Seeking Structure for the Collection of Rieger-Bernays Permutation models

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Abstract

Given any structure for the language of set theory and a permutation of its domain one obtains a new structure for the language of set theory. These are *Rieger-Bernays* models. Thus we have a surjection from the full symmetric group of the domain to a family of structures. Both groups and families-of-structures have natural topologies and it is natural to wonder what sensible things can be said about the way in which the topologies on this group and on this family interact. The techniques of Rieger-Bernays models first came to attention in an NF context but this material can also be applied in a ZF context too, and quite a lot of what follows below will carry over naturally.

This paper was begun in 2003/4 when Olivier Esser was in Cambridge on a postdoctoral fellowship working with Thomas Forster on the problem mentioned in the above abstract. It is designed in the first instance as a briefing paper for the second author's Ph.D. students. It had been hoped that it would fairly swiftly turn into something that the authors could publish, to justify the postdoc funding for 2003/4 and justify further appeals for funds in 2004/5, but the problems turned out to be far too intractable.

1 Problems to look at

1. Given a model of NF, form a topology from the $\sim_n$ equivalence classes as in GPG. What happens?

2. Show that if ϕ is a $\forall^*\exists^*$-formula that is consistent with NF then $NF \vdash \Diamond\phi$;

3. Show that If Φ is a consistent $\exists^*\forall^*$ sentence then $\{\sigma : V^\sigma \models \Phi\}$ contains a permutation of finite support;

4. Obtain a sensible classification of unstratified formulæ so we can prove theorems with the flavour: if ϕ of syntactic nastiness $> n$ then $\{\sigma : V^\sigma \models \sigma\} \cap C(J_n) = \emptyset$.

5. Complete J_0 wrt the various topologies?

6. Saturated model of the theory of the term/atomic model.
7. Is there an order structure on J_0 given by $\phi^\sigma \to \phi^\tau$?

2 Definitions, background and notation

$\{x : \phi\}$ is the extension of the property or formula ϕ. Use of this notation carries no guarantee that its denotation is a set.

In the following, V is a model of *NF*, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{N}$ is the collection of *concrete* (standard) natural numbers.

The reader is assumed to know what a stratification is, what it is for a formula to be stratified, and to know what a Rieger-Bernays permutation model is.

If 'x_i' is a variable in a formula ϕ and s is a stratification of ϕ then we write 's_i' for the type of the variable 'x_i' according to the stratification s.

For a tuple $\vec{x}$, $\mathrm{Diff}(\vec{x})$ says that all the entries in $\vec{x}$ are distinct.

2.1 Other notations, mainly concerning groups

The transposition swapping a and b is (a, b); the ordered pair of a and b is $\langle a, b\rangle$. We try to remember to write $x \cdot y$ for x composed with y so that we can reserve juxtaposition to signify functional application only. Arbitrary compositions of (pairwise disjoint) transpositions we will write in the style $\prod_{\ldots}(---, ***)$. The usual notation for the full symmetric group on a set V is $\Sigma(V)$ and 1_V is the identity element. In this context V is of course the universal set, and we will write $\Sigma(V)$ as J_0. This is because of a notation j which we will now explain ...

We define j, a function acting on permutations, so that $j(\sigma)(x) =: \sigma``x$. Thus if $G \subseteq \Sigma(X)$, $j^n``G$ is that subgroup of $\Sigma(\mathcal{P}^n(X))$ corresponding to the obvious action of G on $\mathcal{P}^n(X)$. To folk with a normal upbringing in algebra this seems a perverse way to think: surely it is more natural to think of G as acting on $\mathcal{P}(X)$ as well as on X? In some ways this is true, but at times when one has to be explicit the j^n notation will come in handy. In particular we will write J_n for $j^n``\Sigma(V)$, so that J_0 is the group of all permutations of V. If x is fixed by everything in J_n we say it is n-**symmetric**.

For two sets a and b, we write $a \sim_n b$ if there is a permutation σ with $j^n\sigma(a) = b$, and we say a and b are n-**equivalent**. Normally in set theory we say a set is definable if there is a formula with one free variable whose unique extension it is. Here we are a little more specific:

A set $a \in V$ is said to be: **definable** if there exists a *stratified* formula ϕ, without parameters, such that $a = \{x : \phi(x)\}$; it is n-**definable**, where $n \in \mathbb{N}$, if there is a *stratified* formula ϕ, such that $\phi(a) \land \forall b\ \phi(b) \to a \sim_n b$. Informally, this means that the *equivalence class* of a modulo n is definable.

De we perhaps mean $\phi(a) \land \forall b\ a \sim_n b \to \phi(b)$?

$C_G(H)$, the **centraliser** of H in G, is $\{x \in G : (\forall y \in H)(x \cdot y = y \cdot x)\}$. The subscript will be omitted when it is clear what the enveloping group is. The centralisers $C_{J_0}(J_n)$ for $n \in \mathbb{N}$ are of immediate and enduring importance as they contain precisely the permutations that (thought of as their graphs) are n-symmetric sets, as we shall see below. In fact, the assertion that every set is n-symmetric for some $n \in \mathbb{N}$ or other is equivalent to the $L_{\omega_1,\omega}$ formula

$$(\forall\sigma)(\bigvee_{n\in\mathbb{N}})(\forall\tau \in J_n)(\tau\sigma\tau^{-1} = \sigma)$$

Proof: If x is not n-symmetric for any $n \in \mathbb{N}$ then the transposition $(x, V \setminus x)$ is not in any of the centralisers $C_{J_0}(J_n)$ for $n \in \mathbb{N}$. ■

We need the concept of a *setlike* permutation. In general, σ being a permutation of the carrier set M of a model $\mathcal{M}$ does not ensure that $j^n(\sigma)$ is also a permutation of M, since it might not be everywhere defined. If $j^n(\sigma)$ is nevertheless a permutation of M for all n we say that σ is **setlike**. In ZF the axiom scheme of replacement ensures that any definable permutation is setlike, but this doesn't hold in Zermelo set theory or in NF.

Much of what we say about permutations does not depend on the permutations being sets, but works equally well if they are merely setlike.

Many results in this paper comes in two versions. The first version would be for a given concrete natural number n. The second variant concerns the case where n is not fixed, and may require additional assumptioms, some of them strong, such as that that everything is symmetric.

If we use Quine pairs then every set is also an ordered pair, and indeed every set is a *set* of ordered pairs. This has the effect that for any set x and any permutation σ, the exponential notation x^σ makes sense. The exponential of course signifies conjugation by a group element, so that x^σ is $\{\langle\sigma(u), \sigma(v)\rangle : \langle u, v\rangle \in x\}$. 'set' italicised

Lemma 1 *For sufficiently large $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that, for all x and for all σ, $j^{n+k}\sigma(x) = x^{j^n\sigma}$.*

Quite what n and k are will depend on our definition of ordered pair. If we are using Quine pairs then $n =$?? and $k = 1$

Corollary 2 *For a small fixed k, depending only on the implementation of ordered pair, the following are equivalent for a permutation π:*

1. *π, considered as a set of ordered pairs, is $n + k$-symmetric;*
2. *$\pi \in C_{J_0}(J_n)$.*

If we are using Quine pairs then $k = 1$.

2.2 Motivation

A word or two is in order on the reasons for this project. At the time of writing, quite a lot of relative consistency results have been obtained by using Rieger-Bernays permutations, but the methods used in each case have been fairly *ad hoc* and—beyond corollary 33 below (which states that any consistency result that can be obtained by permutation methods at all can be obtained by considering involutions only)—no general results are known. There remain outstanding some conjectures of a general character, like the conjecture of the second author that every unstratified universal-existential sentence that is consistent relative to NF can be shown to be consistent by means of a permutation model, and it is this kind of result that we hope to be able to prove by means of refined analysis of the structure on the family of permutation models.

3 Two Fundamental Lemmas

3.1 Coret's Lemma

Lemma 3 *If s is a stratification of Φ and σ is any setlike permutation,*

$$\Phi(x_1, \ldots, x_k) \longleftrightarrow \Phi((j^{s_1}\sigma)(x_1), \ldots, (j^{s_k}\sigma)(x_k)).$$

The proof is standard and will be omitted. The following observation we will probably not exploit: we make it because—in virtue of the severe constraints it places on the nature of the equivalence classes $[x]_n$—it enables one to clarify one's thoughts about them.

Remark 4 *For any set x and any $n \in \mathbb{N}$ the n-equivalence class of x (that is, $[x]_n$) is either a singleton or is of size $T^n|V|$.*

This is proved in [1]. It's all to do with the fact that all normal subgroups of J_0 (and therefore all normal subgroups of any of its isomorphic copies J_n) have large index. J_n acts on $[x]_n$, and this gives us a homomorphism from J_n to $\Sigma([x]_n)$. The kernel of this homomorphism is a normal subgroup of J_n.

3.2 Henson's lemma

To state this lemma we need a construct due to Henson [6]. The idea is Henson's but the notation here is new. 'H' for Henson.

Definition 5 $H_0(\tau) =: 1_V$; $H_{n+1}(\tau) =: (j^n\tau) \cdot H_n(\tau)$.

The following equalities follow easily

Lemma 6

1. $j(H_n(\tau)) = H_n(j\tau)$ *(H_n commutes with j);*
2. $j(H_n(\tau)) = H_{n+1}(\tau) \cdot \tau^{-1}$;
3. $H_n(j\tau) = H_{n+1}(\tau) \cdot \tau^{-1}$;
4. $\tau \cdot H_n(\tau)^{-1} = j(H_{n-1}(\tau))^{-1}$;

5. $H_n((j\tau)^{-1} \cdot \tau)) = (j^{n+1}\tau)^{-1} \cdot \tau$.

The first equation asserts that H_n commutes with the endomorphism j. It doesn't commute with arbitrary endomorphisms: it doesn't commute with conjugation by an arbitrary element of J_0 for example. Henson wrote 'τ_n' for our '$H_n(\tau)$', and his notation has stuck. We prefer ours because his, by using subscripts, precludes their being used in their traditional rôle of indicating *sequences* of permutations: $\langle \tau_n : n < \omega \rangle$. Furthermore '$H_n(\tau)$', by putting the '$n$' in a subscript, makes it easier for the reader to remember that 'n' is not a quantifiable variable: objects in argument position (like the 'n' in '$H(n, \tau)$') should always be genuinely quantifiable variables, whereas numerals in superscript or subscript position (one thinks of examples like '$y = \mathcal{P}^n(x)$') might well not be substitutable for quantifiable variables. Variables that range over $\mathbb{N}$ (rather than $\mathbb{IN}$) will typically appear only in subscript or superscript position and not in argument position.

This is a good reason for writing '$H_n(\tau)$' rather than '$H(n, \tau)$'

We equivocate (as is customary) between curried and uncurried functions so that we can write '$H_n(\tau, x)$'. However we will not carry uncurrying to the extent of writing '$H\ n\ \tau\ x$' because the 'n' is not a quantifiable variable.

H is not very nice: $H_n : \Sigma(V) \to \Sigma(V)$ is a group homomorphism if $n = 0$ or 1, but apparently not otherwise. This means that $H_n(\tau)^{-1} \neq H_n(\tau^{-1})$ in general.

Not only that, but in these other cases it appears that $H(n)$ is not even injective and H_n“$\Sigma(V)$ is not a group, and the condition $H_2(\sigma) = H_2(\tau)$ doesn't seem to say anything sensible about τ and σ.

It seems to be open whether $H_n : J_0 \to J_0$ is injective or surjective. Does H_n“J_0 generate J_0?

Let's get out of the way the observation that H_1“J_0 (which is just J_0, since H_1 is the identity) is transitive on k-tuples. It's banal but it is useful.

Remark 7 *If $\langle a_1 \ldots a_k \rangle$ and $\langle a_1 \ldots a_k \rangle$ are k-tuples of distinct things then there is $\pi \in J_0$ sending each a_i to b_i.*

If $\vec{a}$ and $\vec{b}$ are disjoint we just take π to be the product of the transpositions $\prod_{i<k}(a_i, b_i)$. If they aren't disjoint then we might have some longer cycles. However, it is clear that the strategy "always send a_i to b_i and send b_i to a_i unless it is an a_j" will (with a "looping back" clause) give a canonical permutation π sending $\vec{a}$ to $\vec{b}$. (Notice that π might not be an involution.) Clearly it is necessary and sufficient for the tuples to be tuples of distinct things.

Remark 8 *$AC_2 \to$ there is a permutation model in which H_2 is not injective.*

Proof: It is an old result of the second author's that if AC_2 holds we can find a permutation model in which there is a nontrivial $\in$-automorphism π of order 2. For such a π we can argue as follows:

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$$H_2(\pi) \quad = \quad j(\pi) \cdot \pi \quad =^1 \quad j(\pi^{-1}) \cdot \pi \quad =^2 \quad 1_V$$

(1) holds because $\pi = \pi^{-1}$; (2) holds because $\pi = j(\pi)$. But clearly we have $H_2(1_V) = 1_V$ so in these circumstances H_2 is not injective. ■

So H_2^{-1}"$\{1_V\}$ might not be a singleton. What structure does it have? All we can see at this stage is that it contains all *concrete* powers of all its elements. If $j(\pi) \cdot \pi = 1'_V$ then $j(\pi^2) \cdot \pi^2 = 1'_V$ and so on by induction. (The induction works only in the metalanguage, since it isn't stratified.)

Remark 9 *There is a permutation model in which H_2 not surjective.*

Proof:
Consider the transposition $(\emptyset, \{\emptyset\})$. Suppose it is $H_2(\pi)$, with a view to deriving a contradiction.

1: $j\pi \cdot \pi\emptyset = \{\emptyset\}$	so
2: $\pi\emptyset = \{\pi^{-1}\emptyset\}$	and
3: $j\pi \cdot \pi\{\emptyset\} = \emptyset$	so
4: $\pi\{\emptyset\} = \emptyset$	so, by 2 and 4
5: $\pi\emptyset = \{\{\emptyset\}\}$	

For larger n we prove by induction that

$$\pi \cdot \iota^{2n}\emptyset = \iota^{2n+2}\emptyset$$

and

$$\pi \cdot \iota^{2n+1}\emptyset = \iota^{2n-1}\emptyset$$

Now let Π be $\{\pi^n\emptyset : n \in Z\}$, the orbit of the empty set under π. I think this is precisely the Zermelo naturals, which—by a result of Holmes [**?**]—need not be a set. A bit of hand calculation shows that, for n in $\mathbb{N}$:

$$\pi^{-n}\emptyset = \{\pi^{n-1}\emptyset\}$$

and

$$\pi^n\emptyset = \{\pi^{-n}\emptyset\}$$

and we can formally prove in NF the stratified version:

$$(\forall n \in \mathbb{N})(\pi^{-Tn}\emptyset = \{\pi^{n-1}\emptyset\} \wedge \pi^{Tn}\emptyset = \{\pi^{-n}\emptyset\})$$

The induction is easy: The induction hypothesis tells us that $\pi^{-Tn-1}\emptyset = \pi^{-1}\{\pi^{n-1}\emptyset\}$ and the RHS is $j\pi\{\pi^{n-1}\emptyset\}$ which is of course $\{\pi^n\emptyset\}$. The other conjunct is analogous.

