Congruences between modular forms over imaginary quadratic fields

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July 31, 2007

Outline:

Apply
congruences involving Eisenstein cohomology classes over an imaginary quadratic field F to

(A) construct elts in Selmer groups of Hecke characters of F,

(B) prove modularity of residually reducible $\mathrm{Gal}(\overline{F}/F)$-representations (joint work in progress with K. Klosin).

Modular Forms over F:

- real analytic functions on $\mathbf{H}_3 = \mathbf{C} \times \mathbf{R}_{>0}$

- automorphic representations of $\mathsf{GL}_2(\mathbf{A}_F)$

- cohomology classes in $H^i(\Gamma\backslash\mathbf{H}_3, R)$, where $\Gamma \subset \mathsf{SL}_2(\mathcal{O}_F)$ and R is an $\mathcal{O}_F$-algebra.

Setup:

- $p > 3$ prime, $p \nmid d_F, p \nmid$ #Cl(F), fix $F \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$, Σ finite set of places; $\mathcal{O}$ ring of integers in (suff. large) finite extension K of $\mathbf{Q}_p$, put $G_F :=$ Gal$(\overline{\mathbf{Q}}/F)$,

- $\chi : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ unramified Hecke character s.t. $\chi_\infty(z) = z^2$,

- Let $\mathbf{T}$ be a finite $\mathcal{O}$-algebra generated by Hecke operators $T_v, v \notin \Sigma$, acting on weight 2 ordinary cuspidal automorphic forms on $\mathrm{GL}_2(\mathbf{A}_F)$ of some level and character,

- Call ideal $\mathbf{I}_\chi \subset \mathbf{T}$ generated by $\{T_v - (1 + (\chi \cdot \mathrm{Nm})(\pi_v)) | v \notin \Sigma\}$ the **Eisenstein ideal** associated to χ.

Theorem 1.

$$\mathsf{val}_p \#(\mathbf{T}/\mathbf{I}_\chi) \geq \mathsf{val}_p \left(\frac{L(0,\chi)}{\Omega^2} \right).$$

Sketch of proof: use integral structure from cohomology of a symmetric space S associated to GL_2/F (connection to automorphic forms via Eichler-Shimura-Harder)

- use Harder's theory to construct cohomology class $\mathsf{Eis}\,\omega_\chi \in H^1(S,K)$ with eigenvalues $1 + (\chi \cdot \mathsf{Nm})(\pi_v)$ and integral restriction to the boundary

- bound denominator δ of $\mathrm{Eis}\,\omega_\chi$ from below by L-value using analytic methods

- prove congruence of $\delta \cdot \mathrm{Eis}\,\omega_\chi$ with cuspidal class in $H_c^1(S,\mathcal{O})$ by ruling out congruence with torsion in $H_c^2(S,\mathcal{O})$. $\square$

Theorem 2 (B, Harcos). *Let π be a regular algebraic cuspidal automorphic representation with central character ω such that $\omega^c = \omega$. Then there exists a continuous irreducible*

$$\rho_\pi : \mathsf{Gal}(\overline{\mathbf{Q}}/F) \to \mathsf{GL}_2(\overline{\mathbf{Q}}_p)$$

such that $L_v(s,\pi) = L_v(s,\rho_\pi)$

$$\forall v \nmid p d_F \mathsf{cond}(\pi)\mathsf{cond}(\pi^c).$$

Remark. *This strengthens a result of Taylor who showed $L_v(s,\pi) = L_v(s,\rho_\pi)$ for a set of v of Dirichlet density one.*

Our proof initially follows the strategy of Taylor but results of Weissauer and Laumon allow us to simplify his proof:

Sketch of proof:

- lift many quadratic twist $\pi \otimes \mu$ to cuspidal automorphic representation Π^μ of $\mathrm{GSp}_4(\mathbf{A}_{\mathbf{Q}})$ (Harris, Soudry, Taylor),

- use congruences to higher weight forms, results of Weissauer/Laumon, and theory of pseudorepresentations to get

  $$\tau^\mu : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_4(\overline{\mathbf{Q}_p})$$

  such that $L^S(s, \tau^\mu) = L^S(s, \mathrm{I}_F^{\mathbf{Q}}(\pi \otimes \mu))$,

- show existence of $\rho^\mu : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}_p})$ such that $\tau^\mu \cong \mathrm{ind}_F^{\mathbf{Q}}(\rho^\mu)$,

- prove existence of ρ such that $\rho^\mu \oplus (\rho^\mu)^c \cong (\rho \otimes \mu) \oplus (\rho \otimes \mu)^c$,

- Let $\{\alpha_v, \beta_v\}$ inverse roots of Hecke poly of π, $\{\gamma_v, \delta_v\}$ inverse roots of Frobenius poly of ρ, get

$$\{\gamma_v \mu(v), \delta_v \mu(v), \gamma_{v^c} \mu(v^c), \delta_{v^c} \mu(v^c)\}$$

$$= \{\alpha_v \mu(v), \beta_v \mu(v), \alpha_{v^c} \mu(v^c), \beta_{v^c} \mu(v^c)\},$$

consider μ_1, μ_2 such that $\mu_1(v) = \mu_1(v^c)$ and $\mu_2(v) \neq \mu_2(v^c)$.

