] \setminus U$ is closed, bounded and does not have Riemann length.

K175*

No comments.

K176

(i) Observe that $f(t_j) \leq f(b)$. Recall that an increasing sequence bounded above tends to limit.

(ii) Chose $n(j)$ and x_j so that $n(1) = 1$, $x_1 = t_{n(1)}$, $n(2j) > n(2j-1)$, $x_{2j} = s_{n(2j)} > x_{2j-1}$, $n(2j+1) > n(2j)$, $x_{2j+1} = t_{n(2j)+1} > x_{2j}$ $[j \geq 1]$. Consider the limit of $f(x_j)$.

Let $a = 0$, $b = 2$, $f(t) = 0$ for $t < 1$, $f(t) = 1$ for $t \geq 1$ $t_j = 1 - j^{-1}$, $s_j = 1$.

(vii) If $x \notin E \cup \{a\}$, then $f(x-) = f(x+)$. But $f(x-) \leq f(x+)$ so $f(x) = f(x+)$. Thus $\tilde{f} = \bar{f}$.

If $x < y$ then, choosing $x < x_n < y < y_n < b$ with x_n, y_n strictly decreasing with $x_n \to x$, $y_n \to y$, we have $f(x_n) \leq f(y_n)$ so $\tilde{f}(x) \leq \tilde{f}(y)$. A similar but simpler argument applies when $y = b$. Thus $\tilde{f}$ is increasing. If $x \in [a, b)$ we can find $x_n \to x$ with x_n strictly decreasing and $x_n \notin E \cup \{a\}$. Thus $\tilde{f}(x_n) = f(x_n) \to \tilde{f}(x)$ and $\tilde{f}$ is right continuous.

(ix) Observe that $g\big((n\pi)^{-1}\big) = 0 \to 0$ and $g\big(((2n+\frac{1}{2})\pi)^{-1}\big) = 1 \to 1$.

K177

(iii) The function f is Riemann integrable with respect to H if and only if $f(t) \to f(0)$ as $t \to 0-$.

Observe that, if $\mathcal{D}$ is a dissection, then

$$\begin{aligned} S_H(f, \mathcal{D}) - s_H(f, \mathcal{D}) &\geq S_H(f, \mathcal{D} \cup \{0\}) - s_H(f, \mathcal{D} \cup \{0\}) \\ &= \limsup_{t \to 0,\, t<0} \Big(\sup_{x \in (t,0]} f(x) - \inf_{x \in (t,0]} f(x) \Big) \end{aligned}$$

so if $I^*(f, H) = I_*(f, H)$ then $f(t) \to f(0)$ as $t \to 0-$.

Conversely if $f(t) \to f(0)$ as $t \to 0-$, then, taking $\mathcal{D}_N = \{-N^{-1}, 0\}$ we have

$$S_H(f, \mathcal{D}_N),\ s_H(f, \mathcal{D}_N) \to f(0).$$

K178

(i) By adding a constant, we may suppose that $u(t) \to 0$ as $t \to -\infty$ We know that u is continuous except at a countable set of points x_j and and that, writing $\lambda_j = u(x_j) - \sup_{t<x_j} u(t)$, $\sum_{j=1}^{\infty} \lambda_j$ converges (indeed

$$\sum_{j=1}^{\infty} \lambda_j \leq \lim_{t\to\infty} f(t).$$

Without loss of generality we may suppose $\lambda_1 \geq \lambda_2 \geq \dots$. Define $u_n(t)$ so that

$$u(t) = u_n(t) + \sum_{j=1}^{n} H(x - x_j).$$

We observe that $u_n(t) \geq u_{n+1}(t) \geq 0$ so $u_n(t) \to U(t)$ for some $U : \mathbb{R} \to \mathbb{R}$. Since u_n is increasing for each n, U is increasing. We observe that

$$0 \leq \inf_{s<x<t} u_n(t) - u_n(s) \leq \lambda_n$$

and

$$0 \leq u_{n+1}(t) - u_{n+1}(s) \leq u_n(t) - u_n(s)$$

for any fixed t and s with $t > s$, so

$$\inf_{s<x<t} U(t) - U(s) = 0$$

and U is continuous.

A very much simpler argument shows that $\sum_{j=1}^{n} H(x-x_j)$ converges for all x.

(ii) Suppose that f and g are bounded increasing left continuous functions with $0 \leq f(t) - g(t) \leq \epsilon$ for all t and g is Riemann-Stieljes integrable with respect to G. Then

$$I^*(f,G) - I_*(f,G) \leq \epsilon(\lim_{t\to\infty} G(t) - \lim_{t\to-\infty} G(t)).$$

By (i), we can write

$$f(t) = v(t) + \sum_{j=1}^{\infty} \lambda_j \tilde{H}(t - x_j)$$

where $\tilde{H}(t) = 0$ for $t \leq 0$, $\tilde{H}(t) = 1$ for $t > 0$, $\lambda_j > 0$ and $\sum_{j=1}^{\infty} \lambda_j$ converges. By the first paragraph we need only show that the function f_N given by

$$f_N(t) = v(t) + \sum_{j=1}^{N} \lambda_j \tilde{H}(t - x_j)$$

is Riemann-Stieljes integrable. We know that v is Riemann-Stieljes integrable so we need only show that $t \mapsto \tilde{H}(t - x_j)$ is a Riemann-Stieljes integrable function and this follows much the same pattern as in K177.

(iii) f will be Riemann-Stieljes integrable if and only if no discontinuity of f coincides with that of G.

If f and G share a discontinuity x_0 then

$$\begin{aligned} S(f,\mathcal{D}) - s(f,\mathcal{D}) &\geq S(f,\mathcal{D}\cup\{x_0\}) - s(f,\mathcal{D}\cup\{x_0\}) \\ &\geq \inf_{s<x_0<t}(G(t)-G(s)) \inf_{s<x_0<t}(f(t)-f(s)) > 0 \end{aligned}$$

so f is not Riemann-Stieljes integrable with respect to G.

If f and G have no points of discontinuity in common, the argument of (ii) will work if appropriately modified.

K179

(ii) The condition is 'G strictly increasing'.

If G is not strictly increasing, choose $a < b$ such that $G(a) = G(b)$. Take $f(x) = \max(0, 1-4(b-a)^{-1}|x-(a+b)/2|$ to see that the theorem fails.

Suppose G is strictly increasing. If f is bounded continuous positive function with $f(x_0) \neq 0$, observe that there exists an $\epsilon > 0$ such that $|f(t) - f(x_0)| \leq f(x_0)/2$ so $f(t) \geq f(x_0/2$ for $|x_0 - t| \leq 2\epsilon$. Now

$$\int_{\mathbb{R}} f(x)\, dG(x) \geq \frac{(G(x_0+\epsilon) - G(x_0-\epsilon))f(x_0)}{2} > 0.$$

(iii) Same condition as (ii).

Suppose G is strictly increasing. If $f(x_0) \neq 0$ there exists an $\epsilon > 0$ such that $|f(t)-f(x_0)| \leq |f(x_0)|/2$ for $|x_0-t| \leq \epsilon$. Choose g continuous so that g vanishes outside $(x_0 - \epsilon, x_0 + \epsilon)$, $g(t)f(x_0) \geq 0$ for all t and $g(X_0)f(x_0) = 1$. By part (ii), $g(t)f(t) = 0$ for all t which is impossible. Thus f is identically zero.

K180

(i) Observe that

$$\sum_{j=1}^{n}|g(x_j)-g(x_{j-1})|=\sum_{j=1}^{n}g(x_j)-g(x_{j-1})=g(b)-g(a).$$

(ii) Observe that

$$\sum_{j=1}^{n}|(f+g)(x_j)-(f+g)(x_{j-1})|\leq\sum_{j=1}^{n}|f(x_j)-f(x_{j-1})|$$
$$=\sum_{j=1}^{n}|g(x_j)-g(x_{j-1})|.$$

(iii) Observe that

$$|f(x)|\leq|f(a)|+|f(b)-f(x)|+|f(x)-f(a)|.$$

(iv) Observe that, setting $x_0=0$, $x_{N+2}=1$ and $x_j=\big((N+4-j+\frac{1}{2})\pi\big)^{-1}$, we have

$$\sum_{j=1}^{N}|f(x_j)-f(x_{j-1})|\geq\pi^{-1}\sum_{j=5}^{N+5}j^{-1}\to\infty$$

as $N\to\infty$.

K181

(ii) Given $\epsilon > 0$, we can find $a = x_0 \leq y_0 \leq x_1 \leq \ldots x_n \leq y_n = t$ such that

$$\sum_{j=1}^{n} F(y_j) - F(x_j) \geq F_+(t) - \epsilon$$

and $a = x'_0 \leq y'_0 \leq x'_1 \leq \ldots x_m \leq y'_m = t$ such that

$$\sum_{j=1}^{n} F(x'_j) - F(y'_j) \geq F_-(t) - \epsilon.$$

Let $a = z_0 \leq z_1 \leq \ldots z_N = t$ with

$$\begin{aligned}\{z_0,\, z_1,\, z_2\, \ldots\, z_N\} = &\{x_0,\, x_1,\, x_2\, \ldots\, x_n\} \cup \{y_0,\, y_1,\, y_2\, \ldots\, y_n\} \\ &\cup \{x'_0,\, x'_1,\, x'_2\, \ldots\, x'_m\} \cup \{y'_0,\, y'_1,\, y'_2\, \ldots\, y'_m\}\end{aligned}$$

Then

$$\begin{aligned}F(t) - F(a) &= \sum_{j=1}^{N} F(z_j) - F(z_{j-1}) \\ &= \sum_{F(z_j)-F(z_{j-1})>\geq 0} F(z_j) - F(z_{j-1}) + \sum_{F(z_j)-F(z_{j-1})<0} F(z_j) - F(z_{j-1}),\end{aligned}$$

$$F_+(t) \geq \sum_{F(z_j)-F(z_{j-1})>\geq 0} F(z_j) - F(z_{j-1}) \geq F_+(t) - \epsilon$$

and

$$F_-(t) \geq -\sum_{F(z_j)-F(z_{j-1})>\leq 0} F(z_j) - F(z_{j-1}) \geq F_-(t) - \epsilon$$

so $|F(t) - F_+(t) + F_+(t)| \leq 2\epsilon$. Since ϵ is arbitrary, $F(t) = F_+(t) - F_-(t)$.

The proof that $V_F = F_+(t) + F_-(t)$ is similar.

(iii) Observe that, if $a = x_0 \leq y_0 \leq x_1 \leq \ldots x_n \leq y_n = t$ then

$$\begin{aligned}G_+(t) &= \sum_{j=1}^{n} G_+(y_j) - G_+(x_j) + \sum_{j=1}^{n} G_+(x_j) - G_+(y_{j-1}) \\ &\geq \sum_{j=1}^{n} G_+(y_j) - G_+(x_j) \\ &= \sum_{j=1}^{n} (f(y_j) - f(x_j)) + (G_-(y_j) - G_-(x_j)) \\ &\geq \sum_{j=1}^{n} f(y_j) - f(x_j)\end{aligned}$$

so $G_+(t) \geq F_+(t)$. Exactly the same calculation gives $G_+(t) - G_-(s) \geq F_+(t) - F_-(s)$ for $t \geq s$ so $G_+ - F_+$ is increasing.

K182

(ii) If F is right continuous at t then, since $F = F(a) + F_+ - F_-$, either both F_+ and F_- are right continuous or neither is (which is impossible by (i)).

(iii) Write

$$G_+(t) = \int_a^t \max(F'(x), 0)\,dx,\ G_-(t) = -\int_a^t \min(F'(x), 0)\,dx.$$

Then, by the fundamental theorem of the calculus, G_+ is differentiable with $G'_+(t) = \max(F'(t), 0)$ so G_+ is increasing. Similarly G_- is decreasing. Further $G'_+(t) - G'_-(t) = F'(t)$, so using the fundamental theorem of the calculus, $F = F(a) + G_+ - G_-$.

Now let $\epsilon > 0$. Since F' is continuous on $[a, t]$, it is uniformly continuous so we can find an integer N such that

$$|F'(x) - F'(y)| < \epsilon \text{ whenever } |x - y| \leq (t - a)/N.$$

Let us an integer r with $1 \leq r \leq n$ 'good' if there exists an $x \in [(r-1)/N, rN]$ with $|F'(x)| > \epsilon$. Let us call the other integers r with $1 \leq r \leq n$ 'bad'. Observe that, if r is good, then F' is single signed on $x \in [(r-1)/N, rN]$. Thus, using the mean value theorem

$$\begin{aligned}\sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| &\geq \sum_{j\text{ good}} \int_{x_{j-1}}^{x_j} |F'(x)|\,dx \\ &\geq \int_a^t |F'(x)|\,dx - \sum_{j\text{ bad}} \epsilon(t-a)/N \\ &\geq \int_a^t |F'(x)|\,dx - \epsilon(t-a)\end{aligned}$$

so that

$$V_F(t) \geq \int_a^t |F'(x)|\,dx = G_+(t) + G_-(t)$$

and $F_+(t) = G_+(t)$, $F_-(t) = G_-(t)$ as stated.

K183

Observe that $f_{\alpha,\beta}$ is well behaved away from 0, so this exercise is about behaviour near 0. The behaviour is different according as $\beta > 0$, $\beta = 0$ or $\beta < 0$.

(A) If $\beta > 0$, we have

$$|x|^\alpha \sin(|x|^\beta) = |x|^{\alpha+\beta} + |x|^{\alpha+2\beta}\epsilon(|x|)$$

with $\epsilon(|x|) \to 0$ as $x \to 0$.

If $\alpha + \beta > 1$, $f_{\alpha,\beta}$ is differentiable at 0 with derivative 0. We check that $f'_{\alpha,\beta}$ is continuous at 0 so $f_{\alpha,\beta}$ is everywhere differentiable with continuous derivative (so of bounded variation).

If $1 \geq \alpha + \beta > 0$, $f_{\alpha,\beta}$ is not differentiable at 0 but is continuous there (so everywhere continuous). By inspection of $f'_{\alpha,\beta}$, or otherwise, $f_{\alpha,\beta}$ is of bounded variation.

If $\alpha + \beta = 0$, $f_{\alpha,\beta}$ is not continuous at 0 but is of bounded variation.

If $0 > \alpha + \beta$ then $f_{\alpha,\beta}(x) \to \infty$ so $f_{\alpha,\beta}$ is not continuous and not of bounded variation.

(B) If $\beta = 0$. $f_{\alpha,0} = |x|^\alpha \sin 1$. The function is differentiable everywhere with continuous derivative if and only if $\alpha > 1$, and continuous everywhere if and only if $\alpha > 0$, of bounded variation if and only if $\alpha \geq 0$.

(C) If $\beta < 0$ then the function differentiable at 0 so everywhere if and only if $\alpha > 1$. Derivative is then 0. Now

$$f'_{\alpha,\beta}(x) = \alpha x^{\alpha-1} \sin x^\beta + \beta x^{\alpha+\beta-1} \cos x^\beta$$

so derivative is continuous if and only if (consider $x = (n\pi/2)^{-1}$) $\alpha + \beta - 1 > 0$.

The function is continuous at 0 (and so everywhere) if and only if $\alpha > 0$.

If $\alpha \leq 0$, inspection shows that $f_{\alpha,\beta}$ is not of bounded variation. If $\alpha > 0$ then by differentiating (or rescaling) we see that in each interval $I_n = [((n+1)\pi)^{1/\beta}, (n\pi)^{1/\beta}]$ the function $f_{\alpha,\beta}$ either increases and then decreases or decreases and then increases. The total variation V_n of $f_{\alpha,\beta}$ in the interval I_n is thus $2\sup_{t\in I_n} |f_{\alpha,\beta}(t)|$. It follows that

$$2(n\pi)^{\alpha/\beta} \geq V_n \geq 2((n+1)\pi)^{\alpha/\beta}$$

so $f_{\alpha,\beta}$ is of bounded variation if and only if $\sum_{n=1}^\infty n^{\alpha/\beta}$ converges, i.e. $\alpha/\beta < -1$, i.e. $\alpha > -\beta$, i.e. $\alpha + \beta > 0$.

K184*

No comments.

K185

(i) Observe that, in a self explanatory notation,

$$S(f,\mathcal{D},G+F)=S(f,\mathcal{D},F)+S(f,\mathcal{D},G)$$

and

$$\begin{aligned}S(f,\mathcal{D}_1\cup\mathcal{D}_2,G+F)&=S(f,\mathcal{D}_1\cup\mathcal{D}_2,F)+S(f,\mathcal{D}_1\cup\mathcal{D}_2,G)\\&\geq S(f,\mathcal{D}_1,F)+S(f,\mathcal{D}_2,G)\end{aligned}$$

so

$$I^*(f,F+G)=I^*(f,F)+I^*(f,G)$$

and similarly $I_*(f,F+G)=I_*(f,F)+I_*(f,G)$.

(ii) Rewrite as

$$\int_{\mathbb{R}} f(x)\,dF_1(x)+\int_{\mathbb{R}} f(x)\,dG_2(x)=\int_{\mathbb{R}} f(x)\,dF_2(x)+\int_{\mathbb{R}} f(x)\,dG_1(x).$$

and observe that by part (i)

$$\int_{\mathbb{R}} f(x)\,dF_1(x)+\int_{\mathbb{R}} f(x)\,dG_2(x)=\int_{\mathbb{R}} f(x)\,d(F_1+G_2)(x)$$

and

$$\int_{\mathbb{R}} f(x)\,dF_2(x)+\int_{\mathbb{R}} f(x)\,dG_1(x)=\int_{\mathbb{R}} f(x)\,d(F_2+G_1)(x)$$

K186*

No comments.

K187

The main problem is the behaviour of $\theta(y)$ for y close to 1.

Observe that writing $t = 1 - s$ we have, using appropriate Taylor theorems,

$$\begin{aligned}(1-(1-s)^2)^{-1/2} &= (2s-s)^{-1/2}\\ &= 2^{-1/2}s^{-1/2}(1-s/2)^{-1/2}\end{aligned}$$

$$2^{-1/2}s^{-1/2}(1-\epsilon(s)s)$$

with $\epsilon(s) \to 0$ as $\epsilon \to 0$. Thus $\int_y^1 \frac{1}{(1-t^2)^{1/2}}\,dt$ converges and

$$\begin{aligned}\int_y^1 \frac{1}{(1-t^2)^{1/2}}\,dt &= -\int_0^{1-y} \frac{1}{(1-(1-s)^2)^{1/2}}\,ds\\ &= 2^{1/2}(1-y)^{1/2} + (1-y)^{3/2}\delta(1-y)\end{aligned}$$

for $1 > y > 0$, where $\delta(s) \to 0$ as $s \to 0$.

This proves the existence of ω and shows that

$$\sin(\omega - u) = 1 - u^2/2 + \eta(u)u^2$$

for $\omega > u > 0$ where $\eta(u) \to 0$ as $u \to 0$. Thus in (iii) we know that the extended function sin is differentiable at ω with $\sin'(\omega) = 0$. If $\omega > u > 0$

$$\begin{aligned}\sin'(\omega - u) &= -(1-(\sin\omega - u)^2)\\ &= -(1-(1-u^2/2+\eta(u)u^2)^2)^{1/2} = u + \beta(u)u\end{aligned}$$

and

$$\sin'(\omega + u) = -\sin'(\omega + u) = -u - \beta(u)u$$

with $\beta(u) \to 0$ as $u \to 0+$. Thus sin is twice differentiable at ω with

$$\sin''(\omega) = -1 = -\sin\omega.$$

K188

The points at issue are very similar to those of K187.

K189

(iv) Observe that

$$\pi \int_1^R f(x)^2\,dx = \pi \int_1^R x^{-2}\,dx = \pi(1-R^{-1}) \to \pi$$

but

$$2\pi \int_1^R f(x)(1+f'(x)^2)^{1/2}\,dx \geq 2\pi \int_1^R x^{-1}\,dx = 2\pi \log R \to \infty$$

as $R \to \infty$.

(v) We have

$$\operatorname{Vol} K = \pi \int_0^\infty x^{-1}\,dx = \infty$$

but

$$\begin{aligned}
\operatorname{Vol}(K' \setminus K) &= \pi \int_0^\infty (x^{-1/2}+x^{-1})^2 - (x^{-1/2})^2\,dx \\
&= \pi \int_0^\infty (x^{-2} + 2x^{-3/2})\,dx \\
&= \pi \lim_{R\to\infty} \big((1-R^{-1}) + 4(1-R^{-1/2})\big) = 5\pi.
\end{aligned}$$

K190

We have

$$\begin{aligned}
\int_{\gamma_1} f(\mathbf{x})\,d\mathbf{s} &= \int_{\gamma_2} f(\mathbf{x})\,d\mathbf{s} \\
&= \int_{\gamma_3} f(\mathbf{x})\,d\mathbf{s} \\
&= \int_{\gamma_4} f(\mathbf{x})\,d\mathbf{s} \\
&= \int_0^1 \cos(2\pi t)(-2\pi\sin(2\pi t), 2\pi\cos(2\pi t))\,dt \\
&= 2\pi \int_0^1 (-\cos(2\pi t)\sin(2\pi t), \cos^2(2\pi t))\,dt \\
&= \pi \int_0^1 (\sin(4\pi t), 1+\cos(4\pi t))\,dt \\
&= (0,\pi)
\end{aligned}$$

Also

$$\int_{\gamma_5} f(\mathbf{x})\,d\mathbf{s} = -\int_{\gamma_1} f(\mathbf{x})\,d\mathbf{s} = (0,-\pi)$$

and

$$\int_{\gamma_6} f(\mathbf{x})\,d\mathbf{s} = 2\int_{\gamma_1} f(\mathbf{x})\,d\mathbf{s} = (0,2\pi).$$

K191

(ii) We have

$$\frac{u_n(k+1)}{u_n k} = \frac{p(n-k)}{(1-p)k}$$

so $u_n(k+1) > u_n k$ if $p(n-k) > (1-p)k$, i.e. if $np > k$ and $u_n(k+1) < u_n k$ if $np < k$ (and, if $np = k$, then $u_n(k+1) = u_n k$). We take k_0 so that $np \geq k_0 - 1$ and $np \leq k_0$.

(ii) If $k \geq n(p + \epsilon/2)$ then

$$\frac{u_n(k+1)}{u_n k} = \frac{p(n-k)}{(1-p)k} \leq \frac{p(1-p-\epsilon/2)}{(1-p)(p+\epsilon/2)}$$

Now $1 = \sum_{j=0}^{n} u_n(j) \geq u_n(k)$ so, writing $k_1 = k_1(n)$ for the least integer greater than $n(p+\epsilon/2)$, we see that

$$u_n(k) \leq \left(\frac{p(1-p-\epsilon/2)}{(1-p)(p+\epsilon/2)}\right)^{k-k_1}$$

for all $k \geq k_1$

If $k_2 = k_2(n)$ is the greatest integer less than $n(p+\epsilon/2)$ we have

$$\sum_{k \geq np+n\epsilon} u_n(k) \leq \sum_{k \geq k_2} \left(\frac{p(1-p-\epsilon/2)}{(1-p)(p+\epsilon/2)}\right)^{k-k_1}$$

$$\leq \left(\frac{p(1-p-\epsilon/2)}{(1-p)(p+\epsilon/2)}\right)^{k_2(n)-k_1(n)} \left(1 - \frac{p(1-p-\epsilon/2)}{(1-p)(p+\epsilon/2)}\right)^{-1} \to 0$$

as $n \to \infty$.

Similarly

$$\sum_{k \leq np-n\epsilon} u_n(k) \to 0$$

so

$$\Pr\{|N_n - np| \geq \epsilon n\} = \sum_{|k-np| \geq n\epsilon} u_n(k) \to 0$$

as $n \to \infty$.

(iii) We have

$$\mathbb{E}X_j = p1 + (1-p)0 = p$$

and

$$\mathbb{E}X_j^2 = p1 + (1-p)0 = p$$

so $\operatorname{var} X_j = p - p^2 = p(1-p)$.

The sum of the expectations is the expectation of the sum so

$$\mathbb{E}N_n = \mathbb{E}\sum_{j=1}^{n} X_j = \sum_{j=1}^{n} \mathbb{E}X_j.$$

For independent random variables, the sum of variances is the variance of the sum so

$$\operatorname{var} N_n = np(1-p)$$

and Thchebychev gives

$$\Pr\{|N_n - np| \geq \epsilon n\} \leq \frac{\operatorname{var} N_n}{(\epsilon n)^2} = \frac{p(1-p)}{\epsilon^2 n} \to 0$$

as $n \to \infty$.

K192

Recall that (A), (B) and (C) give

$$d(x,y) = (d(x,y) + d(y,x))/2 \geq d(x,x)/2 = 0.$$

(i) Set $x = y$ in (C)$'$ to obtain (B). (C) now follows from (B) and (C)$'$.

(ii) Setting $d'(x,y) = -d(x,y)$, we see that d' is a metric space. If $d(x,y) \geq 0$ for x, $y \in X$ then X has exactly one point.

K193

By $(A)'$, $x \sim x$.

By (B), $x \sim y$ implies $y \sim x$.

By (C) if $x \sim y$ and $y \sim z$ then

$$0 = d(x, y) + d(y, z) \geq d(x, z) \geq 0,$$

so $d(x, z) = 0$ and $x \sim z$.

Observe $x \sim x'$, $y \sim y'$ gives

$$d(x, y) = d(x', x) + d(x, y) + d(y, y') \geq d(x', y')$$

and $d(x', y') \geq d(x, y)$ so $d(x, y) = d(x', y')$. Thus $\tilde{d}$ is well defined.

K194

(i) The result is true with $A_1 = 1$. We use induction on n to prove it. The result is trivially true for $n = 1$. Suppose it is true for $n = m$. If $\tau \in S_{m+1}$ then writing $\sigma = (m+1, \tau(m+1))$ and $\tau' = \sigma^{-1}\tau$ we see that τ' fixes $m+1$ so is generated by at most m elements of the form (ij) (transpositions). Thus $\tau = \sigma\tau'$ is is generated by at most $m+1$ transpositions. Any shuffle of m cards needs at most m swaps.

(ii) Observe that $(1i)(1j) = (ij)$ and use (i).

(iii) Observe that

$$(123\dots n)^k(12)(123\dots n)^{-k} = (k\,k+1)$$

so any transposition of the form $(k\,k+1)$ is a distance at most $2n+1$ from e. Next observe that

$$(k\,k+1)(1\,k)(k\,k+1) = (1\,k+1)$$

so any transposition of the form $(1\,k)$ is a distance at most $n(2n+1)$ from e. Thus, using (ii) and (i), the diameter of S_n is at most $2n^2(2n+1)$ with respect to X_3.

(iv) Observe that $a^r b = ba^{-r}$. Let N be the integer part of $(n-1)/2$. If $N \geq r$ then, by induction, the product of r elements of the form a, b, a^{-1}, must have one of the forms a^u with $|u| \leq r$ or ba^v with $|v| \leq r-1$. Looking at a^N we see that the diameter of D_n is at least N.

(v) S_n has $n!$ elements and D_n has $2n$. There appears to be little relationship between size and diameter.

K195

(i) We wish to prove the statement $P(N)$. If $|x| \geq 4^{-k}$ for some integer k. and

$$x = \sum_{j=1}^{N} x_j \text{ with } |x_j| \leq 4^{-n(j)} \text{ for some integer } n(j) \geq 0 \ [1 \leq j \leq N],$$

then $\sum_{j=1}^{N} 2^{-n(j)} \geq 2^{-k}$.

If $x = \sum_{j=1}^{1} x_j$ then $x = x_1$ so $P(1)$ is true.

Suppose now that $P(r)$ is true for all $1 \leq r \leq N-1$. If $|x| \geq 4^{-k}$ and

$$x = \sum_{j=1}^{N} x_j \text{ with } |x_j| \leq 4^{-n(j)}$$

then if $n(j) \leq k$ for any j we are done. Thus we need only consider the case $n(j) \geq k+1$ for all $1 \leq j \leq N$. Let M be the smallest m such that

$$\sum_{j=1}^{m} |x_j| \geq 4^{-k-1}.$$

We know that

$$\sum_{j=1}^{M} |x_j| \leq \sum_{j=1}^{M-1} |x_j| + 4^{-k-1} \leq 2 \times 4^{-k-1}$$

so

$$\sum_{M+1}^{N} |x_j| \geq |x| - \sum_{j=1}^{M} |x_j| \geq 2 \times 4^{-k-1}$$

. Applying the inductive hypothesis to

$$x' = \sum_{j=1}^{M} |x_j| \text{ and } x'' = \sum_{M+1}^{N} |x_j|$$

we see that

$$\sum_{j=1}^{N} 2^{-n(j)} = \sum_{j=1}^{M} 2^{-n(j)} + \sum_{j=M+1}^{N} 2^{-n(j)} \geq 2^{-k-1} + 2^{-k-1} = 2^{-k}.$$

Thus $P(N)$ is true and the induction is completed.

(ii) The key point is that $d(x,0) = 0$ implies $x = 0$. But if $x \neq 0$ then $|x| \geq 4^{-k}$ for some k and so, by (i), $d(x,0) \geq 2^{-k}$.

(iii) If $4^{-k+1} > |x| \geq 4^{-k}$ then taking $x = \sum_{j=1}^{1} x_j$ with $x_1 = x$, we see that $2^{-k+1} \geq d(x,0)$. We also know, by (i), that $d(x,0) \geq 2^{-k}$.

Thus

$$4|x|^2 \geq d(x,0) \geq 4^{-1}|x|^2.$$

(iv) I think the graph is quite complex with f constant on intervals of the form $[4^{-k}(1-\eta), 4^{-k}]$ (for some $\eta > 0$) and on other intervals which form a dense set (cf the 'devil's staircase' in K251).

K196

(i) and (iv) are false. Let $X = \mathbb{R}$ with usual metric, $X_1 = \{x : x \leq 0\}$, $X_2 = \{x : x > 0\}$, $f(x) = 0$ for $x \in X_1$ and $f(x) = 1$ for $x \in X_2$.

Parts (ii) and (iii) have one line solutions for the sufficiently sophisticated. Here are longer solutions.

(ii) True. If $x \in X_1 \cup X_2$ then, without loss of generality, we may suppose $x \in X_1$. Thus there exists a $\delta_1 > 0$ such that, if $z \in X$ and $d(x, z) < \delta_1$, we have $z \in X_1$. Since $f|_{X_1}$ is continuous, we know that, given $\epsilon > 0$ there exists a $\delta_2(\epsilon) > 0$ such that $d(x, z) < \delta_2(\epsilon)$ implies $\rho(f(x), f(z)) < \epsilon$ for all $z \in X_1$. Now $d(x, z) < \min(\delta_1, \delta_2(\epsilon))$ implies $\rho(f(x), f(z)) < \epsilon$ for all $z \in X$.

(iii) True. If $x \in X \setminus X_j$, the argument of (ii) shows that f is continuous at x. If $x \in X_1 \cap X_2$, then given $\epsilon > 0$ there exists a $\eta_j(\epsilon) > 0$ such that $d(x, z) < \eta_j(\epsilon)$ implies $\rho(f(x), f(z)) < \epsilon$ for all $z \in X_j$ and $d(x, z) < \min_{j=1,2} \eta_j(\epsilon)$ implies $\rho(f(x), f(z)) < \epsilon$ for all $z \in X$.

Set $\mathbf{f}(x) = \mathbf{g}(x)$ when $\|\mathbf{g}(x)\| < 1$, $\mathbf{f}(x) = \|\mathbf{g}(x)\|^{-1}\mathbf{g}(x)$ otherwise. Let

$$X_1 = \{x \in X : \|\mathbf{g}(x)\| \leq 1\} \text{ and } X_2 = \{x \in X : \|\mathbf{g}(x)\| \geq 1\}$$

and apply part (iii).

K197

(i) Consider $X = \{1, 2\}$ with the discrete metric and $A = \{1\}$.

(iii) By (ii), f is continuous. If A, $B \neq \emptyset$ choose $a \in A$, $b \in B$ and apply the intermediate value theorem to show that there exists a c with $f(c) = 1/2$.

(iv) If A is closed, then $\{t \in [0, 1) : (1-t)\mathbf{a} + t\mathbf{b} \in A\}$ is closed. If A is open, then $\{t \in [0, 1) : (1-t)\mathbf{a} + t\mathbf{b} \notin A\}$ is closed.

K198

Suppose $\bar{A}$ satisfies (i). Then, if $y_m \in \bar{A}$ and $y_m \to y$, we can find $x_m \in A$ with $d(x_m, y_m) < 1/m$ so $x_m \to y$ and $y \in \bar{A}$. Thus $\bar{A} \in \mathcal{F}$. But, if $F \supseteq A$ and F is closed, we have $F \supseteq \bar{A}$ Thus $\bar{A}$ satisfies (ii).

Suppose $\bar{A}$ satisfies (ii). Then, since $\bar{A}$ is the intersection of closed sets, $\bar{A} \in \mathcal{F}$ and automatically $F \supseteq \bar{A}$ whenever $F \in \mathcal{F}$ so $\bar{A}$ satisfies (iii).

Suppose $\bar{A}$ satisfies (iii). If $x_n \in A$ and $x_n \to x$ then $x_n \in \bar{A}$ so since $\bar{A}$ is closed $x \in \bar{A}$. Suppose $x \in \bar{A}$, then (writing $B(x, 1/n)$ for the open ball centre x radius $1/n$) if $B(x, 1/n) \cap A = \emptyset$ $\bar{A} \setminus B(x, 1/n)$ is a strictly smaller closed set containing A contradicting our assumption. Thus $B(x, 1/n) \cap A \neq \emptyset$ and we can find $x_n \in B(x, 1/n) \cap A$. We have $x_n \in A$ and $x_n \to x$.

(iii$'$ B° is an open set with $B \supseteq B^\circ$ such that, if U is open and $B \supseteq U$, we have $B^\circ \supseteq U$.

$\operatorname{Int} \mathbb{Q} = \emptyset$, $\operatorname{Int} \mathbb{Z} = \emptyset$, $\operatorname{Int}\{1/n \,:\, n \geq 1\} = \emptyset$, $\operatorname{Int}[a, b] = \operatorname{Int}(a, b) = \operatorname{Int}[a, b) = (a, b)$.

$\operatorname{Cl} \mathbb{Q} = \mathbb{R}$, $\operatorname{Cl} \mathbb{Z} = \mathbb{Z}$, $\operatorname{Cl}\{1/n \,:\, n \geq 1\} = \{0\} \cup \{1/n \,:\, n \geq 1\}$, $\operatorname{Cl}[a, b] = \operatorname{Cl}(a, b) = \operatorname{Cl}[a, b) = [a, b]$.

Suppose $f(\bar{A}) \subseteq \overline{f(A)}$ for every A. If $x_n \to x$ but $f(x_n) \nrightarrow f(x)$, then we can find a $\delta > 0$ and $n(j) \to \infty$ such that $\rho(f(x_{n(j)}), f(x)) > \delta$. Now take

$$A = \{x_{n(j)} \,:\, j \geq 1\}$$

to obtain a contradiction ($x \in \bar{A}$ but $f(x) \notin \overline{f(A)}$). Thus f is continuous.

Conversely, if f is continuous and $x \in \bar{A}$, then we can find $x_n \in A$ such that $x_n \to x$ and so, by continuity, $f(x_n) \to f(x)$. Thus $f(\bar{A}) \subseteq \overline{f(A)}$.

Let $X = \{1, 2\}$ with $d(1, 2) = d(2, 1) = 1$ and $d(1, 1) = d(2, 2) = 0$. Take $y = 1$.

K199

Observe that $\operatorname{Cl} U \supseteq \operatorname{Int}(\operatorname{Cl} U))$ and $\operatorname{Cl} U$ is closed so that

$$\operatorname{Cl} U \supseteq \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} U)).$$

However, since U is open, and $\operatorname{Cl} U \supseteq U$, we have $\operatorname{Int}(\operatorname{Cl} U)) \supseteq U$ so

$$\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} U)) \supseteq \operatorname{Cl} U$$

and $\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} U)) = \operatorname{Cl} U$.

It follows that, If B is any set,

$$\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl}(\operatorname{Int} B))) = \operatorname{Cl}(\operatorname{Int} B)$$

and by taking complements, if C is any set

$$\operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} C))) = \operatorname{Int}(\operatorname{Cl} C)$$

Since $\operatorname{Cl}\operatorname{Cl} A = \operatorname{Cl} B$ and $\operatorname{Int}\operatorname{Int} A = \operatorname{Int} B$, the only possible distinct sets that we can produce are A, $\operatorname{Int} A$, $\operatorname{Cl} A$, $\operatorname{Cl}\operatorname{Int} A$, $\operatorname{Int}\operatorname{Cl} A$, $\operatorname{Int}\operatorname{Cl}\operatorname{Int} A$ and $\operatorname{Cl}\operatorname{Int}\operatorname{Cl} A$.

The set

$$A = \{-3\} \cup (-2,-1] \cup (\mathbb{Q} \cap [0,1]) \cup ((2,4) \setminus \{3\})$$

shows that these 7 can all be distinct.

K200

(i) (C) Take $\lambda = (\max(2, \|\mathbf{x}\|))^{-1}$, for example.

(iii) Since Γ is absorbing we can define

$$n(\mathbf{x}) = \inf\{t \geq 0 \,:\, t^{-1}\mathbf{x} \in \Gamma\}.$$

We note that $n(\mathbf{x}) \geq 0$. Our object is to show that $\|\mathbf{x}\| = n(\mathbf{x})$ defines the required norm.

If $\lambda \geq 0$ the definition gives $n(\lambda\mathbf{x}) = \lambda n(\mathbf{x})$. The fact that Γ is symmetric now gives

$$n(\lambda\mathbf{x}) = |\lambda| n(\mathbf{x})$$

for all $\lambda \in \mathbb{R}$.

Condition (B) tells us that $\mathbf{0} \in \Gamma$, and then (D)$'$ together with convexity tells us that, if $\mathbf{x} \neq \mathbf{0}$, there exists a $\mu > 0$ such that $t\mathbf{x} \notin \Gamma$ for all $t \geq \mu$. Thus $n(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Suppose that $\mathbf{x}$, $\mathbf{y} \neq \mathbf{0}$. Then, if $\epsilon > 0$,

$$\big((1+\epsilon)^{-1} n(\mathbf{x})^{-1}\mathbf{x},\, (1+\epsilon)^{-1} n(\mathbf{y})^{-1}\mathbf{y}\big) \in \Gamma$$

and so, by convexity,

$$(1+\epsilon)^{-1}\frac{1}{n(\mathbf{x})+n(\mathbf{y})}(\mathbf{x}+\mathbf{y}) = (1+\epsilon)^{-1}\frac{n(\mathbf{x})^{-1}n(\mathbf{y})^{-1}}{n(\mathbf{x})^{-1}+n(\mathbf{y})^{-1}}(\mathbf{x}+\mathbf{y})$$
$$= \frac{n(\mathbf{y})^{-1}}{n(\mathbf{x})^{-1}+n(\mathbf{y})^{-1}}(1+\epsilon)^{-1}n(\mathbf{x})^{-1}\mathbf{x} + \frac{n(\mathbf{x})^{-1}}{n(\mathbf{x})^{-1}+n(\mathbf{y})^{-1}}(1+\epsilon)^{-1}n(\mathbf{y})^{-1}\mathbf{y} \in \Gamma.$$

Since ϵ is arbitrary, $n(\mathbf{x}+\mathbf{y}) \leq n(\mathbf{x}) + n(\mathbf{y})$.