So Π contains the empty set and is closed under singleton. Now let X be any other set that contains the empty set and is closed under singleton. We prove by induction on n that $\pi^n\emptyset$ and $\pi^{-n}\emptyset$ are in X, so Π is minimal with this property and is indeed the Zermelo naturals. Now consider the map that sends $2n$ to $\pi^n\emptyset$ and $2n+1$ to $\pi^{-n}\emptyset$.

This (i think!) gives rise to a bijection f between $\mathbb{N}$ and Π satisfying $f(Tn+1) = \{f(n)\}$.

So if the transposition $(\emptyset, \{\emptyset\})$ is a value of H_2 then the Zermelo naturals are a set. This is known to be independent of any stratified extension of NF by a result of Holmes [**?**] ■

Remark 10 *There are permutation models in which H_2 is not transitive, even on single elements.*

Proof: The hard case is sending x to $V \setminus \{x\}$. Given x we seek π such that $H_2(\pi, x) = V \setminus \{x\}$. Clearly $\pi(x)$ must be 1-equivalent to $V \setminus \{x\}$ so it must be the complement of a singleton: $\pi(x) = V \setminus \{z\}$ for some z. But then $\pi``(V \setminus \{z\}) = V \setminus \{x\}$ so $\pi(x)$ must be z. So $z = V \setminus \{z\}$, a Quine antiatom. But we can get rid of Quine antiatoms by permutations. ■

Lemma 11 *Let $\vec{a}$ and $\vec{b}$ be two n-tuples, with* $\mathrm{Diff}(\vec{a})$ *and* $\mathrm{Diff}(\vec{b})$. *For any natural number k, we can find a permutation σ such that $H_k(\sigma, a_i) \sim_k b_i$, $i = 1, \ldots n$.*

Proof:

(Notice that we are not assuming that $\vec{a}$ and $\vec{b}$ are disjoint.) Let τ be any permutation such that $\tau(a_i) = b_i$, $i = 1, \ldots n$ and let $\sigma =: j\tau^{-1} \cdot \tau$. We now have $H_k(\sigma) = j^k(\tau^{-1}) \cdot \tau$. Thus $H_k(\sigma)$ sends each a_i to something k-equivalent to b_i. ■

Now comes the version of this lemma for non fixed n; this version works only if everything is symmetric.

Lemma 12 *Suppose that every set in V is symmetric. Given two tuples $\vec{a}$ and $\vec{b}$ such that* $\mathrm{Diff}(\vec{a})$ *and* $\mathrm{Diff}(\vec{b})$, *then:*

$$\exists n \in \mathbb{N}.\ \forall k \geq n.\ H_k(\sigma, a_i) = b_i \quad i = 1, \ldots n$$

Where should σ first appear in the statement of this lemma?

Proof: First we pick n large enough for all the $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ to be n-symmetric. We then apply lemma 11 and the fact that, for $k \geq n$, the $\sim_k$-equivalence classes of $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are singletons. ■

Actually what this construction proves is that H_n acts transitively on the n-equivalence classes of tuples, in the sense that given $\langle a_1 \ldots a_k \rangle$ and $\langle b_1 \ldots b_k \rangle$, there is something in H_n sending the a's to something coordinatewise n-equivalent to the b's. This is just the group theoretic expression of the fact that the modal logic of "there is a permutation model in which..." obeys Fine's principle, which is theorem 41 below.

The behaviour of $H_n(\sigma)$ appears now to be much more complicated than we first expected. We do not know for instance if we can ask that $H_k(\sigma, a_i) = b_i$ in lemma 11 instead of simply asking that they are $\sim_k$.

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At all events $H_n(\tau)$ is at least *well-defined* as long as τ is setlike, and this enables us to state the following result.

Lemma 13 Henson

If s is a stratification of Φ and τ is any setlike permutation of V then

$$(\forall \vec{x})V \models (\Phi(\vec{x})^\tau \longleftrightarrow \Phi(H_{s_1}(\tau, x_1) \ldots H_{s_n}(\tau, x_n))).$$

In the case where Φ is closed and stratified, we infer that, if V is a model of NF and τ a setlike permutation, then

$$V \models \Phi \longleftrightarrow \Phi^\tau.$$

■

3.3 Some questions about H

We are unable at present to determine whether or not:

1. H_n is injective;
2. H_n is surjective;
3. H_n"J_0 is a group;
4. H_n"J_0 acts transitively on unordered k-tuples;

It is the fourth of these that is of most concern to us, since it is related to the question of whether, for any n-tuple $\vec{a}$ and any n-ary relation ϕ, there is a permutation model in which $\phi(\vec{a})$ holds. (This is true if ϕ is stratified: see remark 39.)

some old thor'ts, now largely superseded.

It might be worth trying to prove that H_2"J_0 acts transitively on singletons. That is to say, given x and y, is there π such that π"$\pi(x) = y$. There's a crude cardinality/degrees of freedom argument that says this should be possible. We put $\langle x, y\rangle$ into π. This uses up only one degree of freedom, so we have lots left to ensure that π"$y = y$.

Try $\pi =: (x, y)$. This works if $x \in y \longleftrightarrow y \in y$. But suppose $x \in y \longleftrightarrow y \notin y$? Then i think we look for z and w where $z \in y \longleftrightarrow x \in y$ and $w \in y \longleftrightarrow y \in y$ and set $\pi = (x, y, z, w)$. Then $\pi(x) = y$ and π"$y = y$ so $H_2(\pi, x) = y$. Can such w and z always be found? Well, they can unless y is a singleton or the complement of a singleton. This give rise to at least six cases: (i) $y = \{x\}$, (ii) $y = V \setminus \{x\}$, (iii) $y = \{y\}$, (iv) $y = V \setminus \{y\}$, (v) $y = \{z\}$, (vi) $y = V \setminus \{z\}$ for some z other than x or y.

We deal with (i) by setting $\pi =: (z, x, \{z\})$ for a random z.

(ii) Works only if there is a Quine antiatom. This is why. We want π"$\pi(x) = V \setminus \{x\}$. Clearly $\pi(x)$ must be $V \setminus \{z\}$ for some z. But then π"$(V \setminus \{z\} = V \setminus \{x\}$. But then $\pi(x)$ must be z so $z = V \setminus \{z\}$. So H_2 might not act transitively even on singletons!

Let us abuse notation slightly by writing 'H_n' for H_n"J_0

I think i may be able to prove that H_2 acts transitively on k-tuples. Let $A = \{a_1 \ldots a_n\}$ and $B = \{b_1 \ldots b_n\}$. We seek π s.t. for each $i \leq n$, $H_2(\pi, a_i) = b_i$. In other words, $\pi``(\pi(a_i)) = b_i$ for each i.

Let us try to get an involution that does it. We seek a set $C = \{c_i \ldots c_k\}$, a set disjoint from $A \cup B$ (We'll worry later about what to do if $A \cap B \neq \emptyset$) such that if we set $\pi =: \prod_{i<k}(a_i, c_i)$, then for each $j \leq k$, $b_j = \pi``c_j$—which is to say: $b_j = (c_j \setminus (A \cup C)) \cup \{z : (\bigvee_{i \leq k}(a_i \in c_j \wedge z = c_i \vee c_i \in c_j \wedge z = a_i)\}$

What we want is that, for all unordered k-tuples A and B, there is an unordered k-tuple C such that, for each $b_j \in B$, $b_j = (c_j \setminus (A \cup C)) \cup \{z : (\bigvee_{i \leq k}(a_i \in c_j \wedge z = c_i \ \vee \ c_i \in c_j \wedge z = a_i)\}$

equivalently:

$c_j = (b_j \setminus (A \cup C)) \cup \{z : (\bigvee_{i \leq k}(a_i \in b_j \wedge z = c_i \ \vee \ c_i \in b_j \wedge z = a_i)\}$

This is $\forall^*\exists^*\forall^*$

How might this work? Given A and B, how do i compute c_1? We obtain c_1 from b_1 by deleting all the a_j from it (OK so far) and inserting all the corresponding c_j. So we'd better go and compute them. We also have to identify all the c_j in b_1 and replace them by the corresponding a_j. Clearly we have here two good reasons why this algorithm cannot be relied upon to terminate.

Let us consider a specific example. We have an ordered pair $\langle a_1, a_2 \rangle$, which we want sent to $\langle b_1, b_2 \rangle = \langle \{\emptyset, a_2\}, \{\emptyset, a_1\} \rangle$. We now find that $c_1 = \{\emptyset, c_2\}$ and $c_2 = \{\emptyset, c_1\}$. Some of this impasse happens beco's we want π to be an involution. If we are planning on π turning out to be a product of 3-cycles, $\{a_1, c_1, d_1\}$, $\{a_2, c_2, d_2\}$ then we find instead that $c_1 = \{\emptyset, d_2\}$ and $c_2 = \{\emptyset, d_1\}$. Which is perfectly possible, beco's $b_1 \cup b_2$ is not V. It goes to show that we mustn't expect to be able to take π to be an involution.

Perhaps taking π to consist of a product of disjoint m-cycles for $m >> k$ is the key. Might be some nasty case-hacking, depending on whether $\bigcap B$ is finite or empty or something, and on whether-or-not $\bigcup B = V$.

We expect something like the following to be true. If s and t are tuples of the same length that are each "tuples of distinct objects" in a suitable sense and σ is any as-it-were stratification, namely a function from naturals below $|s|$ to natural numbers, then there is a permutation π such that, for each coordinate s_i of s, $H_{\sigma(i)}(s_i) = t_i$. We believe this beco's of theorem 41

Olivier says that perhaps we should consider, for each such σ, the relations $s \rightarrow_\sigma t$ which holds iff there is a permutation π such that, for each coordinate s_i of s, $H_{\sigma(i)}(s_i) = t_i$. This relation is obviously reflexive, but it is not obviously transitive or symmetrical.

To underline how big a mess we're in, consider the case where σ is just the constant function $\lambda n.2$. Is $\rightarrow_\sigma$ transitive? This raises apparently very simple questions like: is $H_2``(\Sigma(V))$ a subsemigroup? If it is, then $\rightarrow_{\lambda n.2}$ is transitive.

Why is this so hard??

A question from Olivier:

Let $\phi(a_1, ..., a_n)$ be a stratified formula and s a stratification giving 'a_i' the types s_i. Is the following true?

$$(\forall a_1 \ldots a_n, b_1, ..., b_n)(a_i \sim_{s_i} b_i \to (\phi(a_1, ..., a_n) \longleftrightarrow \phi(b_1, ..., b_n))$$

Clearly not. Try "$x \in y$" but is there a strengthening of Coret's lemma obtainable by weakening the assumption?

We have not taken into account any stratification information. Can we require π to be in J_1 if $a_i \sim_1 b_i$ for each i? I think it's fairly clear that we can't. (I should prove this really) But i think this is the wrong generalisation. We should try thinking of remark 7 as the case where all the variables are of type 1. The type 2 case would be

> If $\langle a_1 \ldots a_k \rangle$ and $\langle a_1 \ldots a_k \rangle$ are k-tuples of distinct things then there is $\pi \in H_2$"(J_0) sending each a_i to b_i. (So that $\pi = j\sigma \cdot \sigma$ for some σ.)

Is this provable? If it is, we should consider:

> If $\langle a_1 \ldots a_k \rangle$ and $\langle a_1 \ldots a_k \rangle$ are k-tuples of distinct things then there is $\pi \in H_n$"(J_0) sending each a_i to b_i.

This is just the question whether or not H_n"J_0 acts transitively on k-tuples. So let Φ_n be the assertion:

> Let $\langle a_1 \ldots a_k \rangle$ and $\langle b_1 \ldots b_k \rangle$ be k-tuples with a "stratification" s making a_i and b_i the same type for each i, so that $s(i)$ is the type of both a_i and b_i, with $s(i) \leq n$ for all i. Then there is a permutation σ such that, for each $i < k$, $H_{s(i)}(\sigma, a_i) = b_i$.

Think about the case Φ_2 for the moment. Certainly, if $(\forall i, j \leq k)(a_i \notin b_j)$ then the same transposition will work. If this condition is violated then we have some work to do. I suspect it will turn out to be connected to the universal-existential conjecture. It may be that if the Universal-existential conjecture holds then Φ_n holds for all n. My guess is that Φ_2 is consistent and is perhaps provably consistent using only permutations of finite support.

One interesting feature about this general version is the question of what becomes of the distinctness condition. Since we intend to move a_i to b_i by $H_{s(i)}(\sigma)$—and what this permutation is will vary with i—we might be able to send a_i and a_j to distinct things even if $a_i = a_j$! However, we do not have complete freedom, because clearly conditions like

$$a_i = a_j \wedge s(i) = s(j). \to b_i = b_j$$

will have to be respected. There is also the following consideration. Let ψ be a stratified formula with two free variables, of type 3 and of type 5. Suppose further that there can be no x such that $\psi(x, x)$, but that $\psi(a, b)$ holds for some a and b. Let c be arbitrary. If Φ_5 held we would be able to find σ such that $H_3(\sigma, a) = c$ and $H_5(\sigma, b) = c$. But then we would have $V^\psi \vdash \phi(c, c)$, contradicting the assumption that $\psi(x, x)$ never holds.

What this reflection tells us is that if we can find such a ψ then Φ_5 above fails. In fact we do know there are such ϕ. Marcel and Maurice found one years ago and i put it in Jaune 5. However their example is quite complicated and doesn't refute Φ_n for n small enough. My guess is that Φ_2 is probably consistent.