$\square$

Theorem 3 (Urban). *If π is ordinary at $v \mid p$ then $\rho_\pi|_{D_v}$ is ordinary.*

(A): Construct elements in Selmer groups

(Strategy of Ribet, Wiles, Skinner-Urban,...)

For $\lambda : G_F \to \mathcal{O}^*$, a continuous p-adic Galois character, let $\mathcal{O}_\lambda$ be the free rank one $\mathcal{O}$-module with G_F-action by λ.

Selmer group Sel$(F, \lambda) \subset H^1(G_F, \mathcal{O}_\lambda \otimes K/\mathcal{O})$,
unramified away from p

$$\begin{cases} \text{ordinary at } v \mid p & \text{if } p \text{ split,} \\ \text{crystalline} & \text{if } p \text{ inert.} \end{cases}$$

Let χ_p be the p-adic Galois character associated to χ by Weil, ϵ the p-adic cyclotomic character.

Theorem 4.

$$\mathrm{val}_p\#(\mathbf{T}/\mathbf{I}_\chi) \leq \mathrm{val}_p\#\mathrm{Sel}(F, \chi_p\epsilon).$$

Idea of proof:

- For $\pi \equiv \mathrm{Eis}\,\omega_\chi \mod \pi_K$ use techniques of Wiles/Urban to construct suitable lattice for ρ_π;
  get an element of $H^1(G_F, \mathcal{O}_{\chi_p\epsilon} \otimes K/\mathcal{O})$,
- properties of $\rho_\pi \Rightarrow$ element of $\mathrm{Sel}(F, \chi_p\epsilon)$.

$\square$

Remarks:

- Bloch-Kato conjecture implies that

$$\mathrm{val}_p \# \mathrm{Sel}(F, \chi_p^{-1}) = \mathrm{val}_p \left(\frac{L(0, \chi)}{\Omega^2} \right).$$

- By generalised Cassels-Tate pairing $\mathrm{Sel}(F, \chi_p \epsilon) \cong \mathrm{Sel}(F, \chi_p^{-1})$.

- Certain cases of p-primary Bloch-Kato have been proven from Main Conjecture for imaginary quadratic fields, including this critical, weight -2 case (Kato, Guo, Han, Tsuji).

- Interesting about our method is that we just work with congruences over F.

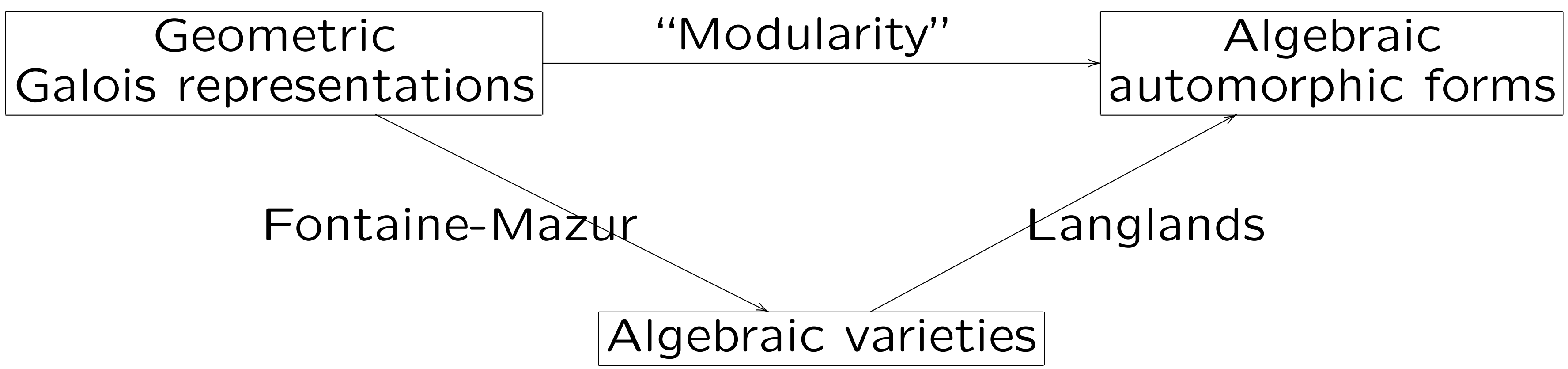
(B): Modularity of Galois representations

Let L be a totally real number field. Call an irreducible continuous representation

$$\rho : G_L \to \mathsf{GL}_2(\overline{\mathbf{Q}}_p)$$

geometric if ρ is unramified outside a finite set of places and potentially semistable at primes dividing p.

Conjecture (Fontaine-Mazur, 1995) ρ is geometric $\Leftrightarrow$ ρ is isomorphic to a (Tate twist) of the Galois representation associated to a Hilbert modular newform.