(iv) A necessary and sufficient condition is

$$\Gamma_B \subseteq K\Gamma_A$$

where $K\Gamma_A = \{K\mathbf{x} \,:\, \mathbf{x} \in \Gamma_A\}$ and Γ_C is the closed unit ball of the norm $\| \, \|_C$.

K201

Observe that

$$\mathbf{x}_n = (1, 2^{-1}, 3^{-1}, \dots, n^{-1}, 0, 0, \dots) \in E$$

but that

$$\mathbf{x}_n \to \mathbf{x} = (1, 2^{-1}, 3^{-1}, \dots, m^{-1}, (m+1)^{-1}, (m+2)^{-1}, \dots) \notin E.$$

If $\mathbf{a}(j) \in F$ and $\mathbf{a}(j) \to \mathbf{a}$, then

$$|a_{2m}| = |a_{2m} - a(j)_{2m}| \leq \|\mathbf{a} - \mathbf{a}(j)\| \to 0$$

as $j \to 0$ so $a_{2m} = 0$ for all m. Thus $\mathbf{a} \in F$.

Suppose that $(V, \| \ \|)$ is a normed space and E is a finite dimensional subspace. Let $\mathbf{e}_1$, $\mathbf{e}_2, \dots,$ $\mathbf{e}_N$ be a basis for E. Then

$$\left\| \sum_{j=1}^{N} x_j \mathbf{e}_j \right\|_* = \sum_{j=1}^{N} |x_j|$$

is a norm on E. But all norms on a finite dimensional space are Lipschitz equivalent so there exists $K \geq 1$ with $K\|\mathbf{x}\| \geq \|\mathbf{x}\|_* \geq K^{-1}\|\mathbf{x}\|$.

If $\mathbf{y}_n \in E$, $\mathbf{y} \in V$ and $\|\mathbf{y}_n - \mathbf{y}\| \to 0$, then $\mathbf{y}_n$ is a Cauchy sequence in $(V, \| \ \|)$ so a Cauchy sequence in $(E, \| \ \|)$ so, by Lipschitz equivalence, a Cauchy sequence in $(E, \| \ \|_*)$ so, since $(E, \| \ \|_*)$ is complete, converges in $(E, \| \ \|_*)$ to some $\mathbf{z} \in E$. Since $\|\mathbf{y}_n - \mathbf{z}\|_* \to 0$, it follows, by Lipschitz equivalence, that $\|\mathbf{y}_n - \mathbf{z}\| \to 0$. By the uniqueness of limits, $\mathbf{y} = \mathbf{z} \in E$. Thus E is closed.

K202

(i) If

$$\lambda'\mathbf{y} + \mathbf{e}' = \lambda\mathbf{y} + \mathbf{e}$$

with λ, $\lambda' \in \mathbb{R}$ and $\mathbf{e}$, $\mathbf{e}' \in E$ then

$$(\lambda' - \lambda)\mathbf{y} = \mathbf{e} - \mathbf{e}'$$

so, applying T to both sides,

$$(\lambda' - \lambda)T\mathbf{y} = 0$$

so $\lambda = \lambda'$ and $\mathbf{e} = \mathbf{e}'$.

Also

$$\mathbf{x} - \frac{T\mathbf{x}}{T\mathbf{y}}\mathbf{y} \in E.$$

If $T = 0$ then $N = E$.

(ii) Observe that the statement that

$$B(\mathbf{y}, \delta) \cap N \neq \emptyset$$

is equivalent to the statement that there exists an $\mathbf{f} \in N$ such that

$$\|\mathbf{y} - \mathbf{f}\| < \delta$$

and (setting $\mathbf{e} = -\lambda\mathbf{f}$) this is equivalent to saying that

$$\|\lambda\mathbf{y} + \mathbf{e}\| < \delta\lambda$$

for some $\mathbf{e} \in N$ and $\lambda \neq 0$.

(iii) If T is continuous then since the inverse image of a closed set under a continuous function is closed N is closed.

If N is closed pick a $\mathbf{y} \notin N$ and observe that there exists a $\delta > 0$ such that

$$\|\lambda\mathbf{y} + \mathbf{e}\| \geq \delta|\lambda$$

for all λ and $\mathbf{e} \in N$.

By part (i), if $\mathbf{x} \in E$ we have

$$\mathbf{x} = \lambda\mathbf{y} + \mathbf{e}$$

with $\mathbf{e} \in N$, so, by (ii),

$$\|\mathbf{x}\| \geq \delta|\lambda| \text{ whilst } T\mathbf{x} = \lambda T\mathbf{y}.$$

Thus

$$\|T\mathbf{x}\| \leq \delta^{-1}|T\mathbf{x}|\|\mathbf{x}\|$$

for all $\mathbf{x}$ and T is continuous.

(iv) We have $N = \{\mathbf{a} \in s_{00} \,:\, \sum_{j=1}^{\infty} a_j = 0\}$.

(a) If we use the norm $\| \|_\infty$, then N not closed. Take

$$e_j(n) = \begin{cases} 1 & \text{if } j = 1 \\ -n^{-1} & \text{if } 2 \leq j \leq n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{e}(n) \in N$, but $\mathbf{e} \to (1, 0, 0, \dots) \notin N$. Thus N is not closed.

(b) If we use the norm $\| \|_w$, N not closed. Observe that $\sum_{j=2}^{\infty} j^{-1}$ diverges so (since $j^{-1} \leq 1$) we can find $N(n)$ such that

$$n \leq \sum_{j=2}^{N(n)} j^{-1} \leq n+1.$$

Set $A(n) = \sum_{j=2}^{N(n)} j^{-1}$. Take

$$e_j(n) = \begin{cases} 1 & \text{if } j = 1 \\ -A(n)^{-1}j^{-1} & \text{if } 2 \leq j \leq N(n) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{e}(n) \in N$ but $\mathbf{e} \to (1, 0, 0, \dots) \notin N$. Thus N is not closed.

(c) If we use the norm $\| \|_1$, N is closed. If $\mathbf{e}(n) \in N$ and $\|\mathbf{e}(n) - \mathbf{e}\|_1 \to 0$ then (remember all sums over a finite number of non-zero terms)

$$\left| \sum_{j=1}^{\infty} e_j \right| = \left| \sum_{j=1}^{\infty} e_j - \sum_{j=1}^{\infty} e_j(n) \right| = \|\mathbf{e}(n) - \mathbf{e}\|_1 \to 0$$

so $\sum_{j=1}^{\infty} e_j = 0$.

(d) If we use the norm $\|_2$, N not closed. Consider the same example as (a).

(e) If we use the norm $\| \|_u$ then N is closed. We can use much the same proof as (c).

[To see T not continuous in (a), (b) and (d) we can consider the norms of $\mathbf{e} - \mathbf{e}(n)$ and $T(\mathbf{e} - \mathbf{e}(n))$. In (c) and (e) simple inequalities give $\|T\| \leq 1$, indeed $\|T\| = 1$.]

K203

The key observation is that, since $2^{1/2}$ is irrational, $x + y2^{1/2} = x' + y'2^{1/2}$ for x, y, x', $y' \in \mathbb{Q}$ if and only if $x = x'$ and $y = y'$.

If p_n, q_n are positive integers with $p_n/q_n \to 2^{1/2}$ (in the usual Euclidean metric) then $N\big((-p_n, q_n)\big) \to 0$ but $\|(-p_n, q_n)\| \to \infty$.

On the other hand

$$N\big((x, y)\big) = |x + y2^{1/2}| \le |x| + 2|y| \le 2\|(x, y)\|$$

for $(x, y) \in \mathbb{Q}^2$.

K204*

No comments.

K205

Pick $x_n \in F_n$. If $m \geq n$ then

$$d(x_n, x_m) \leq \operatorname{diam}(F_n) \to 0$$

as $n \to \infty$. Thus x_n is Cauchy and we can find $x \in X$ with $d(x_n, x) \to 0$ as $n \to \infty$.

Since $x_n \in F_m$ for all $n \geq m$ and F_m is closed, $x \in F_m$ for each m so $x \in \bigcap_{j=1}^{\infty} F_j$. On the other hand, if $y \in \bigcap_{j=1}^{\infty} F_j$, then since x, $y \in F_n$ for each n,

$$d(x, y) = \operatorname{diam}(F_n) \to 0$$

so $d(x, y) = 0$ and $x = y$. Thus $\bigcap_{j=1}^{\infty} F_j$ contains exactly one point.

(a) Take $X = (0, 1]$ with the usual Euclidean metric d and $F_n = (0, 1/n]$.

(b) Take $X = \mathbb{Z}$ with the discrete metric d (so $d(m, n) = 1$ for $m \neq n$). Let $F_n = \{m : m \geq n\}$.

(c) Take $F_n = \{x : x \geq n\}$.

K206

(i) If ρ is a metric, then $\rho(x,y) > 0$ for $x \neq y$ so f is strictly increasing.

If f is strictly increasing, then it is easy to check that ρ is a metric.

(ii) If f is not continuous at x, then without loss of generality, we may suppose that f is not left continuous and there exist $x_1 < x_2 < \dots$ with $x_j < x$ and $x_j \to x$ together with a $\delta > 0$ such that $f(x) - \delta \geq f(x_j)$ for all j. Since

$$\sum_{j=1}^{N} \rho(x_{j+1}, x_j) = f(x_{N+1}) - f(x_1) < f(x) - f(x_1),$$

we have $\sum_{j=1}^{\infty} \rho(x_{j+1}, x_j)$ convergent and x_j Cauchy with respect to ρ.

If $t < x$ then there exists an N such that $t > x_N$ so $\rho(t, x_j) \geq \rho(t, x_N) > 0$ for $j \geq N$ so $\rho(t, x_j) \nrightarrow 0$. If $t > x$, then $\rho(t, x_j) \geq \rho(t, x) > 0$ for all j so $\rho(t, x_j) \nrightarrow 0$. If $t = x$, then $\rho(t, x_j) \geq \delta > 0$ for all j so $\rho(t, x_j) \nrightarrow 0$. Thus the x_j form a non-convergent Cauchy sequence for ρ.

If $f(t) \nrightarrow \infty$ then a similar argument shows that, if $x_j = j$, the x_j form a non-convergent Cauchy sequence for ρ. A similar argument covers the case $f(t) \nrightarrow -\infty$.

(iii) We must have

$$g(t) + g(s) = d(t, 0) + d(0, -s) \geq d(t, -s) = g(t + s)$$

for all t, $s \geq 0$,

$$g(t) + g(s) = d(t, 0) + d(0, s) \geq d(t, s) = g(t - s)$$

for all $t \geq s \geq 0$ and

$$g(0) = d(0, 0) = 0.$$

We check that these conditions are sufficient.

(iv) If $2^{-n-1} \leq x \leq 2^{-n}$, then

$$2^{n+1} g(x) \geq 2^{n=1} g(2^{-n-1}) = \sum_{j=1}^{2^{n+1}} g(2^{-n-1}) \geq g\left(\sum_{j=1}^{2^{n+1}} 2^{-n-1}\right) = g(1)$$

so $g(x) \geq 2^{-n-1} g(1) \geq x 2^{-1} g(1)$. Take $K = 2^{-1} g(1)$.

If we take $g(x) = x^{1/2}$ we see that no relation $d(x, y) \leq L|x - y|$ will hold in this case.

(v) Since g is increasing and $g(t) \geq 0$, $g(t) \to l$ as $t \to 0+$ for some $l \geq 0$. If $l > 0$ then $d(x, y) > l$ for all $x \neq y$ so any Cauchy sequence for d is eventually constant and so converges.

If $l = 0$ then, since g is decreasing, given any $\delta > 0$ we can find and $\epsilon > 0$ such that $\epsilon > g(s) \geq 0$ implies $|s| < \delta$. Thus if x_n is Cauchy for d, it follows that x_n is Cauchy for the Euclidean metric so there exists an $x \in \mathbb{R}$ with $|x_n - x| \to 0$ and so $d(x_n, x) = g(|x_n - x|) \to 0$.

(vi) Sufficiency by direct verification. Not necessary. Let θ be the discrete metric with $\theta(x, y) = 1$ for $x \neq y$. Then $g(t) = 0$ for $t \neq 1$, $g(1) = 1$ gives a metric but $g(1/2) + g(1/2) = 0$ and $g(1) = 1$.

K207

(i) Write $\mathbf{e}_j$ for the vector with 1 in the jth place and 0 elsewhere. Then

$$\|a_j\mathbf{e}_j\| \leq \|\mathbf{a}\|$$

and so

$$|a_j| \leq \|\mathbf{a}\|\|\mathbf{e}_j\|^{-1}.$$

K208

We have

$$\sum_{j=1}^{N}(a_j+b_j)^2 \leq \left(\left(\sum_{j=1}^{N} a_j^2\right)^{1/2} + \left(\sum_{j=1}^{N} a_j^2\right)^{1/2}\right)^2$$
$$\leq \left(\left(\sum_{j=1}^{\infty} a_j^2\right)^{1/2} + \left(\sum_{j=1}^{\infty} a_j^2\right)^{1/2}\right)^2,$$

so, since an increasing sequence bounded above converges we have $\sum_{j=1}^{\infty}(a_j+b_j)^2$ convergent. Moreover

$$\sum_{j=1}^{\infty}(a_j+b_j)^2 \leq \left(\left(\sum_{j=1}^{\infty} a_j^2\right)^{1/2} + \left(\sum_{j=1}^{\infty} a_j^2\right)^{1/2}\right)^2.$$

(ii) To show completeness, suppose $\mathbf{a}(n)$ a Cauchy sequence in l^2. We have

$$|a_j(n) - a_j(m)| \leq \|\mathbf{a}(n) - \mathbf{a}(m)\|_2$$

so $a_j(n)$ is Cauchy with respect to the usual Euclidean distance and there exists an $a_j \in \mathbb{R}$ such that $|a_j(n) - a_j| \to 0$.

Now observe that,

$$\begin{aligned}
\left(\sum_{j=1}^{M} a_j^2(n)\right)^{1/2} &\leq \left(\sum_{j=1}^{M}(a_j - a_j(n))^2\right)^{1/2} + \left(\sum_{j=1}^{M} a_j(n)^2\right)^{1/2}\\
&\leq \left(\sum_{j=1}^{M}(a_j - a_j(n))^2\right)^{1/2} + \|\mathbf{a}(n)\|_2\\
&\leq \left(\sum_{j=1}^{M}(a_j - a_j(n))^2\right)^{1/2} + \sup_{r\geq 1}\|\mathbf{a}(r)\|_2\\
&\to \sup_{r\geq 1}\|\mathbf{a}(r)\|_2
\end{aligned}$$

as $n \to \infty$. (Recall that a Cauchy sequence is bounded.) Thus

$$\left(\sum_{j=1}^{M} a_j^2\right)^{1/2} \leq \sup_{r\geq 1}\|\mathbf{a}(r)\|_2$$

for all M and so $\mathbf{a} \in l^2$.

Finally, if n, $m \geq N$

$$\begin{aligned}\left(\sum_{j=1}^{M}(a_j^2 - a_j(n))^2\right)^{1/2} &\leq \left(\sum_{j=1}^{M}(a_j^2 - a_j(m))^2\right)^{1/2} + \left(\sum_{j=1}^{M}(a_j^2(n) - a_j(m))^2\right)^{1/2}\\ &\leq \left(\sum_{j=1}^{M}(a_j^2 - a_j(m))^2\right)^{1/2} + \|\mathbf{a}(n) - \mathbf{a}(m)\|_2\\ &\leq \left(\sum_{j=1}^{M}(a_j^2 - a_j(m))^2\right)^{1/2} + \sup_{r,,s\geq N}\|\mathbf{a}(s) - \mathbf{a}(r)\|_2\end{aligned}$$

so, allowing $m \to \infty$,

$$\left(\sum_{j=1}^{M}(a_j^2 - a_j(n))^2\right)^{1/2} \leq \sup_{r,s\geq N}\|\mathbf{a}(s) - \mathbf{a}(r)\|_2$$

and, allowing $M \to \infty$,

$$\|\mathbf{a} - \mathbf{a}(n)\|_2 \leq \sup_{r,s\geq N}\|\mathbf{a}(s) - \mathbf{a}(r)\|_2$$

for all $n \geq N$. Thus

$$\|\mathbf{a} - \mathbf{a}(n)\|_2 \to 0$$

as $n \to \infty$.

(iii) We have (by Cauchy-Schwarz in N dimensions)

$$\sum_{j=1}^{N}|a_j b_j| \leq \left(\sum_{j=1}^{N} a_j^2\right)^{1/2}\left(\sum_{j=1}^{N} a_j^2\right)^{1/2} \leq \|\mathbf{a}\|_2\|\mathbf{b}\|$$

so, since an increasing sequence bounded above converges, we have $\sum_{j=1}^{\infty} a_j b_j$ absolutely convergent and so convergent.

K209

(ii) we have

$$-\log\left(\frac{x}{p}+\frac{y}{q}\right) \leq -\frac{1}{p}\log x - \frac{1}{q}\log y$$

for all $x\,y > 0$. Setting $x = X^p$, $y = Y^Q$, multiplying by -1 and taking exponentials gives the inequality.

(iii) By (ii),

$$|f(t)g(t)| \leq \frac{|f(t)|^p}{p} + \frac{|g(t)|^q}{q}.$$

Now integrate.

K210

(i) Observe that $(p-1)q = p$, so, taking $g(t) = |f(t)|^{p-1}$, we have

$$\int_a^b |f(t)|^p \, dt \leq A \left(\int_a^b |f(t)|^{(p-1)q} \right)^{1/q} = A \left(\int_a^b |f(t)|^p \, dt \right)^{1/q},$$

whence the result.

(ii) We have

$$\begin{aligned}
\int_a^b |(f(t)+h(t))g(t)| \, dt &\leq \int_a^b |f(t)g(t)| \, dt + \int_a^b |h(t)g(t)| \, dt \\
&\leq \left(\left(\int_a^b |f(t)|^p \, dt \right)^{1/p} + \left(\int_a^b |h(t)|^p \, dt \right)^{1/p} \right) \left(\int_a^b |g(t)|^q \, dt \right)^{1/q}.
\end{aligned}$$

Now use (i).

(iii) To see that $(C([a,b]), \| \;\|_p)$ is not complete follow the proof that $(C([a,b]), \| \;\|_1)$ is not complete.

(iv) We work in $C([a,b])$. $\|fg\|_1 \leq \|f\|_p \|g\|_q$. If $\|fg\|_1 \leq A\|g\|_q$ for all g then $\|f\|_p \leq A$.

K211

It may be helpful to look at K209.

(iv) If $\| \; \|_*$ is derived from an inner product, then
$$\|\mathbf{a}+\mathbf{b}\|_*^2 + \|\mathbf{a}-\mathbf{b}\|_*^2 = 2(\|\mathbf{a}\|^2 + (\|\mathbf{a}\|^2).$$
Taking
$$\mathbf{a} = (1,0,0,0\dots),\ \mathbf{b} = (0,1,0,0,\dots),$$
we see that this fails for $\| \; \|_p$ unless $p=2$.

In the case $C([0,1], \| \; \|_p)$ we look at (for example)
$$f(x) = g(1-x) = \max(1-4x, 0).$$
Observe that
$$\int_0^1 f(x)^p \, dx = \int_0 1/4(4t)^p \, dt.$$

(v) We consider $C([a,b])$. It is easy to check that
$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$$
and (by setting $g=1$) that if
$$\|fg\|_1 \leq A\|g\|_\infty$$
for all continuous g then $\|f\|_1 \leq 1$. If $f \in C([a,b])$ and there exists an $x \in [a,b]$ with $|f(x)| \geq A+\epsilon$. Without loss of generality, we may suppose $f(x) \geq A+\epsilon$. By continuity we can find a $\delta > 0$ such that $f(t) \geq A = \epsilon/2$ for $t \in [a,b] \cap (x-\delta, x+\delta)$. Choose a non-zero $g \in C([a,b])$ with $g(t) = 0$ for all $t \notin [a,b] \cap (x-\delta, x+\delta)$. Then
$$|fg\|_1 > (A+\epsilon/2)\|g\|_1.$$
Thus Hölder and reverse Hölder hold for $p=1$, $q=\infty$ and $p=\infty$, $q=1$.

K212

(i) $p = 1$. A square with sides at $\pi/4$ to the axes.

$p = 2$. A disc.

$p = \infty$ A square with sides parallel to the axes.

Other values of p give smooth boundary (as does $p = 2$).

(ii) If $1 < p < \infty$, then the chain rule shows that f_p differentiable except at $(0, 0)$. Since

$$\frac{f_p(x, 0)}{x} = \operatorname{sgn} x$$

does not tend to a limit as $x \to 0$, f_p is not differentiable at $(0, 0)$.

If x, $y > 0$ then $f_1(x, y) = x + y$ so f_1 is differentiable on $\{(x, y) : x, y > 0\}$. On the other hand

$$\frac{f_1(h, y) - f_1(0, y)}{h} = \operatorname{sgn} h$$

does not tend to a limit as $h \to 0$ so f_1 is not differentiable at $(0, y)$. Similar arguments show that f_1 is differentiable at the points (x, y) with both $x \neq 0$ and $y \neq 0$ and only there.

If $x > y \geq 0$ then $f_\infty(x, y) = x$ so f_1 is differentiable on $\{(x, y) : x > y \geq 0\}$. However

$$\frac{f_\infty(x + h, x) - f_\infty(x, x)}{h} = \begin{cases} 1 & \text{if } h > 0, \\ -1 & \text{if } h < 0, \end{cases}$$

so f_∞ is not differentiable at (x, x) if $x > 0$ and a similar argument shows that f_∞ is not differentiable at $(0, 0)$. Similar arguments show that f_∞ is differentiable at the points (x, y) with $x \neq y$ and only there.

(iii) If $\|\mathbf{x}\|_s = 1$ then $|x_j| \leq 1$ and $|x_j|^s \leq |x_j|^r$ so

$$1 = \|\mathbf{x}\|_s^s \leq \|\mathbf{x}\|_r^r$$

so $\|\mathbf{x}\|_r \geq 1$.

$\mathbf{x} = (1, 0, 0, \dots, 0)$.

(iv) We have

$$\sum_{j=1}^n |x_j|^r \leq \left(\sum_{j=1}^n |x_j|^{rp} \right)^{1/p} n^{1/q}.$$

Now set $rp = s$ (so $q(s - r) = s$).

(vi) If $s > r$ we can find an α with $r^{-1} > \alpha > s^{-1}$. If $b_j(N) = j^{-\alpha}$ for $j \leq N$, $b_j(N) = 0$ otherwise, then $\|\mathbf{b}(N)\|_s / \|\mathbf{b}(N)\|_r \to \infty$ as $N \to \infty$.

If $b_j = j^{-\alpha}$, then $\mathbf{b} \in l^r$ but $\mathbf{b} \notin l^s$.

(vii) Argument similar to (iii).

Consider $g_n = \max(1 - nx, 0)$.

Essentially same but there is scale change and we have the formula

$$(b-a)^{1/r}\|f\|_s \leq (b-a)^{1/s}\|f\|_r.$$

K213

(ii) First look at maps from $(\mathbb{R}^n, \|\ \|_\infty)$ to $(\mathbb{R}^m, \|\ \|_\infty)$ Observe that

$$\|T\mathbf{x}\|_\infty = \max_{1\le i\le m}\left|\sum_{j=1}^n a_{ij}x_j\right| \le \max_{1\le i\le m}\sum_{j=1}^n |a_{ij}|\|\mathbf{x}\|_\infty$$

and that, if we set $y_j(i) = \operatorname{sgn} a_{ij}$ we have

$$\|T\mathbf{y}(i)\|_\infty = \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{ij}|\|\mathbf{y}(i)\|_\infty.$$

Thus $\|T\| = \max_{1\le i\le m}\sum_{j=1}^n |a_{ij}|$.

Now look at maps from $(\mathbb{R}^n, \|\ \|_1)$ to $(\mathbb{R}^m, \|\ \|_\infty)$. Observe that

$$\|T\mathbf{x}\|_\infty = \max_{1\le i\le m}\left|\sum_{j=1}^n a_{ij}x_j\right| \le \max_{1\le i\le m}\sum_{j=1}^n |a_{ij}||x_j| \le \max_{1\le i\le m}\max_{1\le j\le n}|a_{ij}|\|\mathbf{x}\|_1$$

and that, if we set $y_j(k) = 1$ if $j = k$, $y_j(k) = 0$, otherwise, then

$$\|T\mathbf{y}(k)\|_\infty = \max_{1\le i\le m}|a_{ik}| = \max_{1\le i\le m}|a_{ik}|\|\mathbf{y}(k)\|_1$$

Thus $\|T\| = \max_{1\le i\le m}\max_{1\le j\le n}|a_{ij}|$.

Now look at maps from $(\mathbb{R}^n, \|\ \|_1)$ to $(\mathbb{R}^m, \|\ \|_1)$. Observe that

$$\begin{aligned}
\|T\mathbf{x}\|_1 &= \sum_{i=1}^m\left|\sum_{j=1}^n a_{ij}x_j\right| \\
&\le \sum_{i=1}^m\sum_{j=1}^n |a_{ij}||x_j| \\
&= \sum_{j=1}^n\sum_{i=1}^m |a_{ij}||x_j| \\
&= \sum_{j=1}^n |x_j|\sum_{i=1}^m |a_{ij}| \\
&\le \max_{1\le j\le n}|\sum_{i=1}^m |a_{ij}|\|\mathbf{x}\|_1
\end{aligned}$$

and that, if we set $y_k(j) = 1$ if $k = j$, $y_k(j) = 1$, otherwise then

$$\|T\mathbf{y}(j)\|_1 = \sum_{i=1}^m |a_{ij}| = \sum_{i=1}^m |a_{ij}\|\mathbf{y}(j)\|_1.$$

Thus $\|T\| = \max_{1\le j\le n}|\sum_{i=1}^m |a_{ij}|$.

If we look at maps from $(\mathbb{R}^n, \| \ \|_\infty)$ to $(\mathbb{R}^m, \| \ \|_1)$ we have the obvious bound

$$\|T\| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$$

but looking at the maps T_1 and T_2 with matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

we see that, in the first case the bound is attained but in the second case it is not since

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

and $|x+y| + |x-y| \leq 2\max(|x|, |y|)$. However, if we really need a to find $\|T\|$ we can obtain it as a solution of a linear programming problem.

K214

Suppose a, $b \in X$. If $\epsilon > 0$, we can find a $y \in E$ such that

$$d(a, E) \geq d(a, y) - \epsilon.$$

Thus

$$d(b, E) \leq d(b, y) \leq d(a, b) + d(a, y) \leq d(a, b) + d(a, E) + \epsilon.$$

Since ϵ is arbitrary

$$d(b, E) \leq d(a, b) + d(a, E)$$

and, similarly, $d(a, E) \leq d(a, b) + d(b, E)$. Thus $|f(a) - f(b)| \leq d(a, b)$ and f is continuous.

Suppose $\mathbf{k} \in K$, $\mathbf{l} \in L$. In the previous paragraph we saw that

$$d(\mathbf{k}, M) \leq d(\mathbf{k}, \mathbf{l}) + d(\mathbf{l}, M)$$

so, by definition,

$$d(\mathbf{k}, M) \leq d(\mathbf{k}, \mathbf{l}) + \rho'(L, M)$$

so, allowing $\mathbf{l}$ to range freely,

$$d(\mathbf{k}, M) \leq d(\mathbf{k}, L) + \rho'(L, M)$$

and so, by definition again,

$$d(\mathbf{k}, M) \leq \rho'(K, L) + \rho'(L, M)$$

so

$$\rho'(K, M) \leq \rho'(K, L) + \rho'(L, M).$$

It follows at once that ρ obeys the triangle inequality. Also ρ is symmetric and positive. If $\rho(E, F) = 0$ then $\rho'(E, F) = 0$ and, if $\mathbf{e} \in E$, we can find $\mathbf{f}_n \in F$ such that $d(\mathbf{f}_n, \mathbf{e}) \to 0$. Since F is closed $\mathbf{e} \in F$ so $F \supseteq E$. Similarly $E \supseteq F$ so $E = F$. Thus ρ is a metric.

The continuous image of a closed bounded set in $\mathbb{R}^n$ is closed and bounded. If $|f_n(\mathbf{x}) - f(\mathbf{x})| < \epsilon$ then

$$\rho'(f(K_n), F(K)),\ \rho'(f(K), F(K_n)) < \epsilon.$$

No. Let $K = [-\pi, \pi]$ and $f_n(x) = \sin nx$.

K215

The intersection of closed bounded sets is closed and bounded. Choose $\mathbf{x}_n \in K_n$. Since K_1 is closed and bounded, the theorem of Bolzano–Weierstrass says that we can pick $n(j) \to \infty$ such that $\mathbf{x}_{n(j)} \to \mathbf{x}$ for some $\mathbf{x} \in K_1$. Since $\mathbf{x}_{n(j)} \in K_m$ for j sufficiently large and K_m is closed $\mathbf{x} \in K_m$ for all m so $\mathbf{x} \in K$.

If $\rho(K, K_n) \nrightarrow 0$ then we can find a $\delta > 0$ such that there exists $\mathbf{x}_n \in K_n$ with $d(\mathbf{x}_n, K) > \delta$ (recall $K_N \supseteq K_{N+1}$). By the argument of the first paragraph we can find $n(j) \to \infty$ and $\mathbf{x} \in K$ with $d(\mathbf{x}_{n(j)}, \mathbf{x}) \to 0$ which is absurd.

K216

(i) and (iii). It is helpful to observe that $d(e, f) = \rho(\{e\}, \{f\})$.

(ii) Take $y \in Y$ if and only if $d(y, E) \leq 2/N$. Observe that the set of possible Y is finite.

K217

Since E is closed, $r = \inf\{\|\mathbf{z} - \mathbf{y}\| : \mathbf{y} \in E\} > 0$. If $1 > \epsilon' > 0$ we can find $\mathbf{y}_0 \in E$ such that

$$\|\mathbf{z} - \mathbf{y}_0\| > r(1 - \epsilon')$$

Set $\mathbf{x}_0 = \mathbf{z} - \mathbf{y}_0$. Then $\|\mathbf{x}_0\| < r(1 + \epsilon')$ and

$$\|\mathbf{x}_0 - \mathbf{e}\| = \|\mathbf{z} - (\mathbf{y}_0 + \mathbf{e})\| \geq r$$

for all $\mathbf{e} \in E$.

If we set $\mathbf{x} = \|\mathbf{x}_0\|^{-1}\mathbf{x}_0\|$ then $\|\mathbf{x}\| = 1$ and

$$\|\mathbf{x} - \mathbf{e}\| \geq \|\mathbf{x}_0\|^{-1} r = (1 + \epsilon')^{-1} > 1 - \epsilon$$

for all $\mathbf{e} \in E$ if we choose ϵ' appropriately.

Let $\mathbf{x}_1$ be any vector of norm 1. Let E_1 be the subspace generated by $\mathbf{x}_1$. Now proceed inductively choosing $\mathbf{x}_{n+1}$ of norm 1 with $\|\mathbf{x}_{n+1} - \mathbf{e}\| \geq 1/2$ for all $\mathbf{e} \in E_n$ and taking E_{n+1} be the subspace generated by $\mathbf{x}_{n+1}$ and E_n.

K218

(i) If $\kappa_n \nrightarrow 0$ we can find a $\delta > 0$ and $n(j) \to \infty$ with $\kappa_{n(j)} \geq \delta$. If $u_{n(j)}(j) = \delta$, $u_k(j) = 0$ if $k \neq n(j)$, then $\mathbf{u}(j) \in E$ but the sequence has no convergent subsequence.

Suppose $\kappa_n \to 0$ and $\mathbf{x}_n \in E$. Set $n(0,j) = n(j)$. Then since $[-\kappa_r, \kappa_r]$ is closed and bounded we can use Bolzano–Weierstrass to find sequences $n(r,j)$ and $y_r \in [-\kappa_r, \kappa_r]$ such that

(a) $n(r,j)$ is a subsequence of $n(r,j-1)$ and $n(r,j) > n(r-1,j)$.

(b) $x_r\big(n(r,j)\big) \to x_r$ as $j \to 0$.

(c) $|x_j\big(n(r,j)\big) - x_j| \leq 2^{-r}$ for all $1 \leq j \leq r$.

Observe that $\mathbf{x} \in E$. Now consider $\mathbf{x}_{n(r,r)}$. We have

$$\begin{aligned}\|\mathbf{x}(n(r,r)) - \mathbf{x}\| &\leq \max\big(\max_{1\leq j\leq r} |x_j\big(n(r,j)\big) - x_j|, \sup_{j\geq r+1} |\kappa_j|\big)\\ &\leq \max(2^{-r}, \sup_{j\geq r+1} |\kappa_j|) \to 0\end{aligned}$$

so $\mathbf{x}(n(r,r)) \to \mathbf{x}$ as $r \to \infty$.

(ii) If $\sum_{n=1}^{\infty} \kappa_n$ diverges, we can find $N(j)$ such that $\sum_{n=j}^{N(j)} \kappa_n > 1$. If $u_k(r) = \kappa_k/(\sum_{n=j}^{N(j)} \kappa_n)$ for $j \leq k \leq N(j)$, $u_k(r) = 0$ otherwise, then $\mathbf{u}(k) \in E$ but the sequence has no convergent subsequence.

The proof of the positive result resembles that in (i).

K219

(i) Take complements.

(ii) If not we can find x_n such that $B(x_n, 1/n)$ does not lie in any U. By the Bolzano–Weierstrass property we can find $n(j) \to 0$ and $x \in X$ such that $x_{n(j)} \to x$. Now there must exist a $U \in \mathcal{U}$ with $x \in U$. Since U is open there exists a $\delta > 0$ such that $B(x, \delta) \subseteq U$. But we can find a J such that $n(J) > 2\delta^{-1}$ and $\|x_{n(J)} - x\| < \delta/2$ so

$$B(x_{n(J)}, 1/n(J)) \subseteq B(x, \delta) \subseteq U$$

which contradicts our initial assumption.

(iii) By Lemma 11.22 we can find $y_1, y_2, \ldots y_M$ such that $\bigcup_{m=1}^{M} B(y_m, \delta) = X$ and by (ii) we can find $U_m \in \mathcal{U}$ with $B(y_m, \delta) \subseteq U_m$. Thus $\bigcup_{m=1}^{M} U_m = X$.

K220

(i) Take contrapositives.

(ii) If the space has property (B) then, taking $\mathcal{U}$ to be the collection of $B(y, \delta_y)$, we can find y_1, y_2, $\dots$, y_M such that $X = \bigcup_{m=1}^{M} B(y_m, \delta_{y_m})$. Now set $N = \max_{1 \leq m \leq M} N_m$. We have $x_N \notin B(y_m, \delta_{y_m})$ for each $1 \leq m \leq M$ and this gives a contradiction.

(iii) Combine the results of this question with that of K219.

K221

$\| \ \|_B$ is not a norm since $\|1\|_A = 0$ but $1 \neq 0$. The rest can be checked to be norms.

Recall that Lipschitz equivalent metrics are either both complete or neither complete.

$$\|f\|_\infty \leq \|f\|_A \leq 2\|f\|_\infty$$

so $\| \ \|_\infty$ and $\| \ \|_A$ are equivalent. Looking at $f_n(x) = (n^{-2} + (x - \frac{1}{2})^2)^{1/2}$ we see that f_n is Cauchy. If it has a limit f in the uniform norm then it is a pointwise limit of f_n so $f(x) = |x - \frac{1}{2}|$. But $f \notin C^1([0,1])$.

$$\|f\|_D \leq \|f\|_C \leq 2\|f\|_D$$

so $\| \ \|_C$ and $\| \ \|_D$ are equivalent. If f_n is Cauchy in $\| \ \|_C$ then both f_n and f_n' are Cauchy in $\| \ \|_\infty$ converge uniformly to continuous f and F and standard theorems tell us that f is differentiable with $f' = F$. Thus $f \in C^1([0,1])$ and $\|f_n - f\|_C \to 0$.

By considering $f_N(x) = \sin Nx$ we see that $\| \ \|_D$ is not Lipschitz equivalent to $\| \ \|_\infty$ or $\| \ \|_1$. By considering $f_N(x) = x^N$ we see that $\| \ \|_\infty$ is not equivalent to $\| \ \|_1$ (or observe that one norm is complete and the other is not).

K222

(i) By the inverse function rule
$$\frac{d}{dx}\sin^{-1}x = \frac{1}{(1-x^2)^{1/2}}$$
for $|x| < 1$. By the binomial expansion (or by some rigorous Taylor expansion)
$$\frac{1}{(1-t)^{1/2}} = 1 + \sum_{n=1}^{\infty}\prod_{r=1}^{n}\frac{(2r-1)}{2r}t^n$$
for $|t| < 1$. Thus
$$\frac{d}{dx}\sin^{-1}x = 1 + \sum_{n=1}^{\infty}\prod_{r=1}^{n}\frac{(2r-1)}{2r}x^{2n}$$
for $|x| < 1$. But we may integrate term by term within the radius of convergence so (since $\sin^{-1}0 = 0$)
$$\sin^{-1}x = x + \sum_{n=1}^{\infty}\prod_{r=1}^{n}\frac{(2r-1)}{2r}\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{2^{-2n}}{2n+1}\binom{2n}{n}x^{2n+1}$$
for $|x| < 1$.

(ii) The power series has radius of convergence 1 so the only other points to consider are $x = \pm 1$. Taking logarithms (cf K96) shows that
$$\prod_{r=1}^{n}\frac{(2r-1)}{2r} = \prod_{r=1}^{n}\left(1-\frac{1}{2r}\right) \to n^{-1/2}$$
so
$$\sum_{n=1}^{\infty}\frac{1}{2n+1}\prod_{r=1}^{n}(2r-1)2r$$
converges by comparison with $n^{-3/2}$ so we actually have (by the Weierstrass M-test) uniform convergence on $[-1,1]$ so the power series is continuous on $[-1,1]$ so, since $\sin^{-1}$ is also continuous on $[-1,1]$, we have equality at $x = 1$ and $x = -1$ as well.

K223

(i) $f_n(x) = (n^{-2} + x^{-2})^{1/2}$ (with positive square root) will do.

(ii) Use (i).

(iii) If f_n is Cauchy in $\| \|_*$ then both f_n and f_n' are Cauchy in $\| \|_\infty$ converge uniformly to continuous f and F but then we know that f is differentiable with $f' = F$. Thus $f \in C^1([0,1])$ and $\|f_n - f\|_C \to 0$.