What this does mean is that if we can prove things like Φ_2 then we know that sufficiently simple type-raising operations can consistently have fixed points

> Randall says:
>
> Let f be a permutation. I want to be able to have a permutation which swaps $f(x)$ with $jf(x)$, giving a permutation model in which $f = j(f)$. If in addition $f^n = 1_V$, it will follow that $H_n(f) = 1_V$ in the permutation model.
>
> Suppose that g is a period-2 map on Quine atoms and the identity on non-Quine-atoms. Consider the map taking $g(x)$ to $jg(x)$; does this have period 2? This sends Quine atoms to themselves and non-Quine-atoms to their images under $jg(x)$ (which will not be Quine atoms); it will be period 2, so the permutation works.
>
> So in the resulting permutation model we have $H_2(g) = 1_V$; it seems to be provable that H_2 is possibly not injective. For H_n, let g be a period n map acting nontrivially only on Quine atoms. The map which sends $g(x)$ to $jg(x)$ acts as the identity on Quine atoms and acts as jg on non-Quine-atoms (sending them to non-Quine-atoms) so it is a period n permutation of the universe. The permutation model generated by this map should have the map g equal to its image under j, and further has g a period n map, so $H_n(g) = 1_V$ Does this make sense? I could be hallucinating, of course.
>
> Randall
>
> Suppose $H_2(\sigma) = H_2(\tau)$ and $\sigma^2 = \tau^2 = 1$.
>
> Then σ and τ agree on wellfounded sets. We prove this by induction. Suppose σ and τ agree on everything in X. Then
>
> $$j\sigma \cdot \sigma \cdot \sigma(X) = j\tau \cdot \tau \cdot \sigma(X)$$
>
> so (beco's σ is an involution)
>
> $$j\sigma(X) = j\tau \cdot \tau \cdot \sigma(X)$$
>
> but σ and τ agree on members of X so
>
> $$j\tau(X) = j\tau \cdot \tau \cdot \sigma(X)$$
>
> $$X = \tau \cdot \sigma(X)$$
>
> so (beco's τ is an involution)
>
> $$\tau(X) = \sigma(X)$$
>
> The same proof actually shows that if $H_n(\sigma) = H_n(\tau)$ and they are both involutions then they agree on all wellfounded sets. We prove by $\in$-induction that if X is wellfounded then σ and τ agree on everything in $\bigcup^{\leq n} X$ (Actually we don't mean this exactly: we mean

$\sigma``X = \tau``X$ (call this set X_1) and $j\sigma``X_1 = (j\tau)``X_1$ (call this set X_2) and $j^2\sigma``X_2 = (j^2\tau)``X_2$ (call this set X_3) and so on, out to n ...

Suppose $H_n(\sigma) = H_n(\tau)$, and assume the induction hypothesis for X. Then

$$H_n(\sigma) \cdot \sigma(X) = H_n(\tau) \cdot \sigma(X)$$

$$j^n\sigma \cdot j^{n-1}\sigma \cdots \sigma \cdot \sigma(X) = j^n\tau \cdot j^{n-1}\tau \cdots \tau y \cdot \sigma(X)$$

$$j^n\sigma \cdot j^{n-1}\sigma \cdots j\sigma(X) = j^n\tau \cdot j^{n-1}\tau \cdots \tau \cdot \sigma(X)$$

But now by induction hypothesis we can replace all the σs on the LHS by τs getting

$$j^n\tau \cdot j^{n-1}\tau \cdots j\tau(X) = j^n\tau \cdot j^{n-1}\tau \cdots \tau\sigma(X)$$

and the τs all cancel getting

$$X = \tau \cdot \sigma(X)$$

4 Structures on the symmetric group on the universe

4.1 Twisted-conjugation

When one has a group G and an injective homomorphism $f : G \to G$ then G acts on itself by **twisted-conjugation**: Twisted-conjugate σ by τ to get $f(\tau)\cdot\sigma\cdot\tau^{-1}$. (A rather special case of this is known from the theory of automorphic forms, and we are endebted to Tony Scholl for explaining some of this to us: see below.) Twisted-conjugation of J_0 using j will turn out to be a very important operation, and hereafter when we speak of twisted-conjugation it is that operation we have in mind.

Applying H_n to the result of twisted-conjugating σ by τ doesn't give us the result of twisted-conjugating $H_n(\sigma)$ by τ: it gives us $(j^n\tau)^{-1} \cdot H_n(\sigma) \cdot \tau$ which isn't quite the same thing. (Tho' this equality might yet turn out to be useful!)

4.1.1 Twisted-conjugation is not left-distributive

It is known that ordinary conjugation is left-distributive, but it turns out that twisted-conjugation isn't: write $x * y$ for $(jx) \cdot y \cdot x^{-1}$, then

$(x * y) * (x * z) =$

$((jx) \cdot y \cdot x^{-1}) * ((jx) \cdot z \cdot x^{-1}) =$

$j^2x \cdot jy \cdot (jx)^{-1} jx \cdot z \cdot x^{-1} \cdot x \cdot y^{-1} \cdot (jx)^{-1}$

$j^2x \cdot jy \cdot z \cdot y^{-1} \cdot (jx)^{-1}$

$(jx) * (y * z)$

4.1.2 automorphic forms

A message from Tony Scholl:

E/F a cyclic Galois field extension, $Gal(E/F) = \langle\sigma\rangle$, G an algebraic group over F (GL_n is already rather interesting). Then x and y in $G(E)$ are said to be σ-conjugate if $x = \sigma(z) \cdot y \cdot z^{-1}$ for some $z \in G(E)$.

If j is actually an automorphism of G, you can write everything in terms of "usual" structures by replacing G by the semidirect product of G and the subgroup J of $Aut(G)$ generated by j. Then (modulo questions of left and right action) the twisted conjugate of $y \in G$ is just conjugation of (y, j) in the semidirect product:

$$(x, 1)(y, j)(x^{-1}, 1) = (x \cdot y \cdot j(x)^{-1}, j)$$

As the set of (y, j) is not a subgroup you shouldn't expect nice things to happen. I have no idea if there is any literature about the algebra here but I suspect there is no useful structure (other than anything which is essentially obvious) unless you make some further hypotheses on G or j.

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In earlier draughts of this work I used to call this operation *skew*-conjugation. I might yet go back to it: I moved to adopt the 'twisted' notation because it is earlier, but twisted conjugation in the theory of automorphic forms doesn't seem—at the moment at least—to have the same meaning as the operation here. And skew-conjugation gives rise to a verb to skew-conjugate, which twisted-conjugation doesn't.

4.2 Topologies on J_0

Review Gaughan here [5].

Olivier: can you have a look at Gaughan and tell me what you think?

The group structure on J_0 isn't obviously relevant here. For example the two transpositions $(\emptyset, V)$ and $(\emptyset, \{\emptyset\})$ are conjugate but give rise to nonisomorphic models, since $V^{(\emptyset,\{\emptyset\})}$ must contain a Quine atome whereas $V^{(\emptyset,V)}$ might not.

Naturally one wonders if there is other structure on J_0. One obvious thing to look at is topologies on $\Sigma(V)$. There are obvious topologies on permutation groups: If G is a group of permutations of a set X, then we can take pointwise or setwise stabilisers as basis elements. Or again, for each x, y in X, the set of permutations sending x to y.

DEFINITION 14 *The* **Stone topology** ST *on* J_0 *is defined in the following way:*

$$\{\sigma \; : \; V^\sigma \models \phi(h_1, \ldots, h_n)\} \tag{1}$$

is open for every stratified ϕ and $h_1, \ldots, h_n \in V$ where $h_1, \ldots, h_n$ have the same level of stratification.

Now comes the version for a given concrete natural number:

Definition 15 *The n-***Stone topology** *ST on J_0 is defined by:*

$$\{\sigma \ : \ V^\sigma \models \phi(h_1, \ldots, h_n)\} \tag{2}$$

is open for every stratified ϕ and $h \in V$ and $s(h_i) = n$, s being a stratification of the formula ϕ.

At first sight, one might have the impression that ... could require that the formula ϕ have only one parameter by playing with the Quine pair. This is not true. Although we still do not know if the topologies generated by these two definitions are the same, we can actually prove that the two subbases which they define are distinct by the following proposition:

Proposition 16 *There exists a stratified formula $\phi(a, b)$ together with two sets a, b such that, for any formula $\psi(c)$ with one free variable and for any set c, we have:*

$$\{\sigma \ : \ V^\sigma \models \phi(a, b)\} \neq \{\sigma \ : \ V^\sigma \models \psi(c)\}$$

Proof: Let a, b any sets with $a \neq b$. Consider the following set

$$U = \{\sigma \ : \ V^\sigma \models \forall x \in a. \perp \wedge \forall x. x \in b\}$$

We can easily show that:

$$U = \{\sigma \ : \ \sigma(a) = \emptyset \wedge \sigma(b) = V\}$$

Now let $\psi(c)$ be a stratified formula with one free variable and a set c and define

$$W = \{\sigma \ : \ V^\sigma \models \psi(h)\}$$

We will show that $W \neq U$. Suppose that $s(h) = n$, where s is a stratification of the formula ψ. By lemma 13 (Henson's lemma), we have:

$$W = \{\sigma \ : \ V \models \psi(H_n(\sigma, h))\}$$

Suppose without loss of generality that $c \neq a$ (otherwise replace a by b). Consider a permutation τ such that $H_n(\tau, h) \sim_n H_n(\sigma, h)$ and $\tau(a) \neq \emptyset$. This exists by lemma 11: notice that, for a permutation τ, $\tau(a) = \emptyset \leftrightarrow \tau(a) \sim_n \emptyset \leftrightarrow H_n(\tau, a) = \emptyset$; this is also true for $\emptyset$ replaced by V. Now $\tau \in W$ but $\tau \notin U$, which shows our result. ■

For each ϕ with (say) k free vbls, we can take a basis whose elements are, for each k-tuple $\vec{a}$, the set

$$\{\sigma : V^\sigma \models \phi(\vec{a})\}$$

This is the **ϕ-topology**.

The ϕ-topology is a refinement of the n-Stone topology for some n depending on ϕ.

DEFINITION 17 *The* **Henson topology** *HT on J_0 is defined by:*

$$\{\sigma \ : \ \exists k \in \mathbb{N}.\ \forall n \geq k\ H_n(\sigma, a) \sim_n b\}$$

is open for $a, b \in V$

and the corresponding notion for a concrete n:

DEFINITION 18 *The* **n-Henson topology** *HT_n on J_0 is given by taking $\{\sigma \ : \ H_n(\sigma, a) \sim_n b\}$ to be open for $a, b \in V$.*

We can easily check that the open set generating the Stone, n-Stone and the n-Henson topology forms a *base* of topology. This is not true for the open set in the Henson topology where we have only a *subbase*.

PROPOSITION 19 *ST (resp. ST_n) is coarser than HT (resp. HT_n)*

Proof: We will show this proposition for a fixed concrete natural number n, the other version of this proposition is similar.

Let U be a a basic open set for the ST_n topology, we have

$$U = \{\sigma \ : \ V^\sigma \models \phi(h_1, \ldots, h_k)\}$$

where the level of stratification of h_i $(i = 1, \ldots, n)$ is n for a $n \in \mathbb{N}$. So by lemma 13 (Henson's lemma)

$$U = \{\sigma \ : \ V \models \phi(H_n(\sigma, h_1), \ldots, H_n(\sigma, h_k))\}$$

Let $\sigma \in U$ and write $r_i = H_n(\sigma, h_i)$ $(i = 1, \ldots, n)$. Consider

$$W = \{\tau \ : \ V \models \bigwedge_{1 \leq i \leq n} H_n(\tau, h_i) \sim_n r_i\}$$

Clearly W is open for the HT_n topology with $\sigma \in W$ and $W \subset U$. This shows that U is open for the HT_n topology. ■

PROPOSITION 20 *If every set is n-definable in V, then $HT_n = ST_n$.*

Proof: Let U be an open set of the subbase of the HT_n topology. We have $U = \{\sigma \ : \ H_n(\sigma, a) \sim_n b\}$. By definition of the HT_n topology, we can find a formula $\phi(x)$, with one free variable, such that $\phi(b)$ and such that $\forall x\ \phi(x) \to x \sim_n b$ and where the level of stratification of b in $\phi(b)$ is at most n. Consider the formula $\phi(a)$. Suppose that $V^\sigma \models \phi(a)$, then $V \models \phi(H_n(\sigma, a))$ and then $H_n(\sigma, a) \sim_n b$. Reciprocally, if $H_n(\sigma, a) \sim_n b$, we have $V \models \phi(H_n(\sigma, a))$ and thus $V^\sigma \models \phi(a)$. This shows that U is open for the ST_n topology and achieves the proof.

■

The version of this proposition for a non fixed natural number n becomes:

Proposition 21 *If every set is definable in V (i.e. V is a term model) then HT = ST*

Proof: The fact that V is a term model ensures that everything is symmetric; this implies that, for every set, its n-equivalence class is a singleton for n large enough. The proof is then a very easy adaptation of the previous one ∎

The following theorem gives a nice characterization of the notion of n-definability in terms of topology.

Theorem 22 *For $n \in \mathbb{N}$, we have $HT_n = ST_n$ iff every set is n-definable.*

Proof: The implication from right to left is proposition 20.

Suppose HT_n=ST_n, and suppose $a \in V$. We will prove that a is n-definable, i.e. there is a stratified formula $\psi(x)$ with one free variable x with $\psi(a)$ and such that $\forall b\ b \sim_n a \to \psi(b)$.

Consider $U = \{\sigma \ :\ H_n(\sigma, \emptyset) \sim_n a\}$, an open set for the HT_n topology, and let $\tau \in U$. Let W be a basic open set for the ST_n topology with $\tau \in W$ and $W \subseteq U$. We have

$$W = \{\sigma \ :\ V^\sigma \models \phi(h_1, \ldots, h_k)\}$$

for some stratified ϕ and $h_1, \ldots, h_k \in V$, where the level of stratification of h_i in ϕ is n. By lemma 13 (Henson's lemma), we have also

$$W = \{\sigma \ :\ V \models \phi(H_n(\sigma, h_1), \ldots, H_n(\sigma, h_k))\}$$

We claim that one of the h_i must be $\emptyset$. Suppose otherwise that all the $h_i \neq \emptyset$. Use lemma 11 to find a σ with $H_n(\sigma, h_k) \sim_n H_n(\tau, h_k)$ and $H_n(\sigma, \emptyset) \not\sim_n H_n(\tau, \emptyset)$. Then $\sigma \in W$ and $\sigma \notin U$ a contradiction. Suppose thus that $h_1 = \emptyset$. The fact that $W \subset U$ can be rewritten as follow:

$$\forall \tau\ \phi(H_n(\tau, \emptyset), H_n(\tau, h_2), \ldots, H_n(\tau, h_k)) \to H_n(\tau, \emptyset) \sim_n a$$

Using lemma 11 this last formula can be rewritten as:

$$\forall x_1, \ldots, x_n\ \mathrm{Diff}(x_1, \ldots, x_n) \wedge \phi(x_1, \ldots, x_n) \to x_1 \sim_n a$$

Using this last property, we conclude that the formula $\psi(x)$ defining a we are looking for is

$$\exists x_2, \ldots, \exists x_n\ \mathrm{Diff}(x, x_2, \ldots, x_n) \wedge \phi(x, x_2, \ldots, x_n)$$

∎

It is here that is necessary that all h_i have the same level of stratification in the definition of the n-Stone topology. We would need a more general version of lemma 11 to handle the more general case.

In the case where n is not fixed, we can prove the following theorem. This give a nice characterization of term models in the case where everything is symmetric.