Geometric
Galois representations
"Modularity"
Algebraic
automorphic forms
Fontaine-Mazur
Langlands
Algebraic varieties

Results: $\Rightarrow$ is proven using Serre's Conjecture (Theorem of Khare-Wintenberger) + Modularity Lifting Theorems

Work of Wiles, Diamond, Fujiwara, Skinner-Wiles[3] (ordinary representations), Taylor, L. Berger-Breuil, Kisin, Colmez, Emerton...
Skinner-Wiles[1,2] study residually reducible representations.

Back to setup for F=imaginary quadratic:

- Assume in addition p splits, $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, let G_Σ be the Galois group of the maximal extension of F unramified outside Σ.

- Put $\tau = \overline{\chi_p \epsilon} : G_\Sigma \to k^\times$, k residue field of $\mathcal{O}$; note $\tau|_{I_\mathfrak{p}} = \bar{\epsilon}^{-1}|_{I_\mathfrak{p}}$ and $\tau|_{I_{\bar{\mathfrak{p}}}} = \bar{\epsilon}|_{I_{\bar{\mathfrak{p}}}}$.

- Assume

  - The τ^{-1}-eigenspace of the p-part of the class group of the splitting field $F(\tau)$ of τ is trivial,

  - If $v \in \Sigma$, then either τ is ramified at v or $\tau^{-1}(\mathrm{Frob}_v) \neq \# k_v$.

Theorem 5. *Let* $\rho, \rho' : G_\Sigma \to \mathrm{GL}_2(k)$ *be two Galois representations satisfying*

- $\rho, \rho' = \begin{pmatrix} 1 & * \\ & \tau \end{pmatrix}$,

- ρ, ρ' *have scalar centraliser.*

Then $\rho' \cong \rho$.

Proof. Careful analysis of Galois structure of local and global units of $F(\tau)$. □

Fix a Galois representation

$$\rho_0 = \begin{pmatrix} 1 & * \\ 0 & \tau \end{pmatrix} : G_\Sigma \to \mathsf{GL}_2(k)$$

as in the Theorem.

Goal: study deformations of ρ_0:

Definition. *An $\mathcal{O}$-deformation of ρ_0 is a local complete Noetherian $\mathcal{O}$-algebra $(A, k, \mathfrak{m}_A)$ and $\rho : G_\Sigma \to \mathsf{GL}_2(A)$ such that $\rho_0 = \rho \mod \mathfrak{m}_A$.*

Question: Are they (under appropriate conditions) all modular, i.e. arise from Galois representations attached to cuspforms of $\mathsf{GL}_2(\mathbf{A}_F)$?

Existence of modular deformations:

Corollary(of Theorems 1,2,3).
If $p \mid \frac{L(0,\chi)}{\Omega^2}$ then there exists an ordinary cuspform π and $\rho \cong \rho_\pi$ such that $\overline{\rho} = \rho_0$, i.e. an irreducible ordinary deformation.

Corollary. ρ_0 splits when restricted to $D_{\overline{\mathfrak{p}}}$.

Existence of a reducible deformation:

Put $\Psi = \chi_p \epsilon : G_\Sigma \to \mathcal{O}^*$.

Definition. *For T a finite set of places of F let $L_\Psi(T)$ be the maximal abelian pro-p extension of $F(\Psi)$ unramified outside T such that* $\mathrm{Gal}(F(\Psi)/F)$ *acts on* $\mathrm{Gal}(L_\Psi(T)/F(\Psi))$ *by Ψ^{-1}.*

A result of Greenberg shows:

Proposition 6.
$\mathrm{Gal}(L_\Psi(\Sigma \backslash \{\bar{\mathfrak{p}}\})/F(\Psi))$ is $\mathbf{Z}_p$-torsion.

This means that there exists no reducible ordinary $\mathcal{O}$-deformation of ρ_0, since an ordinary representation of the form $\begin{pmatrix} 1 & * \\ 0 & \Psi \end{pmatrix}$ has to split when restricted to $I_{\overline{\mathfrak{p}}}$.

This is different from the situation over $\mathbf{Q}$ studied by Skinner and Wiles in [SW1].

However, the result of Greenberg also tells us:

Proposition 7.
The $\mathbf{Z}_p$-rank of Gal$(L_\Psi(\Sigma)/F(\Psi))$ is 1.

This means that we can define a *nearly ordinary* reducible deformation of ρ_0 of the form

$$\rho^{\mathrm{Eis}} = \begin{pmatrix} 1 & * \\ 0 & \Psi \end{pmatrix}$$

which does not split when restricted to $I_{\overline{\mathfrak{p}}}$.

Note that this deformation is not de Rham at $\overline{\mathfrak{p}}$.

Question:

Are there conditions under which the universal ordinary deformation ring R^{ord} is a discrete valuation ring and generated as an $\mathcal{O}$-algebra by traces (studied over $\mathbf{Q}$ for non-ordinary deformations by Bellaïche, Chenevier, Calegari)?

Then this would imply an $R^{\mathrm{ord}} = T$ theorem, i.e. prove the modularity of residually reducible Galois representations.

Thank you.

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