K224

Let $h_n : [-1,1] \to \mathbb{R}$ be given by

$$h_n(x) = \begin{cases} 2^{n+3}(x+1) & \text{for } x \in [-1, -1+2^{-n-3}], \\ 1 & \text{for } x \in [-1+2^{-n-3}, -2^{-n-3}] \\ -2^{n+3}x & \text{for } x \in [-2^{-n-3}, 0] \\ -h(-x) & \text{for } x \in [0,1]. \end{cases}$$

Let $g_n : [-1,1] \to \mathbb{R}$ be given by

$$g_n(x) = \int_{-1}^{1} h_n(t)\,dt$$

By the fundamental theorem of analysis, g_n is differentiable with continuous derivative h_n so $|g_n'(x)| \leq 1$ for all x and $|g_n(x)| = 1$ outside three intervals of total length 2^{-n-1}. We observe that $g_n(x) \to 1 - |x|$ uniformly on $[-1,1]$

We define $k_n : \mathbb{R} \to \mathbb{R}$ to be the 1-periodic function with $k_n(x) = 2^{-1}g_n(2(x-1))$ for $x \in [-1,1]$ and define $f_n : [0,1] \to \mathbb{R}$ by $f_n = n^{-1}k_n(nx)$ for $x \in [0,1]$.

Observe that $f_n \in \mathcal{A}$, $\|f_n\|_\infty \to 0$ as $n \to 0$ but $I(f_n) \to 0$ and $I(0) = 1$. Thus I is not continuous with respect to the uniform norm.

However if g_n, $g \in \mathcal{A}$ and $\|g_n - g\|_* \to 0$ then $g_n'(x) \to g(x)$ uniformly, so $(1-(g_n'(x))^4)^2 \to (1-(g'(x))^4)^2$ uniformly, so

$$(1-(g_n'(x))^4)^2 + g_n(x)^2 \to (1-(g'(x))^4)^2 + (x)^2$$

uniformly as $n \to 0$ so $I(g_n) \to I(g)$ and I is continuous with respect to $\|\ \|_*$.

(iii) Done in (i).

(iv) Since I is continuous it follows that, if $f_N \to f$ with respect to $\|\ \|_*$ then $If_n \to If$, so $If = 0$ which is impossible. (If $If = 0$ then $\int_{-1}^{1} f(x)^2\,dx = 0$ so $f = 0$ (since f is continuous) but $I(0) = 1$.)

K225

(i) If $f_{N,\epsilon}(x) = \epsilon \sin 2\pi N x$ then

$$\begin{aligned} I(f_{N,\epsilon}) &= \int_0^1 \big(1 - 2\epsilon(2\pi N)^4(\cos 2\pi N x)^4 \\ &\qquad + \epsilon^2(2\pi N)^8(\cos 2\pi N x)^8 + \epsilon^2(\sin 2\pi N x)^2\big)\,dx \\ &= 1 - A\epsilon N^4 + B\epsilon^8 N^8 + C\epsilon^2 \end{aligned}$$

with

$$\begin{aligned} A &= 2(2\pi)^4 \int_0^1 (\cos 2\pi x)^4\,dx > 0 \\ B &= (2\pi)^8 \int_0^1 (\cos 2\pi x)^8\,dx > 0 \\ C &= (2\pi)^2 \int_0^1 (\sin 2\pi x)^2\,dx > 0. \end{aligned}$$

Also

$$\|f_{N,\epsilon}\|_* = (1 + 2\pi N)\epsilon.$$

If we take $g_N = f_{N,N^{-4/3}}$ then $\|g_N\|_* \to 0$, and (when N is sufficiently large) $I(g_N) < 1 = I(0)$. If we take $h_N = f_{1,N^{-1}}$, then $\|h_N\|_* \to 0$, and (when N is sufficiently large) $I(h_N) > 1 = I(0)$.

(ii) Observe that $g(t) = 6(t-1)(t-2) = 6t^2 - 18t + 12$ has zeros at 1 and 2. Thus $f(t) = 2t^3 - 9t^2 + 12t$ has stationary points at 1 and 2 so (from our knowledge of cubics or by looking more closely) has a unique zero at 0. Setting $P(t) = f(t)^2$ gives a function of the required type.

(iii) There exists a $\delta > 0$ such that $P(t) > 0$ for all $0 < |t| < \delta$. Thus, if $|f'(t)| < \delta^{1/2}$, the function $h : [0,1] \to \mathbb{R}$ defined by

$$h(x) = P(1 - f'(x)^2) + f(x)^2 - P(1)$$

is continuous and positive so

$$J(f) - I(0) = \int_0^1 h(x)\,dx \geq 0$$

with equality only if $h(x) = 0$ for all $x \in [0,1]$ i.e. only if $f = 0$. Thus if $\|f\|_* < \delta^{1/2}$ we have $J(f) \geq 0$ with equality if and only if $f = 0$.

Use the f_n of K224.

(iv) This is really no more complicated than finding a smooth function $u : \mathbb{R} \to \mathbb{R}$ with $0 < u(x) < 1$ for all t and $\inf_{x\in\mathbb{R}} u(x) = 0$, $\sup_{x\in\mathbb{R}} u(x) = 1$.

Set $v(x) = \frac{1}{2} + \pi^{-1}\tan^{-1} x$ and $G(s,t) = v(s)$.

K226

(iv) It remains true that d_∞ is a metric and that the continuous functions form a closed subspace of $\mathcal{B}(E)$ and that the uniform limit of continuous functions is continuous since the proofs do not use the completeness of (Y,ρ).

Suppose that Y is not complete, so we can find a Cauchy sequence $y(n)$ in Y which does not converge. Let $E = e$ and $f_y(e) = y$. (Note $\mathcal{B}(E) = \mathcal{C} = C(E)$ and the members of $C(E)$ are precisely the f_z.) Then $d_\infty(f_y, f_z) = \rho(y,z)$ so the $f_{y(n)}$ do not converge. $\mathcal{B}(E)$, $\mathcal{C}(E)$ and $C(E)$ are not complete and the general principle of convergence fails.

K227

Observe that, if $0 < v < u$, then $e^{vx}e^{-ux}x^{\gamma-1} \to 0$ as $x \to 0$ (exponentials beat powers) so we can find an A such that

$$0 \leq e^{-ux}x^{\gamma-1} \leq Ae^{-vx}$$

for $x \geq 1$ so, by comparison, $\int_1^\infty e^{-ux}x^{\gamma-1}\,dx$ converges. Also

$$0 \leq e^{-ux}x^{\gamma-1} \leq x^{\gamma-1}$$

so, by comparison, $\int_0^1 e^{-ux}x^{\gamma-1}\,dx$ converges. Thus $\int_0^\infty e^{-ux}x^{\gamma-1}\,dx$ converges.

Our theorem on differentiating under a finite integral show that

$$\frac{d}{du}\int_{1/n}^{n} e^{-ux}x^{\gamma-1}\,dx = \int_{1/n}^{n} e^{-ux}x^{\gamma}\,dx.$$

The argument of our proof on differentiating under an infinite integral now shows that

$$\frac{d}{du}\int_{0}^{\infty} e^{-ux}x^{\gamma-1}\,dx = \int_{0}^{\infty} e^{-ux}x^{\gamma}\,dx.$$

Now integrate by parts, to get

$$\int_\epsilon^X e^{-ux}x^\gamma\,dx = \left[-u^{-1}x^\gamma e^{-ux}\right]_\epsilon^X + \gamma u^{-1}\int_\epsilon^X e^{-ux}x^{\gamma-1}\,dx$$

and, taking limits,

$$\phi_\gamma'(u) = \gamma u^{-1}\phi_\gamma(u).$$

Thus

$$\frac{d}{du}\big(u^{-\gamma}\phi_\gamma(u)\big) = 0$$

and, by the constant value theorem,

$$\phi_\gamma(u) = A_\gamma u^\gamma.$$

(ii) We have

$$\phi_n(u) = \frac{1}{(n-1)!}u^\gamma.$$

(iii) The arguments including the proofs of convergence for the appropriate integrals are similar to but simpler than those of (i).

$$\psi'(u) = -\int_0^\infty xe^{-ux}e^{-x^2/2}\,dx = \left[-e^{-ux}e^{-x^2/2}\right]_{-\infty}^{\infty} - u\psi(u) = -u\psi(u)$$

so

$$\frac{d}{du}\big(\psi(u)e^{u^2/2}\big) = 0$$

and $\psi(u) = Ae^{-u^2/2}$ for some constant A.

(iv) To obtain (i) make the substitution $s = ux$. To obtain (iii) make the substitution $y = x - u$.

K228

Since $f^{(n+1)} \to g$ uniformly on the closed interval with end points a and x, we have

$$\int_a^x g(t)\,dt = g(x) - g(a).$$

Since g is continuous, the fundamental theorem of analysis gives

$$g(x) = g'(x)$$

so

$$\frac{d}{dx}\big(e^{-x}g(x)\big) = 0$$

so, by the constant value theorem, $g(x) = Ce^x$ for some C.

No, $f(x) = 1 + e^x$ has the property.

K229

(i) Since exponentials beat polynomials, $f_n(x) \to 0$ for $x \neq 0$. But $f_n(0) = 0 \to 0$, so $f_n \to 0$ pointwise for all α, $\beta > 0$.

(ii) We observe that (if $x \in (0,1)$)

$$f_n'(x) = n^\alpha x^{\beta-1}(1-x)^{n-1}\big(\beta - (\beta+n)x\big)$$

so, examining the sign of f_n', we see that there is a unique maximum at $x = \beta/(\beta+n)$. Thus

$$0 \leq f(x) \leq f\left(\frac{\beta}{\beta+n}\right) = n^{\alpha-\beta}\left(\frac{\beta}{1+\beta/n}\right)^\beta \left(1 - \frac{\beta}{\beta+nr}\right)^n$$

so, remembering that $(1+\gamma n^{-1})^n \to e^\gamma$, we see that $f_n \to 0$ uniformly if and only if $\beta > \alpha$.

(iii) We have

$$\begin{aligned}\int_0^1 f_n(x)\,dx &\geq \int_{1/n}^{2/n} f_n(x)\,dx \\ &\geq \int_{1/n}^{2/n} n^\alpha (n^{-1})^\beta \left(1-\frac{2}{n}\right)^n\,dx \\ &= n^{\alpha-\beta-1}\left(1-\frac{2}{n}\right)^n\end{aligned}$$

so $\int_0^1 f_n(x)\,dx \not\to 0$ if $\alpha \geq \beta+1$.

Now suppose $\alpha < \beta+1$. Choose γ with

$$1 > \gamma > \frac{\alpha}{\beta+1}.$$

We have

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= \int_0^{n^{-\gamma}} f_n(x)\,dx + \int_{n^{-\gamma}}^1 f_n(x)\,dx \\ &\leq \int_0^{n^{-\gamma}} n^\alpha x^\beta\,dx + \int_{n^{-\gamma}}^1 n^\alpha (1-x)^n\,dx \\ &= \frac{n^{\alpha-(\beta+1)\gamma}}{\beta+1} + \frac{n^\alpha}{n+1}(1-n^{-\gamma})^{n+1} \\ &= \frac{n^{\alpha-(\beta+1)\gamma}}{\beta+1} + \frac{n^\alpha}{n+1}(1-n^{-\gamma})\big((1-n^{-\gamma})^{n^\gamma}\big)^{n(1-\gamma)} \to 0,\end{aligned}$$

as $n \to \infty$.

[An alternative approach is to show that, when $\alpha = \beta+1$,

$$\infty > \limsup_{N\to\infty} \int_0^1 f_n(x)\,dx \geq \liminf_{N\to\infty} \int_0^1 f_n(x)\,dx > 0$$

and deduce the other cases.]

K230

(i) Suppose that $g_n \to g$ uniformly and $x_n \to x$. Let $\epsilon > 0$. By the continuity of g at x, we can find a $\delta > 0$ such that $|g(t) - g_n(t)| < \epsilon/2$ for $|x - t| < \delta$ and $t \in [a, b]$. Now there exists an N such that $|x_n - x| < \delta$ for $n \geq N$ and an M such that $\|g_n - g\|_\infty < \epsilon/2$ for $n \geq M$. Thus if $n \geq \max(N, M)$

$$|g_n(x_n) - g(x)| \leq |g_n(x_n) - g(x_n)| + |g(x_n) - g(x)| < \epsilon.$$

If $g_n \nrightarrow g$ uniformly then we can find a $\delta > 0$ and $n(j) \to \infty$ such that

$$|g_{n(j)}(x_{n(j)}) - g(x_{n(j)})| > \delta.$$

By Bolzano–Weierstrass we can find $j(k) \to \infty$ such that $x_{n(j(k))} \to y$ say. Set $y_{n(j(k))} = x_{n(j(k))}$, $y_r = y$ otherwise. Then $y_r \to y$ but, since

$$\begin{aligned}|g_{n(j)}(y_{n(j)}) - g(y)| &\geq |g_{n(j)}(x_{n(j)}) - g(x_{n(j)})| - |g(x_{n(j)}) - g(y)| \\ &\geq \delta - |g(x_{n(j)}) - g(y)| \to \delta\end{aligned}$$

we have $g_r(y_r) \nrightarrow g(y)$.

(ii) 'Only if' fails if g is not assumed continuous. Let $[a, b] = [-1, 1]$ and $g_n(x) = g(x) = H(x)$ with $H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x > 0$. Then $g_n \to g$ uniformly but $g(1/n) \nrightarrow g(0)$.

However, the 'if' part still works. Suppose $g_n \nrightarrow g$ uniformly as before and define $\delta > 0$ as before. Now define $n(j)$ and $m(j)$ inductively as follows. Set $m(0) = 0$. For each $j \geq 1$ choose $n(j) > m(j-1)$ and $x_{n(j)}$ such that

$$|g_{n(j)}(x_{n(j)}) - g(x_{n(j)})| > \delta.$$

Now we know that $g_n(x_{n(j)}) \to g(x_{n(j)})$ so we can find $m(j) > n(j)$ such that

$$|g_{m(j)}(x_{n(j)}) - g(x_{n(j)})| < \delta/2,$$

and so, in particular

$$|g_{n(j)}(x_{n(j)}) - g_{m(j)}(x_{n(j)})| > \delta/2.$$

By Bolzano–Weierstrass we can find $j(k) \to \infty$ such that $x_{n(j(k))} \to y$ say. Set

$$y_{n(j(k))} = y_{m(j(k))} = x_{n(j(k))},$$

and $y_r = y$ otherwise. Since

$$|g_{n(j)}(y_{n(j)}) - g_{m(j)}(y_{m(j)})| = |g_{n(j)}(x_{n(j)}) - g_{m(j)}(x_{n(j)})| > \delta/2 > 0$$

we know that $g_r(y_r)$ does not converge.

(ii) The proof 'only if' in (i) still works. However 'if' part fails (even if g_n is continuous). Let $(a, b) = (0, 1)$ and $g_n(x) = \max(1 - nx, 0)$. If $x \in (0, 1)$, then we can find a $\delta > 0$ such that $[x - \delta, x + \delta] \subseteq (0, 1)$. Since $g_n \to 0$ uniformly on $[x - \delta, x + \delta]$ we apply part (i) to show that $x_n \to x$ implies $g_n(x_n) \to g(x)$. However g_n does not converge uniformly on $[0, 1]$.

K231

(i) Since f is continuous on $[0, 1-\delta]$, it attains its bounds so there exists a $y \in [0, 1-\delta]$ with $f(y) \geq f(x) \geq 0$. Since $0 \leq f(y) < 1$, we have

$$n\int_0^{1-\delta} f(x)^n\,dx \leq nf(y)^n \to 0.$$

Observe that (if $f'(1) \neq 0$) $f'(1) > 0$ since $f(1) > f(1-x)$ for $x \in (0, 1)$. Given any ϵ with $f'(1) > \epsilon > 0$ we can find a $1 > \delta > 0$ such that

$$|f(x) - 1 - f'(0)(1-x)| = |f(x) - f(1) - f'(1)(1-x)| \leq \epsilon(1-x)$$

and so

$$1 - (f'(1) - \epsilon)(1-x) \leq f(x) \leq 1 - (f'(1) - \epsilon)(1-x)$$

for $x \in [1-\delta, 1]$. Thus

$$\begin{aligned}\int_{1-\delta}^1 (1 - (f'(1) - \epsilon)(1-x))^n\,dx &\leq \int_{1-\delta}^1 f(x)^n\\ &\leq \int_{1-\delta}^1 (1 - (f'(1) - \epsilon)(1-x))^n\,dx.\end{aligned}$$

Now

$$\begin{aligned}n\int_{1-\delta}^1 (1 - A(1-x))^n\,dx &= n\int_0^\delta (1 - At)^n\,dt\\ &= \frac{n}{A(n+1)}(1 - (1 - A\delta)^{n+1}) \to A^{-1}\end{aligned}$$

so, using the first paragraph,

$$n\int_0^1 f(x)^n\,dx \to \frac{1}{f'(1)}.$$

(ii) Suppose $g(y) = 1$ for some $y \in (0, 1)$. By the argument of Rolle's theorem, $g'(y) = 0$, so given any $\epsilon > 0$, we can find a $\delta > 0$ with $[y-\delta, y+\delta] \subseteq (0, 1)$ such that

$$g(t) > 1 - \epsilon|y - t|$$

for all t with $|y - t| \leq \delta$. Thus

$$\begin{aligned}n\int_0^1 g(x)^n\,dx &\geq n\int_{y-\delta}^{y+\delta} (1 - \epsilon|y-t|)^n dt\\ &= \frac{2n}{\epsilon(n+1)}(1 - (1 - A\delta)^{n+1}) \to \epsilon^{-1}.\end{aligned}$$

Since ϵ was arbitrary,

$$n\int_0^1 g(x)^n\,dx \to \infty$$

as $n \to \infty$.

K232

Observe that, if η is fixed with $1/3 > \eta > 0$,

$$\begin{aligned}\int_{-1}^{1} S_{m,\eta}(x)\,dx &\geq \int_{-\eta^{1/2}/2}^{\eta^{1/2}/2} S_{m,\eta}(x)\,dx \\ &\geq \int_{-\eta^{1/2}/2}^{\eta^{1/2}/2} \left(1+\frac{3\eta}{4}\right)\,dx \to \infty\end{aligned}$$

but, if $|x| \geq 2\eta^{1/2}$, then

$$|S_{m,\eta}(x)| \leq (1-3\eta)^n \to 0$$

as $m \to \infty$. Thus $T_{m,\eta}(t) \to 0$ uniformly on $[-1, -2\eta^{1/2}] \cup [2\eta^{1/2}, 1]$ as $m \to \infty$.

(vi) If $F : [-1, 1] \to \mathbb{R}$ is continuous, then it is uniformly continuous and, given any $\epsilon > 0$, we can find an integer N such whenever $|t-s| \leq N^{-1}$ and t, $s \in [-1, 1]$ we have $|F(t) - F(s)| < \epsilon/4$. Thus if h is the simplest piecewise linear function on $[-1, 1]$ with $h(r/N) = F(r/N)$ for ${}_N \leq r \leq N$ we have $|h(t) - F(t)| < \epsilon$ for all $t \in [-1, 1]$.

K233

Define $F : [-1,1] \to \mathbb{R}$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \in [0,1] \\ f(-x) & \text{if } x \in [-1,0]. \end{cases}$$

Since F is continuous, we can find polynomials Q_n with $Q_n(t) \to F(t)$ uniformly on $[-1,1]$. Set $R_n(t) = (Q_n(t) + Q_n(-t))/2$. Then

$$\begin{aligned} |R_n(t) - F(t)| &= |(Q_n(t) + Q_n(-t))/2 - (F(t) + F(-t))/2| \\ &\leq |Q_n(t) - F(t)|/2 + |Q_n(-t) - F(-t)|/2 \to 0 \end{aligned}$$

uniformly on $[-1,1]$. Also, since $R_n(t) = R_n(-t)$, we can find a polynomial P_n with $P_n(t^2) = Q_n(t)$.

Let $g : [-1,1] \to \mathbb{R}$ be given by $g(x) = x$. Then

$$\begin{aligned} |g(1) - Q(1^2)| + |g(-1) - Q((-1^2))| &= |g(1) - Q(1)| + |Q(1)) - g(-1)| \\ &\geq ||g(1) - g(-1)| = 2. \end{aligned}$$

K234

Write $X_n(t) = Y_1 + Y_2 + \cdots + Y_n$ where $Y_j = 1$ if the jth trial is a success and $Y_j = 0$ otherwise. Observe that (since the expectation of the sum of random variable is the sum of the expectations, and the variance of the sum of independent random variables is the sum of the variances)

$$\begin{aligned}\mathbb{E}X_n(t) &= \sum_{j=1}^{n} \mathbb{E}Y_j = nt\\ \operatorname{var} X_n(t) &= \sum_{j=1}^{n} \operatorname{var} Y_j = nt(1-t).\end{aligned}$$

We know that there exists an M such that $|f(t)| \leq M$ for all $t \in [0,1]$ and we know that given $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(t) - f(s)| \leq \epsilon$ whenever t, $s \in [0,1]$ and $|t-s| \leq \delta$. By Tchebychev's inequality

$$\begin{aligned}\Pr\left(\left|\frac{X_n(t)}{n} - t\right| \geq \delta\right) &= \Pr\left(\left|\frac{X_n(t)}{n} - \mathbb{E}\left(\frac{X_n(t)}{n}\right)\right| \geq \delta\right)\\ &= \frac{\operatorname{var}(X_n(t)/n)}{\delta^2}\\ &= \frac{t(1-t)}{n\delta^2}\\ &\leq \frac{1}{n\delta^2}\end{aligned}$$

so, writing $\mathbb{I}_A$ for the indicator function of A,

$$\begin{aligned}\mathbb{E}f\left(\frac{X_n(t)}{n}\right) - f(t)\mathbb{E}&\left(\left(f\left(\frac{X_n(t)}{n}\right) - f(t)\right)\mathbb{I}_{[t-\delta,t+\delta]\cup[0,1]}\left(\frac{X_n(t)}{n}\right)\right)\\ &\qquad + \mathbb{E}\left(\left(f\left(\frac{X_n(t)}{n}\right) - f(t)\right)\mathbb{I}_{[0,1]\setminus[t-\delta,t+\delta]}\left(\frac{X_n(t)}{n}\right)\right)\\ &\leq \epsilon \Pr\left(\left|\frac{X_n(t)}{n} - t\right| \leq \delta\right) + 2M \Pr\left(\left|\frac{X_n(t)}{n} - t\right| > \delta\right)\\ &\leq \epsilon + \frac{2M}{n\delta^2}.\end{aligned}$$

Allowing $n \to \infty$ and observing that ϵ is arbitrary gives the required result.

We have

$$p_n(t) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right)\Pr(X_n(t) = r) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right)\binom{n}{r} t^r(1-t)^{n-r},$$

so p_n is polynomial of degree n.

Since

$$\binom{n}{r}t^r(1-t)^{n-r} \geq 0,\ \sum_{r=0}^{n}\binom{n}{r}t^r(1-t)^{n-r} = 1$$

and

$$\sum_{r=0}^{n}\binom{n}{r}\frac{r}{n}t^r(1-t)^{n-r} = t$$

we can use Jensen's inequality (or we could just quote K144 (iv)).

K235

(i) Suppose that f_n does not converge uniformly to f. Since the $f_n(x)$ are decreasing this means that there exists a $\delta > 0$ and $x_n \in I$ such that $f_n(x_n) > f(x)+\delta$. By the theorem of Bolzano–Weierstrass we can find a subsequence $n(j) \to \infty$ and an $x \in I$ such that $x_{n(j)} \to x$. Since $f_n(x) \to f(x)$ we can find an N such that $f(x) + \delta/2 > f_N(x)$ Since $f - f_N$ is continuous we can find an $\eta > 0$ such that $\delta > f_N(y) - f(y)$ for all $y \in I$ with $|y - x| < \eta$. Choosing a J such that $|x_{n(J)} - x| < \eta$ we obtain a contradiction.

(ii) False without (a). Let $I = [0, 1]$, $f_n(1/r) = 1$ for all $r \geq n$, $f_n(x) = 0$ otherwise.

False without (b). Let $I = [0, 1]$, $f_n(x) = x^n$.

False without (c). Witch's hat.

False without (d). Let $I = (0, 1)$, $f_n = 1/(nx)$.

(iii) Easy to check $n = 0$.

If $t \in [-1, 1]$ and $0 \leq p_{n-1}(t) \leq p_n(t) \leq |t|$ then

$$\begin{aligned} p_{n+1}(t) - p_n(t) &= p_n(t) - p_{n-1}(t) + \frac{1}{2}(p_n(t)^2 - p_{n-1}(t)^2) \\ &= \left(1 - \frac{p_n(t) + p_n(t)}{2}\right)(p_n(t) - p_{n-1}(t)) \\ &\leq (1 - |t|)(p_n(t) - p_{n-1}(t) \geq 0 \end{aligned}$$

and

$$p_{n+1}(t) \leq \frac{t^2 + |t|}{2} = \frac{|t|(|t| + 1)}{2} \leq |t|.$$

The result follows by induction.

Since $p_n(t)$ is increasing and bounded above, $p_n(t) \to g(t)$ say. We have

$$\begin{aligned} 0 &= p_n(t) - p_{n-1}(t) - \frac{1}{2}(t^2 - p_{n-1}(t)) \\ &\to g(t) - g(t) - \frac{1}{2}(t^2 - g(t)) = -\frac{1}{2}(t^2 - g(t)) \end{aligned}$$

so (since $g(t) \geq 0$) $g(t) = |t|$.

Setting $f = -g$, $f_n = -p_n$ we see that part (i) applies and $p_n(t) \to |t|$ uniformly as $n \to \infty$.

(v) By induction p_n is a polynomial.

(vi) Can extend to any bounded closed set I in $\mathbb{R}^n$ or any metric space satisfying the Bolzano–Weierstrass condition.

K236

(ii) Proceed inductively. Write $p_n(x) = x^n$. Set $s_0 = 1$. When s_0, s_1, $\dots$, s_{n-1} have been defined set

$$s_n = p_n - \sum_{j=0}^{n-1} \frac{\langle p_n, s_j\rangle}{\langle s_j, s_j\rangle}.$$

(This is the Gramm-Schmidt technique.)

(iii) To prove uniqueness observe that

$$\sum_{j=0}^{n} b_j x^j = \sum_{j=0}^{n} a_j x^j$$

implies $b_n = a_n$ and use induction. A similar induction proves existence.

Observe that $Q = \sum_{j=0}^{n-1} c_j x^j$.

(iv) Write $\mathcal{P}_n$ for the set of polynomials of degree or less. Since the polynomials are uniformly dense, we have, using (iii),

$$\begin{aligned}\inf_{\mathbf{a}\in\mathbb{R}^{n+1}} \left\| \sum_{j=0}^{n} a_j s_j - f \right\|_2 &= \inf_{P\in\mathcal{P}_n} \|P - f\|_2 \\ &\leq (b-a)^{1/2} \inf_{P\in\mathcal{P}_n} \|P - f\|_\infty \to 0\end{aligned}$$

as $n \to \infty$.

(v) We have

$$\begin{aligned}\|\sum_{j=0}^{n} b_j s_j - f\|_2^2 &= \|f\|_2^2 + \sum_{j=0}^{n} (b_j^2 \|s_j\|_2^2 - 2b_j\langle f, s_j\rangle) \\ &= \|f\|_2^2 + \sum_{j=0}^{n} (b_j\|s_j\|_2 - \langle f, s_j\rangle \|s_j\|_2^{-1})^2 - \sum_{j=0}^{n} (\langle f, s_j\rangle \|s_j\|_2^{-1}) \\ &= \left\| f - \sum_{j=0}^{n} \frac{\langle f, s_j\rangle}{\|s_j\|_2^2} s_j \right\|_2^2 + \sum_{j=0}^{n} \|s_j\|_2^2 \left(b_j - \frac{\langle f, s_j\rangle}{\|s_j\|_2^2}\right)^2 \\ &= \left\| f - \sum_{j=0}^{n} \hat{f}(j) s_j \right\|_2^2 + \sum_{j=0}^{n} \|s_j\|_2^2 (b_j - \hat{f}(j))^2\end{aligned}$$

The result of the first sentence can now be read off.

The result of the second sentence follows from part (iv).

(vi) Observe that $s_n(t)q(t)w(t)$ is single signed (that is always positive or always negative) and only zero at finitely many points. Thus

$$\int_a^b s_n(t)q(t)w(t)\,dt \neq 0$$

and, by part (iii), q must be a polynomial of degree at least n. Thus $k \geq n$ and we have exhibited n distinct zeros t_1, t_2, $\ldots$, t_n in (a, b). Since s_n can have at most n distinct zeros we are done.

K237

Observe (by Leibniz rule or induction or otherwise) that, if $n > r \geq 0$, then $\frac{d^r}{dx^r}(x^2-1)^n$ has (x^2-1) as a factor and so vanishes when $x = \pm 1$. Thus, if $n \geq m \geq 0$, integrating by parts n times gives

$$\begin{aligned}
&\int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m \, dx \\
&\quad = \left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m \, dx \\
&\quad = -\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^m \, dx \\
&\quad = \dots \\
&\quad = (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^{m+n}}{dx^{m+n}}(x^2-1)^m \, dx.
\end{aligned}$$

Thus

$$\int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m \, dx = \begin{cases} 0 & \text{if } n \neq m \\ (2n)! \int_{-1}^{1} (1-x^2)^n \, dx & \text{if } n = m. \end{cases}$$

Making the substitution $x = \sin\theta$, we have

$$\int_{-1}^{1} (1-x^2)^n \, dx = \int_{-\pi/2}^{\pi/2} \cos^{2n+1}\theta \, d\theta.$$

If $I_n = \int_{-\pi/2}^{\pi/2} \cos^n \theta \, d\theta$, and $n \geq 2$ then integration by parts gives

$$\begin{aligned}
I_n &= \int_{-\pi/2}^{\pi/2} \cos^{n-1}\theta \cos\theta \, d\theta \\
&= \left[\cos^{n-1}\theta \sin\theta\right]_{-\pi/2}^{\pi/2} + (n-1)\int_{-\pi/2}^{\pi/2} \cos^{n-2}\theta \sin^2\theta \, d\theta \\
&= (n-1)\int_{-\pi/2}^{\pi/2} \cos^{n-2}\theta (1-\cos^2\theta) \, d\theta \\
&= (n-1)I_{n-2} - (n-1)I_n
\end{aligned}$$

so $nI_n = (n-1)I_{n-2}$. Since $I_1 = 2$ we have

$$\gamma_n = (2n)! I_{2n+1} = (2n)! \times 2 \times \prod_{j=0}^{n-1} \frac{2n-2j}{2n+1-2j} = \frac{2^{n+1}(n!)^{2n}}{(2n+1)!}.$$

By expanding $(x^2-1)^n$ and differentiating n times we see that p_n is a polynomial of degree n with leading coefficient $(2n)!/(n!)$. Take $c_n = (n!)/(2n)!$.

There are several ways of doing the last sentence including integrating by parts along the lines of the first part of the question.

K238

(i) Observe that $P-\sum_{j=1}^n P(x_j)e_j$ is a polynomial of degree at most $n-1$ which vanishes at the n points x_k and so must be the zero polynomial.

$$a_j = \int_{-1}^1 e_j(t)\,dt.$$

(ii) Long division (more exactly, induction on the degree k of p) shows that, if q is a polynomial of degree r, any polynomial p of degree k can be written as $p = sq+r$ where s has degree at most $k-s$ if $k \geq s$ and is zero, otherwise, and r has degree at most $k-1$).

Observe that $\sum_{j=1}^n a_j Q(x_j)p_n(x_j) = \sum_{j=1}^n 0 = 0$.

(iv) Let $p(t) = \prod_{j=1}^n (t-y_j)$. Then p is polynomial of degree exactly n with leading coefficient 1 such that

$$\int_1^1 p(t)q(t)\,dt = \sum_{j=1}^n b_j p(y_j)q(y_j) = 0$$

whenever q has degree at most $n-1$ (and so, in particular, when $q = p_r$ with $0 \leq r \leq n-1$). Thus $p = p_n$.

K239

(i) The argument of K236 shows that $s_n + \lambda s_{n-1}$ must change sign at least $n-1$ times in (a, b). It changes sign only at zeros of odd order and has exactly n zeros if multiple roots are counted multiply so, by counting, it must have n simple zeros in (a, b).

(ii) Let the common root be y. We have $P(x) = (x-y)p(x)$ and $Q(x) = (x-y)q(x)$. If $p(y) = 0$, then P has a multiple root at 0 and we set $\mu = 1$, $\lambda = 0$. If not then $(q(y)/p(y))p - q$ has a root at y and $(q(y)/p(y))P - Q$ has a multiple root at y.

If s_n and s_{n-1} have a common root at y then there exist λ and μ both non-zero (since s_n and s_{n-1} do not have multiple roots) such that $\mu s_n + \lambda s_{n-1}$ and so $s_n + \mu^{-1}\lambda s_{n-1}$ has multiple roots which is impossible.

(iii) Without loss of generality we may suppose that P has no real roots in the open interval (x, y). Thus both P and Q are single signed and, without loss of generality, we may suppose that P and Q are positive in the open interval (x, y). Since a continuous function on a closed bounded interval is bounded and attains its bounds we can find $s_1,\ s_2,\ s_3 \in [x, y]$ such that

$$Q(s_1) \geq Q(s) \geq Q(s_2) > 0 \text{ and } P(s_3) \geq P(s) \geq 0$$

for $s \in [x, y]$.

It follows that the set of real numbers

$$E = \{t \geq P(s_3)/Q(s_1) \,:\, P(s) - tQ(s) \geq 0 \text{ for some } s \in [x, y]\}$$

is a non-empty and bounded above by $P(s_3)/Q(s_2)$. Thus E has a supremum λ' say.We observe that, for each $n \geq 1$ there exists an $w_n \in [x, y]$ such that

$$P(w_n) - (\lambda' - n^{-1})Q(w_n) \geq 0.$$

By the Bolzano–Weierstrass theorem there exists a sequence $n(j) \to \infty$ and an $w \in [x, y]$ such that $w_{n(j)} \to w$. By continuity

$$P(w) - \lambda' Q(w) = 0.$$

We note that this implies $w \in (x, y)$. Since

$$P(t) - \lambda' Q(t) \leq 0$$

for all $t \in [x, y]$ we see that w is a multiple root. Set $\lambda = -\lambda'$.

Last sentence much as for (ii).

K240

Fix $M \geq 1$ temporarily. Observe that (if $|x| < M/2$) we have

$$\left|\frac{1}{(x-n)^{-2}}\right| \leq \frac{4}{n^2}$$

so by Weierstrass M-test $\sum_{n=M}^{N}(x-n)^2$ converges uniformly. We write

$$F_{M,N}(x) = \sum_{n=M}^{N} \frac{1}{(x-n)^2} + \frac{1}{(x+n)^2}.$$

Since the uniform limit of continuous functions is continuous, f_M is continuous.

$$\left|\frac{-2}{(x-n)^3}\right| + \left|\frac{-2}{(x+n)^3}\right| \leq \frac{8}{n^3}$$

so by the Weierstrass M test $f'_{M,N}$ converges uniformly on $[-M/2, M/2]$ to limit g_M say. Thus f_M is differentiable with derivative g_M on $(-M, M)$. Since M was arbitrary we are done.

If $y \notin \mathbb{Z}$ we can find a $\delta > 0$ such that $B(y, 2\delta) \cap \mathbb{Z} = \emptyset$ and an integer M such that $|y| + 2\delta < M/4$. Thus since the sum of two differentiable functions is differentiable

$$F(x) = F_M(x) + \sum_{n=0}^{M-1} \frac{1}{(x-n)^2} + \frac{1}{(x+n)^2}$$

is well defined, continuous and differentiable on $B(y, 2\delta)$. Since y was chosen arbitrarily, we are done.

(iii) Fix $x \notin \mathbf{Z}$.

$$0 = \sum_{n=-N}^{N} \frac{1}{(x-n)^2} - \sum_{n=-N+1}^{N+1} \frac{1}{\big((x+1)-n\big)^2} \to F(x) - F(x+1)$$

as $N \to 0$, so $F(x) = F(x+1)$.

K241

(i) Since g is continuous on $[0,1]$ it is bounded, with $|g(x)| \leq K$ say for all $x \in [0,1]$. Now use periodicity.

If we set $f(x) = x^{-1}$ for $x \in (0,1]$ and define $f(x) = f(x-n)$ whenever $x - n \in (0,1]$ then f is periodic but not bounded. If we set $h(x) = x$ for all x then h is continuous but not bounded.

(ii) Observe that

$$|g(x)| = \frac{1}{4}\left(\left|g\left(\frac{x}{2}\right)\right| + \left|g\left(\frac{x+1}{2}\right)\right|\right) \leq \frac{K}{2}$$

and repeat.

(iii) Use L'Hôpital or power series manipulation

$$\begin{aligned}
\frac{\pi^2}{(\sin \pi x)^2} &= \frac{\pi^2}{(\pi x - (\pi x)^3/3! + \epsilon_1(x)x^3)^2} \\
&= \frac{1}{x^2(1 - (\pi x)^2/6 + \epsilon_2(x)x^3)^2} \\
&= \frac{1}{x^2}\left(1 + \frac{2(\pi x)^2}{6} + \epsilon_3(x)x^3\right) \qquad = \frac{1}{x^2} + \frac{\pi^2}{3} + \epsilon_3(x)x
\end{aligned}$$

where $\epsilon_j(x) \to 0$ as $x \to 0$.

Observe that $g(x) \to g(0)$ as $x \to 0$ so g is continuous at 0. Now use periodicity.

(iv) Observe that, if $2x \notin \mathbb{Z}$, then

$$\begin{aligned}
&g\left(\frac{x}{2}\right) + g\left(\frac{x+1}{2}\right) \\
&= \sum_{n=-\infty}^{\infty}\left(\frac{1}{(\frac{x}{2}-n)^2} + \frac{1}{(\frac{x}{2}-n+\frac{1}{2})^2}\right) - \pi^2(\operatorname{cosec}^2(\pi x/2) - \operatorname{cosec}^2(\pi(x+1)/2) \\
&= \sum_{n=-\infty}^{\infty}\left(\frac{4}{(x-2n)^2} + (x-2n+1)^2\right) - \pi^2\frac{\cos^2(\pi x/2) - \sin^2(\pi x/2)}{\cos^2(\pi x/2)\sin^2(\pi x/2)} \\
&= 4g(x).
\end{aligned}$$

The result holds for all x by continuity.

By (i) and (ii) $g(x) = 0$ for all x. Thus if $x \notin \mathbb{Z}$ we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2} = \pi^2 \operatorname{cosec}^2(\pi x)$$

and looking at $x = 0$,

$$\frac{\pi^2}{3} = f_1(0) = 2\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

K242

(i) $f_n'(x) = x(x^2+n^2)^{-1/2}$. Thus

$$f_n'(x) \to \begin{cases} 1 & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x \in [-1,0). \end{cases}$$

We also have

$$\big|f_n - |x|\big| = \frac{n^{-2}}{|x| + (x^2+n^2)^{1/2}} \leq \frac{1}{n} \to 0$$

uniformly on $[-1,1]$.

(ii) We could take $g_n(x) = n\max(1 - |x-(2n)^{-1}|, 0)$ and $f_n(x) = \int_0^x g_n(t)\,dt$. f_n tends pointwise to a (continuous from below at 0) Heaviside function.

(iii) Demand uniform convergence of the derivatives.

K243*

No comments.