Theorem 23 *Suppose that everything is symmetric. Then HT = ST iff everything in V is definable i.e. V is a term model.*

Proof: The left to right part is already given by proposition 21. The right to left part is a corollary of theorem 23. Just notice that if a is n-symmetric then a is definable iff it is k-definable for $k \geq n$ ■

LEMMA 24 *If ϕ is a stratified formula, then twisted-conjugation is a continuous action of $\Sigma(V)$ with respect to the ϕ-topology.*

Proof: The basis element of the ϕ-topology corresponding to the tuple $\langle a_1 \ldots a_n \rangle$ is of course

$$\{\sigma : V^\sigma \models \phi(a_1 \ldots a_n)\}$$

If we twisted-conjugate it by τ we get

$$\{(j\tau) \cdot \sigma \cdot \tau^{-1} : V^\sigma \models \phi(a_1 \ldots a_n)\}$$

which is

$$\{\sigma : V^{(j\tau)^{-1} \cdot \sigma \cdot \tau} \models \phi(a_1 \ldots a_n)\}$$

which (by lemma 13 (Henson's lemma) is

$$\{\sigma : V \models \phi(H_{s_1}((j\tau)^{-1} \cdot \sigma \cdot \tau, a_1), \ldots H_{s_n}((j\tau)^{-1} \cdot \sigma \cdot \tau, a_n))\}$$

Now when we expand $H_{s_i}((j\tau)^{-1} \cdot \sigma \cdot \tau, a_i)$ using definition 5, all the internal τs cancel, leaving $(j^{s_i}\tau)^{-1} \cdot H_{s_i}(\sigma) \cdot \tau$, so the set is

$$\{\sigma : V \models \phi((j^{s_1}\tau)^{-1} \cdot H_{s_i}(\sigma) \cdot \tau(a_1), \ldots (j^{s_n}\tau)^{-1} \cdot H_{s_n}(\sigma) \cdot \tau(a_n))\}$$

Since the exponent on the j is s_i we can apply Coret's lemma (lemma 3), to delete the $j^{s_i}(\tau)$ getting

$$\{\sigma : V \models \phi(H_{s_1}(\sigma) \cdot \tau(a_1) \ldots H_{s_n}(\sigma) \cdot \tau(a_n))\}$$

and by lemma 13 (Henson's lemma) this is

$$\{\sigma : V^\sigma \models \phi(\tau(a_1) \ldots \tau(a_n))\}$$

which is the basis element corresponding to the tuple $\langle \tau(a_1) \ldots \tau(a_n) \rangle$. ■

Notice that twisted-conjugation not only sends open sets to open sets, but sets with are open-in-virtue-of-ϕ to sets with are open-in-virtue-of-ϕ.

Thomas to Olivier

> Olivier: just check that i've got this right: for each stratified ϕ the ϕ-topology is finer than the n-Stone topology where n is the degree of stratification of ϕ. Is this not correct? And the n-stone topology is finer than the stone topology? So lemma 24 holds for all those topologies? Do we want to drop all reference to the ϕ-topology? Would it be simpler?

Olivier to Thomas

The annoying point is that I have assumed that all parameters have the same level of stratification while no assumption of this kind is made for the ϕ-topology. I don't think that a set such that $\{x : \phi(a, b)\}$ where a and b have the same level of stratification can be decribed by a formula where all parameters has the same level of stratification (I am probably able to show this, I have to think about it).

I agree that this restriction that the parameters have the same level of stratification should be removed in the definition of the Stone topology but due to lack of understanding H_n I am not able to prove results without this assumption.

In these topologies, it is the level of stratification on which the parameters appears that seems to be relevant and not the number of types we have to use to stratify the formula. Also this give rise to notions which closed by $\wedge$ and $\vee$ which is not the case for the latter one.

Olivier

However it turns out that this is no use to us in the endeavour to find structure on J_0 that tells us about the family of permutation models: the continuous action by twisted-conjugation moves things around only within twisted-conjugacy classes, and—as we are about to see—this action corresponds to the identity on all the quotient structures imposed on the set of models, since V^σ and V^π are isomorphic iff σ and π are twisted-conjugate. But we're getting ahead of ourselves.

Now what are these quotient structures?

5 The quotient structures

There are various obvious equivalence relations on $\Sigma(V)$ arising from the $\in$-structure that we will have reduce J_0 by in order to get anything sensible: after all, even if $\sigma \neq \tau$, V^τ and V^σ might still be elementarily equivalent.

1. The first notion, $\sim_{(i)}$, is isomorphism via an internal permutation.
2. The second notion, $\sim_{(ii)}$, is isomorphism via a setlike permutation. (I briefly thought there was another entry due in this list after isomorphism by a setlike permutation, namely isomorphism by an arbitrary external permutation. But any external permutation that is an isomorphism is perforce setlike.)
3. Finally there is elementary equivalence.

These equivalence relations give rise to surjections onto various quotients, which may or may not have topological structure of their own. (the third has a well-known Stone topology defined above). Natural to ask if these quotient maps are cts with respect to any natural topologies on $\Sigma(V)$.

Remark 25 *$V^\sigma \simeq V^\tau$ (i.e., $\sigma \sim_{(i)} \tau$) iff σ and τ are twisted-conjugate.*

Proof:
$x \in \tau(y)$ iff $\sigma(x) \in j\sigma \cdot \tau(y)$ iff $\sigma(x) \in j\sigma \cdot \tau \cdot \sigma^{-1} \cdot \sigma(y))$ which is to say that σ is an isomorphism between V^τ and $V^{(j\sigma)\cdot\tau\cdot\sigma^{-1}}$. ■

In particular, if we skew-conjugate σ by σ we get $j\sigma$, so the skew-conjugacy classes are closed under j. (So, at least from the point of view of permutation models, σ and $j\sigma$ are the same, and it might seem that we could have stuck with thinking of different actions of the one group rather than lots of new permutations, but i think this is the wrong moral to draw).

The first equivalence relation is thus definable internally; the second one clearly won't be. If we assume the axiom of counting then the third equivalence relation becomes internally definable too.

There is an exactly similar theorem for setlike permutations, saying that, if τ and σ are setlike permutations so that V^τ and V^σ are isomorphic (where the isomorphism is setlike), then τ and σ are twisted-conjugated by a setlike permutation.)

The following observation is worth flagging

Lemma 26 *τ is an isomorphism between V^σ and $V^{j\tau^{-1}\cdot\sigma\cdot\tau}$*

Proof:

$$x \in j\tau^{-1} \cdot \sigma \cdot \tau(y)$$

iff

$$\tau(x) \in \sigma \cdot \tau(y)$$

■

For what topologies on the quotient is the quotient map continuous? Quite possibly none ...

6 Iteration of the Rieger-Bernays construction

Fundamental to this endeavour is an understanding of the fate of atomic subformulæ of an arbitrary expression $\phi \in \mathcal{L}(\in, =)$ in contexts like

$$V^\tau \models (Q\sigma)(\ldots \phi^\sigma \ldots)$$

That is to say we must consider

$$(x \in \sigma(y))^\tau$$

which is what becomes of the atomic formula '$x \in y$' under two applications of Rieger-Bernays. The 'Q' signals a quantifier. It doesn't matter which of '$\exists$' or '$\forall$' it is but the 'σ' must be bound. This is because an object that V^τ believes to be a permutation might not be a permutation in the base model. This means that expressions like '$V^\tau \models \phi^\sigma$' are not to be read in a casual intuitive way, but have to be treated with extreme caution and read literally. Of course the 'σ' has to range over permutations, so some of the dots are consumed in a clause saying that σ is a permutation.

For n sufficiently large, '$(\sigma$ is a permutation$)^\tau$' is equivalent to '$H_n(\tau, \sigma)$ is a permutation', by lemma 13 (Henson's lemma). Also '$x \in \sigma(y)$' is a stratified formula with three free variables, 'x', 'y' and 'σ' of types $n-1$, n and $n+1$ respectively for $n \in \mathbb{N}$ sufficiently large. So we can apply Henson's lemma to

$$(x \in \sigma(y))^\tau \tag{3}$$

to obtain

$$H_{n-1}(\tau, x) \in H_{n+1}(\tau, \sigma)(H_n(\tau, y)) \tag{4}$$

for n sufficiently large.

Next we want to reletter '$H_{n+1}(\tau, \sigma)$' as 'σ'. When is this legitimate? For a start, 'σ' will have to be a bound variable. On top of that we will have to check that all occurrences of 'σ' have this prefix. But this is easy: every occurrence of 'σ' occurs within the scope of '$V^\tau \models \ldots$' and therefore acquires this prefix when we "lower" the τ.

Once we have done the relettering, we have

$$H_{n-1}(\tau, x) \in \sigma \cdot H_n(\tau)(y) \tag{5}$$

This preservation-under-relettering that we have just performed is the **First Relabelling Factoid**.

Next we apply the rule $u \in v \longleftrightarrow \pi(u) \in (j\pi)(v)$ to formula 6, taking π to be $H_{n-1}(\tau)^{-1}$, getting

$$H_{n-1}(\tau)^{-1} \cdot H_{n-1}(\tau, x) \ \in \ (jH_{n-1}(\tau)^{-1}) \cdot \sigma \cdot H_n(\tau)(y) \tag{6}$$

which is of course

$$x \ \in \ (jH_{n-1}(\tau)^{-1}) \cdot \sigma \cdot H_n(\tau)(y) \tag{7}$$

Now $H_n(\tau) = j(H_{n-1}(\tau)) \cdot \tau$ so
$H_n(\tau)^{-1} = \tau^{-1} \cdot j(H_{n-1}(\tau))^{-1}$ whence
$\tau \cdot H_n(\tau)^{-1} = j(H_{n-1}(\tau)^{-1})$. So formula 6 becomes

$$x \in \tau \cdot H_n(\tau)^{-1} \cdot \sigma \cdot H_n(\tau)y \tag{8}$$

Now things become a little bit trickier. Conjugation by $H_n(\tau)$ permutes the set of permutations so we might hope that—since 'σ' is bound—we can reletter '$H_n(\tau)^{-1} \cdot \sigma \cdot H_n(\tau)$' as '$\sigma$'. This will be all right as long as every occurrence of 'σ'

within the scope of the '$V^\tau \models$' is inside an occurrence of '$H_n(\tau)^{-1} \cdot \sigma \cdot H_n(\tau)$'. That will be true for all occurrences of 'σ' that are inside ϕ of course, but there may be occurrences of 'σ' within the dots that are *not* inside occurrences of '$H_n(\tau)^{-1} \cdot \sigma \cdot H_n(\tau)$'. Therefore the projected relettering replaces all such occurrences of 'σ' by '$H_n(\tau) \cdot \sigma \cdot (H_n(\tau))^{-1}$'. If the dots are saying merely that σ is a permutation with particular properties then it might be that those properties are preserved by conjugation by $H_n(\tau)$. If those properties are captured by a stratified formula then they will be preserved by conjugation by $H_n(\tau)$ as long as $\tau \in J_n$ for large enough n.

This is the **Second Relabelling Factoid** and it is much more delicate than the first, since it works only in those cases where the dotted information about σ is preserved by conjugation by suitably restricted kinds of permutations. However, when it *does* work, we can simplify formula 6, to wit:

$$x \in \sigma(y))^\tau$$

to

$$x \in \tau \cdot \sigma(y).$$

■

Notice that we have relettered 'σ' a couple of times but never 'τ'.

Lemma 27 *The Iteration-Relabelling lemma*
If the information in the dots is preserved by conjugation then

$$V^\tau \models (Q\sigma)(\ldots \phi^\sigma \ldots)$$

is equivalent to

$$(Q\sigma)(\ldots \phi^{\tau \cdot \sigma} \ldots)$$

■

Corollary 28
Let $A()$ and $B()$ be one-place predicates appropriate for permutations and $A()$ additionally a predicate whose extension is normal (a union of conjugacy classes). Then

$$(\exists\tau)(B(\tau) \wedge V^\tau \models [(\exists\sigma)(A(\sigma) \wedge \phi^\sigma)]) \longleftrightarrow (\exists\tau)(\exists\sigma)(B(\tau) \wedge A(\sigma) \wedge V^{\tau\cdot\sigma} \models \phi)$$

Corollary 29
Let $\Diamond_k \phi$ mean that there is a permutation σ that is a product of k involutions such that $V^\sigma \models \phi$. As before, $\Diamond\phi$ means that there is a permutation σ such that $V^\sigma \models \phi$. Then

$$\Diamond_1 \Diamond_k \phi \longleftrightarrow \Diamond_{k+1} \phi$$

Proof:

Assume $\Diamond_1 \Diamond_k \phi$. That is to say

$$(\exists \tau)(\tau^2 = 1 \wedge V^\tau \models \Diamond_k \phi)$$

or

$$(\exists \tau)(\tau^2 = 1 \wedge V^\tau \models (\exists \sigma)(I(k, \sigma) \wedge (\phi)^\sigma)$$

... where $I(k, \sigma)$ says that σ is a product of k involutions. The property of being a product of k involutions is normal—preserved under conjugation.

To establish this equivalence we process

$$V^\tau \models (\exists \sigma)(I(k, \sigma) \wedge (\phi)^\sigma)$$

and by the manipulations above this becomes

$$(\exists \sigma)(I(k, \sigma) \wedge (\phi)^{\tau\sigma})$$

so the whole expression becomes

$$(\exists \tau)(\tau^2 = 1 \wedge (\exists \sigma)(I(k, \sigma) \wedge (\phi)^{\tau\sigma})$$

which is

$$(\exists \tau)(\exists \sigma)(\tau^2 = 1 \wedge (I(k, \sigma) \wedge (\phi)^{\tau\sigma})$$

... replacing '$\tau\sigma$' by 'π'

$$(\exists \pi)(I(k + 1, \pi) \wedge (\phi)^\pi)$$

$$\Diamond_{k+1} \phi$$

■

Notice that with some straightforward modifications to this proof we can obtain a demonstration of

COROLLARY 30 $\Diamond_k \Diamond_1 \phi \longleftrightarrow \Diamond_{k+1} \phi$

We can also prove

COROLLARY 31 $\Diamond \phi \longleftrightarrow \Diamond_k \phi$, *for* $k > 8$.

(The '8' is beco's of Nathan's April Rainer Theorem—every permutation is a product of at most 8 involutions; see [1].)

This gives as a corollary my old result that if we can prove that ϕ remains true in V^σ if it was true in V then we can prove that there is no permutation proof of the consistency of $\neg\phi$.

In general, if A is a normal generating subset of $Symm(V)$ then if ϕ can be proved consistent using permutations then a permutation model can be obtained by iterating the Rieger-Bernays method using at each stage only permutations in A.

The following result is an example of the kind of thing I want more of.

We can probably delete from here ...

Lemma 32 *Let ϕ be an arbitrary formula and let A define a normal subset of J_0. Then if*

$$NF \vdash \phi \to (\forall \sigma)(A(\sigma) \to \phi^\sigma)$$

then $NF \vdash \{\tau : \phi^\tau\}$ is closed under postmultiplication by things in A.

Proof:
The 'σ' in "$\phi \to (\forall \sigma)(A(\sigma) \to \phi^\sigma)$" ranges over permutations, and we have to make this explicit in the processing that is to come.

Suppose we have proved

$$\phi \to (\forall \sigma)(\sigma \in J_0 \to A(\sigma) \to \phi^\sigma)$$

Then we have also proved

$$(\forall \tau)(\phi \to (\forall \sigma)(\sigma \in J_0 \to A(\sigma) \to \phi^\sigma))^\tau$$

which is

$$(\forall \tau)(\phi^\tau \to (\forall \sigma)((\sigma \in J_0)^\tau \to A(\sigma)^\tau \to \phi^*)) \tag{9}$$

where ϕ^* is a complex expression obtained from ϕ by replacing every atomic subformula $x \in y$ by $H_{n-1}(\tau, x) \in H_{n+1}(\tau, \sigma) \cdot H_n(\tau)(y)$

Now every occurrence of σ anywhere in formula 9 is in an occurrence of $H_{n+1}(\tau, \sigma)$ so we can reletter to get

$$(\forall \tau)(\phi^\tau \to (\forall \sigma)((\sigma \in J_0) \to A(\sigma) \to \phi^{**}))$$

where ϕ^{**} is a complex expression obtained from ϕ by replacing every atomic subformula

$$x \in y$$

by

$$H_{n-1}(\tau, x) \in \sigma \cdot H_n(\tau)(y)$$

or, alternatively, by

$$x \in j(H_{n-1}(\tau))^{-1} \cdot \sigma \cdot H_n(\tau)(y)$$

Now $j(H_{n-1}(\tau))^{-1}$ is $\tau \cdot j(H_n(\tau))^{-1}$ so we can write this as

$$x \in \tau \cdot (\sigma^{H_n(\tau)})(x)$$

Now we want to reletter $\sigma^{H_n(\tau)}$ as σ but to do this we need to be sure that $A(\sigma) \longleftrightarrow A(\sigma^{H_n(\tau)})$. For this it will be sufficient that A is normal—a union of conjugacy classes (tho' all we really need is that it should be preserved by conjugation by members of H_n"J_0. It might be worth checking that this gives us no extra freedom!) Assuming this we have

$$\phi^\tau \to (\forall \sigma)((\sigma \in J_0) \to A(\sigma) \to \phi^{\tau\sigma})$$

or in short

$$(\forall\tau)(\phi^\tau \to (\forall\sigma)(A(\sigma) \to \phi^{\tau\sigma}))$$

which says that $\{\tau : \phi^\tau\}$ is closed under postmultiplication by things in A.■

Corollary 33

- *Suppose $A(\ )$ is a stratified formula whose extension is a normal subset of J_0, such that every permutation is a product of things in A. Then if $NF \vdash \phi \to (\forall\sigma)(A(\sigma) \to \phi^\sigma)$ then $NF \vdash \phi$.*
- *In particular, if we can prove that no involution will enable us to prove the consistency of ϕ, then ϕ cannot be proved consistent by Rieger-Bernays methods.*

Notice that we cannot strengthen part 1 of corollary 33 (keeping the same conditions on A) to

Then if $NF \vdash (\phi \to (\forall\sigma)(A(\sigma) \to \phi^\sigma)) \to \phi$ then $NF \vdash \phi$.

except for the trivial cases where ϕ is already a theorem, since a single application of Peirce's law would yield $NF \vdash \phi$!

We could show that involutions will aways do the trick if we could show that $(\forall\sigma)(\exists\tau)((j\sigma\tau\sigma^{-1})^2 = 1)$ but that seems hopeless. ... to here.

Still groping my way through here ...

The complementation permutation is a product of two permutations both of which commute with everything in J_1 and with each other. Let $A(\sigma)$ say that σ is the permutation that swaps finite sets with their complements. Let $B(\tau)$ say that τ is swaps infinite coinfinite sets with their complements. Suppose we are in a model where duality fails. Then the following is true

$$(\exists\tau)(\exists\sigma)(B(\tau) \wedge A(\sigma) \wedge \phi^{\tau\sigma})$$

holds for some ϕ wot is false (That is, complementaion has changed the truth-value of something). We now apply the lemma to obtain

$$(\exists\tau)(B(\tau) \wedge V^\tau \models (\exists\sigma)(A(\sigma) \wedge \phi^{\tau\sigma}))$$

This works beco's $\sigma^{\tau_n} = \sigma$. This is true beco's σ and τ commute (since their supports are disjoint) and σ commutes with j of anything. But we don't seem to get anything out of it. I had been sort-of hoping that we would find that precisely one of $\{\sigma, \tau\}$ changes the truth-value of ϕ ...

Let's apply this relettering manipulation to the situation where there is a definable permutation that we have in mind and we want to see what holds in the permutation model it gives rise to. What does V^τ believe happens in the model arising from, say, the Ackermann permutation? The first thing to note is that V^τ is isomorphic to $V^{j^n\tau}$ for n suf large. So let us consider

$$(x \in \sigma(y))^\tau$$

where σ is a permutation definable without parameters in a stratified way. The first stage of relettering goes through as before. The second stage relies on the-property-of-being-σ being stratified. Some details are in order.

Recall that we replace every atomic subformula $x \in y$ in $(\phi^\sigma)^\tau$ by $H_{n-1}(\tau, x) \in H_{n+1}(\tau, \sigma) \cdot H_n(\tau)(y)$.

Now every occurrence of σ anywhere in formula 9 is in an occurrence of $H_{n+1}(\tau, \sigma)$ so we can reletter to get

$$H_{n-1}(\tau, x) \in \sigma \cdot H_n(\tau)(y)$$

or, alternatively, by

$$x \in j(H_{n-1}(\tau))^{-1} \cdot \sigma \cdot H_n(\tau)(y)$$

Now $j(H_{n-1}(\tau))^{-1}$ is $\tau \cdot j(H_n(\tau))^{-1}$ so we can write this as

$$x \in \tau \cdot (\sigma^{H_n(\tau)})(x)$$

Now we want to reletter $\sigma^{H_n(\tau)}$ as σ but to do this we need to be sure that $A(\sigma) \longleftrightarrow A(\sigma^{H_n(\tau)})$. This is where we need the collection of permutations-that-are-A to be closed under conjugation by things in J_n and where—by the same token—τ must be in J_n. Thus we discover that

$$V^\tau \models \phi^\sigma$$

(where '$x = \sigma(y)$' is a homogeneous formula without parameters) is equivalent to

$$V^{j^n \tau \cdot \sigma} \models \phi$$

...for n large enough.

This in turn means that:

Corollary 34

If *we can obtain a model of ϕ by recourse to a chain of models $\langle \mathcal{M}_0, \mathcal{M}_1 \ldots \mathcal{M}_k \rangle$ where*
(i) $\langle V, \in \rangle = \mathcal{M}_0$ and $\mathcal{M}_k \models \phi$ and
(ii) each $\mathcal{M}_{i+1}$ is a permutation model arising from a permutation in $\mathcal{M}_i$ definable by a stratified formula without parameters;

then *we can find a permutation τ such that $V^\tau \models \phi$; and the method of proof above will reveal at least one such τ that is definable by a stratified formula without parameters.*

Here is another way to do it.

Corollary 35

Let $A_0 \ldots A_{k-1}$ be stratified properties of permutations and let P_k be ϕ. For $0 \leq i \leq k$ let P_{i-1} be $(\exists \sigma)(A_{i-1}(\sigma) \wedge V^\sigma \models P_i)$. Then

$$NF \vdash P_1 \to \Diamond \phi$$

While we are about it, the manipulations of lemma 27 (the iteration-relabelling lemma) enable one to prove something that—although unsurprising—is worth establishing: *if you can make ϕ true in a permutation model when it wasn't true in the base model, then in that permutation model you can find another permutation model in which ϕ is once again false.*

Corollary 36 $NF \vdash \phi \to (\forall\tau)(V^\tau \models (\exists\sigma)(\phi^\sigma))$

Proof: One transforms

$$\phi \to (\forall\tau)(V^\tau \models (\exists\sigma)(\phi^\sigma))$$

to

$$\phi \to (\forall\tau)((\exists\sigma)(\phi^\sigma))^\tau$$

and then to

$$\phi \to (\forall\tau)((\exists\sigma)(\phi^\sigma))^\tau$$

and then—by the devices of the lemma 27 (the iteration-relabelling lemma)—to

$$\phi \to (\forall\tau)(\exists\sigma)(\phi^{\tau\cdot\sigma})$$

Now this last is clearly a theorem, since one just takes σ to be τ^{-1} ■

6.1 The Modal Logic

It will be sometimes useful to think of invariance modally: thus ϕ is T-invariant iff $T \vdash \Diamond\phi \longleftrightarrow \Box\phi$.

We start with the propositional logic. Any proof in *NF* of a wff ϕ translates uniformly to a proof of ϕ^τ *with τ free*. This shows that the rule of necessitation holds. $\Box$ clearly distributes over $\to$ since it is a secret $\forall$; $\Box p \to p$ follows similarly.

In this section S is a normal subsemigroup of J_0, and $\Box\phi$ will say that $(\forall\sigma \in S)(\phi^\sigma)$. 'Subsemigroup' sounds a bit complicated, but we have this extra generality because we don't (apparently) need to consider inverses, since the inverse operation on J_0 doesn't seem to have any *logical* meaning.

6.1.1 The K4 axiom

The K4 axiom is $\Box p \to \Box\Box p$.

Lemma 37 $\Box\phi(\vec{x}) \to \Box\Box\phi(\vec{x})$

We will assume the antecedent:

$$\Box\phi(\vec{x})$$

and gradually transform it into the result of prefixing another $\Box$ on the front. The antecedent is

$$(\forall \sigma \in S)(\phi(\vec{x})^{\sigma})$$

Now S is a normal subsemigroup so, for any σ in S, we must have $\sigma^{H_n(\tau)} \in S$ (by normality); if, additionally, we have $\tau \in S$ then $\tau \cdot \sigma^{H_n(\tau)} \in S$ (by composition).

So we infer in particular that

$$(\forall \tau \in S)(\forall \sigma \in S)(\phi(\vec{x})^{\tau \cdot \sigma^{H_n(\tau)}})$$

Now let's look at the atomic formulæ inside this expression. All atomic formulæ

$$x \in y$$

in ϕ have become

$$x \in \tau \cdot \sigma^{H_n(\tau)}(y),$$

which is

$$x \in \underline{\tau \cdot H_n(\tau)^{-1}} \cdot \sigma \cdot H_n(\tau)(y)$$

Now we use lemma 6 part 4 to rewrite the underlined part as $j(H_{n-1}(\tau))^{-1}$ thereby obtaining

$$x \in j(H_{n-1}(\tau))^{-1} \cdot \sigma \cdot H_n(\tau)(y)$$

which is of course

$$H_{n-1}(\tau)x \in \sigma \cdot H_n(\tau)(y)$$

Therefore, so far, we have manipulated $\Box\phi(\vec{x})$ into a formula which is like ϕ but has two leading quantifiers $(\forall \tau \in S)(\forall \sigma \in S)$ and all occurences of $x \in y$ have been replaced by $H_{n-1}(\tau)x \in \sigma \cdot H_n(\tau)(y)$. In particular we can specialise the universal quantifier over the variable 'σ' to hold only for σ of the form $H_{n+1}(\tau, \sigma)$. We have thus obtained from $\Box\phi(\vec{x})$ the formula which is like ϕ but has two leading quantifiers $(\forall \tau \in S)(\forall \sigma \in S)$ and all occurences of $x \in y$ have been replaced by $H_{n-1}(\tau)x \in H_{n+1}(\tau, \sigma) \cdot H_n(\tau)(y)$.

$$(\forall \tau \in S)(\forall H_{n+1}(\tau, \sigma) \in S)(\phi(\vec{x})^{\tau \cdot H_{n+1}(\tau,\sigma)^{H_n(\tau)}})$$

but this is just

$$(\forall \tau \in S)(\forall \sigma \in S)((\phi(\vec{x})^{\sigma})^{\tau})$$

which is

$$\Box\Box\phi(\vec{x})$$

■

6.1.2 The K5 axiom

The K5 axiom is $\Diamond\Box p \to \Box p$. It seems that in order to prove the K5 axiom we need the subsemigroup to be not only normal but closed under j. Are there any normal subsemigroups closed under j—other than $Symm(V)$ itself of course? I do not know of any. But the extra generality does no harm, and who knows, someone might find one! As before '$\Diamond\phi$' means '$(\exists\sigma \in S)(\phi^\sigma)$' and the box is the dual.

Lemma 38 will announce to the world that '$\Diamond$' defined in this way obeys the K5 axiom.

Lemma 38
If j"$S \subseteq S$ then $\Diamond\phi \to \Box\Diamond\phi$

Proof:
$\Diamond\phi \to \Box\Diamond\phi$ is of course $(\exists\sigma \in S)(\phi^\sigma) \to (\forall\tau \in S)(\exists\sigma \in S)((\phi^\sigma)^\tau)$. We will assume the antecedent and infer the consequent. The consequent is

$(\forall\tau \in S)((\exists s)((s \text{ is a permutation in } S) \to \phi^s))^\tau$

and this is

$(\forall\tau \in S)(\exists s)((H_{n+1}(\tau, s) \text{ is a permutation in } S) \to \phi^*)$

where ϕ^* is the result of substituting '$H_{n-1}(\tau, x) \in H_{n+1}(\tau, s) \cdot H_n(\tau)(y)$' for '$x \in y$' throughout ϕ. Being-a-permutation-in-S is a stratified property (S is normal), and we exploit the convention that Greek letters range over permutations to reletter '$H_{n+1}(\tau, \sigma)$' as 'σ' and simplify to

$(\forall\tau \in S)(\exists\sigma \in S)(\phi^{**})$

where ϕ^{**} is the result of substituting '$H_{n-1}(\tau, x) \in \sigma \cdot H_n(\tau)(y)$' for '$x \in y$' throughout ϕ. Now

$H_{n-1}(\tau, x) \in \sigma \cdot H_n(\tau)(y)$ can be rearranged successively to

$H_{n-1}(\tau, x) \in \sigma \cdot (j^n\tau)^{-1} \cdot H_{n-1}(\tau)(y)$ and then to

$x \in j(H_{n-1}(\tau))^{-1} \cdot \sigma \cdot (j^n\tau)^{-1} \cdot H_{n-1}(\tau)(y)$

so these elementary manipulations reveal that ϕ^{**} is in fact equivalent to $\phi^{j(H_{n-1}(\tau))^{-1} \cdot \sigma (j^n\tau)^{-1} H_{n-1}(\tau)}$

Now $j(H_{n-1}(\tau))^{-1}\sigma(j^n\tau)^{-1}H_{n-1}(\tau)$ is twisted-conjugate to $\sigma(j^n\tau)^{-1}$. Therefore $V^{j(H_{n-1}(\tau))^{-1} \cdot \sigma \cdot (j^n\tau)^{-1} H_{n-1}(\tau)}$ and $V^{\sigma(j^n\tau)^{-1}}$ are isomorphic, and $H_{n-1}(\tau)$ is an isomorphism. Specifically we have, for any formula ψ,

(i) $V^{j(H_{n-1}(\tau))^{-1}\sigma \cdot (j^n\tau)^{-1} H_{n-1}(\tau)} \models \psi(H_{n-1}(\tau, x_0) \ldots H_{n-1}(\tau, x_k))$ iff

(ii) $V^{\sigma \cdot (j^n\tau)^{-1}} \models \psi(\vec{x}_0 \ldots x_k)$

so ϕ^{**} is equivalent to

$\phi^{\sigma \cdot (j^n\tau)^{-1}}$

So the question now becomes: Can we find, for any permutation τ in S, a permutation σ in S, such that $\phi^{\sigma \cdot (j^n\tau)^{-1}}$? If π is a permutation such that ϕ^π then so is $\pi \cdot (j^n\tau) \cdot (j^n\tau)^{-1}$, so as long as there is such a π the desired σ will be $\pi(j^n\tau)$. Our assumption was $\Diamond\phi$ so there is such a π in S. Now is $\pi \cdot (j^n\tau)$ in S? It will be if S is closed under postmultiplication by things in j^n"S. This will be true, for example, if j"$S \subseteq S$. ■

This proves the K5 axiom.