K244

Observe that

$$|t|^{-1/2}v(t) = |t|^{1/2}\frac{v(t)-v(0)}{t} \to 0 \times v'(0) = 0.$$

(ii) Standard results (chain rule, product rule etc) show that g is continuous except possibly at points $(x', 0)$. Observe that since u is non-zero only a bounded closed set we can find a K such that $|u(t)| \leq K$ for all t. Thus, if $y \neq 0$

$$|g(x, y) - g(x', 0)| = |g(x, y)| \leq K|y^{-1/2}v(y)| \to 0$$

as $y \to 0$. Thus g is everywhere continuous.

(iii) Standard results (chain rule, product rule etc) show that g has continuous partial derivatives on the open set

$$A = \{(x, y) \in \mathbb{R}^2 \, : x, \, y \neq 0\}.$$

Also we know that g is identically zero on the open set

$$B = \{(x, y) \in \mathbb{R}^2 \, : \, x^2|y| < 1\} \cup \{(x, y) \in \mathbb{R}^2 \, : \, x^2|y| > 3\}$$

and so, trivially, has continuous partial derivatives there. The only point not in $A \cup B$ is $(0, 0)$. But $g(x, 0) = 0$ for all x and $g(0, y) = 0$ for all y so partial derivatives exist at $(0, 0)$

(iv) If $y \neq 0$, and $|y| \leq 1/9$

$$\begin{aligned} G(y) &= \int_0^1 g(x, y)\, dx \\ &= |y|^{-1/2}v(y) \int_0^1 u(x|y|^{-1/2})\, dx \\ &= v(y) \int_0^{y^{1/2}} u(s)\, ds = v(y), \end{aligned}$$

and $G(0) = 0 = v(0)$ so $G'(0) = v'(0)$.

Since $(x, 0) \in B$ for $x \neq 0$, we have $g_{,2}(x, 0) = 0$ for $x \neq 0$ (and for the reasons given in the last sentence of (iii), $g_{,2}(0, 0) = 0$ as well). Thus $\int_0^1 g_{,2}(x, 0)\, dx = 0$.

We do not have $g_{,2}$ continuous at the origin. (More vividly, look at the behaviour of $g(x, y)$ when x is fixed and small and y runs from $x^2/9$ to x^2.)

K245

By comparison, $\int_0^\infty |f_n(t)|\, dt$ converges and so, since absolute convergence implies convergence, $\int_0^\infty f_n(t)\, dt$ converges. Given $\epsilon > 0$, we can find an X such that $\int_X^\infty g(t)\, dt \leq \epsilon/2$ and then

$$\left| \int_X^\infty f_n(t)\, dt \right| \leq \int_X^\infty |f_n(t)|\, dt \leq \int_X^\infty g(t)\, dt \leq \epsilon/2.$$

Similarly $\int_0^\infty f(t)\, dt$ converges and

$$\left| \int_X^\infty f(t)\, dt \right| \leq \epsilon/2.$$

Thus

$$\begin{aligned}
&\left| \int_0^\infty f_n(t)\, dt - \int_0^\infty f_n(t)\, dt \right| \\
&\leq \left| \int_X^\infty f_n(t)\, dt - \int_X^\infty f_n(t)\, dt \right| + \left| \int_X^\infty f_n(t)\, dt \right| + \left| \int_X^\infty f(t)\, dt \right| \\
&\leq \left| \int_X^\infty f_n(t) - f(t)\, dt \right| + \epsilon \to \epsilon
\end{aligned}$$

as $n \to 0$ and so, since ϵ was arbitrary, the result follows.

The proof above works under the conditions of the last sentence.

K246

(ii) If follows by comparison and (using Dini's theorem) dominated convergence (K245).

To prove only if, observe that if any of the $\int_0^\infty f_n(t)\,dt$ fails to converge then so does $\int_0^\infty f(t)\,dt$. If $\int_0^\infty f_n(t)\,dt$ converges to a_n then we can find X_n such that

$$\int_0^{X_n} f_n(t)\,dt \geq a_n - 2^{-n}$$

and so

$$\int_0^{X_n} f(t)\,dt \geq a_n - 2^{-n}$$

. Thus if $\int_0^\infty f(t)\,dt$ converges so do all the $\int_0^\infty f_n(t)\,dt$ and the values a_n of the integrals form an increasing sequence bounded above so converging to a limit A. We have

$$A \geq \int_0^\infty f_n(t)\,dt \geq \int_0^X f_n(t)\,dt \to \int_0^X f(t)\,dt$$

by part (i). Since X is arbitrary $\int_0^\infty f(t)\,dt$ converges. The fact that $\int_0^\infty f_n(t)\,dt \to \int_0^\infty f(t)\,dt$ now follows from the 'if' part.

K247

(i) Observe that
$$0 \leq \frac{2^{-j} d_j(x,y)}{1+d_j(x,y)} \leq 2^{-j}$$
so the sum converges and d is well defined. To prove the triangle inequality, observe that, if d is metric,
$$\begin{aligned}
&\frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} - \frac{d(x,z)}{1+d(x,z)} \\
&= A\big(d(x,y)(1+d(y,z))(1+d(x,z)) + d(y,z)(1+d(x,y))(1+d(x,z)) \\
&\qquad - d(x,z)(1+d(x,y))(1+d(y,z))\big) \\
&= A\big(d(x,y)+d(y,z)-d(x,z)+d(x,y)d(x,z)+d(y,z)d(x,z) \\
&\qquad + d(x,z)d(x,y)d(y,z)\big) \\
&\geq 0
\end{aligned}$$
for some $A = A(x,y,z) > 0$.

(ii) Suppose that x_n is Cauchy for d. Then, since $2^j d(a,b) \geq d_j(a,b)$, the sequence x_n is Cauchy for d_j so there exists a $z_j \in X$ with $d_j(x_j, z_j) \to 0$ for each j. We observe that $d_{j+1}(x_n, z_{j+1}) \geq d_j(x_n, z_{j+1})$ so $d_j(x_j, z_{j+1}) \to 0$ and by the uniqueness of limits $z_{j+1} = z_j$. Thus
$$z_1 = z_2 = \cdots = z_j = \cdots = z, \text{ say.}$$
Since
$$\begin{aligned}
0 \leq d(x_n, z) &\leq \sum_{j=1}^{N} 2^{-j} d_j(x_n, z) + \sum_{j=N+1}^{\infty} 2^{-j} \\
&= \sum_{j=1}^{N} 2^{-j} d_j(x_n, z) + 2^{-N} \to 2^{-N}
\end{aligned}$$
and N is arbitrary, $d(x_n, z) \to 0$ as $n \to 0$.

(iii) Observe that any Cauchy sequence for (X, d_*) is either eventually constant (and then converges to that constant value) or is a subsequence of $1/n$ and so converges to 0_*. Thus (X, d_*) is complete.

However $1/n$ is a Cauchy sequence in (X, d) which does not converge. (Observe that $d(1/n, 1/m) \geq 1/m$ when $n \geq 2m$ and $d(1/n, 0_*) = d(1/n, 0_{**}) = 2$.)

Take $d_1 = d_*$, $d_j = d_{**}$ otherwise.

(iv) Use the general principle of uniform convergence. If f_j is Cauchy then there is an f_j such that $f^{(j)} \to f_j$. Now use the theorem on interchanging derivative and limit.

(v) Could take
$$d(f,g)=\sum_{j=1}^{\infty}\frac{2^{-j}\sup_{|z|\leq 1-j^{-1}}|f(z)-g(z)|}{1+2^{-j}\sup_{|z|\leq 1-j^{-1}}|f(z)-g(z)|}.$$

K248*

No comments.

K249*

No comments

K250

(i) Suppose that the result is false for some j. Then we can find $g_k \in D$ such that $\|g_k\| \leq 1/k$ but $\|g_k^{(j)}\|_\infty \geq 1$. Since $\|g_k\| \to 0$ but $g_k^{(j)}$ does not converge uniformly to 0 as $k \to \infty$ we have a contradiction.

(ii) Use the fact that we have a norm.

K251*

No comments.

K252*

No comments.

K253

(v) Observe that, given any $\epsilon > 0$, we can find

$$0 = x_0 < y_0 < x_1 < y_1 < \cdots < x_{n-1} < y_{n-1} < x_n < y_n = 1$$

such that $f(y_{j-1}) = f(x_j)$ for $1 \leq j \leq n$ and $\sum_{j=0}^{n}(y_j - x_j) \leq \epsilon$. Thus, if $|f(x)-f(y)| \leq K|x-y|$ for all x, $y \in [0,1]$, we have $1 = f(1)-f(0) \leq K\epsilon$ which is impossible if ϵ is chosen sufficiently small.

K254

(i) Observe that $r_n(x) = x^n$ is a strictly increasing, differentiable function. It thus has a differentiable inverse $r_{1/n}$ whose derivative is given by the rule for differentiation of inverses

$$(r_n^{-1})'(x) = \frac{1}{r_n'(r_n^{-1}(x))} = \frac{r_n^{-1}(x)}{r_n(nr_n^{-1}(x))} = \frac{r_n^{-1}(x)}{nx}.$$

(ii) Take nn'th powers of both sides.

(iii) (f) is trivial but must be checked. $r_n(x) = (r_1(x))^n = x^n$.

(iv) $r_{-\alpha}(x) = (r_\alpha(x))^{-1}$.

K255

The key here is bounding $|r_\alpha(x) - r_\beta(x)|$ with α, $\beta \in \mathbb{Q}$. Since
$$|r_\alpha(x) - r_\beta(x)| = r_\alpha(x)|1 - r_{\beta-\alpha}(x)|.$$
we have to bound $r_\alpha(x)$ and $|1 - r_\gamma(x)|$ (again for $\gamma \in \mathbb{Q}$).

If $N \geq \alpha$ we have $N - \alpha > 0$ so $r_{N-\alpha}(x) \geq 1$ so $r_N(x) \geq r_{\alpha(x)} \geq 0$ and so
$$0 \leq r_\alpha(x) \leq r_N(x) = x^N \leq R^N$$
for all $0 < x \leq R$.

If $\gamma \geq 0$ then $r_\gamma(1) = 1$ and r_γ is increasing so
$$|1 - r_\gamma(x)| \leq \max\big(r_\gamma(R) - 1, 1 - r_\gamma(R^{-1})\big)$$
for all $R^{-1} \leq x \leq R$. We obtain a similar formula for $\gamma \leq 0$ and combining this with the previous paragraph we obtain
$$\bigstar \qquad |r_\alpha(x) - r_\beta(x)| \leq R^N \max\big(r_{|\alpha-\beta|}(R) - 1, 1 - r_{|\alpha-\beta|}(R^{-1})\big)$$
whenever $N \geq \alpha$ and $R \geq x \geq R^{-1}$.

Since $r_{1/n}(x) \to 1$ as $n \to 0$ (proof by, e.g. observing that $(1+\delta)^n \geq 1 + n\delta$ so $1 \leq r_{1/n}(1+\delta) \leq 1 + \delta/n$ for $\delta > 0$) it follows that $r_\gamma(x) \to 1$ as $\gamma \to 0$ through values of $\gamma \in \mathbb{Q}$ (proof by e.g. observing that if $1/n \geq \gamma \geq 0$ and $x \geq 1$ then $r_{1/n}(x) \geq r_\gamma(x) \geq 1$).

(ii) Observe that $r'_{\alpha_n} \to r'_\alpha$ uniformly on $[a, b]$.

K256

(v) Since

$$f'(y) = y^{1-1/n} - y^{1-1/(n+1)} \geq 0$$

for $y \geq 1$, f is increasing on $[1, \infty)$. But $f(1) = 0$ so $f(y) \geq 0$ for $y \geq 1$. Thus

$$n(y^{1/n} - 1) \geq (n+1)(y^{1/(n+1)} - 1) \geq 0$$

and so, since a decreasing sequence bounded below tends to a limit, $n(y^{1/n} - 1) \to L$.

(iii) If $1/n \geq x \geq 1/(n+1)$ we also have

$$\begin{aligned}\frac{P(x) - P(0)}{x} &\geq \frac{P(1/(n+1)) - P(0)}{x}\\ &\geq \frac{n}{n+1} \times \frac{P(1/(n+1)) - P(0)}{1/(n+1)} \qquad \to 1 \times L = L\end{aligned}$$

as $n \to \infty$.

(viii) To deal with $1 > a > 0$ repeat argument or use chain rule on the maps $t \mapsto -t$ and $t \mapsto a^t$.

If $L_a = 0$ then P_a is constant, if $L_a > 0$ then P_a is strictly increasing and if $L_a < 0$ then P_a is strictly decreasing. It follows that $L_1 = 0$, $L_a > 0$ for $a > 1$ and $L_a < 0$ for $a > 0$.

(ix) By the chain rule,

$$\frac{d}{dt}(a^\lambda)^t = \frac{d}{dt}a^{\lambda t} = \lambda L_a a^{\lambda t} = \lambda L_a (a^\lambda)^t$$

so $L_{a^\lambda} = \lambda L_a$.

Choose $\lambda = L_2^{-1}$ and set $e = 2^{L_2^{-1}}$.

(x) We step outside the context of the question and observe that $e^t = \exp t$ so

$$L_a = L_{e^{\log a}} = \log a L_e = \log a.$$

K257

(i) Observe that

$$\begin{aligned}\cosh x\cos x &= \frac{e^x+e^{-x}}{2}\frac{e^{ix}+e^{-ix}}{2}\\ &= \frac{1}{4}\big(e^{2^{1/2}\omega x}+e^{2^{1/2}\omega^3 x}+e^{2^{1/2}\omega^5 x}+e^{2^{1/2}\omega^7 x}\big)\end{aligned}$$

with $\omega = e^{i\pi/4}$. Now use the Taylor expansion of each of the four terms.

(ii) Since $\cosh x\cos x$ is even the Taylor series has only even powers. The coefficient of x^{2n} is

$$\begin{aligned}\sum_{r=0}^{n}\frac{(-1)^r}{(2r)!(2n-2r)!} &= \frac{1}{(2n)!}\sum_{r=0}^{n}\binom{2n}{2r}(-1)^r\\ &= \frac{1}{(2n)!}\frac{(1+i)^{2n}+(1-i)^{2n}}{2} \quad = \begin{cases}0 & \text{if } n \text{ is odd}\\ \frac{(-1)^{n/2}2^{n/2}}{(2n)!} & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

(iii) Set $f(x) = \cosh x\cos x$. Our main problem is to calculate $f^{(n)}(x)$ and then to bound it. Using Leibniz's rule gives a calculation for $f^{(r)}(0)$ which resembles the calculation for (ii). Also

$$|f^{(n)}(x)| \leq \sum_{r=0}^{n}\binom{n}{r}\cosh x \leq 2^n\cosh R$$

for $|x| \leq R$. This gives an estimate for the remainder

$$|R_n(f,x)| \leq \frac{(2R)^n\cosh R}{(n-1)!}$$

for $|x| \leq R$ (or something similar) so $R_n(f,x) \to 0$.

(iv) Solve $y^{(4)} - y = 0$ with $y(0) = 1$, $y'(0) = y''(0) = y^{(3)}(0) = 0$. Remember to check the radius of convergence of the resulting power series.

K258

Since $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence at least R, we know that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Thus if $|h| < R - |z_0|$ we know that

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} |a_n| \binom{n}{r} |z_0|^{n-r} |h|^r = \sum_{n=0}^{\infty} |a_n| (|z_0| + |h|)^n.$$

Write $c_{n,r} = |a_n| \binom{n}{r} |z_0|^{n-r} |h|^r$ if $r \leq n$, $c_{n,r} = 0$ otherwise. By Fubini's theorem for sums,

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} c_{n,r} = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} c_{n,r}$$

so

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n (z_0 + h)^n &= \sum_{n=0}^{\infty} \sum_{r=0}^{n} a_n \binom{n}{r} z_0^{n-r} h^r \\
&= \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} a_n \binom{n}{r} z_0^{n-r} h^r \\
&= \sum_{r=0}^{\infty} h^r \sum_{n=r}^{\infty} \binom{n}{r} z_0^{n-r} \\
&= \sum_{r=0}^{\infty} b_r h^r
\end{aligned}$$

with

$$b_r = \sum_{m=0}^{\infty} \binom{n+m}{m} z_0^m.$$

Thus $\sum_{r=0}^{\infty} b_r z^r$ converges for $|z| < R - |z_0|$ and $\sum_{n=0}^{\infty} a_n (z_0 + h)^n = \sum_{r=0}^{\infty} b_r h^r$.

Let $2\delta = R - |z_0|$. Since $\sum_{r=0}^{\infty} b_r \delta^r$ converges we can find an M such that $|b_r| \leq M\delta^{-r}$. It follows that, if $|w| \leq \delta/2$,

$$\begin{aligned}
\left| \frac{g(z_0 + w) - g(z_0)}{w} - b_1 \right| &= \left| \sum_{r=2}^{\infty} b_r w^{r-1} \right| \\
&\leq |w| \sum_{k=0}^{\infty} b_{k-2} |w|^k \\
&\leq |w| M \delta^{-2} \sum_{k=0}^{\infty} (w|\delta^{-1})^k \\
&\leq |w| M \delta^{-2} \sum_{k=0}^{\infty} 2^{-k} \\
&\leq 2M\delta^{-2} |w| \to 0
\end{aligned}$$

as $|w| \to 0$.

K259

(i) We could set $A_0 = 4$, $B_0 = 1$, $C_0 = \pi/2$, $B_n = 2^{n+1}\big((n!)^2 + 1 + \sum_{r=0}^{n-1} |f_r^{(n)}(x)|$, $A_n = B_n^{n-1}$ and take C_n so that $\sin^{(n)} C_n = 1$ for $n \geq 1$.

Observe that

$$|f_n^{(r)}(x)| \leq 2^{-n-1}$$

for all $n > r$ and all x so, by the Weierstrass M-test, $\sum_{n=r+1}^{\infty} f_n^{(r)}(x)$ converges uniformly and

$$\left| \sum_{n=r+1}^{\infty} f_n^{(r)}(x) \right| \leq 2^{-r}.$$

Thus $\sum_{n=1}^{\infty} f_n^{(r)}(x)$ converges uniformly for each r and $f = \sum_{n=1}^{\infty} f_n$ is infinitely differentiable with

$$f^{(r)}(x) = \sum_{n=1}^{\infty} f_n^{(r)}(x).$$

Also

$$f^{(r)}(0) \geq f^{(r)}(0) - \sum_{n=1}^{r-1} |f_n^{(r)}(0)| - \sum_{n=r+1}^{\infty} |f_n^{(r)}(0)| \geq (n!)^2.$$

K260

(i) Just observe that

$$\sum_{n=0}^{\infty} |a_n z^n| \leq \sum_{n=0}^{\infty} |a_n|(R-\epsilon)^n.$$

if $|z| \leq R - \epsilon$.

(ii) Since $|na_n z^n| \geq |a_n z^n|$, the radius of convergence does not increase. Now observe that, if $|z| < R$ and we choose $r = (2R+|z_0|)/3$, we have $\sum_{n=0}^{\infty} a_n r^n$ convergent so $a_n r^n \to 0$ so we can find an $M \geq 0$ with $|a_n r^n| \leq M$ for all n and so $|a_n| \leq M r^{-n}$. Now set $\rho = (R + 2|z_0|)/3$. Since $r > \rho$, $n(\rho/r)^n \to 0$ as $n \to 0$ so there exists a K with $|na_n| \leq K\rho^{-n}$ for all $n \geq 0$. Thus $|na_n z^n| \leq K(|z|/\rho)^n$ and, by comparison, $\sum_{n=0}^{\infty} na_n z^n$ converges absolutely and so converges.

Thus $\sum_{n=0}^{\infty} na_n z^n$ has radius of convergence R and so $\sum_{n=1}^{\infty} na_n z^{n-1}$ has radius of convergence R. The result for $\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}$ follows.

(iii) and (iv). Observe that

$$n(n-1)\binom{n-2}{j} = \frac{n(n-1)(n-2)!}{j!(n-2-j)!} \geq \frac{n!}{j!(n-j)!} = \binom{n}{j}.$$

so

$$\begin{aligned} |(z+h)^n - z^n - nz^{n-1}h| &= \left| \sum_{j=0}^{n-2} \binom{n}{j} z^j h^{n-j} \right| \\ &\leq \sum_{j=0}^{n-2} \binom{n}{j} |z|^j |h|^{n-j} \\ &\leq n(n-1)|h|^2 \sum_{j=0}^{n-2} \binom{n-2}{j} |z|^j |h|^{n-j-2} \\ &\leq n(n-1)(|z|+|h|)^{n-2}. \end{aligned}$$

Suppose $\delta > 0$ and $|z| + |h| < r - \delta$. By parts (ii) and (i), there exists an $A(\delta)$ independent of h and z such that

$$\sum_{n=2}^{\infty} n(n-1)|a_n|(|z|+|h|)^n \leq A(\delta).$$

Thus

$$\begin{aligned}
\left|\left(\sum_{n=0}^{N} a_n(z+h)^n \right.\right. & \left.\left. - \sum_{n=0}^{N} a_n z^n\right) - h\sum_{n=1}^{N} n a_n z^{n-1}\right| \\
&= \left|\sum_{n=0}^{N} a_n\big((z+h)^n - z^n - n z^{n-1}h\big)\right| \\
&\leq |h|^2 \sum_{n=2}^{N} n(n-1)|a_n|(|z|+|h|)^n \\
&\leq A(\delta)|h|^2
\end{aligned}$$

and, allowing $N \to \infty$,

$$\left|\left(\sum_{n=0}^{\infty} a_n(z+h)^n - \sum_{n=0}^{\infty} a_n z^n\right) - h\sum_{n=1}^{\infty} n a_n z^{n-1}\right| \leq A(\delta)|h|^2$$

(v) Divide by h and allow $|h| \to 0$.

K261

We try, formally, $y = \sum_{j=0}^{\infty} a_j x^j$ obtaining, on substituting in the Legendre equation,

$$(1-x^2)\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} - 2x\sum_{j=1}^{\infty} j a_j x^{j-1} + l(l+1)\sum_{j=0}^{\infty} a_j x^j = 0.$$

Rearranging gives

$$\sum_{j=0}^{\infty}\Big(\big(-j(j-1)-2j+l(l+1)\big)a_j + (j+2)(j+1)a_{j+2}\Big)x^j = 0$$

so we must have

$$\big((j+1)(j+2)-l(l+1)\big)a_j = (j+2)(j+1)a_{j+2}$$

that is

$$a_{j+2} = \frac{(j+l+2)(j-l)}{(j+2)(j+1)}a_j.$$

This gives

$$y(x) = a_0\sum_{r=0}^{\infty}\frac{\prod_{k=0}^{r}(l-2k)(l+2k+1)}{(2r)!}x^{2r} + a_1\sum_{r=0}^{\infty}\frac{\prod_{k=0}^{r}(l-2k-1)(l+2k+2)}{(2r+1)!}x^{2r+1}$$

where a_0 and a_1 can be chosen freely. When l is not a positive integer we observe that

$$\frac{(j+l+2)(j-l)}{(j+2)(j+1)} = \frac{(1+(l+2)j^{-1})(1-j^{-1})}{(1+2j^{-1})(1+j^{-1})} \to 1$$

so careful application of the ratio test to the sums of even and odd powers shows that the power series has radius of convergence 1 and justifies our formal manipulation for $|x|<1$.

If l is positive integer we see that one of the two infinite sums in the first paragraph actually terminates to give a polynomial of degree l. The other does not terminate and the argument above shows it has radius of convergence 1,

If $v(x) = (x^2-1)^n$ then $v'(x) = 2nx(x^2-1)^{n-1}$ so

$$(x^2-1)v'(x) = 2nxv(x)$$

as required. Using Leibniz's rule we have

$$\frac{d^{n+1}}{dx}(1-x^2)v'(x) = (1-x^2)v^{(n+2)}(x) - 2(n+1)xv^{(n+1)}(x) + n(n+1)v^{(n)}(x)$$

and
$$\frac{d^{n+1}}{dx}2nxv'(x) = 2nxv^{(n+1)}(x) + 2n(n+1)v^{(n)}(x).$$
Thus differentiating the final equation of the first sentence of this paragraph $n+1$ times gives
$$(1-x^2)v_n''(x) - 2xv_n'(x) + n(n+1)v(x) = 0$$
as required. Observe that v_n is the p_n of K237.

K262

(this is a special case of Bessel's equation.)

We try, formally, $w = \sum_{j=0}^{\infty} a_j z^j$ obtaining, on substituting in the given equation,

$$z^2 \sum_{j=2}^{\infty} j(j-1)a_j z^{j-2} + z \sum_{j=1}^{\infty} j a_j z^{j-1} + (z^2-1)\sum_{j=0}^{\infty} a_j z^j = 0$$

that is to say

$$a_0 + \sum_{j=2}^{\infty} \big(-(j-1)(j+1)a_j - a_{j-2}\big) z^j = 0$$

Thus $a_0 = 0$ and

$$a_j = -\frac{a_{j-2}}{(j-1)(j+1)}$$

for $j \geq 2$. It follows that $a_{2j} = 0$ for all $j \geq 0$ and $a_{2j+1} = a_1 \frac{(-1)^j}{2^{2j}(j!)^2(j+1)}$. Thus

$$w(z) = a_1 \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j}(j!)^2(j+1)} z^{2j+1}$$

Since

$$\frac{1}{(j-1)(j+1)} \to 0$$

the ratio test shows that our power series has infinite radius of convergence so our solution is valid everywhere.

We note that we can only choose one constant a_1 freely. Observe that setting $z = 0$ in the original equation shows that automatically $w(0) = 0$.

Let us try and find a solution $w = \sum_{j=0}^{\infty} b_j z^j$ for the equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - 1)w = z^2.$$

We obtain

$$b_0 + (3b_2 - b_0 - 1)z^3 \sum_{j=3}^{\infty} \big((j-1)(j+1)b_j - b_{j-2}\big) z^j = 0$$

so, equating coefficients, $b_0 = 0$, $b_2 = 1/3$ and

$$b_j = -\frac{b_{j-2}}{(j-1)(j+1)}$$

for $j \geq 3$. Thus

$$w(z) = b_1 W(z) + \sum_{j=1} \frac{(-1)^j}{(2j+1)\prod_{k=1}^{j-1}(2k+1)^2} z^{2j}$$

with
$$W(z)=\sum_{j=0}\frac{(-1)^j}{2^{2j}(j!)^2(j+1)}z^{2j+1}.$$
(We can talk about a particular integral plus a complementary function.) As before the solution is valid everywhere.

If we try and find a solution $w=\sum_{j=0}^{\infty}c_jz^j$ for
$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-1)w=z,$$
looking at the coefficient of z gives $0=1$ which is impossible.

K263

By Lemma 11.81 we have (provided $|x| < 1$)

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} a_n x^n$$

with

$$a_n = (-1)^n \frac{1}{n!} \prod_{j=0}^{n-1} (\tfrac{1}{2} - j).$$

We observe that (if $z \neq 0$)

$$\frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \frac{|\frac{1}{2} - n||z|}{n} \to |z|$$

so, by the ratio test, $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1. Since power series can be multiplied within their circles of convergence

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)^2 = \sum_{n=0}^{\infty} c_n z^n$$

for $|z| > 1$ with

$$1 + x = \sum_{n=0}^{\infty} c_n x^n$$

for all $|x| < 1$ and x real. By the uniqueness of power series (or we could use K268 below) we have $c_0 = c_1 = 1$ and $c_j = 0$ otherwise so

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)^2 = 1 + z$$

has radius of convergence infinity.

Take $K > 1$. Observe that $(1+z)^{-1/2}$ has power series radius of convergence 1 (proof much as above). Since $(1 + K^{-1}z)^{1/2}$ has power series has radius of convergence K the function $(1+z)^{-1/2}(1+K^{-1}z)^{1/2}$ has power series radius of convergence at least 1. Since power series are continuous within their circle of convergence and

$$(1-z)^{-1/2}(1+K^{-1}z)^{1/2}$$

is not continuous at -1, the power series $\sum_{n=0}^{\infty} c_n z^n$ of $(1-z)^{-1/2}(1+K^{-1}z)^{1/2}$ has radius of convergence exactly 1. Now

$$\begin{aligned} |\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} c_n z^n &= (1+z)^{1/2}(1+z)^{-1/2}(1+K^{-1}z)^{1/2} \\ &= (1+K^{-1}z)^{1/2} \\ &= \sum_{n=0}^{\infty} a_n K^{-n} z^n \end{aligned}$$

for $|z| < 1$. So we have an example of two power series of radius of convergence 1 whose product has radius of convergence K for any $K > 1$. An example with radius of convergence infinity was given in the first paragraph, An example with radius of convergence 1 is given by considering $(1+z)^{-1/2}(1+z)^{-1/2}$. General result obtained by rescaling (i.e. looking at $\sum_{j=0}^{\infty} b_j R^{-j} z^j$ for appropriate R.

[For enthusiasts. Observe that (if we ignore convergence) to obtain

$$\sum_{j=0}^{\infty} u_j z^j \sum_{j=0}^{\infty} v_j z^j = \sum_{j=0}^{\infty} w_j z^j$$

we merely have to solve

$$\sum_{j=0}^{n} u_{n-j} v_j = w_n$$

Thus we can ensure that $u_{2n} = (2n)!$ and $v_{2n+1} = (2n+1)!$ (so $\sum_{j=0}^{\infty} u_j z^j$ and $\sum_{j=0}^{\infty} v_j z^j$ certainly have radius of convergence 0 and $\sum_{j=0}^{\infty} w_j z^j$ is any series we please (so with any radius of convergence we please).]

Repeating much the same arguments, if $L > K > 1$, then $(1+z)^{-1}(1+K^{-1}z)^{1/2}$ has power series with radius of convergence 1 and $(1+z)(1+L^{-1}z)^{1/2}$ has power series with radius of convergence L and their product has power series with radius of convergence K. On the other hand, if $K > L > 1$ then $(1+z)^{-1}(1+L^{-1}z)^{-1/2}(1+K^{-1}z)^{1/2}$ has power series with radius of convergence 1 and $(1+z)(1+L^{-1}z)^{1/2}$ has power series with radius of convergence L and their product has power series with radius of convergence K. If $K > 1$ then $(1+z)^{-1}(1+K^{-1}z)^{-1/2}$ has power series with radius of convergence 1 and $(1+K^{-1}z)^{-1/2}$ has has power series with radius of convergence K and their product has power series with radius of convergence K. Remaining cases (infinite radius of convergence) dealt with similarly.

K264

If we write

$$\mathbf{S}_n(t) = \sum_{j=1}^{n} \mathbf{a}_j(t),$$

then

$$\begin{aligned}\sum_{j=p}^{q} \lambda_j \mathbf{a}_j(t) &= \sum_{j=p}^{q} \lambda_j (\mathbf{S}_j(t) - \mathbf{S}_{j-1}(t)) \\ &= \lambda_{q+1}\mathbf{S}_q(t) + \sum_{j=p}^{q} (\lambda_j - \lambda_{j+1})\mathbf{S}_j(t) - \lambda_p \mathbf{S}_{p-1}(t)\end{aligned}$$

and so

$$\left\| \sum_{j=p}^{q} \lambda_j \mathbf{a}_j(t) \right\| \leq \lambda_{q+1} K + \sum_{j=p}^{q} (\lambda_j - \lambda_{j+1}) K + \lambda_p K = 2\lambda_p K$$

for all $q \geq p \geq 1$ and so, by the general principle of uniform convergence, $\sum_{j=1}^{\infty} \lambda_j \mathbf{a}_j(t)$ converges uniformly.

Observe that, since x^n is a decreasing sequence,

$$\left| \sum_{j=p}^{q} b_j x^j \right| \leq 2x^p \sup_{q \geq p} \left| \sum_{j=p}^{q} b_j x^j \right| \leq 2 \sup_{q \geq p} \left| \sum_{j=p}^{q} b_j x^j \right| \to 0$$

as $p \to \infty$, so by the general principle of uniform convergence, $\sum_{j=1}^{\infty} b_j x^j$ converges uniformly.

We may thus integrate term by term to get

$$\int_0^1 \left(\sum_{n=0}^{\infty} b_n x^n \right) dx = \sum_{n=0}^{\infty} \int_0^1 b_n x^n \, dx = \sum_{n=0}^{\infty} \frac{b_n}{n+1}.$$

If $\sum_{n=0}^{\infty} b_n/(n+1)$ converges, then the power series $\sum_{n=0}^{\infty} b_n x^n/(n+1)$ has radius of convergence at least 1 so the power series $\sum_{n=0}^{\infty} b_n x^n$ has radius of convergence at least 1 and so $\sum_{n=0}^{\infty} b_n x^n$ converges uniformly on $[0, 1-\epsilon]$ and

$$\int_0^{1-\epsilon} \left(\sum_{n=0}^{\infty} b_n x^n \right) dx = \sum_{n=0}^{1-\epsilon} \int_0^1 b_n x^n \, dx = \sum_{n=0}^{\infty} \frac{b_n (1-\epsilon)^{n+1}}{n+1}$$

for all $1 > \epsilon > 0$. By the result of the last paragraph but one, $\sum_{j=1}^{\infty} b_j x^{j+1}/(j+1)$ converges uniformly on $[0, 1]$. The function

$$\sum_{j=1}^{\infty} b_j \frac{x^{j+1}}{j+1}$$

is thus defined and continuous on $[0, 1]$ so

$$\sum_{j=1}^{\infty} \frac{b_j x^j}{j+1} \to \sum_{j=1}^{\infty} \frac{b_j}{j+1}$$

as $x \to 1-$. Thus

$$\int_0^{1-\epsilon} \left(\sum_{n=0}^{\infty} b_n x^n \right) dx \to \sum_{j=1}^{\infty} \frac{b_j}{j+1}$$

as $\epsilon \to 0+$.

Taking $b_j = (-1)^j$ we know, by the alternating series test, that $\sum_{j=0}^{\infty} b_j/(j+1)$ converges and $\sum_{j=0}^{\infty} b_j x^j = (1+x)^{-1}$ for $0 \leq x < 1$. Also

$$\int_0^1 \frac{1}{1+x}\, dx = [\log(1+x)]_0^1 = \log 2.$$

Taking $b_{2j} = (-1)^j$, $b_{2j+1} = 0$ we know, by the alternating series test, that $\sum_{j=0}^{\infty} b_j/(j+1)$ converges and $\sum_{j=0}^{\infty} b_j x^j = (1+x^2)^{-1}$ for $0 \leq x < 1$. Also

$$\int_0^1 \frac{1}{1+x^2}\, dx = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}.$$

Not good methods. Rate of convergence painfully slow.

Observe that $\log a \log_a b = \log b$

$$\sum_{j=0}^{\infty} (-1)^{j+1} \log_{2^{j+1}} e = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(j+1)\log 2} = \frac{\log 2}{\log 2} = 1.$$

K265

Can be done in several ways. Here is one. Write $p_n = \Pr(X = n)$. Observe that, since $p_n \geq 0$ and

$$0 \geq \frac{1-t^n}{1-t} = 1 + t + \cdots + t^{n-1} \leq n$$

for $1 > t > 0$, we have

$$\sum_{j=0}^{N} p_j \frac{1-t^j}{1-t} \leq \frac{\sum_{j=0}^{M} p_j - \sum_{j=0}^{N} p_j t^j}{1-t} \leq \sum_{j=0}^{M} j p_j \leq \mathbb{E}X$$

for $M \geq N$ and so

$$\sum_{j=0}^{N} p_j \frac{1-t^j}{1-t} \leq \frac{\phi_X(1) - \phi_X(t)}{1-t} \leq \mathbb{E}X$$

for all $N \geq 0$ and all $0 < t < 1$. Allowing $t \to 1-$ we have

$$\sum_{j=0}^{N} j p_j \leq \liminf_{t \to 1-} \frac{\phi_X(1) - \phi_X(t)}{1-t} \leq \limsup_{t \to 1-} \frac{\phi_X(1) - \phi_X(t)}{1-t} \leq \mathbb{E}X.$$

Allowing $N \to \infty$ gives the result.

K266

Just as in K82, if $m \geq n$,

$$\begin{aligned}
&\left|\prod_{j=1}^{n}(1+a_j(z)) - \prod_{j=1}^{m}(1+a_j(z))\right| \\
&\qquad \leq \exp\left(\sum_{j=1}^{n}|a_j(z)|\right)\left(\exp\left(\sum_{j=n+1}^{m}|a_j(z)|\right)-1\right) \\
&\qquad \leq \exp\left(\sum_{j=1}^{\infty}M_j\right)\left(\exp\left(\sum_{j=n+1}^{\infty}M_j\right)-1\right) \to 0
\end{aligned}$$

as $n \to \infty$ so, by the general principle of uniform convergence, it follows that $\prod_{j=1}^{N}(1+a_j(z))$ converges uniformly.

K267

(i) We have

$$|z^2n^{-2}| \leq R^2n^{-2}$$

for all $|z| \leq R$ so, by the previous question, $S_N(z)$ converges uniformly on $D(R) = \{z : |z| < R\}$. Since the uniform limit of continuous functions is continuous, S is continuous on $D(R)$ for each $R > 0$ and so on all of $\mathbb{C}$.

(ii) Observe that (if z is not an integer)

$$\begin{aligned}\frac{S_N(z+1)}{S_N(z)} &= \frac{(N!)^2 \prod_{r=-N}^{N}(r-(1+z))}{(N!)^2 \prod_{r=-N}^{N}(r-z)}\\ &= \frac{N+1-z}{-N-z} = \frac{1-(z-1)/N}{-1-z/N} \to -1.\end{aligned}$$

Thus $S(z+1) = -S(z)$ for z not an integer. Extend to all z by continuity.

Recall from K82 that, if $\sum_{n=1}^{\infty}|a_n|$ converges, and $a_n \neq -1$ for all n then $\prod_{n=1}^{\infty}(1+a_n) \neq 0$. Thus, if $N > |z|$ we know that $\prod_{n=N}^{\infty}(1 - z^2n^{-2}) \neq 0$ and so $S(Z) = 0$ if and only if $z\prod_{n=1}^{N-1}(1-z^2n^{-2}) = 0$ i.e. if and only if z is an integer.

[We might guess that $S(z) = A\sin \pi z$. Near 0, $S(z)$ behaves like z so, if the guess is correct, $A = \pi^{-1}$.]

K268

(i) Take $R > r > 0$ As usual we note that $\sum_{n=0}^{\infty} a_n r^n$ converges, so $a_n r^n \to 0$ and there exists an M such that $|a_n| \leq M r^{-n}$. Thus, if $|z| < r/2$,

$$\begin{aligned}
\left|\sum_{n=0}^{q} a_n z^n\right| &= \left|\sum_{n=N}^{q} a_n z^n\right| \\
&\geq |a_N z^N| - \sum_{n=N+1}^{q} |a_n z^n| \\
&= |z|^N \left(|a_N| - |z| \sum_{n=N+1}^{q} |a_n z^{n-N-1}|\right) \\
&\geq |z|^N \left(|a_N| - M|z|r^{-N-1} \sum_{n=N+1}^{q} (|z|r)^{n-N-1}\right) \\
&\geq |z|^N (|a_N| - 2M|z|r^{-N-1}).
\end{aligned}$$

Thus if we set $\delta = \min(r/2, |a_N| r^{N+1}/4M)$ we have

$$\left|\sum_{n=0}^{q} a_n z^n\right| \geq \frac{|a_N||z|^N}{2}$$

for all $q \geq N$ and so

$$\left|\sum_{n=0}^{\infty} a_n z^n\right| \geq \frac{|a_N||z|^N}{2}.$$

K269

(i) Proof of K268 works.