A result of Scroggs [1951] says that if there are infinitely many non-elementarily-equivalent possible worlds and the logic is at least S5 then it is precisely S5. For each concrete k we can prove that there is a permutation model containing precisely k Quine atoms. Let τ_k be the permutation[1]

$$\prod_{x \in V} (\{\{x\}\}, V \setminus \{\{x\}\}) \cdot \prod_{0 \leq i < k} (k, \{k\})$$

The left-hand product kills all Quine atoms, and the right-had product adds precisely k Quine atoms. So the V^{τ_k} are pairwise elementarily inequivalent.

■

6.2 The Modal Predicate Logic

For this section '$\Diamond$' has once again its old meaning of "there is a permutation" rather than "there is a permutation in S".

The modal principles that leap to mind are the Barcan formula and the converse Barcan formula, and Fine's Principle. The conjunction of the Barcan and converse Barcan formulas is $\forall x \Box \Phi(x) \longleftrightarrow \Box \forall x \Phi(x)$ and under this interpretation this follows simply from the fact that like quantifiers commute.

Fine's principle H (see Fine [3]). H is the following:

$$(\forall \vec{x})(\forall \vec{y})(\mathrm{Diff}(\vec{x}) \wedge \mathrm{Diff}(\vec{y}) \rightarrow (\Box \Phi(\vec{x}) \longleftrightarrow \Box \Phi(\vec{y})))$$

where $\mathrm{Diff}(\vec{x})$ is the conjunction of all inequations between the $\vec{x}$. The clauses with "Diff" are needed because otherwise we could falsify H trivially by taking Φ to be an open formula that compelled some arguments to be distinct, or to be identical. Later we will write '$(\forall \mathrm{Diff} \vec{x})(\ldots)$' and '$(\exists \mathrm{Diff} \vec{x})(\ldots)$' for the obvious restricted quantifiers.

Remark 39 *Let ϕ be stratified; then*
$NF \vdash (\forall \vec{x})(\mathrm{Diff}(\vec{x}) \wedge \phi(\vec{x}) \rightarrow (\forall \vec{y})(\mathrm{Diff}(\vec{y}) \rightarrow \Diamond \phi(\vec{y})))$

Proof: Suppose
$V \vdash \phi(x_1 \ldots x_k)$ where x_i is of type $s(i)$ in a stratification s of ϕ. Let τ be any permutation.

Then, by Coret's lemma (lemma 3), we infer

$$V \vdash \phi((j^{s(1)+1}\tau(x_1) \ldots j^{s(k)+1}\tau(x_k)).$$

Note that $j^{s(i)+1}\tau = H_{s(i)}(\tau) \cdot \tau$ so we have

$$V \vdash \phi((H_{s(1)}(\tau, x_1) \ldots H_{s(k)}(\tau, x_k))$$

which, by Henson's lemma (lemma 13) is

$$V^\tau \vdash \phi(\tau(x_1) \ldots \tau(x_k))$$

[1] Now, Dear Reader, you see why i prefer the '$H(n, \tau)$' notation to the 'τ_n' notation: it frees the subscript place for its usual *rôle* of *ad hoc* nonce notations.

We aleady know from remark 7 that we can choose τ to send $\vec{x}$ to any $\vec{y}$ we like, as long as they are all different, so we infer

$$V^{\tau} \vdash \phi(y_1 \dots y_k)$$

■

Lemma 40

$$\exists \vec{x}(\mathrm{Diff}(\vec{x}) \wedge \Box\phi(\vec{x})) \quad \rightarrow \quad \forall \vec{x}(\mathrm{Diff}(\vec{x}) \rightarrow \phi(\vec{x})).$$

Proof: Let us suppose the antecedent true and let $\vec{a}$ be a set of witnesses to it. Let $\vec{y}$ be an arbitrary tuple (of the right length) of distinct things so that $\mathrm{diff}(\vec{y})$. Let τ be a permutation sending each a_i to y_i. Since $\vec{a}$ and $\vec{y}$ are sequences without repetitions, this will be possible. We do not care what τ does to the $\vec{y}$ (only what it does to the $\vec{a}$), and in any case it might not be possible to take it to be the product of the transpositions (a_i, y_i) since $\vec{y}$ and $\vec{a}$ may overlap.

We have assumed

$$\Box\phi(\vec{a})$$

so in particular

$$V^{j\tau^{-1}\cdot\tau} \models \phi(\vec{a})$$

We recall that τ is an isomorphism between V and $V^{j\tau^{-1}\cdot\tau}$ and it sends $\vec{a}$ to $\vec{y}$ giving

$$V \models \phi(\tau\vec{a})$$

which is to say

$$\phi(\vec{y})$$

■

The idea now is that we argue as follows.

We have just proved

$$(\forall \mathrm{Diff}\vec{a})(\forall \mathrm{Diff}\vec{y})(\Box\phi(\vec{a}) \rightarrow \phi(\vec{y}))$$

so we can have infer

$$\Box(\forall \mathrm{Diff}\vec{a})(\forall \mathrm{Diff}\vec{y})(\Box\phi(\vec{a}) \rightarrow \phi(\vec{y}))$$

and we can push the box inside (because like quantifiers commute) to obtain

$$(\forall \mathrm{Diff}\vec{a})(\forall \mathrm{Diff}\vec{y})\Box(\Box\phi(\vec{a}) \rightarrow \phi(\vec{y})).$$

Now we can distribute the box over the arrow:

$$(\forall \mathrm{Diff}\vec{a})(\forall \mathrm{Diff}\vec{y})(\Box\Box\phi(\vec{a}) \rightarrow \Box\phi(\vec{y}))$$

Now we use lemma 37 which tells us that $\Box\phi(\vec{x}) \rightarrow \Box\Box\phi(\vec{x})$ to get

$$(\forall \mathrm{Diff}\vec{a})(\forall \mathrm{Diff}\vec{y})(\Box\phi(\vec{a}) \to \Box\phi(\vec{y}))$$

NOTE: when we get round to thinking about how much of this modal stuff restricts to normal subsemigroups of J_0 we will want to know which permutations of the form $j\tau^{-1} \cdot \tau$ belong to the normal subsemigroup we have in mind.

We can now prove H.

Theorem 41

$$\exists\vec{x}(\mathrm{Diff}(\vec{x}) \wedge \Box\phi(\vec{x})) \quad \to \quad \forall\vec{x}(\mathrm{Diff}(\vec{x}) \to \Box\phi(\vec{x})).$$

Let $\vec{a}$ be a witness to the antecedent as before, so $\Box\phi(\vec{a})$. Then, by lemma **??**, $\Box\Box\phi(\vec{a})$. Now apply lemma 40 with $\Box\phi(\vec{a})$ instead of $\phi(\vec{a})$. ■

6.3 Embedding Relations

Embedding relations between permutation models are relations between τ and σ which assert

$$(\exists f : V \to V)(f \text{ is 1-1} \wedge (\forall\vec{x})(\Phi(\vec{x})^\sigma \longleftrightarrow \Phi(f`\vec{x})^\tau))$$

where Φ is some suitable predicate. As long as the first-order theory of an embedding relation is invariant we can pretend that we are looking at J_0 from outside. It would be grossly pathological if the individual permutation models made differing allegations about the family of all permutation models, for then we could see from outside that some were right and others wrong! Fortunately this does not arise.

Proposition 42 *The first-order theory of any embedding relation is invariant.*

The proof in Forster [1987a] has not been improved on so far, and there is little point in recapitulating it here. I stated and proved it only for first-order theories of embedding relations: in fact the same proof will work for the nth-order theory, for each n. Although we now know that the theory of any embedding relation is invariant, we still have absolutely no idea what any of them contain! The next result makes this explicit.

Proposition 43 *Consider a language extending set theory with new primitives for various embedding relations as above. Let $\Phi(\ )$ be some expression with one free (Greek) variable in this language, and let $A(\ )$ be normal and stratified.*

Let us suppose

$$\vdash (\exists\sigma)(\Phi(\sigma))$$

and

$$\vdash (\forall\sigma)(\Phi(\sigma) \to A(\sigma)).$$

Then

$$\vdash (\forall\sigma)(A(\sigma)).$$

In other words, facts about embedding relations between permutation models tell us nothing about the group-theoretic properties of the permutations involved.

Proof: We have $\vdash (\forall\sigma)(\Phi(\sigma) \to A(\sigma))$ so we will have $\vdash (\forall\tau)(\forall\sigma)(\Phi(\sigma) \to A(\sigma))^\tau$. With the customary relettering for σ, this will become $\vdash (\forall\tau)(\forall\sigma)(\Phi(\tau\cdot\sigma) \to A(\sigma))$. So as long as something is Φ, everything must be A. ■

We find ourselves in a situation very similar to that in which theorem **??** placed us: we would like to have a stronger version (with Φ, A, as above):

$$[(\exists\sigma)(\Phi(\sigma)) \wedge (\forall\sigma)(\Phi(\sigma) \to A(\sigma))] \to (\forall\sigma)(A(\sigma))$$

but, as before, the technique of the proof which gave us the weak version will not deliver the strong. This time I will omit the proof that the strong version is at least invariant.

7 Which subgroups preserve what?

The theme common to the results in this section is that group-theoretic information about how simple a permutation is (in this case, that it's in $C(J_1)$) gives us information about the kind of sentence whose truth-value it can change. This suggests to me a programme:

Given a formula ϕ we want to show things like

- If $\neg\phi$ then the least n such there is $\sigma \in C(J_n)$ such that ϕ^σ is at least $k = k(\phi)$;
- If ϕ then the least n such there is $\sigma \in C(J_n)$ such that $(\neg\phi)^\sigma$ is at least $k = k(\neg\phi)$.

These integers are not always the same. For example, we can always add Quine atoms by using something in $C(J_2)$ (the transposition $(\emptyset, \{\emptyset\})$ commutes with everything in J_2) but to be sure of getting rid of them we need something in $C(J_3)$ (the product of all transpositions $(\{\{x\}\}, V \setminus \{\{x\}\})$ commutes with everything in J_3). (See remark 50 in this connection.) We now want a theorem that tells us how $k(\phi)$ depends on the dysstratification of ϕ. The above illustration of a case where $k(\phi) \neq k(\neg\phi)$ shows that the value of $k(\phi)$ will depend not just on the dysstratification of ϕ but also on its quantifier structure.

Another way of announcing this programme is to say, given a subset $X \subseteq \Sigma(V)$, supply a syntactic characterisation of $\{\phi : (\forall\sigma \in X)(\phi \longleftrightarrow \phi^\sigma)\}$. **Are there any nice theorems of this form?**

Of course this will commit us to having an analysis of dysstratification. Curious fact about this is that as n gets larger, the class Γ_n of formulæ preserved by permutations in $C(J_n)$ gets *smaller* so we are looking at a *shrinking* hierarchy of unstratified formulæ not a growing one. This is clearly a complicated situation.

Thinking aloud. Let S be a set closed under complement (possibly 1-symmetric), and let σ be

$$\prod_{x \in S} (x, V \setminus x)$$

Let ϕ be any sentence whatever, then ϕ^σ is a self-dual formula $\Phi(S)$ where S appears only *after* $\in$.

Now suppose ϕ is a sentence preserved by permutations in $C(J_1)$. Then we have

$$\phi \longleftrightarrow (\forall S)(A(S) \to \Phi(S))$$

where $A(S)$ says that S is 1-symmetric and is closed under complementation, and $\Phi(S)$ is a self-dual expression wherein all occurrences of 'S' are to the right of an $\in$.

But this is merely a fatuous rewriting of the definition. Quite how pointless this exercise is is underlined by the fact that if we had taken instead of $C(J_1)$ the much less natural group of those permutations that consist of products of permutations $(x, V \setminus x)$ we would have that any ϕ preserved by permutations in that group satisfies

$$\phi \longleftrightarrow (\forall S)(A(S) \to \Phi(S))$$

where $A(S)$ just says that S is closed under complement, and that really is just fatuous.

Let's spell out what Φ is. Sse ϕ is in PNF and has a quantifier prefix containing the vbls $x_1 \ldots x_n$, and a matrix M. Then Φ is the formula with the same quantifier prefix and a matrix that is the conjunction of all conditionals of the form

$$C \to M^C$$

where C is one of the 2^n conjunctions of atomic formulæ $x_i \in S$ or $x_i \notin S$ and M^C is obtained from M by replacing every occurrence of '$x_i \in x_j$' by '$x_i \notin x_j$' if C contains '$x_j \in S$' and leaving it alone o/w.

The class of self-dual formulæ $(\forall x)(A(x))$ where there is no occurence of x to the left of an $\in$; is it a natural class?

> If f is a stratified but inhomogeneous operation, definable by a formula using n types, then in order to find σ s.t. V^σ contains a fixed point for f we have to consider things in $C(J_n)$, and similarly to kill off all such fixed points.

(Actually this isn't true. As long as there is an x s.t. $|x| = |f(x)|$ then any permutation extending the bijection between them will do. For all we know, such permutations might be in C_n for n quite small: it all depends on x.)

The formulæ asserting the existence or nonexistence of fixed points for such f are unstratified in fairly uncomplicated ways, so it's not major, but it is a start. What it does do is force us to think about degrees-of-unstratification of formulæ, which is something that we have all managed not to think about for many years.

Conjecture 44 *No permutation in $C(J_1)$ can change the truth-value of any self-dual sentence*

...This seems hard to prove. Consider the self-dual sentence that says that there is [no] Boffa atom. I don't see how to prove that this is invariant under permutations in $C(J_1)$

Here's an illustration that might help. NQA is the self-dual statement that says that there are no Quine atoms or Quine antiatoms:

$$(\forall y)(\exists x_1 x_2)((x_1 \in y \longleftrightarrow x_1 = y) \land (x_2 \in y \longleftrightarrow x_2 \neq y)) \tag{NQA}$$

The dual of this formula is the same formula up to rearrangement of conjuncts *and relettering of variables* (α-conversion). It's now easy to show that NQA^σ is equivalent to NQA as long as $\sigma(x)$ is always either x or $V \setminus x$. Pick up a random y. Then by NQO there are witnesses a_1 and a_2 to the two existentially bound x_1 and x_2. If y is fixed by σ then the witnesses we want for x_1 and x_2 are a_1 and a_2 as before. If y is moved then the witnesses are a_2 and a_1. But y was arbitrary ... UG ...

What did this rely on? It relies on the fact that NQA is $\forall^*\Gamma$ and that the only variables to appear to the right of an '$\in$' are universally quantified and are in the first block. It doesn't seem to rely on $\sigma \in C(J_1)$; the weaker condition $(\forall x)(\sigma(x) = x \quad \lor \quad \sigma(x) = V \setminus x)$ seems to be sufficient.

A tho'rt: The set of sentences preserved by permutations in the group $\{1_V, c\}$ is precisely the set of self-dual sentences. So the set of sentences preserved by permutations in $C(J_1)$ might be smaller. OTOH it might not.

We can move in two directions. For each $X \subseteq \Sigma(V)$ we can ask which expressions are preserved by permutations in X. We can also, for each expression ϕ, ask about the class $\{\sigma \in \Sigma(V) : \phi \longleftrightarrow \phi^\sigma\}$. Does it happen that there is a class Γ of formulæ and a class $X \subseteq \Sigma(V)$ s.t X is the class of permutations preserving Γ and Γ is the class of sentences preserved by permutations in X? e.g. Start with $\{1_V, c\}$. The set of sentences preserved by everything in this is the set of self-dual sentences. But what is the collection of permutations that preserves all self-dual sentences?

Watch the following carefully.

Suppose $A()$ is a nice normal property of permutations such that $A(\sigma) \to \sigma$ preserves formulæ in some class Γ containing all theorems:

$$\phi \to (\forall \sigma)(A(\sigma) \to \phi^\sigma)$$

holds for all $\phi \in \Gamma$. The displayed formula is a theorem and so is in Γ and therefore is preserved by permutations in A, whence

$$(\forall \tau)(A(\tau) \to (\phi \to (\forall \sigma)(A(\sigma) \to \phi^\sigma))^\tau$$

This simplifies (using lemma 27) to

$$(\forall\tau)(A(\tau) \to (\phi^\tau \to (\forall\sigma)(A(\sigma) \to \phi^{\tau\cdot\sigma})))$$

But ϕ^τ is just ϕ so this becomes

$$(\forall\tau)(A(\tau) \to (\phi \to (\forall\sigma)(A(\sigma) \to \phi^{\tau\cdot\sigma})))$$

and eventually

$$(\forall\tau)(\forall\sigma)(A(\tau) \wedge A(\sigma) \to (\phi \to \phi^{\tau\cdot\sigma}))$$

Notice that if we had had a normal predicate B with the same property as A then we could have proved

$$(\forall\tau)(\forall\sigma)(A(\tau) \wedge A(\sigma) \to (\phi \to \phi^{\tau\cdot\sigma}))$$

This doesn't tell us that $G =: \{\sigma : \sigma$ preserves formulæ in $\Gamma\}$ is closed under product but it does tell us that if it contains σ and τ each of which belong to a normal subset of D then their product is in D. That is to say, the union of the normal subsets of D is closed under product. My guess is that this is much less use than one thinks, because D almost certainly has no non-trivial normal subsets.

Consider the following general situation:

There are sets A and B, and a relation $R \subseteq A \times B$. (Secretly A is $\Sigma(V)$, B is the set of all formulæ, and R is "preserves")

There are two functions

$f : \mathcal{P}(A) \to \mathcal{P}(B)$ and $g : \mathcal{P}(B) \to \mathcal{P}(A)$

defined so that $f(A') = \{b \in B : (\forall a \in A')(R(a, b)\}$

and $g(B') = \{a \in A : (\forall b \in B')(R(a, b)\}$.

Clearly $f \cdot g$ and $g \cdot f$ are monotone functions on a CPO and have lots of fixed points. The least fixed point might be the singleton group $\{1_V\}$ so the interesting fixed points are the minimal fixed points above the singletons. Are they groups? They're certainly closed under j, and they all contain 1_V. Are they closed under composition or inverse? Can't see any reason why they should be. The fixed points form a lattice, so the intersection of two minimal nontrivial fixed points must be the trivial fixed point

Imre has persuaded me that the operations $f \circ g$ and $g \circ f$ are idempotent. Let A contain a lot of σs. $g \circ f$ adds any τ that preserves everything preserved by the σs.

Now we add a π that preserves everything preserved by the σs and by τ. Suppose p is preserved by the σs and by τ. Everything preserved by the σs is also preserved by τ so being-preserved-by-the-σs-and-by-τ is the same as being-preserved-by-the-σs. So if π preserves everything-preserved-by-the-σs-and-by-τ then it also preserves everything preserved by the σs, and so should have been added at the same time τ was. Ergo idempotence.

But *is* $\{1_V\}$ a fixed point? Is 1_V the only permutation that preserves everything? If we start with a model of duality then this least fixed point is $\{1_V, c\}$. So some of these allegations are not invariant. We want the invariant ones....

Let f be a stratified and inhomogeneous function, so there there is a formula $\phi(x_1, x_2)$ with two free variables of types n and $n+1$ which says that $x_2 = f(x_1)$. Then $V^\sigma \models (\exists x)(x = f(x))$ iff $V \models (\exists x)(\phi(H_n(\sigma, x), H_{n+1}(\sigma, x))$. This is $V \models (\exists x)(\phi(H_n(\sigma)(x), j^{n+1}\sigma \cdot H_n(\sigma)(x))$. Now $H_n(\sigma)$ is a permutation so we can reletter $H_n(\sigma)(x)$ as x getting the equivalent $V \models (\exists x)(\phi(x, j^{n+1}\sigma(x)))$.

This tells us that $V^\sigma \models (\exists x)(x = f(x))$ iff $j^{n+1}\sigma$ sends some x to $f(x)$.

Now suppose $j^{n+1}\sigma$ commutes with τ, so that if $j^{n+1}\sigma$ sends x to $f(x)$. $f(x)$ will be "less symmetric" than x, in that if x is k-symmetric then $f(x)$ is $k+1$-symmetric. We want τ to be unable to tell apart k-equivalent things (such as x), but sometimes able to tell apart $k+1$-symmetric things (such as $f(x)$). Then $(j^{n+1}\sigma)^\tau$ (which is of course $j^{n+1}\sigma$) will send x to both $f(x)$ and to $\tau(f(x))$—which will be different.

That's the plan. Now for the execution.

I think part of the idea has to be that if you are a permutation that sends x to $f(x)$ then you cannot be k-symmetric for $k < n$. Hmmmmmmm i think we need at least one of the xs to be symmetric.

7.1 The poverty of $C_{J_0}(J_1)$

This is a particularly easy group to understand. And its members do not give any interesting relative consistency proofs. Later we will look at $C_{J_0}(J_n)$ for larger n.

Remark 45

$$C_{J_0}(J_1) \quad \subseteq \quad \{\sigma : (\forall x)(\sigma(x) = x \ \vee \ \sigma(x) = V \setminus x)\} \quad \subseteq \quad C_{J_0}(\{c, 1_V\}).$$

Proof: First inclusion:

Suppose $\sigma \in C_{J_0}(J_1)$. Let τ be any permutation whatever. Then

$$\tau``\sigma(x) = \sigma(x)$$

iff (commutativity)

$$\sigma(\tau``x) = \sigma(x)$$

iff (because σ is a permutation)

$$\tau``x = x$$

So τ fixes $\sigma(x)$ setwise iff it fixes x setwise. But τ was arbitrary. It follows easily that $\sigma(x)$ must be x or $V \setminus x$.

Second inclusion:

Assume $(\forall x)(\pi(x) = x \vee \pi(x) = c(x))$. We will show that $\pi \in C_{J_0}(\{c, 1_V\})$.

If $\pi(x) = c(x)$ then $\pi \cdot c(x) = x$ so $c \cdot \pi(x) = \pi \cdot c(x) = x$.

If $\pi(x) = x$ then $\pi \cdot c(x) = c(x)$ so $c \cdot \pi(x) = c(x) = \pi \cdot c(x)$.

■

Both these inclusions are proper: $\prod_{x \in \iota``V} (x, V \setminus x)$ is a counterexample to the converse of the first inclusion. The second inclusion cannot be reversed because $J_1 \subseteq C_{J_0}(\{c, 1_V\})$.

Let us note now that $(\exists\sigma)(y = \sigma``x)$ is an equivalence relation. Let us write it $\sim_1$, and let us write the equivalence class of x under $\sim_1$ (the orbit of x under J_1) as $[x]$. What we have shown is that, for $\pi \in C_{J_0}(J_1)$ for each x, π must either fix all members of $[x]$ or send them all to their complements. That is, we can code members of $C_{J_0}(J_1)$ by the equivalence classes whose members they fix. If we now identify $[x]$ and $[-x]$ by $\approx$ we see that $C_{J_0}(J_1)$ is precisely the additive part of the boolean ring on $(\mathcal{P}^2(x)/\sim_1)/\approx$.

This apparently trivial fact—that every permutation in J_1 commutes with complementation—turns out to be surprisingly useful.

Antimorphisms:

This next result tells us that—in at least some respects—if $\pi \in C_{J_0}(J_1)$ then V and V^π do not differ greatly.

Remark 46 *If $\pi \in C_{J_0}(J_1)$ then V and V^π have the same antimorphisms and the same automorphisms.*

Proof: σ is an antimorphism of V^π iff $(j\sigma)^{-1} \cdot c \cdot \pi \cdot \sigma = \pi$. In particular σ is an antimorphism of V iff

$$(j\sigma)^{-1} \cdot c \cdot \sigma = 1$$

compose both sides with π

$$\pi \cdot (j\sigma)^{-1} \cdot c \cdot \sigma = \pi$$

Now π commutes with $j\sigma$ because it is in $C(J_1)$, and it commutes with c by remark 45 , so we can push it inside to get

$$(j\sigma)^{-1} \cdot c \cdot \pi \cdot \sigma = \pi$$

which says that σ is an antimorphism of V^π.

■

Theorem 47 *V and V^σ have the same symmetric sets.*

Proof: If x is symmetric then the sequence $\langle H_n(\sigma)^{-1}(x) : n \in \mathbb{N}\rangle$ is eventually constant, and the function sending x to this eventually constant value is an isomorphism. Let's call it σ_∞.

$$x \in y$$

iff for any $n \in \mathbb{N}$

$$(H_n(\sigma))^{-1}(x) \in j(H_n(\sigma))^{-1}(y)$$

By lemma 6 this is

$$(H_n(\sigma))^{-1}(x) \in \sigma \cdot (H_{n+1}(\sigma))^{-1}(y)$$

But, for large enough n, $(H_{n+1}(\sigma))^{-1}(y) = (H_n(\sigma))^{-1}(y) = \sigma_\infty(y)$ so we have

$$\sigma_\infty(x) \in \sigma \cdot \sigma_\infty(y)$$

We need to prove the other direction. Suppose x is an n-symmetric thing in V^σ that is to say $H_n(\sigma, x)$ is n-symmetric ...

So σ_∞ is an isomorphism between the symmetric sets of V and the symmetric sets of V^σ.

Not hard to show that σ_∞ is injective. After all if x and y are both symmetric then for large enough k we have $H_k(\sigma)^{-1}(x) = H_k(\sigma)^{-1}(y)$. But of course $H_k(\sigma)^{-1}$ is injective. ■

Now suppose $V = SYMM$. Must σ_∞ be surjective? No: we might add a Quine atom.

Corollary 48 *There is a canonical antimorphism on the symmetric sets and it is of order 2.*

Proof: If x is symmetric, the sequence $\langle H_n(c, x) : n \in \mathbb{N} \rangle$ is eventually constant. The key idea is that if x is n-symmetric, then so is $c(x)$: $j^n\sigma \cdot c(x) = c \cdot j^n\sigma(x)$ beco's c commutes with everything in J_n, and $j^n\sigma(x) = x$ beco's x is n-symmetric, so $c \cdot j^n\sigma(x) = c(x)$ and $j^n\sigma \cdot c(x) = c(x)$. But the same argument shows that $j^n\sigma \cdot (j^k c)(x) = (j^k c)(x)$ for any $k < n$. Therefore if x is n-symmetric and $k > n$ we have $H_k(c, x) = H_n(c, x)$. (We can "push in" the $j^k(c)$ on the outside of $H_k(c, x)$). ■

Corollary 49 *If V is a model wherein every set is symmetric and $\sigma \in C(J_1)$ then $V \simeq V^\sigma$.*

Proof: If $\sigma \in C(J_1)$ then σ_∞ is a composition of a lot of involutions all of which commute with each other. So it is an involution itself. So it is onto! So if V is a model of $V = SYMM$, so too is V^σ. ■

In particular you cannot add a Quine atom (at least to a model of $V = SYMM$) by a permutation in $C(J_1)$: the simplest permutation that does it is the transposition $(\emptyset, \{\emptyset\})$. (The assumption that $V = SYMM$ may be necessary here: suppose V contains a Quine antiatom, x. Then, in $V^(x, V \setminus x)$, x has become a Quine atom. Admittedly $(x, V \setminus x) \notin C(J_1)$, but perhaps we can tweak it slightly.)

We ought to be able to spice this up to say something like: things in $C(J_1)$ add no new nonsymmetric sets...

But the transposition $(\emptyset, \{\emptyset\})$ is in $C(J_2)$ and adds a nonsymmetric set so this is best possible.

Remark 50 .

(i) No permutation in $C(J_1)$ can change the truth-value of the sentence

$$(\exists x)(x = \{x\} \vee x = V \setminus \{x\})$$

but

(ii) there is a permutation in $C(J_2)$ that does.

Proof:

(i) Suppose $\sigma \in C(J_1)$ and $V^\sigma \models (\exists x)(x = \{x\} \vee x = V \setminus \{x\})$. There are two cases to consider: $x = \{x\}$ and $x = V \setminus \{x\}$. However they are dual, so it is sufficient to consider the first.

Suppose $V^\sigma \models x = \{x\}$. Then either x was a Quine atom to start with, or $\sigma(x) = V \setminus x = \{x\}$. In other words, x was a Quine antiatom. So $V^\sigma \models x = \{x\}$ implies $V \vdash x = \{x\} \vee x = V \setminus \{x\}$. And dually $V^\sigma \models x = V \setminus \{x\}$ implies $V \vdash x = \{x\} \vee x = V \setminus \{x\}$. So $V^\sigma \models (\exists x)(x = \{x\} \vee x = V \setminus \{x\})$ implies $V \models (\exists x)(x = \{x\} \vee x = V \setminus \{x\})$.

For the converse, suppose $V \models (\exists x)(x = \{x\} \vee x = V \setminus \{x\})$. In particular, suppose $V \models x = \{x\}$. If $\sigma \in C(J_1)$ then either σ fixes x (in which case $V^\sigma \models x = \{x\}$) or it swaps it with $V \setminus \{x\}$ (in which case $V^\sigma \models x = V \setminus \{x\}$). Either way $V^\sigma \models (\exists x)(x = \{x\} \vee x = V \setminus \{x\})$. The case where $V \models x = V \setminus \{x\}$ is dual and will be omitted.