(ii) Exercise 5.91.

(iii) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

K270

(i) Observe that

$$\binom{n}{r-1}+\binom{n}{r}=\frac{n!}{(r-1)!(n-r)!}\left(\frac{1}{n-r+1}+\frac{1}{r}\right)$$
$$=\frac{n!}{(r!(n+1-r)!}(n+1)=\binom{n+1}{r}.$$

(ii) Result true for $n=0$. If the result is true for n then

$$\begin{aligned}(x+y)^{n+1}&=x(x+y)^n+(x+y)^n y\\&=x^{n+1}+\sum_{r=1}^{n}\left(\binom{n}{r-1}+\binom{n}{r}\right)x^r y^{n+1-r}+y^{n+1}\\&=\sum_{r=0}^{n+1}\binom{n+1}{x}^r y^{n+1-r}\end{aligned}$$

so the result is true by induction.

(iii) Result true for $n=0$. If true for n then

$$\prod_{q=0}^{n}(x+y-q)=\sum_{r=0}^{n}(x+y-n)\binom{n}{r}\prod_{j=0}^{r-1}(x-j)\prod_{k=0}^{n-r-1}(y-k)$$
$$=\sum_{r=0}^{n}\binom{n}{r}\prod_{j=0}^{r}(x-j)\prod_{k=0}^{n-r-1}(y-k)+\sum_{r=0}^{n}\binom{n}{r}\prod_{j=0}^{r}(x-j)\prod_{k=0}^{n-r}(y-k)$$
$$=\prod_{j=0}^{n}(x-j)+\sum_{r=0}^{n-1}\left(\binom{n}{r}+\binom{n}{r-1}\right)\prod_{j=0}^{r}(x-j)\prod_{k=0}^{n-r}(y-k)+\prod_{k=0}^{n}(y-k)$$
$$=\sum_{r=0}^{n+1}\binom{n+1}{r}\prod_{j=0}^{r-1}(x-j)\prod_{k=0}^{n-r}(y-k)$$

so the result is true by induction.

K271

(i) Since $|w| < 1$, we can find a $\rho > 1$ such that $\rho|w| < 1$. We can now find an $N \geq 1$ such that

$$\left|\frac{x-j}{j}\right| \leq \frac{M+j}{j} = 1 + Mj^{-1} < \rho$$

for all $|x| \leq M$ and all $j \geq N-1$. We observe that

$$\left|\sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=0}^{n-1}(x-j)\right| \leq (N+M)^N = K, \text{ say}$$

for all $|x| \leq M$ and all $n \leq N$. Thus

$$\left|\sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=0}^{n-1}(x-j)w^n\right| \leq K(\rho w)^n$$

for all $|x| \leq M$ and all $n \geq 1$ so by the Weierstrass M-test,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=0}^{n-1}(x-j)w^n$$

converges uniformly. Since the uniform limit of continuous functions is continuous, f is continuous on $[-M, M]$. Since N is arbitrary, f is defined and continuous everywhere. Note that we have shown that the sum is absolutely convergent.

(ii) By Cauchy's lemma (Exercise 5.38) or considering the power series in w we have

$$f(x)f(y) = \sum_{j=0}^{\infty} c_n$$

with

$$c_n = \sum_{r=0}^{n} \binom{n}{r} \prod_{j=0}^{r-1}(x-j) \prod_{k=0}^{n-r-1} (y-k).$$

Exercise K270 now gives $f(x)f(y) = f(x+y)$.

(iii) If $x = 0$ then $x - 0 = 0$ and so $\prod_{j=0}^{n-1}(x-j) = 0$. Thus $f(0) = 1$. It follows that $f(x)f(-x) = f(0) = 1$ so $f(x) \neq 0$. By the intermediate value theorem, $f(x) > 0$ for all x.

(iv) Since $f(x) > 0$, we can define $g(x) = \log f(x)$ and, since f is continuous, g is. Thus by K101, $g(x) = ax$ and $f(x) = e^{ax} = b^x$ for some b.

(v) Since $b = f(1) = 1 + w$, $f(x) = (1+w)^x$ and we are done.

K272

(i) We prove that $A_N \geq |a_N|$ inductively. The result is trivial for $N = 0$. If the result is true for $N = k-1$, we observe that

$$ka_k = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} c_{n,m} P_{n,m}(a_1, a_2, \dots, a_{k-1})$$

and

$$kA_k = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} K\rho^{n+m} P_{n,m}(A_1, A_2, \dots, A_{k-1})$$

where the $P_{n.m}$ are multinomials with all coefficients positive. Thus

$$\begin{aligned} k|a_k| &\leq \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} |c_{n,m}| P_{n,m}(|a_1|, |a_2|, \dots, |a_{k-1}|) \\ &\leq \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} K\rho^{n+m} P_{n,m}(A_1, A_2, \dots, A_{k-1}) = kA_k \end{aligned}$$

and the induction can proceed.

(ii) Formally

$$\begin{aligned} \frac{dw}{dt} &= K \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\rho t)^n (\rho w)^m \\ &= K \left(\sum_{n=0}^{\infty} (\rho t)^n \right) \left(\sum_{m=0}^{\infty} (\rho w)^m \right) \\ &= \frac{K}{(1-\rho t)(1-\rho w)}. \end{aligned}$$

To solve the differential equation we set, informally,

$$(1-\rho w)dw = \frac{K}{1-\rho t} dt$$

obtaining

$$w(1-\rho w/2) = -\frac{K}{\rho} \log(1-\rho t) + C.$$

Since $w(0) = 0$ this gives

$$w(1-\rho w/2) = -\frac{K}{\rho} \log(1-\rho t)$$

and

$$w(t) = \frac{-1 + (1 - k^2(\log(1-\rho t))^2)^{1/2}}{\rho}.$$

(iii) If we now fix w as in the previous paragraph then Exercise K79 tells us that w has a power series expansion

$$w(t) = \sum_{j=0}^{\infty} A_j t^j$$

with radius of convergence $\eta > 0$. (Note $A_0 = 0$ since $w(0) = 0$.) Since w satisfies ★★ our standard methods show that the A_n satisfy ★. By the arguments of (i), $|a_n| \leq A_n$, so, by comparison, $\sum_{n=0}^{\infty} a_n t^n$ has radius of convergence at least η. If we set $u(t) = \sum_{n=0}^{\infty} a_n t^n$, then, since $a_0 = 0$, we have $u(0) = 0$. By continuity, we can find δ with $0 < \delta \leq \eta$ and $|u(t)| < \rho$ for $|t| < \delta$. The required results can now be read off.

K273

(i) Observe that, since f is continuous on $[0, 4\pi]$, it is uniformly continuous on $[0, 4\pi]$. Since f is 2π periodic it is thus uniformly continuous on $\mathbb{R}$. Similarly g is bounded. Thus

$$\begin{aligned}|f * g(t+u) - f * g(t)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(t+u-s) - f(t-s))g(s)\,ds\right|\\ &\leq \|g\|_\infty \sup_{|x-y|\leq|u|}|f(x)-f(y)| \to 0\end{aligned}$$

as $u \to 0$.

(iii) Observe, for example, that making the substitution $x = t - s$ we have

$$f * g(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} f(x)g(t-x)\,dx = g * f(t).$$

Fubini gives

$$\begin{aligned}f * (g * h)(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)\left(\int_{-\pi}^{\pi} g(s-u)h(u)\,du\right)ds\\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t-s)g(s-u)h(u)\,du\,ds\\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t-s)g(s-u)h(u)\,ds\,du\\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi-u}^{\pi-u} f(t-u-v)g(v)h(u)\,dv\,du\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f * g(t-u)h(u)\,du = (f * g) * h(t).\end{aligned}$$

(iv) If we set $G(s,t) = f(t-s)g(s)$, then G has continuous partial derivative $G_{,2}(s,t) = f'(t-s)g(s)$. We may thus apply our theorem on differentiating under the integral to show that $f * g$ is differentiable and $(f * g)' = f' * g$.

(v) We have

$$
\begin{aligned}
|u_n * f(t) - f(t)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} u_n(s)(f(t-s) - f(t))\, ds \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi/n}^{\pi/n} u_n(s)(f(t-s) - f(t))\, ds \right| \\
&\leq \sup_{|u-v| \leq \pi/n} |f(u) - f(v)| \frac{1}{2\pi} \int_{-\pi/n}^{\pi/n} u_n(s)\, ds \\
&= \sup_{|u-v| \leq \pi/n} |f(u) - f(v)| \to 0
\end{aligned}
$$

as $n \to \infty$.

(vii) Take u_n to be a three times continuously differentiable function in (v).

K274

(i) Fubini gives

$$
\begin{aligned}
\widehat{f*g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)\,ds\right)\,dt\\
&\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-int}f(t-s)g(s)\,ds\,dt\\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-int}f(t-s)g(s)\,dt\,ds\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ins}g(s)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)e^{-in(t-s)}\,dt\right)\,ds\\
&\hat{g}(n)\hat{f}(n).
\end{aligned}
$$

(iii) We have

$$|\hat{f}(n)-\hat{g}(n)| = |\widehat{f-g}(n)| \leq \|f-g\|_\infty.$$

(v) If $e*f=f$ for all f then

$$\hat{f}(n) = \widehat{e*f}(n) = \hat{e}(n)\hat{f}(n)$$

for all f and all n. Take $f(t)=e^{imt}$ to see that $\hat{e}(m)=1$ for all m contradicting the Riemann-Lebesgue lemma.

K275

(i) We have

$$\hat{f}_1(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_1(t)\frac{(e^{it}-e^{-it})}{2i}e^{-int}\,dt = \frac{\hat{g}_1(n+1)-\hat{g}_1(n-1)}{2i}$$

Thus, using Riemann-Lebesgue,

$$\begin{aligned} S_n(f_1,0) &= \sum_{j=-n}^{n}\frac{\hat{g}_1(j+1)-\hat{g}_1(j-1)}{2i} \\ &= \frac{\hat{g}_1(n+1)+\hat{g}_1(n-1)-\hat{g}_1(-n+1)-\hat{g}_1(-n-1)}{2i} \to 0. \end{aligned}$$

(ii) If we set

$$g_2(t) = \begin{cases} \frac{f_2(t)}{\sin t} & \text{if } t/\pi \text{ is not an integer,} \\ f_2'(0) & \text{if } t = 2n\pi, \\ -f_2'(\pi) & \text{if } t = (2n+1)\pi, \end{cases}$$

then g_2 is automatically continuous except possibly at points $n\pi$. But,

$$g_2(t) = \frac{f_2(t)-f_2(\pi)}{t-\pi}\frac{t-\pi}{\big(-\sin(t-\pi)\big)} \to -f_2'(\pi) = g_2(\pi)$$

as $t \to \pi$ so g_2 is continuous at π. Similarly g_2 is continuous at all $n\pi$.

(iii) Note that $f_4 \in C_P(\mathbb{R})$. We have

$$\begin{aligned} \hat{f}_4(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_3(t/2)e^{-int}\,dt \\ &= \frac{1}{\pi}\int_{-2\pi}^{2\pi} f_3(s)e^{-i2ns}\,ds \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f_3(s)e^{-i2ns}\,dt = \hat{f}_4(2n). \end{aligned}$$

Observe that f_4 obeys the conditions of (iii).

(iv) Translate.

K276

(i) Let $S(z) = z - f(z)$. We have $S'(z) = 1 - f'(z)$ so $|S'(z)| \leq 1/2$ for $|z| \leq \delta$ and

$$|Tz| \leq |w| + |S(z) - S(0)| \leq |w| + |z| \sup_{|u|\leq\delta} |S'(u)| \leq \delta/2 + \delta/2 = \delta$$

for all $|z| \leq \delta$.

(ii) Observe that, if z, $u \in X$,

$$|Tz - Tu| = |Sz - Su| \leq |z - u|/2.$$

Thus T is a contraction mapping and there is a unique $z_0 \in X$ with $Tz_0 = z_0$ i.e. with $f(z_0) = w$.

(iii) Observe that the continuity of F' at 0 gives us a $\delta > 0$ such that $|F'(z) - 1| \leq 1/2$ for $|z| \leq \delta$.

(iv) Let $F(z) = (g(z) - g(0))/g'(0)$ and apply (iii).

(v) No. $g = 0$ is a counterexample. [However, in more advanced work, it is shown that this is the only counterexample!]

(vi) No. If $g(z) = z^2$, the equation $g(z) = re^{i\theta}$ has two roots $r^{1/2}e^{i\theta/2}$ and $-r^{1/2}e^{i\theta/2}$.

K277

(i) True. w^* is the solution.

(ii) False. $f_2(z)$ is real so $f_2(z) = \delta i$ has no solution for $\delta > 0$.

(iii) and (v) False. For (iii), observe $\Re f_3(z) \geq 0$ so $f_3(z) = -\delta$ has no solution for $\delta > 0$.

(iv) and (vi) True. For (vi), observe that we can find a δ with $1/6 > \delta' > 0$ such that $|F'(z) - 1| \leq 1/3$ for $|z| \leq \delta'$. Let $X = \{z : |z| \leq \delta\}$. Set $\delta = \delta'/3$. If $|w| \leq \delta$ write $Sz = F(z) - 1$ and $Tz = Sz + |z|^{1/2} - w$. Using the mean value inequality, we have

$$|Tz| \leq |w| + |Sz| + |z|^2 \leq \delta'/3 + \sup_{|u|<\delta'} |S'(u)||z| + |z|/3 \leq \delta'$$

whenever $|z| \leq \delta'$ and

$$\begin{aligned}|Tz - Tu| &\leq |Sz - Su| + ||z|^2 - |u|^2| \\ &\leq |z - u|/3 + |z - u|(|z| + |u|) = 2|z - u|/3,\end{aligned}$$

so T restricted to X is a contraction mapping on X. Since X is closed, we are dealing with a complete metric space and we can find $z_0 \in X$ with $Tz_0 = z_0$ i.e. $w = F(z_0) + |z_0|^2$.

K278

(i) Observe that

$$T'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

so, by the mean value inequality,

$$|Ta - Tb| \leq |\lambda||a - b|.$$

Since $\mathbb{R}$ is complete under the usual metric, the contraction mapping theorem applies.

(ii) Much as in (i),

$$|Tx| = |Tx - T0| \leq |x| \sup_{|w| \leq a} |T'(w)| \leq |\lambda||x|$$

so, if $|x| \leq a$, we see, by induction that $|T^n x| \leq |a|$ and $|T^n x| \leq |\lambda|^n |x| \to 0$.

Observe that

$$T'(x) = \frac{F(x)F''(x)}{F'(x)^2}$$

and

$$T''(x) = \frac{F''(x)}{F'(x)} + \frac{F(x)F'''(x)}{F'(x)^2} - \frac{2F(x)(F''(x))^2}{F'(x)^3},$$

so $T(0) = T'(0) = 0$ and $T''(0) = F''(0)/F'(0)$. Thus

$$\left| Tx - \frac{F''(0)}{2F'(0)} x^2 \right| = |Tx - T(0) - T'(0)x - \tfrac{1}{2}T''(0)| \leq \epsilon(x)x^2$$

with $\epsilon(x) \to 0$ as as $x \to 0$. Choose η so that $\epsilon(x) \leq 1$ for $|x| \leq \eta$.

(iii) We have quadratic convergence (roughly doubling of correct decimal places) with Newton rather than linear convergence (increasing the number of correct decimal places by the same amount) as is implied by the contraction mapping theorem. There may be problems if $F'(0)$ is small.

K279

Suppose f has a unique fixed point x. Then $f(g(x)) = g(f(x)) = g(x)$ and, since x is the unique fixed point of f, $g(x) = x$.

Let $X = \{-1, 0, 1\}$, $f(t) = -t$, $g(t) = t$. Then $fg = gf = f$, f has only one fixed point but g has three.

Let $X = \{-1, 1\}$, $f(t) = t$, $g(t) = -t$. Then $fg = gf = f$, f has two fixed points but g has none.

K280

We first prove uniqueness. If $f(x) = x$ and $f(y) = y$ then $d(x, y) = d(f(x), f(y)) \geq Kd(x, y)$ so $d(x, y) = 0$ and $x = y$.

For existence choose $x_0 \in X$ and then choose inductively $x_{n+1} \in X$ such that that $Tx_{n+1} = x_n$ (this uses surjectivity). We observe that

$$d(x_{n+1}, x_n) = d(Tx_{n+2}, Tx_{n+1}) \geq Kd(x_{n+2}, x_{n+1})$$

so

$$K^{-1}d(x_{n+1}, x_n) \geq d(x_{n+2}, x_{n+1})$$

and precisely the same argument as in the contraction mapping theorem shows that $x_n \to z$ for some $z \in X$. Now $z = Tw$ for some w and

$$\begin{aligned} d(z, w) &\leq d(x_n, z) + d(x_n, w) \leq d(x_n, z) + Kd(Tx_n, Tw) \\ &= d(x_n, z) + Kd(x_{n-1}, z) \to 0 \end{aligned}$$

as $n \to \infty$. Thus $d(z, w) = 0$ so $z = w$ and $Tz = z$.

K281

(i) If $T(\mathbf{a}) = \mathbf{a}$ and $T(\mathbf{b}) = \mathbf{b}$, then $\|T(\mathbf{a}) - T(\mathbf{b})\| = \|\mathbf{a} - \mathbf{b}\|$ so $\mathbf{a} = \mathbf{b}$.

(ii) Uniqueness much as in (i). Existence not implied see e.g. second paragraph of K281.

(iii) Let $E = \{-1, 1\}$ be a subset of $\mathbb{R}$. The mapping $T : E \to E$ given by $Tx = -x$ preserves length but has no fixed point. The map $T : E \to E$ given by $Tx = x$ preserve length and has two fixed points.

(The reader may prefer the example of a circle E in $\mathbb{R}^2$. Non-trivial rotations have no fixed points, reflection in a diameter has two.)

(iv) Observe that, since T decreases distance, it is continuous. The map
$$\mathbf{x} \mapsto \|\mathbf{x} - T\mathbf{x}\|$$
is thus continuous and, since E is closed and bounded in $\mathbb{R}^n$, thus attains a minimum at $\mathbf{x}_0$, say. By definition
$$\|\mathbf{x}_0 - T\mathbf{x}_0\| \leq \|T\mathbf{x}_0 - T^2\mathbf{x}_0\|$$
so $T\mathbf{x}_0 = \mathbf{x}_0$.

(v) Let E^* be the closure of the set $\{T^n\mathbf{c} : n \geq 0\}$. Then E^* is is closed bounded set and $T|_{E^*}$ is a distance decreasing map from E^* to itself. By (iv), $T|_{E^*}$ and so T has a fixed point.

K282

Observe that, if $X = [0, \infty)$ with the usual metric, the map f given by $x \mapsto x + 1$ is an isometry but we cannot find an isometry $g : X \to 0$ such that $f \circ g(x) = x$.

We now restrict ourselves to bijective isometries. Since the bijective maps on X form a group under composition we need only show we have a subgroup.

If f and g are isometries then

$$d(f \circ g(x), f \circ g(y)) = d(g(x), g(y)) = d(x, y)$$

so the composition of bijective isometries is a bijective isometry. The identity map is a bijective isometry.

If f is a bijective isometry, f^{-1} is well defined and

$$d(x, y) = d(f \circ f^{-1}(x), f \circ f^{-1}(y)) = d(f^{-1}(x), f^{-1}(y))$$

so f^{-1} is a bijective isometry.

(i) Take X to be the vertices of an isosceles triangle. $G(X)$ is the group of permutations of the vertices (isomorphic to S_3) so non Abelian.

(ii) Take X to be the vertices of a triangle all of whose sides have different length.

(iii) Take X to be a circle.

(iv) Take X to be a regular n-gon and T rotation about the centre of symmetry through $2\pi/n$.

K283

(i) Observe that, since (X,d) has the Bolzano–Weierstrass property, we can find a sequence $m(j)\to\infty$ of strictly positive integers and an $\alpha\in X$ such that $d(a_{m(j)},\alpha)\to 0$. Take a sequence $j(r)\to\infty$ of strictly positive integers such that $m(j(r+1))>3m(j(r))+1$. Then setting $n'(r)=m(j(r+1))-m(j(r))$ we have

$$\begin{aligned}d(a_{n'(r)},a)&\leq d(f^{m(j(r)}a_{n(r)},f^{m(j(r)}a)=d(a_{m(j(r+1))},a_{m(j(r))})\\&\leq d(a_{m(j(r+1))},\alpha)+d(a_{m(j(r))},\alpha)\to 0\end{aligned}$$

as $r\to\infty$.

Taking $n(j)$ to be subsequence of the $n'(j)$ with

$$d(b_{n(r)},b)\to 0,$$

we have $d(a_{n(r)},a)$, $d(b_{n(r)},b)\to 0$. Thus

$$\begin{aligned}d(a,b)&\leq d(f(a),f(b))\leq d(f^{n(r)}(a),f^{n(r)}(b)))\\&\leq d(a,b)+d(a_{n(r)},a)+d(b_{n(r)},b)\to d(a,b)\end{aligned}$$

as $r\to\infty$ so $d(a,b)=d(f(a),f(b))$.

Examples $Y=\mathbb{R}$, ρ usual metric, $g(x)=2x$, $h(x)=1+2x$ for $x\geq 0$, $h(x)=-1+2x$ if $x<0$.

(ii) Since $f(x)=f(y)$ implies $d(x,y)=d(f(x),f(y))=0$ implies $x=y$, f is injective.

To show surjectivity take any $a\in X$. As before we can find a subsequence $n(r)$ with $n(r)\geq 2$ such that $d(a_{n(r)},a)\to 0$. Let $b_r=a_{n(r)-1}$. By the Bolzano–Weierstrass property we can find $b\in X$ and $r(j)\to\infty$ such that $d(b_{r(j)},b)\to 0$. Thus

$$d(a,f(b))\leq d(a,a_{n(r(j))})+d(f(b_{r(j)},f(b))=d(a,a_{n(r(j))})+d(b_{r(j)},b)\to 0$$

as $j\to\infty$.

Example, $Y=[0,\infty)$ with usual metric and $f(x)=x+1$.

K284

Observe that $g \circ f : X \to X$ is expansive so, by K283, $g \circ f$ is bijective and an isometry. Thus g is surjective and (since it is expansive) injective. Thus f is bijective. Since

$$d(x, y) \leq \rho(f(x), f(y)) \leq d(g \circ f(x), g \circ f(y)) = d(x, y),$$

f is an isometric bijection so (Y, ρ) has the Bolzano–Weierstrass property and g is an isometric bijection.

If $X = Y = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$ with the usual metric, then, taking $f = g$ to be rotation through $\pi/4$, we see that f and g need not be inverses.

If $X = Y = [0, \infty)$ with the usual metric and we set $f(x) = 2x$, $g(x) = x + 1$ then f is a distance increasing bijection but not distance preserving and g is distance preserving but not a bijection.

K285

We note that $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ and $\mathbf{x} \mapsto \|\mathbf{f}(\mathbf{x})\|$ are continuous maps on a closed bounded subset of $\mathbb{R}^n$. Thus there exists an M with $\|\mathbf{f}(\mathbf{x})\| \leq M$ for all $\mathbf{x} \in A$.

By convexity, $\mathbf{g}$ maps A into A. We have

$$\|\mathbf{g}(\mathbf{a}) - \mathbf{g}(\mathbf{b})\| = (1-\epsilon)\|\mathbf{f}(\mathbf{a}) - \mathbf{f}(\mathbf{b})\| \leq (1-\epsilon)\|\mathbf{a} - \mathbf{b}\|$$

so $\mathbf{g}$ is a contraction mapping and so, since A is closed and we are dealing with a complete metric space, $\mathbf{g}$ has a fixed point $\mathbf{z}$, say. We observe that

$$\begin{aligned}
\|\mathbf{f}(\mathbf{z}) - \mathbf{z}\| &= \|\mathbf{f}(\mathbf{z}) - \mathbf{g}\mathbf{z}\| \\
&\leq \epsilon\|\mathbf{f}(\mathbf{x})\| + \epsilon\|\mathbf{a}_0\| \\
&\leq (M + \|\mathbf{a}_0\|)\epsilon.
\end{aligned}$$

Thus

$$\inf\{\|\mathbf{f}(\mathbf{a}) - \mathbf{a}\| \,:\, \mathbf{a} \in A\} = 0.$$

But $\mathbf{x} \mapsto \|\mathbf{f}(\mathbf{x}) - \mathbf{x}\|$ is a continuous map on a closed bounded subset of $\mathbb{R}^n$ so attains its infimum. Thus $\|\mathbf{f}(\mathbf{y}) - \mathbf{y}\| = 0$ for some $\mathbf{y} \in A$ which is thus a fixed point.

Let $V = \mathbb{R}$, $A = [-2,-1] \cup [1,2]$ and $Tx = -x$. Then A is closed and bounded and T is distance non-increasing with no fixed point.

Let $V = \mathbb{R}$, $A = (0,1)$ and $Tx = x/2$. Then A is convex and bounded and T is distance decreasing with no fixed point.

Let $V = A = \mathbb{R}$ and $Tx = x + 1$. Then A is closed and convex and T is distance non-increasing with no fixed point.

In final paragraph take

$$A = \{\mathbf{q} \in \mathbb{R}^n \,:\, \sum_{i=1}^n q_i = 1 \text{ and } q_i \geq 0 \text{ for } 1 \leq i \leq n\}$$

and $\mathbf{f}(\mathbf{q}) = \tilde{\mathbf{q}}$ with

$$\tilde{q}_j = \sum_{i=1}^n q_i p_{ij}.$$

We then have A closed, convex and bounded, $\mathbf{f}$ maps A to A and

$$\begin{aligned}\|\mathbf{f}(\mathbf{a}) - \mathbf{f}(\mathbf{b})\|_1 &= \sum_{j=1}^{n} \left|\sum_{i=1}^{n} (a_i - b_i) p_{ij}\right\| \qquad \leq \sum_{j=1}^{n} \sum_{i=1}^{n} |a_i - b_i| p_{ij} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} |a_i - b_i| p_{ij} \\ &= \sum_{i=1}^{n} |a_i - b_i| = \|\mathbf{a} - \mathbf{b}\|_1.\end{aligned}$$

K286

(i) Note that

$$|d(x,Tx) - d(y,Ty) \leq d(x,y) + d(Tx,Ty) \leq 2d(x,y)$$

so S is continuous. Thus since our space has the Bolzano–Weierstrass property S must attain a minimum at z say. If $Tz \neq z$ then

$$\begin{aligned} S(T^{N(z,Tz)}z) &= d\big(T^{N(z,Tz)}z, T(T^{N(z,Tz)}z)\big) \\ &= d\big(T^{N(z,Tz)}z, T^{N(z,Tz)}(Tz)\big) \\ &< d(z,Tz) = S(z) \end{aligned}$$

which is absurd. Thus $Tz = z$.

(ii) Write $f_n(x) = d(T^n x, z)$. Then f_n is continuous (by direct proof or by composition of continuous functions) so $U_n = f_n^{-1}((-\epsilon,\epsilon))$ is open. Since $d(T^{n+1}x, z) = d(T(T^n x), Tz) \leq d(T^n x, z)$ we have $U_n \subseteq U_{n+1}$.

Observe that F is a closed subset of (X,d) and so inherits the Bolzano–Weierstrass property. If $Tx \in U$ then $Tx \in U_n$ for some n so $x \in U_{n+1} \subseteq U$. Thus, if $F \neq \emptyset$ we can apply (i) to $T|_F : F \to F$ to obtain $w \in F$ with $Tw = w$ contradicting uniqueness.

(iii) Thus $U = X$ and, given $x \in X$, we can find an n such that $x \in U_n$ so $d(T^m x, z) < \epsilon$ for $m \geq n$. Since ϵ was arbitrary, the required result follows.

K287

Observe that if $\mathbf{q} \in X$ then

$$\sum_{i=1}^{m} q_i p_{ij} \geq 0 \text{ and } \sum_{j=1}^{m}\sum_{i=1}^{m} q_i p_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{m} q_i p_{ij} = 1.$$

(i) Note all norms on $\mathbf{R}^m$ Lipschitz equivalent. X is a closed (e.g. because the intersection of closed sets of form $E_i = \{\mathbf{x} : x_i \geq 0\}$ and $E = \{\mathbf{x} : \sum_{i=1}^m x_i = 1\}$) and bounded (if $\mathbf{x} \in E$ then $\|\mathbf{x}\| \leq 1$).

(ii) We have

$$\begin{aligned}\|T\mathbf{u} - T\mathbf{v}\|_1 &= \sum_{j=1}^{m} \left| \sum_{i=1}^{m} (u_i - v_i) p_{ij} \right| \\ &\leq \sum_{j=1}^{m}\sum_{i=1}^{m} |u_i - v_i| p_{ij} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} |u_i - v_i| p_{ij} = \|\mathbf{u} - \mathbf{v}\|_1.\end{aligned}$$

(iii) Show that

$$p_{ij}^{(n+1)} = \sum_{k=1}^{m} p_{ik}^{n} p_{kj}$$

and use induction.

Probabilistically, $p_{ij}^{(n)}$ is the probability that, starting at i we are at j after n steps.

(iv) Repeat the calculation of (ii) but observe that if $\mathbf{u} \neq \mathbf{v}$ there must exist I and I' such that $u_I - v_I$ and $u_{I'} - v_{I'}$ are non-zero and have opposite signs. Thus

$$\left| \sum_{i=1}^{m} (u_i - v_i) p_{ij}^{N} \right| < \sum_{i=1}^{m} |u_i - v_i| p_{ij}^{N}$$

for each j.

(vi) $T\boldsymbol{\pi} = \boldsymbol{\pi}$ gives $\pi_1 = \pi_2$ so $\pi_1 = \pi_2 = 1/2$. $T^n\mathbf{q} \to \boldsymbol{\pi}$ if and only if $\mathbf{q} = \boldsymbol{\pi}$. Observe $p_{11}^{(2n+1)} = 0$ and $p_{12}^{(2n)} = 0$ for all n so there is no n with $p_{ij}^{n} \neq 0$ for all i, j.

(vii) $T\boldsymbol{\pi} = \boldsymbol{\pi}$ for all $\boldsymbol{\pi} \in X$. We have $T^n\mathbf{q} = \mathbf{q}$ for all $\mathbf{q} \in X$. Observe that $p_{12}^{(n)} = 0$ for all n.

(viii) $T\boldsymbol{\pi} = \boldsymbol{\pi}$ has as solutions all $\boldsymbol{\pi} \in X$ with $\pi_1 = \pi_2$ and $\pi_3 = \pi_4$.

Given $\mathbf{q} \in X$ set $\pi_1 = \pi_2 = (q_1 + q_2)/2$ and $\pi_3 = \pi_4 = (q_3 + q_4)/2$. Then, $T\mathbf{q} = \boldsymbol{\pi}$ and (if $n \geq 1$) $T^n\mathbf{q} = \boldsymbol{\pi} \to \boldsymbol{\pi}$.

K288

We have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|x_ip_{ij}| = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|x_i|p_{ij} = \sum_{i=1}^{\infty}|x_i| = \|\mathbf{x}\|_1$$

so Fubini's theorem tells us that $\sum_{i=1}^{\infty} x_ip_{ij}$ converges for each j. Further $\sum_{i=1}^{\infty}|x_i|p_{ij}$ converges and

$$\begin{aligned}\sum_{j=1}^{\infty}\left|\sum_{i=1}^{\infty}x_ip_{ij}\right| &\leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}|x_i|p_{ij}\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|x_i|p_{ij} = \|\mathbf{x}\|_1.\end{aligned}$$

Thus $T\mathbf{x}$ is a well defined point of l^1 and $\|T\mathbf{x}\|_1 \leq \|\mathbf{x}\|_1$. (Note that this only shows that $\|T\| \leq 1$.) Since $\alpha_n \to \alpha$ and $\beta_n \to \beta$ imply $\lambda\alpha_n + \mu\beta_n \to \lambda\alpha + \mu\beta$, it is easy to check that T is linear.

Further, using Fubini again, if $\mathbf{q} \in X$

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}q_ip_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}q_ip_{ij} = 1$$

and, trivially, $\sum_{j=1}^{\infty} q_ip_{ij} \geq 0$ so $T\mathbf{q} \in X$.

(ii) If $\mathbf{u}$, $\mathbf{v} \in X$ and $\mathbf{u} \neq \mathbf{v}$ then we can find I and I' such that hat $u_I - v_I$ and $u_{I'} - v_{I'}$ are non-zero and have opposite signs. If $k = \max\{I', I\}$ then as in (iv) of K287, $d(T^{N(k)}\mathbf{u}, T^{N(k)}\mathbf{v}) < d(\mathbf{u}, \mathbf{v})$.

(iii) If $T\mathbf{u} = \mathbf{u}$ and $T\mathbf{v} = \mathbf{v}$ then (ii) shows that $\mathbf{u} = \mathbf{v}$.

If $T\mathbf{h} = \mathbf{h}$ and $h_i \geq 0$ for all i then either $\mathbf{h} = \mathbf{0}$ or setting $\mathbf{k} = \|\mathbf{h}\|^{-1}\mathbf{h}$ we have $\mathbf{k} \in X$ and $T\mathbf{k} = \mathbf{k}$. If $T\boldsymbol{\pi} = \boldsymbol{\pi}$ has a solution in X then the required solutions are $\lambda\boldsymbol{\pi}$ with $\lambda \geq 0$. If not, the only solution is $\mathbf{0}$.

(iv) We must have $\pi_I > 0$ for some I. Take $k = \max\{I, i\}$

$$\pi_i = \sum_{r=1}^{\infty}\pi_r p^{N(k)}p_{ri} \geq \pi_I p^{N(k)}p_{Ii} > 0.$$

(v)(a) We have for $i \geq 2$

$$\pi_i = \tfrac{1}{2}\pi_{i+1} + \tfrac{1}{4}\pi_i + \tfrac{1}{4}\pi_{i-1}.$$

so

$$2\pi_{i+1} - 3\pi_i + \pi_{i-1} = 0$$

and $\pi_n = A\alpha^n + B\beta^n$ with α and β roots of the auxiliary equation $2\theta^2 - 3\theta + 1 = 0$. Thus $\pi_n = A + B(1/2)^n$.

Treating the previous paragraph as purely exploratory we observe that $\pi_n = 2^{-n}$ is a solution.

(b) Proceeding as in (a) we get

$$\pi_{i+1} - 3\pi_i + 2\pi_{i-1} = 0$$

with general solution $\pi_i = A + B2^n$. There are no constants A and B giving $\boldsymbol{\pi} \in X$.

(c) Proceeding as in (a) we get

$$\pi_{i+1} - 2\pi_i + \pi_{i-1} = 0$$

with general solution $\pi_i = A + Bn$. There are no constants A and B giving $\boldsymbol{\pi} \in X$.

K289

(i) Observe that $\pi_J > 0$ and set $\mathbf{h} = \pi_J^{-1}\boldsymbol{\pi}$.

(ii) If $e(j)_k = 1$ if $k = j$ and $e(j)_k = 0$ otherwise, then the $\mathbf{e}(k)$ form a sequence in X with no convergent subsequence. (Observe that $\|\mathbf{e}(k) - \mathbf{e}(l)\|_1 = 2$ for $k \neq l$.) Thus X does not have the Bolzano–Weierstrass property. However (see Exercise K218)

$$Y = \{\mathbf{y} \,:\, h_i \geq |y_i| \text{ for } i \geq 1\}$$

does so, since X is closed, so does $X_J = X \cap Y$. Observe that

$$\boldsymbol{\pi} = \|\mathbf{h}\|_1^{-1}\mathbf{h} \in X_J.$$

Also if $\mathbf{q} \in X_J$ then $T\mathbf{q} \in X$ and

$$\begin{aligned}(T\mathbf{q})_j &= \sum_{i=1}^{\infty} q_i p_{ij} \\ &\leq \sum_{i=1}^{\infty} h_i p_{ij} = h_j\end{aligned}$$

so $T\mathbf{q} \in X_J$. Thus by Exercise K186 $T^n\mathbf{q} \to \boldsymbol{\pi}$ for all $\mathbf{q} \in X_J$.

(iii) In particular $T^n\mathbf{e}(J) \to \pi$. But J is arbitrary.

Now if $\mathbf{q} \in X$ and $\epsilon > 0$ we can find M such that $\sum_{r=1}^{M} q_r > 1 - \epsilon$. Write $\mathbf{u} = \mathbf{q} - \sum_{r=1}^{M} q_r\mathbf{e}(r)$. Then

$$\begin{aligned}\|T^n\mathbf{q} - \boldsymbol{\pi}\|_1 &\leq \left\|T^n \sum_{r=1}^{M} q_r\mathbf{e}(r) - \sum_{r=1}^{M} q_r\boldsymbol{\pi}\right\|_1 + \left\|\left(1 - \sum_{r=1}^{M} q_r\right)\boldsymbol{\pi}\right\|_1 + \|T^n\mathbf{u}\|_1 \\ &\leq \sum_{r=1}^{M} q_r\|T^n\mathbf{e}(r) - \boldsymbol{\pi}\|_1 + \epsilon + \|\mathbf{u}\|_1 \\ &\leq \sum_{r=1}^{M} q_r\|T^n\mathbf{e}(r) - \boldsymbol{\pi}\|_1 + 2\epsilon \\ &\to 2\epsilon\end{aligned}$$

as $n \to \infty$. Since ϵ was arbitrary, $\|T^n\mathbf{q} - \boldsymbol{\pi}\|_1 \to 0$ as required.

K290

For uniqueness. Suppose $Tx = Ty$. Then $T^n x = T^{n-1}Tx = T^{n-1}Ty = T^n y$ so, by the contraction mapping theorem, $x = y$.

For existence. By the contraction mapping theorem we can find an x_0 with $T^n x_0 = x_0$. Now $T^n(Tx_0) = T(T^n x_0) = Tx_0$ so, by uniqueness, $Tx_0 = x_0$.

In question K282, T^n has a fixed point but is not a contraction mapping.

Desired result trivially true for $n = 0$. If true for n then

$$\begin{aligned}
|(T^{n+1}h)(t) - (T^{n+1}k)(t)| &\leq \left|\int_a^t \phi(h(s), s) - \phi(k(s), s)\, ds\right| \\
&\leq \int_a^t |\phi(h(s), s) - \phi(k(s), s)|\, ds \\
&\leq \int_a^t |M(h(s) - k(s))|\, ds \\
&\leq \frac{M^{n+1}}{n!}\|h - k\|_\infty \int_a^t (s - a)^n\, ds \\
&\leq \frac{M^{n+1}}{(n+1)!}\|h - k\|_\infty (t - a)^{n+1}.
\end{aligned}$$

The induction can proceed.

We thus have

$$\|T^n h - T^n k\| \leq \frac{1}{n!} M^n (b - a)^n \|h - k\|_\infty.$$

Since

$$u_n = \frac{1}{n!} M^n (b - a)^n \to 0$$

(observe that $u_{n+1}/u_n \to 0$) we can find an N such that

$$\|T^N h - T^N k\| \leq \frac{1}{2}\|h - k\|_\infty$$

for all h, $k \in C([a, b])$. Thus there exists a unique $f \in C([a, b])$ with

$$f(t) = g(t) + \int_a^t \phi(f(s), s)\, ds$$

and so (if $g(t) = c$) a unique solution of

$$f'(t) = \phi(f(t), t)$$

with initial condition $f(a) = c$.

K291

Consider $\mathbf{R}^n$ with its usual Euclidean norm and recall that it is complete. Set

$$T\mathbf{x} = \mathbf{y}, \text{ with } y_i = \sum_{j=1}^{n} a_{ij} f(x_j) + b_i$$

then, using Cauchy-Schwarz inequality and the mean value inequality, we have

$$\begin{aligned}
\|T\mathbf{u} - T\mathbf{v}\| &= \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(f(u_j) - f(v_j) \right)^2 \right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2 \right) \left(\sum_{j=1}^{n} (f(u_j) - f(v_j))^2 \right) \right)^{1/2} \\
&\leq \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2 \right) \left(\sum_{j=1}^{n} (M|u_j - v_j|^2) \right) \right)^{1/2} \\
&= M\|\mathbf{u} - \mathbf{v}\| \left(\sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \\
&= K\|\mathbf{u} - \mathbf{v}\|
\end{aligned}$$

with $K \geq 0$ and

$$K^2 = M^2 \sum_{j=1}^{n} a_{ij}^2.$$

If $K < 1$ the contraction mapping theorem applies.

K292

A little fiddling around (we could write $y^{-\alpha}dy = dx$ obtaining, in a purely exploratory, non-rigorous manner $(1-\alpha)y^{1-\alpha} = x + A$) shows that, if $0 < \alpha < 1$, we have $y(t) = 0$ for all t, and $y(t) = 0$ for $t < s$, $y(t) = (1-\alpha)^{1/(1-\alpha)}(t-s)^{1/(1-\alpha)}$ for $t \geq s$ are solutions. (We need $s \geq 0$ to satisfy $y(0) = 0$).

If $\alpha \geq 1$ then using the mean value theorem

$$||y_1|^\alpha - |y_2|^\alpha| = \alpha|z|^{\alpha-1}||y_1| - |y_2|| \leq \alpha(\max(|y_1|, |y_2|))^{\alpha-1}$$

and this Lipschitz condition shows that the obvious solution $y(t) = 0$ for all t is unique.

K293

(i) If $u > 0$, then $f(t,u) \geq 0$ for $t > 0$ so, by continuity, $f(u,0) \geq 0$. Similarly $f(t,u) \leq 0$ so $f(u,0) \leq 0$ Thus $f(u,0) = 0$ for $u > 0$. Similarly $f(0,u) = 0$ for $u < 0$ so $f(0,0) = 0$ by continuity.

Observe, just as in the proof of Rolle's theorem, that, if y is constant on $[0,c]$, then $y(t) = 0$ for $t \in [0,c]$. Otherwise, since a continuous function on a closed bounded interval is bounded and attains its bounds, y must have a maximum or minimum at some $\alpha \in (0,c)$ with $y(\alpha) \neq 0$. Without loss of generality, let α be a maximum. Then $y(\alpha) > 0$ so, using the fact that at an interior maximum the derivative is zero,

$$0 = y'(\alpha) = f(\alpha, y(\alpha)) > 0$$

which is impossible. Thus $y(t) = 0$ for $t \in [0,c]$ for all $c > 0$ so $y(t) = 0$ for $t \geq 0$ and, similarly, for all t. Thus $\bigstar$ has at most one solution. Since $y = 0$ is a solution it is the unique solution.

(ii) If $E = \{(x,t) : \|(x,t)\| \geq \delta\}$ with $\delta > 0$ then E is closed and the restriction of f is continuous on each of the three closed sets,

$$E_1 = \{(x,t) \in E \,:\, x \geq t^2\},\ E_2 = \{(x,t) \in E \,:\, |x| \leq t^2\}$$

and

$$E_3 = \{(x,t) \in E \,:\, x \leq -t^2\}$$

and so the restriction $f|E$ is continuous on $E = \bigcup_{i=1}^3 E_i$. Thus f is continuous at every (x,t) with $\|(x,t)\| > \delta$ with $\delta > 0$ and so at every $(x,t) \neq (0,0)$ Since $|f(t,u)| \leq 2\max(|x|,|t|)$, f is continuous at $(0,0)$ (Lots of other ways to do this, many probably better.)

Now

$$y_1(t) = \int_0^t f(s,s^2)\,ds = \int_0^t (-2s)\,ds = -y_0(t)$$

for $t \geq 0$ and a similar calculation applies when $t < 0$. Thus $y_n = (-1)^n y_0$.

K294

Suppose, if possible, that $y(\tau) \leq 0$ for some $\tau > 0$. Let
$$\sigma = \inf\{s \geq 0 \,:\, y(s) \leq 0\}.$$
By continuity, $y(\sigma) = 0$. By the mean value theorem, we can find an s with $0 < s < \sigma$ such that
$$|y(s)|^\beta \leq y'(s) = s^{-1}(y(s) - y(0)) < 0$$
which is absurd. Thus $y(t) > 0$ for all t, so $y'(t) > 0$ for all t, so y is strictly increasing, so $y'(t) \geq y(0)^\beta = 1$ so $y(t) \geq 1 + t$ for all $t \geq 0$.

On (t_j, t_{j+1}), we have
$$y'(t) \geq y(t)^\beta \geq y((t_j)^\beta = 2^{\beta j}$$
so, by the mean value inequality,
$$(t_{j+1} - t_j)2^{\beta j} \leq y(t_{j+1}) - y(t_j)) = 2^j$$
and so
$$t_{j+1} - t_j \leq 2^{j(1-\beta)}.$$
Thus
$$t_r = \sum_{j=0}^{r-1} t_{j+1} - t_j \leq \sum_{j=0}^{r-1} 2^{j(1-\beta)} \leq \frac{1}{1 - 2^{1-\beta}}$$
for all r and so
$$a \leq \frac{1}{1 - 2^{1-\beta}}.$$

(ii) Observe that, if $y' = 1 + y^2$, then $y' \geq y^2$.

(iii) Recall that $\int (w \log w)^{-1}\, dw = \log\log w + a$. We have $y(t) = \exp(\exp t)$ as a solution.

K295

Observe that
$$\frac{y((r+1)h)-y(rh)}{h}\approx y'(rh).$$

Observe that
$$|f(t,u)-f(t,v)|\leq \sup_s |f_{,2}(t,s)||u-v|\leq K|u-v|$$
so we have a (global) Lipschitz condition.

By the chain rule y'' exists and
$$y''(t)=f_{,1}(t,y(t))+y'(t)f_{,2}(t,y(t))=f_{,1}(t,y(t))+f(t,y(t))f_{,2}(t,y(t)).$$

Thus $|y''(t)|\leq L$ for all t and so
$$|y(nh+s)-y(nh)-y'(nh)s|\leq \frac{Ls^2}{2}$$
whence
$$|y((n+1)h)-y(nh)-f(nh,y(nh))h|\leq \frac{Lh^2}{2}$$
and
$$\begin{aligned}
&|y((n+1)h)-y_{n+1}|\leq |y(nh)-y_n|+|y((n+1)h)-y_{n+1}-y(nh)+y_n|\\
&\leq |y(nh)-y_n|+|y((n+1)h)-y(nh)-f(nh,y(nh))h|+|f(nh,y(nh))h-y_{n+1}+y_n|\\
&\leq |y(nh)-y_n|+|y((n+1)h)-y(nh)-f(nh,y(nh))h|+|f(nh,y(nh))h-f(nh,y_n)h|\\
&\leq |y(nh)-y_n|+\frac{Lh^2}{2}+Kh|y(nh)-y_n|\\
&=(1+Kh)|y(nh)-y_n|+\frac{Lh^2}{2}.
\end{aligned}$$

Thus if
$$|y(nh)-y_n|\leq \frac{Lh}{2K}\big((1+Kh)^{n+1}-1\big),$$
it follows that
$$\begin{aligned}
|y((n+1)h)-y_{n+1}|&\leq (1+Kh)\frac{Lh_N}{2K}\big((1+Kh)^{n+1}-1\big)+\frac{Lh^2}{2}\\
&\leq \frac{Lh}{2K}\big((1+Kh)^{n+2}-1\big).
\end{aligned}$$
Since the result is true for $n=0$ it is true inductively.

In particular
$$|y(Nh)-y_N|\leq \frac{Lh}{2K}\big((1+Kh)^{N+1}-1\big).$$

K296*

No comments.

K297

As in K295,

$$|y((n+1)h) - y(nh) - y'(nh)h| \le \frac{Lh^2}{2},$$

and then, much as in K295,

$$|y((n+1)h) - \tilde{y}_{n+1}| \le (1+Kh)|y(nh) - y_n| + \frac{Lh^2}{2} + \epsilon$$

so, by induction,

$$|y(nh) - \tilde{y}_n| \le \left(\frac{Lh}{2K} + \frac{\epsilon}{Kh}\right)\big((1+Kh)^{n+1} - 1\big).$$

Thus

$$|y(a) - \tilde{y}_N| \le \left(\frac{Lh}{2K} + \frac{\epsilon}{Kh}\right)(e^{aK} - 1).$$

The error due to machine inaccuracy increases directly with the number of computations.

(ii) and (iii) Recall that

$$\left(\frac{A}{h} + Bh\right) = \left(\frac{A^{1/2}}{h^{1/2}} - B^{1/2}h^{1/2}\right)^2 + 2A^{1/2}B^{1/2}$$

takes its minimum value $2A^{1/2}B^{1/2}$ when $h = (A/B)^{1/2}$. We want $h = (2\epsilon/L)^{1/2}$ giving minimum error $(2L)^{1/2}K^{-1}\epsilon^{1/2}$.

K298

(i) Set $f(t,y(t)) = g(t)$, $M = L$. Either redo calculations with $K = 0$ or observe that

$$\left|\int_0^a g(t)\,dt - h\sum_{n=0}^{N-1} g(rh)\right| = |y(a) - y_N^{[h_N]}| \leq \frac{Lh_N}{2}\frac{e^{aK}-1}{K}$$
$$\to \frac{aLh_N}{2} = \frac{aMh}{2}$$

as $K \to 0+$. Take $\kappa_a = Ma/2$.

(ii) We have

$$\int_0^a G(t)\,dt - h\sum_{n=0}^{N-1} G(rh) = \int_0^\pi \frac{1+\sin 2Nt}{2}\,dt = \frac{\pi}{2} = \frac{\pi}{2}\|G\|_\infty.$$

(iii) No. Consider $f(t,u) = G(t)$ in (ii).

K299

(i) The result is trivially true for $n=0$. If it is true for n then

$$(uv)^{(n+1)}(x) = ((uv)')^{(n)}(x) = (u'v+uv')^{(n)}(x)$$
$$= \sum_{r=0}^{n}\binom{n}{r}u^{(n+1-r)}(x)v^{(r)}(x) + \sum_{r=0}^{n}\binom{n}{r}u^{(n-r)}(x)v^{(r+1)}(x)$$
$$= u^{(n+1)}(x)v(x) + \sum_{r=1}^{n}\left(\binom{n}{r}+\binom{n}{r-1}\right)u^{(n+1-r)}(x)v^{(r)}(x) + u(x)v^{(n+1)}(x)$$
$$= \sum_{r=0}^{n+1}\binom{n+1}{r}u^{(n+1-r)}(x)v^{(r)}(x)$$

so the induction can proceed.

(ii) Differentiating we get

$$x(1+x^2)^{-1/2}y(x) + (1+x^2)^{1/2}y'(x) = \frac{1+x(1+x^2)^{-1/2}}{x+(1+x^2)^{1/2}}$$

so, multiplying through by $(1+x^2)^{1/2}$ we have

$$(1+x^2)y'(x) + xy(x) = 1.$$

Differentiating n times, using Leibniz's formula, we have

$$(1+x^2)y^{(n+1)}(x)+2nxy^{(n)}(x)+n(n-1)y^{(n-1)}(x)+xy^{(n)}(x)+ny^{(n-1)}(x) = 0$$

for $n \geq 1$. Setting $x=0$, we have

$$y^{(n+1)}(0) + n^2y^{(n-1)}(0) = 0$$

Since $y(0)=0$ and $y'(0)=1$ (by setting $x=0$ in the first two displayed equations in the statement of (ii)) we have $y^{(2n)}(0)=0$ and

$$y^{(2n+1)}(0) = 2^n n!.$$

Set $u_n = y^{(n)}(0)/n!$. Then

$$\frac{|u_{2n+1}x^2|}{|u_{2n-1}|}\frac{x^2}{2n+1}$$

which tends to zero if $|x| \leq 1$ but diverges otherwise. Thus the Taylor series for y has radius 1.

Now $\sum_{j=0}^{\infty} u_n x^n$ may be differentiated term by term with in its circle of convergence so $u(x) = \sum_{j=0}^{\infty} u_n x^n$ is a solution for the given differential equation with the same value as y at 0. Since the Lipschitz conditions apply, $y(t) = u(t)$ for $|t| < 1$ so

$$y(x) = \sum_{j=0}^{\infty}\frac{y^{(n)}(0)}{n!}x^n$$

for $|x| < 1$.

K300

Since K is continuous on the closed bounded subset $[0,1]^2$ of $\mathbb{R}^2$, it is bounded so there exists an M with $M \geq K(s,t)$ for $(s,t) \in [0,1]^2$.

If $f \in C([0,1])$, set

$$(Tf)(t) = g(t) + \lambda \int_0^1 K(s,t)f(s)\,ds.$$

We observe that $Tf \in C([0,1])$ since K is uniformly continuous (because K is continuous on the closed bounded subset $[0,1]^2$) and so

$$\begin{aligned} |(Tf)(u) - (Tf)(v)| &= \left| \int_0^1 (K(u,s) - K(v,s))f(s)\,ds \right| \\ &\leq \|f\|_\infty \sup_{\|\mathbf{x}-\mathbf{y}\| \leq |u-v|} |K(\mathbf{x}) - K(\mathbf{y})| \to 0 \end{aligned}$$

as $u \to v$.

Further, if $|\lambda| < M^{-1}$, T is a contraction mapping on the complete normed space $(C([0,1]), \|\ \|_\infty)$ since

$$\begin{aligned} |(Tf)(t) - (Th)(t)| &= |\lambda| \left| \int_0^1 K(s,t)(f(s) - h(s)\,ds \right| \\ &\leq |\lambda| \int_0^1 |K(s,t)(f(s) - h(s)|\,ds \\ &\leq |\lambda| M \|f - h\|_\infty \end{aligned}$$

so $\|Tf - Th\|_\infty \leq M\|f-h\|_\infty$. The contraction mapping theorem now gives the required result.

K301

(ii) Observe that

$$
\begin{aligned}
&\frac{d}{dt}\begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{vmatrix} \\
&= \begin{vmatrix} u_1'(t) & u_2'(t) & u_3'(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{vmatrix} + \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{vmatrix} + \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1'''(t) & u_2'''(t) & u_3'''(t) \end{vmatrix} \\
&= 0 + 0 + \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1'''(t) & u_2'''(t) & u_3'''(t) \end{vmatrix} \\
&= \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ -a(t)u_1''(t) - b(t)u_1'(t) - c(t)u_1(t) & -a(t)u_2''(t) - b(t)u_2'(t) - c(t)u_2(t) & -a(t)u_3''(t) - b(t)u_3'(t) - c(t)u_3(t) \end{vmatrix} \\
&= -a(t)\begin{vmatrix} u_1'(t) & u_2'(t) & u_3'(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{vmatrix}
\end{aligned}
$$

so $W'(t) = -a(t)W(t)$.

K302

Observe that, since $W(a) \neq 0$, we have $g(a) \neq 0$. Similarly $g(b) \neq 0$. Thus, if g has no zeros in (a, b), g has no zeros in $[a, b]$. Thus f/g is well defined continuous function on $[a, b]$ which is differentiable on (a, b). By Rolle's theorem there exists a $c \in (a, b)$ with

$$0 = \left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - g'(c)f(c)}{(g'(c))^2} = \frac{W(c)}{(g'(c))^2}.$$

Thus $W(c) = 0$ contrary to the hypothesis. Thus there must be a zero of g between any two zeros of f and, similarly, a zero of f between any two zeros of g. [Moreover the zeros must be simple, if $f(c) = f'(c) = 0$, then $W(c) = 0$.]

K303

(i) Without loss of generality suppose that $\lambda_1 \neq 0$. Then without loss of generality we may suppose $\lambda_1 = 1$ and set $\lambda_2 = \lambda$. Then

$$\begin{aligned} W(f_1, f_2)(x) &= \det \begin{pmatrix} f_1 & f_2 \\ f_1 & f_2' \end{pmatrix} \\ &= \det \begin{pmatrix} f_1 + \lambda f_2 & f_2 \\ f_1' + \lambda f_2' & f_2' \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & f_2 \\ 0 & f_2' \end{pmatrix} = 0. \end{aligned}$$

However, if $g_1(x) = (x-1)^2$ for $x \geq 1$ and $g_2(x) = g_1(-x)$ (we could play the same trick with infinitely differentiable functions) then g_1 and g_2 are continuously differentiable and their Wronskian vanishes everywhere but they are not linearly dependent.

(ii) By Exercise 12.24 applied to the vectorial equation

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ -a_1(t)v(t) - a_2(t)y_1(t) \end{pmatrix},$$

it follows that if a_1 and a_2 are continuous (so bounded on $[a, b]$), the equation

$$y''(x) + a_1(x)y'(x) + a_2 y(x) = 0$$

has exactly one solution with $y(a) = A$ and $y'(a) = B$.

If $W(f_1, f_2)(x)$ never vanishes then setting

$$a_1(x) = \big(W(f_1, f_2)(x)\big)^{-1} \det \begin{pmatrix} f_1(x) & f_2(x) \\ f_1''(x) & f_2''(x) \end{pmatrix}$$

and

$$a_2(x) = \big(W(f_1, f_2)(x)\big)^{-1} \det \begin{pmatrix} f_1'(x) & f_2'(x) \\ f_1''(x) & f_2''(x) \end{pmatrix}$$

we have a_1 and a_2 continuous. Thus the equation

$$\bigstar \qquad y''(x) + a_1(x)y'(x) + a_2 y(x) = 0$$

has exactly one solution with $y(a) = A$ and $y'(a) = B$.

But $\bigstar$ is equivalent to

$$W(y, f_1, f_2) = 0$$

and so has f_1 and f_2 as solutions. Since $W(f_1, f_2)(a) \neq 0$ the equations

$$\begin{aligned} \lambda_1 f_1(a) + \lambda_2 f_2(a) &= y(a) \\ \lambda_1 f_1'(a) + \lambda_2 f_2'(a) &= y'(a) \end{aligned}$$

have a solution, so, if y is a solution, the uniqueness result of the previous paragraph shows that

$$y(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x)$$

for all x so f_1 and f_2 are indeed a basis.

Recall from e.g. Lemma 12.8 that if f_1 and f_2 are solutions of

$$y''(x) + b_1(x)y'(x) + b_2y(x) = 0$$

then if $W(f_1, f_2)$ vanishes at one point it vanishes everywhere.

K304

Observe that

$$
\begin{aligned}
0 &= (u''v + 2v'u' + v''u) + p(u'v + v'u) + qvu \\
&= v(u'' + pu' + qu) + v''u + v'(2u' + pu) = v''u + v'(2u' + pu)
\end{aligned}
$$

so, writing $w = v'$ we have

$$
\dagger \qquad w'(x)u(x) + w(x)(2u'(x) + p(x)u(x)) = 0
$$

Observe that † is linear so, if w is a solution, so is Bw. Thus

$$
y(x) = u(x)v(x) = u(x)\left(A + B\int_0^x w(t)\,dt\right)
$$

is a solution. Since it contains two arbitrary constants we would expect it to be the general solution.

(ii) We try $y(x) = v(x)(\exp x)$ obtaining

$$
v''(x) = (2x - 1)v'(x)
$$

so $v'(x) = B(\exp(x^2 - x))$ and

$$
y(x) = A\exp(x) + B\exp(x)\int_0^t \exp(x^2 - x)\,dx.
$$

(iii) $y(x) = (A + Bx)\exp x$.

(iv) If u is a solution, try $y = uv$, obtaining

$$
\begin{aligned}
0 &= (u'''v + 3v'u'' + 3v''u' + v'''u) + p_1(u''v + 2u'v' + v''u) + p_2(u'v + v'u) + p_3vu \\
&= w''u + w'(3u' + p_1u) + w(3u'' + 2u'p_1 + p_2)
\end{aligned}
$$

a second order equation for $w = v'$.

K305

(i) We have
$$\begin{aligned} f &= (u_1''y_1 + 2u_1'y_1' + u_1y_1'' + p(u_1'y_1 + u_1y_1') + qu_1y_1) \\ &\quad + (u_2''y_2 + 2u_2'y_2' + u_2y_2'') + p(u_2'y_2 + u_2y_2') + qu_2y_2 \\ &= 2u_1'y_1' + u_1''y_1 + pu_1'y_1 + 2u_2'y_2' + u_2''y_2 + pu_2'y_2. \end{aligned}$$

(ii) If we differentiate ††, then we get
$$0 = u_1''y_1 + u_1'y_1 + u_2''y_2 + u_2'y_2$$
so † becomes
$$f = u_1'y_1' + u_2'y_2'.$$
In other words, we have the pair of equations
$$\begin{aligned} u_1'y_1' + u_2'y_2' &= f \\ u_1'y_1 + u_2'y_2 &= 0 \end{aligned}$$
so, since W never vanishes,
$$u_1' = -W^{-1}fy_2, \text{ and } u_2' = W^{-1}fy_1.$$

(iii) Performing the integration we get
$$\begin{aligned} y(x) &= A_1y_1(x) + A_2y_2(x) \\ &\quad + y_1(x)\int_x^a W^{-1}(s)f(s)y_2(s)\,dx + y_2(x)\int_b^x W^{-1}(s)f(s)y_1(s)\,dx \end{aligned}$$
and, if y_1 and y_2 are chosen as in our discussion of Green's function, we obtain our standard Green's function solution plus the general solution of the equation with $f = 0$.

(iv) In the case given, we can take $y_1(x) = e^x$ and $y_2(x) = e^{-x}$, obtaining $W(x) = -2$ and
$$u_1'(x) = -\frac{e^{-x}}{1+e^x} = \frac{e^x}{1+e^x} + e^{-x} - 1 \text{ and } u_2'(x) = \frac{e^x}{1+e^x}.$$
Thus
$$u_1(x) = A_1 + \log(1+e^x) - e^{-x} + x \text{ and } u_2(x) = A_2 + \log(1+e^x)$$
and we have the general solution
$$y(x) = A_1e^x + A_2e^{-x} + (e^{-x} - e^x)\log(1+e^x) - e^x + 1.$$

(v) Set $y = y_1u_1 + y_2u_2 + y_3u_3$ subject to
$$\begin{aligned} 0 &= y_1u_1' + y_2'u_2' + y_3u_3' \\ 0 &= y_1'u_1' + y_2'u_2' + y_3'u_3'. \end{aligned}$$
We get
$$f = y_1''u_1 + y_2''u_2 + y_3''u_3.$$

Since the Wronskian never vanishes, we can solve the last three equations to find u_1', u_2' and u_3'. Integrating now gives u_1, u_2, u_3 and thus the solution.

K306

Observe that since K is continuous on the closed bounded set S it is uniformly continuous. Thus

$$\begin{aligned}|(T_K(f))(s) - (T_K(f))(t)| &= \left|\int_0^1 K(s,y) - K(t,y)f(y)\,dy\right| \\ &\leq \|f\|_\infty \sup_{\|\mathbf{z}-\mathbf{w}\|\leq|s-t|} |K(\mathbf{z}) - K(\mathbf{w})| \to 0\end{aligned}$$

as $s \to t$. Thus T_k maps $C(I)$ to $C(I)$. We easily check linearity and

$$|(T_K(f))(s)| \leq \|f\|_\infty \|K\|_\infty$$

so T_K is continuous and $\|T\| \leq \|K\|_\infty$.

(i) True, since $\|T_{K_n} - T_K\| = \|T_{K_n - K}\| \leq \|K_n - K\|_\infty$.

(ii) False. If $K_n(x,y) = \max(0, 1 - n^2(x^2 + y^2))$ then $\|K_n\|_\infty = 1$ for all n but

$$|(T_{K_n}(f))(s)| \leq \int_0^{1/n} |f(y)|\,dy \leq \|f\|_\infty/n$$

so $\|T_{K_n}\| \leq n^{-1} \to 0$.

(iii) True. Since $T_K - T_L = T_{K-L}$, it suffices to show that $T_K = 0$ implies $K = 0$. Suppose $K \neq 0$. Then we can find $(x,y) \in S$ such that $K(x,y) \neq 0$. Without loss of generality suppose $K(x,y) = m > 0$. By continuity we can find a $\delta > 0$ such that $(s,t) \in S$ and $|s-x|$, $|t-y| \leq \delta$ implies $K(s,t) \geq m/2$. Set $f(t) = \max(0, 1 - 2\delta^{-1}|t-y|)$. Then $T_K f(x) > 0$ so $T_K \neq 0$.

(iv) False. Let $Tf(x) = f(1/2)$ for all $[x \in [0,1]$. Then T is linear and continuous with $\|T\| \leq 1$. We claim that $T \neq T_K$ for all K. For consider $f_n(x) = \max(0, 1 - n|x - \frac{1}{2}|)$. $Tf_n = 1$ for all n but, if $n \geq 2$,

$$|T_K f_n| \leq \int_{\frac{1}{2}-1/n}^{\frac{1}{2}+1/n} \|K\|_\infty\,ds = 2\|K\|_\infty/n \to 0$$

as $n \to \infty$.

K307

(ii) We apply Exercise 12.24 to the vectorial equation

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ -a(t)v_1(t) - b(t)y_1(t) \end{pmatrix}$$

to obtain the existence and uniqueness of y_1.

(iii) We have (noting that $\int_t^1 u(x)\,dx = -\int_1^t u(x)\,dx$)

$$\begin{aligned} y'(t) &= H(t,t)f(t) + \int_0^t H_{,2}(s,t)f(s)\,ds - \tilde{H}(t,t)f(t) + \int_t^1 \tilde{H}_{,2}(s,t)f(s)\,ds \\ &= \int_0^t H_{,2}(s,t)f(s)\,ds + \int_t^1 H_{,2}(s,t)f(s)\,ds \end{aligned}$$

and

$$\begin{aligned} y''(t) &= H_{,2}(t,t)f(t) + \int_0^t H_{,22}(s,t)f(s)\,ds - \tilde{H}_{,2}(t,t)f(t) + \int_0^t \tilde{H}_{,22}(s,t)f(s)\,ds \\ &= f(t) + \int_0^t H_{,22}(s,t)f(s)\,ds + \int_0^t \tilde{H}_{,22}(s,t)f(s)\,ds. \end{aligned}$$

Summing gives

$$\begin{aligned} &y''(t) + a(t)y'(t) + b(t)y(t) \\ &\qquad = f(t) + \int_0^t (b(t)H_{,2} + a(t)H_{,2}(s,t) + H_{,22}(s,t))f(s)\,ds \\ &\qquad\quad + \int_0^t (b(t)\tilde{H}_{,2} + a(t)\tilde{H}_{,2}(s,t) + \tilde{H}_{,22}(s,t))f(s)\,ds \\ &\qquad = f(t). \end{aligned}$$

Further

$$y(0) = \int_0^1 \tilde{H}(s,1)f(s)\,ds = \int_0^1 0\,ds = 0$$

and similarly $y(1) = 0$.

K308

We seek a twice continuously differentiable function $y(x) = G(x,t)$ which is four times differentiable (using left and right derivatives at end points) on $[0,t)$ and $(t,1]$ and satisfies $y(0) = y'(0) = 0$,

$$y^{(4)}(x) - k^4 y(x) = 0 \text{ for } x \in [0,t)$$

and $y(1) = y'(1) = 0$,

$$y^{(4)}(x) - k^4 y(x) = 0 \text{ for } x \in (t,1]$$

whilst

$$y'''(t+) - y'''(t-) = 1.$$

This gives

$$y(x) = a \sinh kx + b \sin kx + c \cosh x + d \cos x \text{ for } x \in [0,t]$$

(remember we have continuity at t) and

$$0 = c + d,\ 0 = b + d$$

so

$$y(x) = A(\sinh kx - \sin kx) + B(\cosh kx - \cos kx) \text{ for } x \in [t,1]$$

and, using appropriate symmetries

$$y(x) = C(\sinh k(1-x) - \sin k(1-x)) + D(\cosh k(1-x) - \cos k(1-x)) \text{ for } x \in (t,1].$$

The continuity of the zeroth, first and second derivatives and the condition on the discontinuity for the third gives (subject to the check in the next paragraph)

$$M \begin{pmatrix} A & B & C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \text{ so } \begin{pmatrix} A & B & C & D \end{pmatrix}^T = M^{-1} \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T$$

where

$$M = \begin{pmatrix} \big(\sinh kt - \sin kt\big) & \big(\cosh kt - \cos kt\big) & -\big(\sinh k(1-t) + \sin k(1-t)\big) & -\big(\cosh k(1-t) - \cos k(1-t)\big) \\ k\big(\cosh kt - \cos kt\big) & k\big(\sinh kt + \sin kt\big) & k\big(\cosh k(1-t) + \cos k(1-t)\big) & k\big(\sinh k(1-t) + \sin k(1-t)\big) \\ k^2\big(\sinh kt - \sin kt\big) & k^2\big(\cosh kt - \cos kt\big) & -k^2\big(\sinh k(1-t) - \sin k(1-t)\big) & -k^2\big(\cosh k(1-t) + \cos k(1-t)\big) \\ k^3\big(\cosh kt + \cos kt\big) & k^3\big(\sinh kt + \sin kt\big) & k^3\big(\cosh k(1-t) - \cos k(1-t)\big) & k^3\big(\sinh k(1-t) - \sin k(1-t)\big) \end{pmatrix}.$$

We must check that the four functions

$$\big(\sinh kx - \sin kx\big),\ \big(\cosh kx + \cos kx\big)$$

and

$$\big(\sinh k(1-x) - \sin k(1-x)\big),\ \big(\cosh k(1-x) + \cos k(1-x)\big)$$

are indeed linearly independent. If

$$A\big(\sinh kx - \sin kx\big) + B\big(\cosh kx - \cos kx\big) \\ + C\big(\sinh k(1-x) - \sin k(1-x)\big) + D\big(\cosh k(1-x) - \cos k(1-x)\big) = 0,$$

then writing

$$y(x) = A\big(\sinh kx - \sin kx\big) + B\big(\cosh kx - \cos kx\big)$$

we must have and $y(1) = y'(1) = 0$ that is to say

$$A\big(\sinh k - \sin k\big) + B\big(\cosh k - \cos k\big) = 0$$
$$A\big(\cosh k - \cos k\big) + B\big(\sinh k + \sin k\big) = 0.$$

But

$$\begin{aligned}\det &\begin{pmatrix} \big(\sinh k - \sin k\big) & \big(\cosh k - \cos k\big) \\ \big(\cosh k - \cos k\big) & \big(\sinh k + \sin k\big) \end{pmatrix} \\ &= \big(\sinh^2 k - \sin^2 k\big) - \big(\cosh^2 k - \cos^2 k\big) \\ &= -1 + \cos^2 k - \sin^2 k = \cos 2k - 1.\end{aligned}$$

Thus if $k \neq n\pi$ for some integer n, we have $A = B = 0$ (and so $C = D = 0$).

If $k = n\pi$ then, by direct calculation, the Wronskian of our four functions (which is a multiple of $\det M$) vanishes at 1 so $\det M = 0$ everywhere and our method fails. (The method will also fail for simpler reasons in the case $k = 0$.) [In fact we are in a situation like the last example of Chapter 12 and there is no solution.]

K309

We seek a continuous function $y(x) = G(x,t)$ which is twice differentiable (using left and right derivatives at end points) on $[0,t)$ and $(t,1])$ and satisfies $y(0) = y'(0) = 0$,

$$y''(x) + 2\beta y'(x) + \omega^2 y(x) = 0 \text{ for } x \in [0,t)$$

and

$$y''(x) + 2\beta y'(x) + \omega^2 y(x) = 0 \text{ for } x \in (t,\infty)$$

whilst

$$y'(t+) - y'(t-) = 1.$$

This gives

$$y(x) = 0 \text{ for } x \in [0,t]$$

and [$*$]

$$y(x) = Ae^{-\beta(x-t)}\sinh(\alpha(x-t)) + Be^{-\beta(x-t)}\cosh(\alpha(x-t))$$

for $x \in [t,\infty)$.

Continuity gives $B = 0$ and the condition on the jump of the first derivative gives

$$1 = \alpha A.$$

Thus

$$G(x,t) = \begin{cases} 0 & \text{if } x \leq t \\ \alpha^{-1}e^{-\beta(x-t)}\sinh(\alpha(x-t)) & \end{cases}$$

so, if $\beta > \omega$, the general solution on $[0,\infty)$ of

$$y''(x) + 2\beta y'(x) + \omega^2 y(x) = f(x)$$

is

$$y(t) = \int_0^\infty G(s,t)f(s)\,ds = \alpha^{-1}\int_0^t f(s)e^{-\beta(t-s)}\sinh(\alpha(t-s))\,ds$$

with α the positive square root of $\beta^2 - \omega^2$.

If $\omega > \beta$ the argument proceeds as far as [$*$] in the paragraph above. We then get

$y(x) = Ae^{-\beta(x-t)}\sin(\lambda(x-t)) + Be^{-\beta(x-t)}\cos(\lambda(x-t)) +$ for $x \in [t,\infty)$,

where λ is the positive square root of $\omega^2 - \beta^2$. Arguing much as above we get $B = 0$, $A = \lambda^{-1}$ and the general solution on $[0,\infty)$ of

$$y''(x) + 2\beta y'(x) + \omega^2 y(x) = f(x)$$

as

$$y(t) = \lambda^{-1}\int_0^t f(s)e^{-\beta(t-s)}\sin(\lambda(t-s))\,ds.$$

If $\omega = \beta$ the argument again proceeds as far as [$*$]. We then get

$$y(x) = A(x-t)e^{-\beta(x-t)} + Be^{-\beta(x-t)}.$$

Arguing much as before we get $B = 0$, $A = 1$ and the general solution on $[0, \infty)$ of

$$y''(x) + 2\beta y'(x) + \omega^2 y(x) = f(x)$$

as

$$y(t) = \int_0^t (t - s) f(s) e^{-\beta(t-s)}\, ds.$$

If β is large the pointer takes a long time to return to close to equilibrium. If β is small the pointer overshoots and we have violent oscillation so it also takes a long time before the pointer remains close to equilibrium. As a rule of thumb β close to ω is preferred.

K310

(i) Observe that, if $m \geq n$

$$\|S_n - S_m\| \leq \sum_{r=n+1}^{m} \|T^r\| \leq \sum_{r=n+1}^{m} \|T\|^r \leq \frac{\|T\|^{n+1}}{1-\|T\|} \to 0$$

as $n \to \infty$, so the sequence S_n is Cauchy. Thus we can find an $S \in \mathcal{L}(U,U)$ with $\|S_n - S\| \to 0$.

Now

$$\begin{aligned}\|S\| &\leq \|S_n - S\| + \|S_n\| \leq \|S_n - S\| + \sum_{r=0}^{n} \|T\|^r \\ &\leq \|S_n - S\| + (1-\|T\|)^{-1} \to (1-\|T\|)^{-1}\end{aligned}$$

so $\|S\| \leq (1-\|T\|)^{-1}$. A similar calculation gives

$$\|I - S\| \leq \|T\|(1-\|T\|)^{-1}.$$

(ii) Observe that

$$\begin{aligned}\|S(I-T) - I\| &\leq \|S_n(I-T) - I\| + \|(S-S_n)(I-T)\| \\ &= \|T^{n+1}\| + \|(S-S_n)(I-T)\| \\ &\leq \|T\|^{n+1} + \|S-S_n\|\|I-T\| \to 0\end{aligned}$$

as $n \to 0$, so $\|S(I-T) - I\| = 0$ and $S(I-T) = I$. Similarly $(I-T)S = I$. Thus $I-T$ is invertible with inverse S. Setting $T = I-A$ gives the result.

(iii) Observe that

$$\|A - I\| = \|B^{-1}B - B^{-1}C\| \leq \|B - C\|\|B^{-1}\| < 1$$

so A is invertible. Thus

$$A^{-1}B^{-1}C = I \text{ and } CA^{-1}B^{-1} = BB^{-1}CA^{-1}B^{-1} = BB^{-1} = I$$

so C is invertible with inverse $A^{-1}B^{-1}$.

We observe that

$$\|C^{-1}\| \leq \|B^{-1}\|\|A^{-1}\| \leq \|B^{-1}\|(1-\|I-A\|)^{-1} = \|B^{-1}\|(1-\|I-A\|)^{-1}$$

but

$$1 > \|B-C\|\|B^{-1}\| \geq \|A - I\| \geq 0$$

so

$$\|C^{-1}\| \leq \|B^{-1}\|(1 - \|B-C\|\|B^{-1}\|)^{-1}.$$

Similar arguments give the second inequality.

(iv) If $B \in E$, then part (ii) shows that

$$\Gamma = \{C \in \mathcal{L}(U,U) \,:\, \|B - C\| < \|B^{-1}\|^{-1}\} \subseteq E$$

so E is open. If $C \in \Gamma$ then
$$\|\Theta(C) - \Theta(B)\| \leq \|B\|^{-1}\|\|B - C\|(1 - \|B\|^{-1}\|B - C\|)^{-1} \to 0$$
as $\|C - B\| \to 0$. Thus Θ is continuous.

(iv) Calculations as in (ii).

(v) Use the chain rule or (essentially the same) observe that
$$\begin{aligned}\Theta(B + H) &= (B + H)^{-1} = (B(I + B^{-1}H))^{-1}\\ &= (I + B^{-1}H)^{-1}B^{-1}\\ &= (I - B^{-1}H + \epsilon(B^{-1}H)\|H\|)B^{-1}\\ &= B^{-1} - B^{-1}HB^{-1} + \eta(H)\|H\|\\ &= \Theta(B) + \Phi_B(H) + \eta(H)\|H\|.\end{aligned}$$
where
$$\|\eta(H)\| = \|B^{-1}\epsilon(B^{-1}H)B^{-1}\| \leq \|B^{-1}\|^2\|\epsilon(B^{-1}H)\| \to 0$$
as $\|H\| \to 0$.

If $n = 1$, this reduces to the statement that
$$\frac{d}{dx}cx^{-1} = -cx^{-2}.$$

K311

(ii) If $\|A\| = 0$, then $A = 0$ and there is nothing to prove so we assume $\|A\| \neq 0$. Δ exists since any non-empty set of real numbers bounded below has an infimum. Now choose $\epsilon > 0$. We can find a j such that $\|A^j\|^{1/j} \leq \Delta + \epsilon$. Thus

$$\begin{aligned}\|A^{jk+r}\|^{1/(jk+r)} &\leq \left(\|A^j\|^k \|A\|^r\right)^{1/(jk+r)} \\ &\leq (\Delta + \epsilon)^{jk/(jk+r)} \|A\|^{r/(jk+r)} \to \Delta + \epsilon\end{aligned}$$

for all $0 \leq r \leq j - 1$. Thus

$$\Delta \leq \liminf_{n\to\infty} \|A^n\|^{1/n} \leq \limsup_{n\to\infty} \|A^n\|^{1/n} \leq \Delta + \epsilon$$

and since $\epsilon > 0$ $\|A^n\|^{1/n} \to \Delta$.

(iii) and (iv) If $\rho(A) < 1$, set $\lambda = (\rho(A) + 1)/2$. There exists an N such that $\|A^n\|^{1/n} \leq \lambda$ and so $\|A\|^n \leq \lambda^n$ for $n \geq N$. The kind of arguments used in K310 show that $\sum_{j=0}^{\infty} A^j$ converges and that the limit is the inverse of $I - A$.

If $\rho(A) > 1$ then there exists an N such that $\|A^n\|^{1/n} \geq 1$ for all $n \geq N$ so, by the easy part of the GPC, $\sum_{j=0}^{\infty} A^j$ fails to converge.

K312

(i) $\|(\lambda\alpha)^n\|^{1/n} = |\lambda|\|\alpha^n\|^{1/n}$

(ii) Take a basis $\mathbf{e}_j$ and let β be the unique linear map with $\beta\mathbf{e}_j = \mathbf{e}_{j+1}$ for $1 \leq j \leq m-1$, $\beta\mathbf{e}_m = \mathbf{0}$. We have $\rho(\beta) = 0$.

(iii) The m eigenvectors $\mathbf{e}_j$ with eigenvalues λ_j are linearly independent. (If $\sum_{j=1}^m \mu_j\mathbf{e}_j = \mathbf{0}$ apply $\prod_{j\neq k}(\alpha - \lambda_j\iota)$ to both sides.) They thus form a basis. If the standard basis is $\mathbf{u}_j$, let θ be the unique linear map with $\theta\mathbf{e}_j = \mathbf{u}_j$.

Observe that

$$\|(\theta\alpha\theta^{-1})^n\|^{1/n} = \|\theta\alpha^n\theta^{-1}\|^{1/n} \leq \|\theta^{-1}\|^{1/n}\|\alpha^n\|^{1/n}\|\theta\|^{1/n} \to \rho(\alpha)$$

so $\rho(\theta\alpha\theta^{-1}) \leq \rho(\alpha)$. Similarly

$$\rho(\alpha) = \rho(\theta^{-1}(\theta\alpha\theta^{-1})\theta) \leq \rho(\theta\alpha\theta^{-1})$$

so $\rho(\theta\alpha\theta^{-1}) = \rho(\alpha)$. But if ϕ is a diagonal matrix with respect to the standard basis, ϕ has norm equal to the absolute value of the largest diagonal entry so $\rho(\theta\alpha\theta^{-1}) = \max|\lambda_j|$.

(iv) If $\mathbf{u}_1$ and $\mathbf{u}_2$ form the standard basis let α, β be the linear maps given by

$$\alpha\mathbf{u}_1 = \mathbf{u}_2,\ \alpha\mathbf{u}_2 = \mathbf{0},\ \beta\mathbf{u}_2 = \mathbf{u}_1,\ \beta\mathbf{u}_1 = \mathbf{0}.$$

K313

(i) This is just the formula for change of basis.

(ii) Here and elsewhere we use the fact that
$$m^2 \sup_{1\leq i,j\leq m} |a_{ij}| \geq \|\alpha\| \geq \sup_{1\leq i,j\leq m} |a_{ij}|$$
when (a_{ij}) is the matrix of α with respect to the matrix. (In fact we merely use
$$K \sup_{1\leq i,j\leq m} |a_{ij}| \geq \|\alpha\| \geq K^{-1} \sup_{1\leq i,j\leq m} |a_{ij}|$$
for some K and this follows from the fact that all norms on a finite dimensional space are Lipschitz equivalent.)

Given any $\eta > 0$, we can certainly find c_{ij} with $c_{ij} = b_{ij}$ for $i \neq j$, $|c_{ii} - b_{ii}| < \eta$ and the c_{ii} all distinct.

(iii) Use the notation of (i). Use (ii) to find β_n with m distinct eigenvalues and $\|\beta_n - \beta\| \to 0$. Let $\alpha_n = \theta^{-1}\beta_n\theta$. We know that α_n has the same eigenvalues as β_n and
$$\|\alpha - \alpha_n\| = \|\theta^{-1}(\beta - \beta_n)\theta\| \leq \|\theta^{-1}\|\|\beta - \beta_n\|\|\theta\| \to 0$$
as $n \to 0$.

(iv) If α has m distinct eigenvalues we know that we can find a basis of eigenvectors. With respect to this basis, α is a diagonal matrix D with entries the eigenvalues so
$$\chi_\alpha(t) = \det(tI - D) = \prod_{j=1}^{m}(t - \lambda_j)$$
and $\chi_\alpha(D) = 0$ the zero $m \times m$ matrix. Thus $\chi_\alpha(\alpha) = 0$.

(v) We know that the map from $\mathcal{L}(\mathbb{C}^m, \mathbb{C}^m)$ to $\mathcal{C}^{m^2}$ given by $\alpha \mapsto (a_{ij})$ is continuous so the map $\alpha \mapsto \chi_\alpha(A)$ is continuous. But the inverse map $(a_{ij}) \mapsto \alpha$ is also continuous so the map $\alpha \mapsto \chi_\alpha(\alpha)$ is continuous.

By the previous parts of the question, given any α we can find α_n with $\|\alpha - \alpha_n\| \to 0$ with $\chi_{\alpha_n}(\alpha_n) = 0$, so, by continuity, $\chi_\alpha(\alpha) = 0$

(vi) We need $\alpha\beta - \beta\alpha$ non singular. One (among many ways) of doing this is as follows. Take $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_m$ to be a basis of $\mathbb{C}^m$. Let α be the linear map with $\alpha\mathbf{e}_j = \alpha\mathbf{e}_{j+1}$ for $1 \leq j \leq m-1$, $\alpha\mathbf{e}_m = \alpha\mathbf{e}_1$ and β be the linear map with $\beta\mathbf{e}_j = j\alpha\mathbf{e}_j$ for $1 \leq j \leq m$.

K314

(i) We know that, given any $\epsilon > 0$, we can find $a(\epsilon) \geq 1$ and $b(\epsilon) \geq 1$ such that

$$\|\alpha^r\| \leq a(\epsilon)(\rho(\alpha)+\epsilon)^r \text{ and } \|\beta^r\| \leq b(\epsilon)(\rho(\beta)+\epsilon)^r$$

for all $r \geq 0$. Thus

$$\begin{aligned}
\|(\alpha+\beta)^n\| &= \left\| \sum_{j=0}^{n} \binom{n}{j} \alpha^j \beta^{n-j} \right\| \\
&\leq \sum_{j=0}^{n} \binom{n}{j} \|\alpha^j\| \|\beta^{n-j}\| \\
&\leq a(\epsilon)b(\epsilon) \sum_{j=0}^{n} \binom{n}{j} (\rho(\alpha)+\epsilon)^j (\rho(\beta)+\epsilon)^{n-j} \\
&= a(\epsilon)b(\epsilon)(\rho(\alpha)+\rho(\beta)+2\epsilon)^n
\end{aligned}$$

so, taking n-th roots and allowing $n \to \infty$,

$$\rho(\alpha+\beta) \leq (\rho(\alpha)+\rho(\beta)+2\epsilon).$$

Since $\epsilon > 0$ this gives the result.

(ii) Observe that, given an upper triangular matrix A, we can find a diagonal matrix B with entries in absolute value less than any give $\eta > 0$ such that the diagonal entries of $A+B$ are all distinct. Observe that A and B commute. Now argue as in K313 parts (i) to (iii).

(iii) If α has m distinct eigenvalues, then we may find a basis $\mathbf{e}_1$, $\mathbf{e}_2$, …of eigenvalues vectors with corresponding eigenvalues λ_j with $|\lambda_1| \geq |\lambda_2| \geq \dots$. Since $\|\alpha^n \mathbf{e}_1\|/\|\mathbf{e}_1\| = |\lambda_1|^n$, we have $\rho(\alpha) \geq |\lambda_1|$. However, if $\lambda > |\lambda_1|$ we have

$$\lambda^n \left\| \alpha^n \sum_{j=1}^{m} x_j \mathbf{e}_j \right\| \leq \sum_{j=1}^{m} \left| \frac{\lambda_j}{\lambda} \right| \|\mathbf{e}_j\| \to 0$$

so $\rho(\alpha) \leq \lambda$. Thus $\rho(\alpha) = |\lambda_1|$.

If α does not have n distinct eigenvalues, then, by (ii), we can find β_r such that the eigenvalues of α and $\alpha+\beta_r$ differ by less than $1/r$, $\|\beta_r\| \leq 1/r$ and α and β_r commute.

Observe that, by (i),

$$\rho(\alpha+\beta_r) \leq \rho(\alpha)+\rho(\beta_r) \leq \rho(\alpha)+\|\beta_r\| = \rho(\alpha)+1/r$$

and

$$\rho(\alpha) = \rho(\alpha+\beta_r-\beta_r) \leq \rho(\alpha+\beta_r)+1/r$$

so $\rho(\alpha+\beta_r) \to \rho(\alpha)$ and the desired result follows.

(iv) If α is represented by A with respect to one basis and by B with respect to another, we can find an invertible θ such that $\theta\alpha\theta^{-1}$ is represented by B with respect to the first basis. Now

$$\|(\theta\alpha\theta^{-1})^n\| = \|\theta\alpha^n\theta^{-1}\| \leq \|\theta\|\|\alpha^n\|\|\theta^{-1}\|,$$

so, taking n-th roots and letting $n \to \infty$, we have $\rho(\theta\alpha\theta^{-1}) \leq \rho(\alpha)$ and similarly

$$\rho(\alpha) = \rho(\theta^{-1}\theta\alpha\theta^{-1}\theta) \leq \rho(\theta\alpha\theta^{-1})$$

so $\rho(\theta\alpha\theta^{-1}) = \rho(\alpha)$ and there is no ambiguity.

K315

(i) Observe that

$$\mathbf{x}_n = \alpha^n \mathbf{x}_0 + \sum_{j=0}^{n-1} \alpha^j \mathbf{b}$$

so $\mathbf{x}_n \to (\iota - \alpha)^{-1}\mathbf{b}$.

(ii) Observe that the map $\tau : \mathbb{C}^m \to \mathbb{C}^m$ given by $\tau\mathbf{x} = \mathbf{b} + \alpha\mathbf{x}$ satisfies

$$\|\tau^n \mathbf{x} - \tau^n \mathbf{y}\| = \|\alpha^n(\mathbf{x} - \mathbf{y})\| \leq \|\alpha^n\|\|\mathbf{x} - \mathbf{y}\|.$$

For large enough n we have τ^n a contraction mapping so $\mathbf{x}_n \to \mathbf{c}$ where $\mathbf{c}$ is the unique fixed point given by

$$\tau\mathbf{c} = \mathbf{c}$$

i.e. by $\alpha\mathbf{c} = \mathbf{b}$.

Suppose $\rho(\alpha) > 1$. If $\sum_{j=0}^{n-1} \alpha^j \mathbf{b}$ diverges, then take $\mathbf{x}_0 = \mathbf{0}$. If not, take $\mathbf{x}_0$ to be an eigenvector corresponding to a largest (in absolute value eigenvalue).

Suppose $\rho(\alpha) = 1$. If $\sum_{j=0}^{n-1} \alpha^j \mathbf{b}$ diverges, then take $\mathbf{x}_0 = \mathbf{0}$. If not, then either all of the largest in absolute value eigenvalues are 1 and we consider $\mathbf{x}_0 = \pm\mathbf{e}$ with $\mathbf{e}$ an associated eigenvector to get two fixed points (with necessarily different limits) or one of the largest in absolute value eigenvalues, is not 1. Take $\mathbf{x}_0 = \mathbf{e}$ with $\mathbf{e}$ an associated eigenvector to get a non-convergent sequence.

K316

We require $\rho(I-A)<1$. If $\|I-A\|$ is small, convergence will be rapid with the error roughly multiplied by $\|I-A\|$ at each step.

Jacobi needs $\rho(D^{-1}(U+L))<1$ and Gauss needs $\rho((D-L)^{-1}U)<1$.

Variation needs $\rho(H)<1$ with

$$H=(D-\omega L)^{-1}((1-\omega)D+\omega U)).$$

Noting that the determinant of a triangular matrix is the product of its diagonal elements we have

$$\begin{aligned}\det H &= (\det(D-\omega L))^{-1}\det((1-\omega)D+\omega U)=(\det D)^{-1}\det((1-\omega)D)\\ &= (\det D)^{-1}(1-\omega)^n\det D=(1-\omega)^n.\end{aligned}$$

Now the determinant of a matrix is the product of its eigenvalues (multiple roots counted multiply) (consider the coefficient of t^0 in $\det(tI-A)$) so, if $|\det H|>1$, we have $\rho(H)>1$. Thus the scheme fails for $\omega<0$ or $\omega>2$.

K317

From K310 we know that the map $\alpha \mapsto \alpha^{-1}$ is continuous on the open subset of $\mathcal{L}(U, U)$ where the inverse is defined. Thus thus composition of the maps

$$\mathbf{x} \mapsto D\mathbf{f}(\mathbf{x}),\ \ \alpha \mapsto \alpha^{-1}$$

is continuous.

K318

(i) Observe that, if $m \geq n$,

$$\left\| \sum_{r=0}^{m} \frac{\alpha^r}{r!} - \sum_{r=0}^{n} \frac{\alpha^r}{r!} \right\| \leq \sum_{r=n+1}^{m} \frac{\|\alpha\|^r}{r!} \to 0$$

(since $\sum_{r=0}^{m}(x^r/r!)$ converges for all x) so, by completeness, $\sum_{r=0}^{n} \alpha^r r!$ converges.

(ii) Observe that

$$\begin{aligned}\left\| \sum_{r=0}^{n} \frac{\alpha^r}{r!} \sum_{r=0}^{n} \frac{\beta^r}{r!} - \sum_{r=0}^{n} \frac{(\alpha+\beta)^r}{r!} \right\| &= \left\| \sum_{0\leq r,s\leq n} \frac{\alpha^r}{r!}\frac{\beta^s}{s!} - \sum_{0\leq r+s\leq n\ r,s\geq 0} \frac{\alpha^r}{r!}\frac{\beta^s}{s!} \right\| \\ &= \left\| \sum_{r+s>n,\ 0\leq r,s\leq n} \frac{\alpha^r}{r!}\frac{\beta^s}{s!} \right\| \\ &\leq \sum_{r+s>n,\ 0\leq r,s\leq n} \frac{\|\alpha\|^r}{r!}\frac{\|\beta\|^s}{s!} \\ &\leq \sum_{2n\geq r+s>n,\ r,s\geq 0} \frac{\|\alpha\|^r}{r!}\frac{\|\beta\|^s}{s!} \\ &= \sum_{k=n+1}^{2n} \frac{1}{k!}(\|\alpha\| + |\beta\|)^k \to 0\end{aligned}$$

[Or use results from Chapter 3.]

(iii) Observe that, if $0 < \|h\| < \|\alpha\|/2$ we have

$$\begin{aligned}\left\| \sum_{r=0}^{n} \frac{h\alpha^r}{r!} - \iota - h\alpha \right| &= \left\| \sum_{r=2}^{n} \frac{h\alpha^r}{r!} \right| \\ &\leq \sum_{r=2}^{n} \frac{|h|\|\alpha\|^r}{r!} \\ &\leq |h|^2\|\alpha\|^2 \sum_{r=0}^{\infty} 2^{-r-1} = |h|^2\|\alpha\|^2.\end{aligned}$$

Thus

$$\exp(h\alpha) = \iota + h\alpha + h^2\theta_\alpha(h)h^2$$

where $\|\theta_\alpha(h)\| \leq \|\alpha\|^2$ for $0 < \|h\| < \|\alpha\|/2$. Similarly

$$\exp(h\beta) = \iota + h\beta + h^2\theta_\beta(h)h^2$$

where $\|\theta_\beta(h)\| \leq \|\beta\|^2$ for $0 < \|h\| < \|\beta\|/2$.

We thus have

$$
\begin{aligned}
&\|h^{-2}(\exp(h\alpha)\exp(h\beta)-\exp(h\beta)\exp(h\alpha))-(\alpha\beta-\beta\alpha)\|\\
&\quad=\|h^{-2}\big((\iota+h\alpha+h^2\theta_\alpha(h)h^2)(\iota+h\beta+h^2\theta_\beta(h)h^2)\\
&\qquad-(\iota+h\beta+h^2\theta_\beta(h)h^2)(\iota+h\alpha+h^2\theta_\alpha(h)h^2)\big)\\
&\qquad-(\alpha\beta-\beta\alpha)\|\\
&\quad\leq 2|h|\|\alpha\|\|\theta_\beta(h)\|+\|\beta\|\|\theta_{\alpha(h)}\|+2h^2\|\theta_{\alpha(h)}\|\|\theta_\beta(h)\|\to 0
\end{aligned}
$$

as $h \to 0$. Thus, if $\alpha\beta \neq \beta\alpha$, we have $\exp(h\alpha)\exp(h\beta) \neq \exp(h\beta)\exp(h\alpha)$ for h small.

If $\exp\alpha\exp\beta = \exp(\alpha+\beta)$ and $\exp\beta\exp\alpha = \exp(\beta+\alpha)$ then $\exp\alpha\exp\beta = \exp\beta\exp\alpha$.

(iv) Observe by considering the various terms in the expansion or by induction that

$$
\begin{aligned}
\|(\alpha+\kappa)^n-\alpha^n\| &\leq \sum_{j=1}^{n}\binom{n}{j}\|\kappa\|^j\|\alpha\|^{n-j}\\
&\leq \sum_{j=1}^{n} n\binom{n-1}{j-1}\|\kappa\|^j\|\alpha\|^{n-j}\\
&= \|\kappa\|\sum_{r=0}^{n-1} n\binom{n-1}{r}\|\kappa\|^r\|\alpha\|^{n-1-r}\\
&= n\|\kappa\|(\|\alpha\|+\|\kappa\|)^{n-1}
\end{aligned}
$$

(We shall do something similar in K319.)

Thus

$$
\begin{aligned}
\left\|\sum_{n=0}^{N}\frac{(\alpha+\kappa)^n}{n!}-\sum_{n=0}^{N}\frac{(\alpha)^n}{n!}\right\| &\leq \sum_{n=1}^{N}\frac{1}{n!}\|(\alpha+\kappa)^n-\alpha^n\|\\
&\leq \|\kappa\|\sum_{n=1}^{N}\frac{n}{n!}(\|\alpha\|+\|\kappa\|)^{n-1}\\
&= \|\kappa\|\sum_{n=0}^{N-1}\frac{1}{n!}(\|\alpha\|+\|\kappa\|)^{n}\\
&= \|\kappa\|\exp(\|\alpha\|+\|\kappa\|)
\end{aligned}
$$

so that

$$
\|\exp(\alpha+\kappa)-\exp(\alpha)\| \leq \|\kappa\|\exp(\|\alpha\|+\|\kappa\|)\to 0
$$

as $\|\kappa\| \to 0$. Thus exp is continuous.

K319

(ii) If $\alpha \in \mathcal{L}(U,U)$ and $\beta \in \mathcal{L}(U,U)$ write

$$\Phi_n(\alpha)(\beta) = \beta\alpha^{n-1} + \alpha\beta\alpha^{n-2} + \alpha^2\beta\alpha^{n-3} + \cdots + \alpha^{n-1}\beta.$$

It is easy to check that $\Phi_n(\alpha) : \mathcal{L}(U,U) \to \mathcal{L}(U,U)$ is a continuous linear function with

$$\|\Phi_n(\alpha)\| \leq n\|\alpha\|^{n-1},$$

The next paragraph (not surprisingly) uses K260 (iii). Observe by considering the various terms in the expansion or by induction that

$$\begin{aligned}
\|(\alpha+\kappa)^n - \alpha^n - \Phi_n(\alpha)(\kappa)\| &\leq \sum_{j=2}^{n} \binom{n}{j} \|\kappa\|^j \|\alpha\|^{n-j} \\
&\leq \sum_{j=2}^{n} n(n-1)\binom{n-2}{j-2} \|\kappa\|^j \|\alpha\|^{n-j} \\
&= n(n-1)\|\kappa\|^2 \sum_{r=0}^{n-2} \binom{n-2}{r} \|\kappa\|^r \|\alpha\|^{n-2-r} \\
&= n(n-1)\|\kappa\|^2 (\|\alpha\| + \|\kappa\|)^{n-2}
\end{aligned}$$

Thus Θ_n is differentiable at α with derivative $\Phi_n(\alpha)$.

(iii) Since $\|\Phi_n(\alpha)\| \leq n\|\alpha\|^{n-1}$ we have

$$\sum_{j=1}^{N} \frac{\|\Phi_n(\alpha)\|}{n!} \leq \sum_{j=1}^{N} \frac{n\|\alpha\|^{n-1}}{n!} \leq e^{\|\alpha\|}$$

so $\sum_{j=1}^{N} \frac{\Phi_n(\alpha)}{n!}$ converges in the space of linear maps from $\mathcal{L}(U,U)$ to $\mathcal{L}(U,U)$ in the appropriate operator norm. Call the limit $\Phi(\alpha)$.

We have

$$\begin{aligned}
\left\| \sum_{n=0}^{N} \frac{(\alpha+\kappa)^n}{n!} - \sum_{n=0}^{N} \frac{(\alpha)^n}{n!} - \sum_{n=1}^{N} \frac{\Phi_n(\alpha)}{n!}\kappa \right\| &\leq \sum_{n=1}^{N} \frac{1}{n!} \|(\alpha+\kappa)^n - \alpha^n - \Phi_n(\alpha)(\kappa)\| \\
&\leq \sum_{n=2}^{N} \frac{1}{n!} n(n-1)\|\kappa\|^2 (\|\alpha\| + \|\kappa\|)^{n-2} \\
&= \|\kappa\|^2 \sum_{n=0}^{N-2} \frac{1}{n!} (\|\alpha\| + \|\kappa\|)^n \\
&\leq \|\kappa\|^2 \exp(\|\alpha\| + \|\kappa\|)
\end{aligned}$$

so

$$\|\exp(\alpha+\kappa) - \exp(\alpha) - \Phi(\alpha)\kappa\| \leq \|\kappa\|^2 \exp(\|\alpha\| + \|\kappa\|)$$

and exp is differentiable at α with derivative $\Phi(\alpha)$.

K320

Use the fact that

$$m^2 \sup_{1\le i,j\le m} |a_{ij}| \ge \|\alpha\| \ge \sup_{1\le i,j\le m} |a_{ij}|$$

when (a_{ij}) is the matrix of α with respect to the matrix. (Or merely use

$$K \sup_{1\le i,j\le m} |a_{ij}| \ge \|\alpha\| \ge K^{-1} \sup_{1\le i,j\le m} |a_{ij}|$$

for some K and this follows from the fact that all norms on a finite dimensional space are Lipschitz equivalent.)

If A is upper triangular then A^r is upper triangular with (j,j) th entry the rth power of a_{jj} the (j,j) th entry of A. Thus $\exp A$ is upper triangular with (j,j) th entry $\exp a_{jj}$. Thus

$$\det(\exp A) = \prod \exp a_{jj} = \exp \sum a_{jj} = \exp \operatorname{Trace} A$$

and, since any α has an upper triangular representation and det and Trace depend on α and not the representation chosen,

$$\det(\exp \alpha) = \exp \operatorname{Trace} \alpha.$$

K321

(ii) Note, for example, that, writing $r(B)$ for the rank of B, we have

$$2 - r(A) \geq r(A) - r(A^2) \geq r(A^2) - r(A^3) \geq r(A^3) - r(A^4).$$

Since $A^4 = 0$ and $A^2 \neq 0$ we have $r(A^2) - r(A^3) \geq 1$ so $2 \geq 3$ which is absurd.

(iii) We have

$$\begin{aligned} a^2 + bc &= 1 \\ b(a+d) &= 0 \\ c(a+d) &= 0 \\ d^2 + bc &= 1 \end{aligned}$$

so either $a+d \neq 0$, in which case $b = c = 0$ and $a = d = \pm 1$ so $A = \pm I$ or $a + d = 0$ and, either $a^2 = 1$ in which case $bc = 0$ and

$$A = \pm \begin{pmatrix} 1 & \beta \\ 0 & -1 \end{pmatrix} \text{ or } A = \pm \begin{pmatrix} 1 & 0 \\ \beta & -1 \end{pmatrix}$$

for some β or $a^2 \neq 1$ and

$$A = \begin{pmatrix} a & b \\ b^{-1}(1-a^2) & -a \end{pmatrix}$$

for some a and b.

Observe that

$$1 = \det I = \det A^2 = (\det A)^2$$

so $\det A = \pm 1$.

By inspection the only such matrices with $\det A = 1$ are $\pm I$ and the only such matrices with diagonal entries are

$$A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

(iv) Observe that $S(I) = I$ and that S has a continuous derivative and that $DS(I)(C) = 2C$ so $DS(I)$ is invertible. The result now follows from the inverse function theorem.

(v) The result fails for $Y = 0$ since

$$\begin{pmatrix} 1 & 0 \\ \beta & -1 \end{pmatrix}$$

is a solution for all β.

We show that $DS(Y)$ is not invertible so the the hypotheses of the inverse function theorem fail. Observe that if

$$H = \begin{pmatrix} u & v \\ w & x \end{pmatrix}$$

then

$$\begin{aligned} DS(Y)H &= YH + YH \\ &= \begin{pmatrix} u & v \\ -w & -x \end{pmatrix} + \begin{pmatrix} u & -v \\ w & -x \end{pmatrix} \\ &= 2\begin{pmatrix} u & 0 \\ 0 & -x \end{pmatrix} \end{aligned}$$

so $DS(Y)$ is not bijective. (Consider, for example, $u = x = 0$, $v = w = 1$. Then $H \neq 0$ but $DS(Y)H = 0$.)

(vi) Inspection of cases (along the lines of (v)) shows that, unless $B = \pm I$ then, if $B^2 = I$, there exist C_n with $C_n \neq B$, $\|C_n - B\| \to 0$ and $C_n^2 = I$. If $B = -I$ then the argument of (iv) (or argument from the result of (iv)) shows that G and H exist.

K322

We have

$$\langle \alpha \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, \alpha^T \mathbf{x} \rangle = -\langle \mathbf{x}, \alpha \mathbf{x} \rangle = -\langle \alpha \mathbf{x}, \mathbf{x} \rangle.$$

So $\langle \alpha \mathbf{x}, \mathbf{x} \rangle = 0$.

If $\alpha^2 \mathbf{x} = \lambda \mathbf{x}$, then

$$\lambda \langle \mathbf{x}, \mathbf{x} \rangle = \langle \lambda \mathbf{x}, \mathbf{x} \rangle = \langle \alpha^2 \mathbf{x}, \mathbf{x} \rangle = -\langle \alpha \mathbf{x}, \alpha \mathbf{x} \rangle \leq 0.$$

Since a real cubic has at least one real root, the characteristic equation $\det(t\iota - \alpha) = 0$ must have at least one real root. Thus α has an eigenvector $\mathbf{e}_3$ of norm 1. By the first paragraph $\alpha \mathbf{e}_3 = \mathbf{0}$. We observe that $\mathbf{e}_3$ is an eigenvector of α^2 and so since α^2 is symmetric we can find $\mathbf{e}_1$ and $\mathbf{e}_2$ such that the $\mathbf{e}_j$ form an orthonormal basis. We know that $\alpha \mathbf{e}_1$ is perpendicular to $\mathbf{e}_1$. Also

$$\langle \alpha \mathbf{e}_1, \mathbf{e}_3 \rangle = \langle \mathbf{e}_1, -\alpha \mathbf{e}_3 \rangle = 0$$

Thus $\alpha \mathbf{e}_1 = \mu \mathbf{e}_2$ for some $\mu \in \mathbb{R}$.

Similarly $\alpha \mathbf{e}_2$ is perpendicular to $\mathbf{e}_2$ and $\mathbf{e}_3$ whilst

$$\langle \alpha \mathbf{e}_2, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, -\alpha \mathbf{e}_1 \rangle = -\mu$$

so $\alpha \mathbf{e}_2 = \mu \mathbf{e}_1$.

With respect to the basis $\mathbf{e}_j$ the linear map $\exp \alpha$ has the matrix representation

$$\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & \mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^n = \begin{pmatrix} \sum_{r=0}^{\infty} \frac{(-1)^r \mu^{2r}}{(2r)!} & \sum_{r=0}^{\infty} \frac{(-1)^r \mu^{2r+1}}{(2r+1)!} & 0 \\ -\sum_{r=0}^{\infty} \frac{(-1)^r \mu^{2r+1}}{(2r+1)!} & \sum_{r=0}^{\infty} \frac{(-1)^r \mu^{2r}}{(2r)!} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos \mu & -\sin \mu & 0 \\ \sin \mu & \cos \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so $\exp \alpha$ is a rotation through μ about $\mathbf{e}_3$.

K323

We have

$$\alpha = (\alpha + \alpha^T)/2 + (\alpha - \alpha^T)/2.$$

The first term is symmetric the second antisymmetric. Note that the decomposition is unique (if $\alpha = \theta + \phi$ with θ symmetric and ϕ antisymmetric then $(\alpha + \alpha^T)/2 = \theta$).

If α is as stated

$$\|(\alpha - \alpha^T)/2\| \leq \|(\alpha - \iota)/2\| + \|(\alpha - \iota)^T/2\| = \|\alpha - \iota\| < \epsilon$$

so $\alpha = \iota + \epsilon\beta + \phi$ with β antisymmetric, $\|\phi\|$ symmetric and $\|\beta\| \leq 1$.

Now

$$\iota = \alpha\alpha^T = (\iota + \epsilon\beta + \phi)(\iota - \epsilon\beta + \phi) = \iota + \epsilon(\beta\phi - \phi\beta) - \epsilon^2\beta^2 + 2\phi + \phi^2$$

so

$$2\phi = -\phi^2 + \epsilon(\phi\beta - \beta\phi) + \epsilon^2\beta^2.$$

But

$$\|(\alpha + \alpha^T)/2\| \leq \|(\alpha - \iota)/2\| + \|(\alpha - \iota)^T/2\| = \|\alpha - \iota\| < \epsilon$$

so

$$2\|\phi\| \leq \|\phi\|^2 + 2\epsilon\|\phi\|\beta\| + \epsilon^2\beta^2 \leq \epsilon^2 + 2\epsilon^2 + \epsilon^2 = 4\epsilon^2$$

so $\|\phi\| \leq 2\epsilon^2$. [We can reuse this estimate to show that $\|\phi\| = \epsilon^2/2 + O(\epsilon^3)$.]

K324

We call matrices U with UU^T orthogonal. If, in addition, $\det U = 1$ we say it is special orthogonal.

(i) Observe that

$$1 = \det UU^T = \det U \det U^T = (\det U)^2.$$

(ii) Observe that

$$\left(\sum_{j=0}^{N} \frac{B^j}{j!}\right)^T = \sum_{j=0}^{N} \frac{(B^T)^j}{j!} = \sum_{j=0}^{N} \frac{(-B)^j}{j!}$$

so, since the map $A \mapsto A^T$ is continuous, $(\exp B)^T = \exp(-B) = (\exp B)^{-1}$. (Recall that, if C and D commute, $\exp(C+D) = \exp C \exp D$.)

(iii) The maps $h \mapsto hB$, $A \mapsto \exp A$ and $C \mapsto \det C$ are continuous so their composition θ is. Note if $B^T = -B$ we have $(hB)^T = -hB$ so $\exp hB$ is orthogonal so $\det(\exp hB) = \pm 1$. Since $h \mapsto \det(\exp hB)$ is continuous, the intermediate value theorem tells us that that the map is constant so $\theta(1) = \theta(0)$ and $\det(\exp B) = 1$.

(iv) By (ii) and (iii) $\exp hB$ is special orthogonal so $A(\exp hB) = (\exp hB)A$. But, as in K318, we have

$$\|h^{-1}(I - hB - \exp hB)\| \to 0$$

so

$$\|A(h^{-1}(I - hB - \exp hB)) - (h^{-1}(I - hB - \exp hB)A\| \to 0$$

so

$$\|AB - BA\| \to 0$$

i.e. $AB = BA$.

(v) If $n \geq 2$, we can have $1 \leq I, J \leq n$ with $I \neq J$. Considering B with matrix given by $b_{IJ} = -b_{JI} = 1$, $b_{ij} = 0$ otherwise, we see that $a_{II} = a_{JJ}$ and $a_{IJ} = -a_{JI}$ for all $I \neq J$.

If $n \geq 3$, we can have $1 \leq I, J \leq n$ with I, J and K distinct Considering B with matrix given by $b_{IJ} = -b_{JI} = 1$, $b_{JK} = -b_{KJ} = 1$, $b_{ij} = 0$ otherwise, we see that

$$a_{JK} = \sum_{r=1}^{n} b_{Ir} a_{rK} = \sum_{s=1}^{n} a_{Is} b_{sK} = a_{IK}.$$

On the other hand, if we consider B with matrix given by $b_{IJ} = -b_{JI} = 1$, $-b_{JK} = b_{KJ} = 1$, $b_{ij} = 0$ otherwise, we see that

$$a_{JK} = \sum_{r=1}^{n} b_{Ir} a_{rK} = \sum_{s=1}^{n} a_{Is} b_{sK} = -a_{IK}.$$

The two last results show that $a_{JK} = 0$ for all $J \neq K$.

Thus $A = \lambda I$. Conversely, if $A = \lambda I$, it is easy to see that A is isotropic.

(vi) When $n = 2$ we have shown that

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Let k be the positive square root of $a^2 + b^2$. Then

$$A = k \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

that is A is product of a dilation and a rotation. But the special orthogonal matrices are precisely the rotations so the isotropic matrices are precisely those which are a product of a dilation and a rotation. The isotropic matrices of determinant 1 are the orthogonal matrices.

(vii) If $n = 1$ the special orthogonal matrices consist of one example (1) so all matrices are isotropic.

(viii) If we omit the restriction $\det U = 1$, we get a stronger condition so the new isotropic matrices are a subset of the old. If $n \geq 3$ or $n = 1$ this makes no difference, by inspection.

If $n = 2$, we observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

so under the new definition only matrices of the form λI can be isotropic (and it is easily checked that they are).

K325

The area of the inscribed n-gon is given by

$$A = \sum_{k=1}^{n} r^2 \sin\frac{\theta_k}{2}\cos\frac{\theta_k}{2} = \frac{r^2}{2}\sum_{k=1}^{n}\sin\theta_k$$

subject to

$$\sum_{k=1}^{n}\theta_k = 2\pi.$$

We form the Lagrangian

$$L = \frac{r^2}{2}\sum_{k=1}^{n}\sin\theta_k - \lambda\sum_{k=1}^{n}\theta_k = 2\pi$$

and observe that, at a stationary point,

$$0 = \frac{\partial L}{\partial \theta_k} = \cos\theta_k - \lambda.$$

Thus $\theta_1 = \theta_2 = \cdots = \theta_n$. Inspection (this is methods question) shows that we have the maximum.

The area of the inscribed n-gon is given by

$$A = \sum_{k=1}^{n} r^2 \tan\frac{\theta_k}{2}$$

subject to

$$\sum_{k=1}^{n}\theta_k = 2\pi.$$

We form the Lagrangian

$$L = \sum_{k=1}^{n} r^2 \tan\frac{\theta_k}{2} - \lambda\sum_{k=1}^{n}\theta_k = 2\pi$$

and observe that, at a stationary point,

$$0 = \frac{\partial L}{\partial \theta_k} = \frac{1}{2\cos^2\frac{\theta_k}{2}} - \lambda$$

Thus $\theta_1 = \theta_2 = \cdots = \theta_n$. Inspection (this is methods question) shows that we have the maximum.

K326

Observe that the set
$$E = \{\mathbf{x} \in \mathbb{R}^n \,:\, x_j \geq 0 \text{ for all } j \text{ and } \sum_{j=1} x_j^p = c\}$$
is closed (observe that the map $\mathbf{x} \mapsto \sum_{j=1} x_j^p$ is continuous) and bounded (if $\mathbf{x} \in E$ then $c^{1/p} \geq x_j \geq 0$.) Thus the continuous function $f : E \to \mathbb{R}$ given by $f(\mathbf{x}) = \sum_{j=1} x_j y_j$ attains a maximum. This maximum does not occur if any of the $x_j = 0$ (for, if $x_J = 0$, then if delta is very small and positive and we take $x'_J = \delta$ and take ϵ so that taking $x'_r = x_r - \epsilon$ if $x_r \neq 0$, $x'_r = 0$ if $x_r = 0$ and $r \neq J$ we have $\mathbf{x}' \in E$, we see that $\epsilon = O(\delta^p)$ and $f(\mathbf{x}') > f(\mathbf{x})$). Thus, if the Lagrange method gives us a unique stationary point with $x_j \neq 0$ for all j, it will indeed be the maximum.

Form the Lagrangian
$$L = \sum_{j=1}^n x_j y_j - \lambda \sum_{j=1}^n x_j^p.$$
At a stationary point,
$$0 = \frac{\partial L}{\partial x_j} = y_j - \lambda p x_j^{p-1},$$
so we have indeed got the maximum. Rewriting the last equations we obtain
$$y_j = \lambda p x_j^{p-1}$$
so
$$x_j^p = k y_j^q$$
for some constant k. Since $\sum_{j=1}^n x_j^p = c$, we have $k = c\left(\sum_{j=1}^n y_j^q\right)^{-1}$ and
$$\begin{aligned}\sum_{j=1}^n x_j y_j &= \left(c^{1/p} \sum_{j=1}^n y_j^{1+q/p}\right)\left(\sum_{j=1}^n y_j^q\right)^{-1/p} \\ &= \left(c^{1/p} \sum_{j=1}^n y_j^q\right)\left(\sum_{j=1}^n y_j^q\right)^{-1/p} \\ &= c^{1/p}\left(\sum_{j=1}^n y_j^q\right)^{1/q}.\end{aligned}$$
Thus
$$c^{1/p}\left(\sum_{j=1}^n y_j^q\right)^{1/q} \geq \sum_{j=1}^n t_j y_j$$

whenever $t_j \geq 0$ and $\sum_{j=1}^n t_j^p = c$ with equality if and only if $t_j^p = ky_j^q$.

Thus

$$\left(\sum_{j=1}^n t_j^p\right)\left(\sum_{j=1}^n y_j^q\right)^{1/q} \geq \sum_{j=1}^n t_j y_j$$

with equality if and only if $t_j^p = ky_j^q$.

We assumed that all the y_j where non-zero but a little reflection shows that if $t_j \geq 0$ and $y_j \geq 0$ (and $\mathbf{t}$, $\mathbf{y} \neq \mathbf{0}$) then

$$\left(\sum_{j=1}^n t_j^p\right)\left(\sum_{j=1}^n y_j^q\right)^{1/q} \geq \sum_{j=1}^n t_j y_j$$

with equality if and only if $t_j^p = ky_j^q$.

Thus if $\mathbf{t}$, $\mathbf{y} \neq \mathbf{0}$ then

$$\left(\sum_{j=1}^n |t_j|^p\right)\left(\sum_{j=1}^n |y_j|^q\right)^{1/q} \geq \sum_{j=1}^n |t_j y_j|$$

with equality if and only if $|t_j|^p = k|y_j|^q$.

K327

(iii) The sum of the squares of the four sides of a parallelogram equals the sum of the squares of the two diagonals.

(iv) Observe that

$$\|\mathbf{x}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\|$$

so

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\|.$$

Now interchange $\mathbf{x}$ and $\mathbf{y}$.

Observe that

$$\begin{aligned} 4\langle \mathbf{x}, \mathbf{y} \rangle &= \|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 \\ &\leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 + (\|\mathbf{x}\| - \|\mathbf{y}\|)^2 \\ &= 4\|\mathbf{x}\|\|\mathbf{y}\| \end{aligned}$$

(v) Observe, for example, that the parallelogram law fails for the uniform norm when we consider $f(x) = \max(1-4x, 0)$ and $g(x) = f(1-x)$.

K328

(ii) We have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|^2+\|\mathbf{u}+\mathbf{v}-\mathbf{w}\|^2 &= \|(\mathbf{u}+\mathbf{v})+\mathbf{w}\|^2+\|(\mathbf{u}+\mathbf{v})-\mathbf{w}\|^2\\ &= 2\|\mathbf{u}+\mathbf{v}\|^2+2\|\mathbf{w}\|^2.\end{aligned}$$

Thus

$$\begin{aligned}&4(\langle(\mathbf{u}+\mathbf{w},\mathbf{v}\rangle+\langle(\mathbf{u}-\mathbf{w},\mathbf{v}\rangle)\\ &\quad=\|\mathbf{u}+\mathbf{w}+\mathbf{v}\|^2-\|\mathbf{u}+\mathbf{w}-\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{w}+\mathbf{v}\|^2-\|\mathbf{u}-\mathbf{w}-\mathbf{v}\|^2\\ &\quad=2\|\mathbf{u}+\mathbf{v}\|^2+2\|\mathbf{w}\|^2-2\|\mathbf{u}-\mathbf{v}\|^2-2\|\mathbf{w}\|^2\\ &\quad=8\langle\mathbf{u},\mathbf{v}\rangle\end{aligned}$$

so (1) holds.

Setting $\mathbf{w}=\mathbf{u}$, we obtain (2). Now set $\mathbf{u}=(\mathbf{x}+\mathbf{y})/2$ and $\mathbf{w}=(\mathbf{x}-\mathbf{y})/2$ to obtain

$$\begin{aligned}\langle\mathbf{x},\mathbf{v}\rangle+\langle\mathbf{y},\mathbf{v}\rangle &= \langle\mathbf{u}+\mathbf{w},\mathbf{v}\rangle+\langle\mathbf{u}-\mathbf{w},\mathbf{v}\rangle\\ &= 2\langle\mathbf{u},\mathbf{v}\rangle\\ &= \langle 2\mathbf{u},\mathbf{v}\rangle\\ &= \langle\mathbf{x}+\mathbf{y},\mathbf{v}\rangle.\end{aligned}$$

(iii) A simple induction now gives $\langle n\mathbf{x},\mathbf{y}\rangle=n\langle\mathbf{x},\mathbf{y}\rangle$ for n a positive integer. The remark that

$$\langle-\mathbf{x},\mathbf{y}\rangle=4^{-1}(\|\mathbf{x}-\mathbf{y}\|^2-\|\mathbf{x}+\mathbf{y}\|^2=-\langle\mathbf{x},\mathbf{y}\rangle$$

now gives the result for all integer n. Now we have

$$n\langle mn^{-1}\mathbf{x},\mathbf{y}\rangle=\langle m\mathbf{x},\mathbf{y}\rangle=m\langle\mathbf{x},\mathbf{y}\rangle$$

so

$$\langle mn^{-1}\mathbf{x},\mathbf{y}\rangle=mn^{-1}\langle\mathbf{x},\mathbf{y}\rangle$$

for all integer m and n with $n\neq 0$.

Now observe that the Cauchy-Schwarz inequality holds (see K327 (iv)) so

$$|\langle\lambda\mathbf{x},\mathbf{y}\rangle-\langle\lambda_k\mathbf{x},\mathbf{y}\rangle|=|\langle(\lambda-\lambda_k)\mathbf{x},\mathbf{y}\rangle|\leq\|(\lambda-\lambda_k)\mathbf{x}\|\|\mathbf{y}\|\to 0$$

as $\lambda_k\to 0$. Thus allowing $\lambda_k\to\lambda$ through rational values of λ_k gives the full result.

(vi) Observe that if a norm obeys the parallelogram law on a dense subset it must obey it on the whole space.

(vii) In parts (ii) and (iii) start by proving the identities for $\Re\langle\mathbf{x},\mathbf{y}\rangle$.

K329*

No comments.

K330

(i) Since $|d(e,t)-d(e,s)| \leq d(s,t)$, the function f_e is continuous. Since $|d(e,t)-d(e_0,t)| \leq d(e_0,e)$ it is bounded.

(ii) We have

$$|f_u(t)-f_v(t)| = |d(u,t)-d(v,t)| \leq d(u,v)$$

and

$$|f_u(u)-f_v(u)| = d(u,v)$$

so $\|f_u - f_v\| = d(u,v)$.

(iii) Since $\mathcal{C}(E)$ is complete, the closed subset Y is complete under the uniform norm. We now observe that θ is a distance preserving mapping from (E,d) to $(Y,\tilde{d})$ such that $\theta(E)$ is dense in Y.

K331

(ii) We have $xn^{-1} = M_E(n^{-1}, x) = M(n^{-1}, x) \to M(0, x)$ as $n \to 0$ by the continuity of M so $M(0, x) = 0$ for all $x \neq 0$. In particular $M(0, 1) = M(0, 2)$, which is incompatible with group laws ($ab = cb$ implies $a = b$).

(iii) As for (ii).

(iv) If x_n is Cauchy for d, then $d(x_n, 1)$ is bounded so there exist $A > 1$ such that $A^{-1} \leq x_n \leq A$. Observe that (by the mean value theorem) there exists a $K > 1$ such that $K^{-1}|x-y| \leq d(x, y) \leq K|x-y|$ for all $|x|$, $|y| \in [A^{-1}, A]$. Take $M(x, y) = xy$.

K332

(i) If $E = \{0\}$ then we are done. If not, then since $x \in E$ implies $-x \in E$, we know that $\{x \in E : x > 0\}$ is non-empty.

Let $\alpha = \inf\{x \in E : x > 0\}$. If $\alpha = 0$ then we can find $x_n \in E$ with $0 < x_n < 1/n$. We have

$$E \supseteq \bigcup_{n=1}^{\infty} \{mx_n : m \in \mathbb{Z}\}$$

so, since E is closed, $E = \mathbb{R}$.

If $\alpha > 0$, then since E is closed $\alpha \in E$. Given $e \in E$ we can find $k \in \mathbb{Z}$ such that

$$\alpha > e - k\alpha \geq 0.$$

Since $e - k\alpha \in E$ we have $e - k\alpha = 0$ so $E \subseteq \alpha\mathbb{Z}$ so $E = \alpha\mathbb{Z}$.

(ii) A similar argument shows that either $E = S^1$ or $E = \{\exp 2\pi in/N : 0 \leq n \leq N - 1\}$ for some positive integer N.

(iii) If E is a subset of a one dimensional subspace of $\mathbb{R}^2$ then (i) tells that either

$$E = \{x\mathbf{u} : x \in \mathbb{R}\} \text{ or } E = \{n\mathbf{u} : n \in \mathbb{Z}\}.$$

If not either

(A) $\mathbf{x}_n \neq \mathbf{0}$ with $\|\mathbf{x}_n\| \to 0$, or (B) not.

In case (A) either

$(A)'$ There exists an N such that all the $\mathbf{x}_n$ lie in a one dimensional subspace or $(A)''$ not.

In case $(A)''$, E will be dense in $\mathbb{R}^2$ and so $E = \mathbb{R}^2$.

In case $(A)'$ the argument of (i) tells us that there is a subspace

$$U = \{x\mathbf{u} : x \in \mathbb{R}\} \subseteq E.$$

We are considering the case $U \neq E$. Consider

$$\alpha = \inf\{\|\mathbf{x}\| : \mathbf{x} \in E \setminus U\}.$$

If $\alpha = 0$, E will be dense in $\mathbb{R}^2$ and so $E = \mathbb{R}^2$. If $\alpha \neq 0$ then since U is a subgroup of E, $\|\mathbf{y} - \mathbf{z}\| \geq \alpha$ whenever $\mathbf{y} \in E \setminus U$ and $\mathbf{z} \in U$. Thus, since E is closed, we can find $\mathbf{v} \in E \setminus U$ with $\|\mathbf{v}\| = \alpha$. Suppose $\mathbf{e} \in E$. By simple geometry we can find $x \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $\|\mathbf{e} - x\mathbf{u} - k\mathbf{v}\| \leq \|\mathbf{v}\|/2$ so $\mathbf{e} - x\mathbf{u} - k\mathbf{v} \in U$ so $\mathbf{e} - k\mathbf{v} \in U$. Thus

$$E = \{x\mathbf{u} + k\mathbf{v} : x \in \mathbb{R},\ k \in \mathbb{Z}\}.$$

In case (B) let $\mathbf{u}$ be an element of smallest norm $E \setminus \{\mathbf{0}\}$, write

$$U = \{l\mathbf{u} \,:\, l \in \mathbb{Z}\}$$

and let $\mathbf{v}$ an element of smallest norm in $E \setminus U$. If $\mathbf{e} \in E$ then by simple geometry we can find integers l and k such that $\mathbf{e} - l\mathbf{u} - k\mathbf{v}$ lies in the parallelogram with vertices $(\pm\mathbf{u} \pm \mathbf{v})/2$ and so is $\mathbf{0}$. Thus

$$E = \{l\mathbf{u} + k\mathbf{v} \,:\, l \in \mathbb{Z},\ k \in \mathbb{Z}\}.$$

K333

Second paragraph. Let $X = [-1, 1]$ with the usual metric d. Let $E = X \setminus \{0\}$ and let $Y = E$ with ρ the restriction of d to Y. Let $f(x) = x$ so f is certainly uniformly continuous. Suppose $\tilde{f} : (X, d) \to (Y, \rho)$ is continuous and $\tilde{f}(x) = f(x)$ for all $x \in E$. If $\tilde{f}(x) = y$ then

$$|x - y| = \rho(\tilde{f}(x), \tilde{f}(0)) \to 0$$

as $|x| = d(x, 0) \to 0$ $(x \neq 0)$ which is absurd.

We have the following theorem. If (x, d) is a metric space and E a dense subset of X then if (Y, ρ) is a *complete* metric space any uniformly continuous function $f : E \to Y$ can be extended to a continuous (indeed uniformly continuous) function $\tilde{f} : X \to Y$.

The proof follows a standard pattern. Suppose $x \in X$. Then we can find $x_n \in E$ with $d(x_n, x) \to 0$. Since f is uniformly continuous on E, given any $\epsilon > 0$ we can find a $\delta(\epsilon) > 0$ such that $d(u, v) < \delta$ implies $\rho(f(u), f(v)) < \epsilon$. Choose N such that $d(x_n, x) < \delta/2$ for $n \geq N$. Then $d(x_n, x_m) < \delta$ and $\rho(f(x_n), f(x_m)) < \epsilon$ for n, $m \geq N$. Thus $f(x_n)$ is Cauchy and so converges in (Y, ρ) to limit l_x, say. A similar, but simpler, argument shows that if $y_n \in E$ with $d(y_n, x) \to 0$ then $\rho(f(y_n), l_x) \to 0$. Thus we may define $\tilde{f}(x) = l_x$ without ambiguity. Taking $x_n = x$, we see that $\tilde{f}(x) = f(x)$ for all $x \in E$.

Now suppose $\epsilon > 0$, u, $v \in X$ and $d(u, v) < \delta(\epsilon)/3$. We can find u_n, $v_n \in E$ with $d(u_n, u)$, $d(v_n, v) < \delta(\epsilon)/3$ and $d(u_n, u)$, $d(v_n, v) \to 0$. By the triangle inequality $d(u_n, v_n) < \delta(\epsilon)$ so $\rho(f(u_n), f(v_n)) < \epsilon$ and $\rho(\tilde{f}(u), \tilde{f}(v)) \leq \epsilon$. Thus $\tilde{f}$ is uniformly continuous.

K334

Observe that
$$\frac{f(x)-f(y)}{x-y} \to 0$$
as $y \to x$. Thus f is everywhere differentiable with derivative zero and so is constant.

If we replace $\mathbb{R}$ by $\mathbb{Q}$, then f is uniformly continuous on $\mathbb{Q}$ and so can be extended to a continuous function $\tilde{f}$ on $\mathbb{R}$. If $x, y \in \mathbb{R}$ and $x \neq y$ choose $x_n, y_n \in \mathbb{Q}$ with $x_n \to x$ and $y_n \to y$. Since
$$|f(x_n) - f(y_n)| \leq (x_n - y_n)^2$$
we have
$$|\tilde{f}(x_n) - \tilde{f}(y_n)| \leq (x_n - y_n)^2$$
so, allowing $n \to \infty$,
$$|\tilde{f}(x) - \tilde{f}(y)| \leq (x - y)^2$$
whence $\tilde{f}$ is constant and so f is.

The function in Example 1.3 satisfies the condition:-

Given $x \in \mathbb{Q}$ there exists a $\delta(x) > 0$ with
$$|f(x) - f(y)| \leq (x - y)^2$$
for all $x, \in \mathbb{Q}$ with $|x - y| < \delta(x)$ but the $\delta(x)$ is not uniform.

K335

This is the statement that a complete metric space is totally bounded (see e.g. Section 11.2).

If F_n is finite set such that $\bigcup_{x\in F_n} B(x,1/n) = X$, then $E = \bigcup_{n=1}^{\infty} F_n$ is a countable dense subset.

If we give $\mathbb{Z}$ the usual metric, then, since the sequence $x_n = n$ has no convergent subsequence, ($|x_n - x_m| \geq 1$ for $m \neq n$), $\mathbb{Z}$ does not have the Bolzano–Weierstrass property. However $\mathbb{Z}$ is a countable dense subset of itself and since any Cauchy sequence is eventually constant it is complete.

The space $\mathbb{Q}$ with the usual metric is not complete but is a countable dense subset of itself.

If we give $\mathbb{R}$ the discrete metric ($d(x,y) = 1$ if $x \neq y$) then $B(x,1/2) = \{x\}$ so the only dense subset of $\mathbb{R}$ is $\mathbb{R}$ itself which is uncountable. Since any Cauchy sequence is eventually constant $\mathbb{R}$ with the discrete metric is complete.

K336

(i) and (ii). Let x_n be a sequence of points forming a dense subset. If U is open, consider x_n. If $x_n \notin U$, we do nothing, If $x_n \in U$, then put $n \in \Gamma$. Since U is open we can find a $\delta'_n > 0$ such that $d(x_n, y) > \delta'_n$ for all $y \notin U$. If $U = X$ set $\delta_n = 1$. Otherwise $\inf_{y \notin U} d(x_n, y)$ is a well defined strictly positive number. Set $\delta_n = \inf_{y \notin U} d(x_n, y)/2$.

We claim that $\bigcup_{n \in \Gamma} B(x_n, \delta_n) = U$. Observe that if $n \in \Gamma$ then $B(x_n, \delta_n) \subseteq U$ so $\bigcup_{n \in \Gamma} B(x_n, \delta_n) \subseteq U$. Now suppose $x \in U$. Since U is open we can find a $\delta > 0$ with $1 > \delta$ such that $B(x, \delta) \subseteq U$. Since the sequence x_n is dense we can find an N such that $d(x_N, x) < \delta/4$. Automatically $\inf_{y \notin U} d(x_N, y) > 3\delta/4$ so $\delta_N \geq 3\delta/8$ and $x \in B(x_N, \delta_N)$. Thus $\bigcup_{n \in \Gamma} B(x_n, \delta_n) = U$.

(iii) The space $\mathbb{R}$ with the discrete metric $d(x, y) = 1$ if $x \neq y$ is complete. Any open ball of the form $B(x, r)$ with $r > 1$ is $\mathbb{R}$ itself. Any open ball of the form $B(x, r)$ with $r < 1$ is $\{x\}$. Thus the countable unions of open balls are countable subsets of $\mathbb{R}$ and $\mathbb{R}$ itself. But every set is open so any uncountable set (e.g. $\{x : x > 0\}$) which is not the whole space is a counterexample.

(iv) Observe that $B(x, r) = \bigcup_{n=1}^{\infty} \bar{B}(x, (1 - 2^{-n})r)$. Since the countable union of countable sets is countable, part (ii) shows that every open set U is the countable union of closed balls L_1, L_2, ...say. Set $K_j = \bigcup_{r=1}^{j} L_r$.

K337

(i) If $x \in U$, we can find a $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$. Thus $x \sim x$.

If $x \sim y$, then x, $y \in (a,b) \subseteq U$ so $y \sim x$

If $x \sim y$ and $y \sim z$, then x, $y \in (a,b) \subseteq U$ and y, $z \in (c,d) \subseteq U$. We then have $(a,b)\cup(c,d)$ an open interval (since both (a,b) and (c,d) contain y) lying within U so $x \sim z$.

(ii) Since $x \in [x]$, $[x]$ is non-empty. If $[x]$ is bounded above look at $\alpha = \sup[x]$. If $\alpha \in U$ then we can find $\delta > 0$ such that $(\alpha - 2\delta, \alpha + 2\delta) \in U$. We can find $\gamma \in [x]$ such that $\gamma \geq \alpha - \delta$ so $x \sim \gamma$, $\gamma \sim \alpha - \delta$ and $\alpha - \delta \sim \alpha + \delta$. Thus $x \sim \alpha + \delta$ contradicting the definition of α. By reductio ad absurdum,

$$\{y \in [x] \,:\, y \geq x\} \subseteq [x, \alpha).$$

But we can find $\alpha_n \geq \alpha - 1/n$ with $\alpha_n \sim x$, so we can find $(a_n, c_n) \subset U$ with a, $\alpha_n \in (a_n, c_n)$. Thus

$$\{y \in [x] \,:\, y \geq x\} \supseteq \bigcup_{n=1}^{\infty} [x, c_n) \supseteq \bigcup_{n=1}^{\infty} [x, \alpha_n) = [x, \alpha).$$

Thus

$$\{y \in [x] \,:\, y \geq x\} = [x, \alpha).$$

If $[x]$ is unbounded above we can find $\alpha_n \geq n$ with $\alpha_n \sim x$ so we can find $(a_n, c_n) \subset U$ with x, $\alpha_n \in (a_n, c_n)$. Thus

$$\{y \in [x] \,:\, y \geq x\} \supseteq \bigcup_{n=1}^{\infty} [x, c_n) \supseteq \bigcup_{n=1}^{\infty} [x, \infty)$$

so

$$\{y \in [x] \,:\, y \geq x\} = [x, \infty).$$

(iii) Take $\mathcal{C}$ to be the set of equivalence classes.

(v) Every open interval contains a rational. Thus each element of $\mathcal{C}$ contains a rational which belongs to no other element so $\mathcal{C}$ is countable. (Or say that the map $f\mathbf{Q} \to \mathcal{C}$ given by $f : (q) = [q]$ is surjective.)

If D is an open disc $\{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| = r\}$ and $\|\mathbf{x} - \mathbf{x}_0\| = r$ then any open disc containing $\mathbf{y}$ intersects D but $\mathbf{y} \notin D$.

K338

(i) Observe that

$$\sum_{j=1}^{n}|(f+g)(x_j)-(f+g)(x_{j-1})| \leq \sum_{j=1}^{n}|f(x_j)-f(x_{j-1})|+\sum_{j=1}^{n}|g(x_j)-g(x_{j-1})|$$

so f, $g \in BV$ implies $f+g \in BV$ and $V(f+g) \leq V(f)+V(g)$.

Observe also that if $V(f)=0$ then

$$0 \leq |f(t)-f(0)| \leq |f(1)-f(t)|+|f(t)-f(0)| \leq V(f)=0$$

so $f(t)=f(0)$ for all t.

The proof that BV is complete follows a standard form of argument. Suppose f_n is Cauchy in BV. Arguing as in the previous paragraph

$$|f_p(t)-f_q(t)| \leq |f_p(0)-f_q(0)|+V(f_p-V(f_q)=\|f_p-f_q\|_{BV}$$

so $f_n(t)$ is Cauchy in $\mathbb{R}$ for each t. Thus $f_n \to f(t)$ for some $f(t)$. We now show that $f \in BV$. Since any Cauchy sequence is bounded we can find a K with $K \geq \|f_n\|_{BV}$ for all n. Now suppose $0=x_0 \leq x_1 \leq \dots \leq x_N=1$, Then

$$\begin{aligned}
&\sum_{j=1}^{N}|f(x_j)-f(x_{j-1})| \\
&\quad \leq \sum_{j=1}^{N}|(f-f_n)(x_j)-(f-f_n)(x_{j-1})|+\sum_{j=1}^{N}|f_n(x_j)-f_n(x_{j-1})| \\
&\quad \leq \sum_{j=1}^{N}|(f-f_n)(x_j)-(f-f_n)(x_{j-1})|+K \\
&\quad \to K
\end{aligned}$$

as $n \to \infty$. Thus $f \in BV$. Finally we use an 'irrelevant m argument' to observe that

$$\begin{aligned}
&|(f - f_n)(0)| + \sum_{j=1}^{N} |(f - f_n)(x_j) - (f - f_n)(x_{j-1})| \\
&\le |(f - f_m)(0)| + \sum_{j=1}^{N} |(f - f_m)(x_j) - (f - f_m)(x_{j-1})| + |(f_m - f_n)(0)| \\
&\quad + \sum_{j=1}^{N} |(f_m - f_n)(x_j) - (f_m - f_n)(x_{j-1})| \\
&\le |(f - f_m)(0)| + \sum_{j=1}^{N} |(f - f_m)(x_j) - (f - f_m)(x_{j-1})| + \|f_n - f_m\|_{BV} \\
&\le |(f - f_m)(0)| + \sum_{j=1}^{N} |(f - f_m)(x_j) - (f - f_m)(x_{j-1})| + \sup_{p,q \ge n} \|f_p - f_q\|_{BV} \\
&\to \sup_{p,q \ge n} \|f_p - f_q\|_{BV}
\end{aligned}$$

as $m \to \infty$. Thus $\|f_n - f\|_{BV} \to 0$ as $n \to \infty$.

(ii) If $f(t) = 0$ for $0 \le t \le 1/2$, $f(t) = 1$ for $1/2 < t \le 1$ then $f \in BV$ but $f \notin C$. If $g(t) = t \cos \pi t$ then $g \in C$ but, examining $\sum_{j=1}^{n} |g(1/j) - g(1/(j+1))|$, we see that $g \notin BV$.

$C \cap BV$ is the intersection of two vector spaces and so a vector subspace of BV. If $f \in BV \cap C$ then arguments like those in (i) show that $\|f\|_\infty \le \|f\|_{BV}$. Thus if $f_n \in BV \cap C$ and $f \in BV$ and $\|f_n - f\|_{BV} \to 0$ we know that f_n is Cauchy in $(BV, \| \ \|_{BV})$ and so f_n is Cauchy in $(C, \| \ \|_\infty)$. Thus f_n converges uniformly to some $g \in C$. We now observe that

$$|f_n(t) - f(t)| \le \|f - f_n\|_{BV} \to 0$$

and

$$|f_n(t) - g(t)| \le \|g - f_n\|_\infty \to 0$$

and so $f(t) = g(t)$ for each $t \in [0, 1]$. Thus $f = g$ and $f \in BV \cap C$. Thus $BV \cap C$ is closed in $(BV, \| \ \|_{BV})$.

The mean value inequality shows that C^1 is a subspace of BV. We observe that if g is continuous and $G(t) = \int_{1/2}^{t} g(x)\, dx$ then

$$|G(x_j) - G(x_{j-1})| \le \sup_{t \in [x_{j-1}, x_j]} |g(t)||x_j - x_{j-1}|.$$

Thus if we take

$$g_n(x) = \begin{cases} -1 & \text{if } 0 \le x \le 2^{-1} - 2^{-1}n^{-1}, \\ 4n(x - 1/2) & \text{if } 2^{-1} - 2^{-1}n^{-1} < x < 2^{-1} + 2^{-1}n^{-1}, \\ 1 & \text{if } 2^{-1} + 2^{-1}n^{-1} \le x \le 1, \end{cases}$$

and set $G_n = \int_{1/2}^t g_n(x)\,dx$, $G(t) = |t - 2^{-1}|$ we have $G_n \in C^1$, $G \in BV$ and $\|G - G_n\|_{BV} \to 0$ as $n \to \infty$ but $G \notin C^1$.

(iii) Since $g \in C^1$ there exists an M such that $|g'(t)| \le M$ for all t. Let $\epsilon > 0$. By the definition of $V(g)$. $0 = x_0 \le x_1 \le \cdots \le x_n = 1$ such that

$$\sum_{j=1}^{n} |g(x_j) - g(x_{j-1})| \ge V(g) - \epsilon.$$

Let m be a strictly positive integer. By the staircase property of the function f we can find a finite collection $\mathcal{I}$ of disjoint closed sets of total length $1 - (2/3)^m$ such that f is constant on each $I \in \mathcal{I}$. We can choose $0 = y_0 \le y_1 \le \cdots \le y_N = 1$ so that

$$x_j \in \{y_0,\, y_1,\, \ldots,\, y_N\}$$

for each $0 \le j \le n$ and

either $[y_{r-1}, y_r] \subset I$ for some $I \in \mathcal{I}$ **or** $(y_{r-1}, y_r) \cap I = \varnothing$ for all $I \in \mathcal{I}$

whenever $1 \le r \le N$. We observe that

$$\sum_{r=1}^{N} |g(y_r) - g(y_{r-1})| \ge \sum_{j=1}^{n} |g(x_j) - g(x_{j-1})| \ge V(g) - \epsilon.$$

Let us say that $r \in A$ if $(y_{r-1}, y_r) \cap I = \varnothing$ for all $I \in \mathcal{I}$. We have

$$\begin{aligned}
\sum_{r\in A} |(f+g)(y_r) - (f+g)(y_{r-1})| &\ge \sum_{r\in A} |f(y_r) - f(y_{r-1})| - \sum_{r\in A} |g(y_r) - g(y_{r-1})| \\
&= \sum_{r\in A} (f(y_r) - f(y_{r-1})) - \sum_{r\in A} |g(y_r) - g(y_{r-1})| \\
&= 1 - \sum_{r\notin A} (f(y_r) - f(y_{r-1})) - \sum_{r\in A} |g(y_r) - g(y_{r-1})| \\
&\ge 1 - \sum_{r\in A} |g(y_r) - g(y_{r-1})| \\
&\ge 1 - \sum_{r\in A} M|y_r - y_{r-1}| \\
&\ge 1 - M(2/3)^m = V(f) - M(2/3)^m
\end{aligned}$$

and

$$
\begin{aligned}
\sum_{r\notin A} |(f+g)(y_r) - (f+g)(y_{r-1})| &\geq \sum_{r\notin A} |g(y_r) - g(y_{r-1})| - \sum_{r\notin A} |f(y_r) - f(y_{r-1})| \\
&\geq \sum_{r} |g(y_r) - g(y_{r-1})| - \sum_{r\in A} |g(y_r) - g(y_{r-1})| - \sum_{r\notin A} (f(y_r) - f(y_{r-1})) \\
&\geq V(g) - \epsilon - M(2/3)^m.
\end{aligned}
$$

Thus

$$
\begin{aligned}
V(f+g) &\geq \sum_{r\in A} |(f+g)(y_r) - (f+g)(y_{r-1})| + \sum_{r\notin A} |(f+g)(y_r) - (f+g)(y_{r-1})| \\
&\geq V(f) + V(g) - \epsilon - 2M(2/3)^m.
\end{aligned}
$$

Since ϵ and m can be chosen freely, $V(f+g) \geq V(f) + V(g)$ and so $V(f+g) = V(f) + V(g)$.

In particular, if $g \in C^1$ we have

$$\|g - f\|_{BV} \geq V(f-g) \geq V(f) + V(-g) \geq V(f) = 1$$

so C^1 is not dense in BV.

K339

(iii) $f : \mathbb{R} \to (-1, 1)$ given by $f(x) = (2/\pi)\tan^{-1} x$ will do.

(iv) $f : I \to J$ given by $f(x) = \exp(i\pi x)$ will do. Observe that $(-1, 1)$ is a dense subset of $[-1, 1]$ which is complete. Observe that J is a dense subset of $\{z \in \mathbb{C} : |z| = 1\}$ which is complete.

The Cauchy sequences $x_n = 1 - 1/n$ and $y_n = -1 + 1/n$ have no limits in I. But $|x_n - y_n| \nrightarrow 0$ so they can not have the same limit in the completion.

K340

(i) Suppose, if possible, that f^{-1} is not uniformly continuous. Then we can find an $\epsilon > 0$ and $u_n,\ v_n \in Y$ such that $\rho(u_n, v_n) \to 0$ but $d(f^{-1}(u_n), f^{-1}(v_n)) > \epsilon$. By the Bolzano–Weierstrass property of X, applied twice to obtain a subsequence and then a subsequence of that subsequence, we can find $n(j) \to 0$ and $\alpha,\ \beta \in X$ such that

$$d(f^{-1}(u_{n(j)}), \alpha) \to 0 \text{ and } d(f^{-1}(v_{n(j)}), \beta) \to 0.$$

By the continuity of f, we have $\rho(u_{n(j)}, f(\alpha)) \to 0$ and $\rho(v_{n(j)}, f(\beta)) \to 0$ so (since $\rho(u_n, v_n) \to 0$) we have $f(\alpha) = f(\beta)$. Since f is bijective, $\alpha = \beta$ so

$$\epsilon < d(f^{-1}(u_{n(j)}), f^{-1}(v_{n(j)})) \leq d(f^{-1}(u_{n(j)}), \alpha) + d(f^{-1}(v_{n(j)}), \alpha) \to 0$$

which is absurd.

Suppose $y_n \in Y$. Then $f^{-1}(y_n) \in X$ and by the Bolzano–Weierstrass property of X we can find $x \in X$ and $n(j) \to \infty$ such that $d(f^{-1}(y_{n(j)}), x) \to 0$ and so, by the continuity of f,

$$\rho(y_{n(j)}, f(x)) = \rho\big(f(f^{-1}(y_{n(j)})), f(x)\big) \to 0$$

as $j \to \infty$. Thus Y has the Bolzano–Weierstrass property.

(ii) The fact that f^{-1} is continuous.

(iii) If $f(x) = x^{1/3}$, the mean value theorem shows that $|f(x) - f(y)| \leq |x - y|/3$ if $|x|,\ |y| \geq 1$. Since f is continuous, it is uniformly continuous on $[-2, 2]$ so f is uniformly continuous on $\mathbb{R}$. However $f^{-1}(x) = x^3$ so $f(x + \delta) - f(x) \geq 3x\delta^2 \to \infty$ as $x \to \infty$ for all $\delta > 0$, so f^{-1} is not uniformly continuous.

(iv) Observe that $\rho(x, y) < 1/2$ implies $x = y$ implies $d(f(x), f(y)) = 0$ but that $d(1/n, 0) \to 0$ and $\rho(f^{-1}(1/n), f^{-1}(0)) = \rho(1/n, 0) = 1 \not\to 0$.

K341

Since (X, d) has the Bolzano–Weierstrass property, there exists a K such that $d(x, y) \leq K$ for all x, $y \in X$ (direct proof or use total boundedness), thus, by the Weierstrass M-test, $\sum_{j=1}^{\infty}(2^{-j}d(x, x_j))^2$ converges and $\mathbf{f}(x) \in l^2$.

Observe that

$$\begin{aligned}\|\mathbf{f}(x) - \mathbf{f}(y)\|_2^2 &= \sum_{j=1}^{\infty}(2^{-j}(d(x, x_j) - d(y, x_j))^2 \\ &\leq \sum_{j=1}^{\infty}(2^{-j}d(x, y))^2 \leq d(x, y)^2\end{aligned}$$

so $\mathbf{f}$ is continuous.

If $x \neq y$ then $d(x, y) > 0$. Set $4k = d(x, y)$. We can find an N with $d(x, x_N) < k$ and so with $d(y, x_N) > 3k$ whence $d(x, x_N) \neq d(y, x_N)$ and $\mathbf{f}(x) \neq \mathbf{f}(y)$. Thus $\mathbf{f}$ is continuous. It follows that $\mathbf{f} : X \to \mathbf{f}(X)$ is a bijective continuous function and, by K339 (i) is a homeomorphism.

K342

(i) We can have (X, d) complete but (Y, ρ) not. Let $X = \mathbb{R}$ with the usual metric d, $Y = (-\pi/2, \pi/2)$ with the usual metric ρ, and $f(x) = \tan^{-1}(x)$. Using the mean value inequality $|f(x) - f(y)| \leq |x - y|$.

We can have (Y, ρ) complete but (X, d) not. Let $X = (-4, 4)$ with the usual metric d and $y = [-1, 1]$ with the usual metric ρ. Take $f(x) = \sin x$ and observe that by the mean value inequality $|f(u) - f(v)| \leq |u - v|$.

(ii) If (X, d) is complete (Y, ρ) must be. Let y_n be Cauchy in Y. Since f is surjective we can find $x_n \in X$ with $f(x_n) = y_n$. Since

$$d(x_n, x_m) \leq K^{-1}\rho(f(x_n), f(x_m) = \rho(y_n, y_m),$$

the x_n are Cauchy so we can find $x \in X$ with $d(x_n, x) \to 0$. By continuity, $\rho(y_n, f(x)) = \rho(f(x_n0, f(x)) \to 0$ as $n \to \infty$.

We can have (Y, ρ) complete but $(X, d))$ not. Let $Y = \mathbb{R}$ with the usual metric ρ, and $X = (-\pi/2, \pi/2)$ with the usual metric d $f(x) = \tan(x)$. Using the mean value theorem, $|f(x) - f(y)| \geq |x - y|$.

K343

Observe that

$$\alpha \triangle \beta \cup \beta \triangle \gamma \supseteq \alpha \triangle \gamma$$

so $d(\alpha, \beta) + d(\beta, \gamma) \geq d(\alpha, \gamma)$.

Suppose $\alpha_n = (a_n, b_n)$ forms a Cauchy sequence with respect to d. If there exists a $c > 0$ such that $b_n - a_n > c$ for all n, then we know that there exists an N such that $d(\alpha_n, \alpha_m) < c/2$ for n, $m \geq N$. Thus, provided n, $m \geq N$, we know that $\alpha_n \cap \alpha_m \neq \emptyset$ so $|a_n - a_m| \leq d(\alpha_n, \alpha_m)$. It follows that a_n is Cauchy in the standard Euclidean metric, so $a_n \to a$ for some a. Similarly $b_n \to b$ for some b. Since $b_n - a_n > c$ we have $b - a \geq c > 0$ and, writing $\alpha = (a, b)$, we have $d(\alpha_n, \alpha) \to 0$.

If there does not exist a $c > 0$ such that $b_n - a_n > c$ for all n, then we can find $n(j) \to \infty$ such that $b_{n(j)} - a_{n(j)} \to 0$. If θ is any open interval of length $|\theta|$, $d(\theta, \alpha_{n(j)}) \to |\theta|$. Thus (X, d) is not complete.

However if we take $X^* = X \cup \{\infty\}$ and define d^* by $d^*(\alpha, \beta) = d(\alpha, \beta)$ for α, $\beta \in X$, $d^*(\theta, \infty) = d^*(\infty, \theta) = |\theta|$ if $\theta \in X$ and $d^*(\infty, \infty) = 0$ then d^* is a metric and a Cauchy sequence which consists from some point on of intervals of length greater than some fixed c converges by the arguments above and (since a Cauchy sequence with a convergent subsequence converges) one which does not converges to ∞.

[The reader may prefer to replace ∞ by $\emptyset$.]

K344

(ii) Observe that the open unit ball E centre 1 consists of the odd numbers. If $r \in E$ and $s \notin E$ then $d(r,s) = 1$ so E is closed.

(iii) $d(n+2^k, n) \leq 2^{-k} \to 0$ so the complement of $\{n\}$ is not closed and $\{n\}$ is not open.

(iv) $d(r,s) \leq 2^{-k}$ implies $r \equiv s \mod 2^k$ so $r^2 \equiv s^2 \mod 2^k$ and $d(r^2, s^2) \leq 2^{-k}$. Thus f is continuous.

(v) $d(n+2^k, n) \to 0$ as $k \to \infty$ but $d(2^{n+2^k}, 2^n) = 2^{-n} \nrightarrow 0$. Thus g is nowhere continuous.

(vi) Let $x_n = \sum_{r=0}^{n} 2^{2r}$. If $0 \leq k \leq 2^m$ then (think of the binary expansion of $x - k$) $d(x_n, y) \geq 2^{-m-2}$ for all $n \geq k+2$ so the sequence x_n can not converge. However $d(x_n, x_m) \leq 2^{-2r}$ for all n, $m \geq r$ so the sequence is Cauchy.

K345

Recall that we can find $E : \mathbb{R} \to \mathbb{R}$ infinitely differentiable with $E(x) = 0$ for $x < 0$ and E strictly increasing on $[0, \infty)$. Thus, if we consider $F(x, y) = E(x + 1)E(-x + 1)E(y + 1)E(-y + 1)$, we have F infinitely differentiable, $F(0, 0) \neq 0$ and $F(x, y) = 0$ if $|x| \geq 1$ and/or $|y| \geq 1$.

Set $(x_n, y_n) = (n^{-2}, n^{-4})$ and $\delta_n = (10n)^{-8}$. Note that, if we write

$$A_n = [x_n - 2\delta_n, x_n + 2\delta_n] \times [y_n - 2\delta_n, y_n + 2\delta_n,]$$

no line through the origin can intersect both A_n and A_m for $n \neq m$.

Set

$$G(x, y) = \sum_{m=1}^{\infty} n^{-1} F(4\delta_m^{-1}(x - x_m), 4\delta_m^{-1}(y - y_m)).$$

For each $(u, v) \neq 0$ we can find a $\delta > 0$ and an n such that

$$F(4\delta_m^{-1}(x - x_m), 4\delta_m^{-1}(y - y_m)) = 0$$

for all $m \neq n$ and all $|u - x|$, $|v - y| < \delta$. Thus G is well defined everywhere (observe that $F(4\delta_m^{-1}(-x_m), 4\delta_m^{-1}(-y_m)) = 0$) and infinitely differentiable except at $(0, 0)$.

Since $\delta_n \leq (10n)^{-8}$ we have $G(x, y) \leq n^{-1}$ for $|x|$, $|y| \leq (10n)^{-1}$ so G is continuous at $(0, 0)$. Since no line through the origin can intersect both A_n and A_m for $n \neq m$, $G(\lambda t, \mu t) = 0$ for sufficiently small $|t|$, so G has directional derivative zero at the origin.

However,

$$\frac{|G(x_n, y_n) - G(0, 0)|}{\|(x_n, y_n)\|} = \frac{n^{-1}E(1)^4}{\|(x_n, y_n)\|} \geq nE(1)^4/2 \to \infty,$$

so G is not differentiable at the origin.