(ii) To be sure of killing off all Quine atoms it's not enough to swap every singleton with its complement by means of the permutation σ:

$$\prod_{x \in V} (\{x\}, V \setminus \{x\}) \qquad (\sigma)$$

since σ turns Quine antiatoms into Quine atoms as we have just seen. What we need is the permutation τ that swaps every singleton^2 with its complement:

$$\prod_{x \in V} (\{\{x\}\}, V \setminus \{\{x\}\}) \qquad (\tau)$$

It is no accident that $\sigma \in C(J_1)$ but $\tau \in C(J_2) \setminus C(J_1)$. (The proof that τ abolishes Quine atoms is left as an exercise for the reader. An answer can be found in [**?**]. It also kills off Quine antiatoms.

■

7.1.1 Cycle Types not informative

It's easy to show that information about the cycle type of σ tells us little or nothing about what happens in V^σ.

The key fact here is that V^σ and $V^{j\sigma}$ are isomorphic. Let us say the *ultimate* cycle type of σ is the asymptotic behaviour of the cycle type of $j^n\sigma$ as $n \to \infty$.

In this connection we should record the following observation:

Remark 51 .

1. *if τ has infinite cycles, $j\tau$ has cycles of all sizes;*
2. *if τ has cycles of arbitrarily large finite sizes, then $j\tau$ has infinite cycles;*
3. *if σ has only (bounded) finite cycles whose lengths are in $I \subset \mathbb{N}$ then $j^n\sigma$ eventually has cycles of all sizes that divide $LCM(I)$.*

This means that the family of ultimate cycle types is very restricted. σ might eventually have infinite cycles and cycles of all finite lengths, and lots of all of them. Or it might have only finite cycles, in which case there is a finite bound on the length of the cycles.

It might well be the case that every skew-conjugacy class meets every conjugacy class.

GC ("group choice") is the principle that says that every set of finite-or-countable sets has a selection function. It is precisely what is needed to establish that two permutations of the same cycle type are conjugate. (Two permutations σ and τ have the same cycle type if there is a bijection between the set of cycles of σ and the set of cycles of τ that preserves cardinality.

σ and $j\sigma$ are twisted-conjugate. It seems to be as basic a fact about twisted-conjugacy as $\sigma \sim \sigma$ is about ordinary conjugacy. So far I know of no facts about twisted-conjugacy that are not implied by proposition **??**. It neither implies nor is implied by ordinary conjugacy. (x, y) moves only two things but $j(x, y)$ moves infinitely many so they cannot be conjugate although they are twisted-conjugate; in a universe with no Quine atoms, $(V, \emptyset)$ adds no Quine atoms, but its conjugate $(V, \{V\})$ does, so they are not twisted-conjugate.

One might wonder if there are nontrivial facts about twisted-conjugacy (which are as plausible as the consequences GC has for conjugacy) that one can prove only with GC, or with some more sophisticated principle yet to be discovered. Is there some banal combinatorial condition on σ and τ which a minimal amount of choice will imply is equivalent to skew-conjugacy?

In general we may expect a lot of trouble with twisted-conjugacy: $\sigma \sim_{(i)} \tau$ is not stratified and it will be very difficult to prove theorems about it.

Since $\sim_{(i)}$ is nonempty there is an obvious temptation to quotient J_0 by it and deal instead with the equivalence classes. Unfortunately there is no reason to suppose that the quotient has a natural group structure and if we force matters by killing all permutations of the form $\sigma\tau^{-1}$ where $\sigma \sim_{(i)} \tau$ we kill all commutators, with disasterous results. The proof is as follows:

For any σ and π we have $\sigma \sim_{(i)} (j\pi)^{-1} \cdot \sigma\pi$ so the kernel must contain (i) $\sigma \cdot \pi^{-1} \cdot \sigma^{-1} \cdot j\pi$ and its inverse (ii) $(j\pi)^{-1} \cdot \sigma \cdot \pi \cdot \sigma^{-1}$. Now substituting τ for σ in (ii) and multiplying on the left by (i) we get $\sigma \cdot \pi^{-1} \cdot \sigma^{-1} \cdot \tau \cdot \pi \cdot \tau^{-1}$ for arbitrary σ, τ, π. Since σ can be taken to be 1 the kernel will contain all commutators.

Now let x, y and z be three arbitrary sets. Let τ be $(x, y) \cdot (z, \{x\})$, and σ be (y, z). Then the commutator $\tau \cdot \sigma \cdot \tau^{-1} \cdot \sigma^{-1}$ is $(x, \{x\}) \cdot (y, z)$ and accordingly gives rise to a model containing a Quine atom. 1_V itself might not. Thus we could sometimes compel the kernel to contain two permutations which give rise to two nonisomorphic models. Quotienting out by $\sim_{(i)}$ is therefore an unsatisfactory procedure.

7.2 Automorphisms and skew-centralisers

Remark 52 *The group of twisted-centralisers of π is the group of inner automorphisms of V^π.*

Suppose σ twisted-centralises π. That is to say $(j\sigma)^{-1} \cdot \pi \cdot \sigma = \pi$ or, in longhand,

$$(\forall xy)(x \in \pi(y) \longleftrightarrow x \in (j\sigma)^{-1} \cdot \pi \cdot \sigma(y))$$

But we also have

$$(\forall xy)(x \in (j\sigma)^{-1} \cdot \pi \cdot \sigma(y) \longleftrightarrow \sigma(x) \in \pi \cdot \sigma(y))$$

giving

$$(\forall xy)(x \in \pi(y) \longleftrightarrow \sigma(x) \in \pi \cdot \sigma(y))$$

which is to say that σ is an inner automorphism of V^π. The arrows are reversible. ■

Remark 53 *All twisted-centralisers (that are sets) are strongly cantorian.*

Proof: If σ is in the twisted-centraliser of τ then $\tau \cdot \sigma \cdot \tau^{-1} = j\sigma$, so $\lambda\sigma.\{j^{-1}(\tau \cdot \sigma \cdot \tau^{-1})\}$ is the singleton function restricted to the twisted-centraliser of τ. ■

There is a simple cardinality argument we can now use that suggests that the third equivalence relation (elementary equivalence) really is weaker than the first (isomorphism via an internal permutation). The size of the quotient under elementary equivalence must be $\leq 2^{\aleph_0}$. If every element of this quotient is a twisted-conjugacy class then J_0 is the union of $\leq 2^{\aleph_0}$ strongly cantorian sets. The effort involved in showing that this would compel $\aleph(|J_0|)$ to be a cantorian aleph is probably not worth it, since none of these quotients can be expected to be sets. But it is a straw in the wind.

Now let π be a permutation definable by a homogeneous expression $P(x, y)$ which says that $y = \pi(x)$. (When considering such P there is no extra cost in restricting attention to homogeneous P: any such stratified P *must* be homogeneous beco's of Cantor's theorem.) Then π commutes with everything in J_n. Most set existence theorems give rise to witnesses definable by stratified expressions, and a permutation that is definable in this sense will be in $C_{J_0}(J_n)$ for some n. All such permutations are of course symmetric. Realistically we should not harbour any hopes of proving $\Diamond\Phi$ except by using symmetric permutations.

Remark 54 *For every $n \in \mathbb{N}$, if π is in $C_{J_0}(J_n)$, then $Aut(V)$ is a subgroup of $Aut(V^\pi)$.*

Proof: If σ is an automorphism of the base model, then $\sigma = j^n\sigma$ for all n, so $(j^{n+1}\sigma)^{-1} \cdot j^n\sigma = 1$. So, for any π, $\underline{\pi \cdot (j^{n+1}\sigma)^{-1}} \cdot j^n\sigma = \pi$. Now if π is in $C_{J_0}(J_n)$ for sufficiently large n it will commute with $j^n\sigma$, so we can swap the underlined terms to infer $(j^{n+1}\sigma)^{-1} \cdot \pi \cdot j^n\sigma = \pi$ which we can rewrite as $j((j^n\sigma))^{-1} \cdot \pi \cdot j^n\sigma = \pi$ which says that $j^n\sigma$ is an automorphism of V^π. (See the calculations in the proof of remark 52.) ■

So every automorphism of V gives rise to an automorphism of V^π. Every automorphism of V^π that is in J_n arises in this way, but for all we know there might be automorphisms of V^π that aren't in J_n.

This seems to be telling us that definable permutations cannot get rid of automorphisms!

In this context it is noteworthy that the permutation that uses AC_2 to add an automorphism is not definable in this sense.

7.3 niggles

1. Are all the ϕ-topologies T_0? compact? complete? Is the Stone topology compact? (in the sense that if A is a set of formulæ such that, for every finite subset a of A, there is a permutation making a true, then there is a permutation making A true.
2. What algebraic structure is there on the set of twisted-conjugacy-classes?
3. How much of what we know about twisted-conjugacy arises merely beco's j is an injective homomorphism?
4. GC implies that two permutations are conjugate iff they have the same cycle type; is there a similar sufficient condition for being twisted-conjugate?
5. Does every conjugacy class (except presumably $\{1\}$) meet every twisted-conjugacy class?
6. Prove some theorems about the kinds of sentences whose truth-value one can alter by means of permutations in $C_{J_0}(J_n)$, the centraliser of J_n?

   Peter Neumann has supplied me with material about the structure of these groups, the $C_{J_0}(J_n)$, and it may be that this internal structure will tell us something about the models that these elements point to. At all events it should be possible to show that permutations in $C_{J_0}(J_1)$ do very little.

8 Do all models of *NF* have automorphisms?

The following expression says that we can't get rid of automorphisms.

$$(\exists\pi)(\pi = j(\pi)) \rightarrow (\forall\tau)(\exists\gamma)(\gamma^{j^n\tau} = j\gamma)$$

We haven't much hope of proving this unless γ is definable in terms of π and τ. Try $\gamma = \pi^x$ then

$$(\pi^x)^{j^n\tau} = j(\pi^x) = j(x \cdot \pi \cdot x^{-1}) = jx \cdot j\pi \cdot (jx)^{-1} = \pi^{jx}$$

so sufficient that

$$x \cdot j^n\tau = jx$$

or equivalently

$$(jx) \cdot x^{-1} = j^n\tau$$

Now this last thing cannot be true, because any permutation like $(jx) \cdot x^{-1}$ gives rise to a model isomorphic to V whereas $j^n\tau$ is arbitrary. Unfortunately this doesn't show that we *can* get rid of them, merely that the obvious machinery to show that we *can't* will not work: for one thing $(\pi^x)^{j^n\tau} = \pi^{jx}$ does not imply $x \cdot j^n\tau = jx$ and for another we cannot be sure that $\gamma = \pi^x$. The tougher way would be to try and devise a word $W(\tau, \pi)$ in τ and π and definable things so that if $\pi = j\pi$ then $\forall\tau\ W(\tau, \pi)^{j^n\tau} = j(W(\tau, \pi))$.

Suppose we have a functional formula $\phi(w, \tau, \pi)$ and we know

We'd better assume ϕ to be weakly stratified o/w we won't be able to get started. This will give us

$$(\forall\pi, \tau)(\pi = j\pi \to \phi(w, \tau, \pi) \to w^{j^n\tau} = jw)^\sigma$$

8.1 J_∞

The group of all (inner) $\in$-automorphisms is certainly a subgroup of $J_\infty = \bigcap_{n<\omega} J_n$. J_∞ contains all fixed points of j (if there are any) and therefore all automorphisms of $\langle V, \in\rangle$.

What do we know about J_∞? There is no reason to suppose that it is nontrivial, nor that if it is nontrivial it should be a set. If it is a set it is cantorian. We know it is the nested intersection of ω symmetric groups, but this does not tell us a great deal. We know that it has an external automorphism (j) which we would wish had a fixed point. (would a finite cycle under j be any good? Perhaps not!) For reasons which will emerge below it would be nice if it had nontrivial centre.

We know rather more about the J_∞ of a saturated model of $NF + GC$. In such a model it is certainly nontrivial and we know exactly the cycle types of all its elements and can even show (tho' perhaps we need AxCount for this) that the external automorphism j of J_∞ is locally represented by a conjugacy relation. That is to say, for all σ in J_∞ there is τ in J_∞ s.t. $\tau^{-1}\sigma\tau = j\sigma$.

Let $\langle V, \in\rangle$ be a saturated model, so J_∞ is not trivial. Consider permutations τ, $j\tau$.

Claim:

> $\forall\sigma$ for all sufficiently large n $j^n\sigma$ does one of the following three things:
>
> 1. It has $|V|$ n-cycles for all $n \leq \aleph_0$
> 2. For some $k \in \mathbb{N}$ it has $|V|$ n-cycles for all n that divide k

In fact $j^n\sigma$ is eventually of type (1) above iff there is no finite bound on the length of finite cycles under σ

This will involve a fair amount of hard work. We have to use some sort of pigeon-hole principle. The idea is that, for at least one n, the number of things residing in n-cycles under σ is $|V|$. It looks as if we need to assume that V is not the disjoint union of $\aleph_0$ smaller sets, but all we actually need is for this to be eventually true of $j^n\sigma$. We sketch where to go from here. (Is there a generalisation of Bernstein's lemma (using GC) which says that if $\alpha = \alpha^{\aleph_0} = \Sigma_{i\in\mathbb{N}}\beta_i$ then either some $\beta_i \geq \alpha$ or all $\beta_i \geq_* \alpha$? That would probably do) GC is essential for what follows.

- If there are $|V|$ infinite cycles then pretty soon there are $|V|$ cycles of any length.
- Whatever happens there will very soon be $|V|$ fixed points.
- If σ has $|V|$ fixed points and some n-cycles then $j^k`\sigma$ (with k fairly small) will have $|V|$ n-cycles. Use ordered pairs of things of order n and fixed points.
- Eventually $j^n\tau$ will have $T|V|$ n-cycles if it has any at all.

Eventually we should prove:

$(\forall\tau)(\exists m)(\forall n > m)(j^n\tau$ and $j^{n+1}`\tau$ have the same cycle type)

and GC then gives us

$(\forall\tau)(\exists m)(\forall n > m)(j^n\tau$ and $j^{n+1}`\tau$ are conjugate in $J_0)$

so in particular for $\tau \in J_\infty$ τ and $j\tau$ are conjugate in J_0. Now consider the general case of σ, $\tau \in J_\infty$ conjugate in J_0. $(\forall n < \omega)$, σ, τ are j^n`something in $J_\infty$ so we can argue that $j^{-n}`\sigma$ and $j^{-n}`\tau$ are conjugated by something $\gamma \in J_0$. (We have this beco's GC implies that two things in J_0 of the same cycle type are conjugate) so σ and τ are conjugated by $j^n\gamma \in J_n$. Therefore, by saturation of J_∞ they are conjugated by something in J_∞. That is to say,

$$(\forall\tau \in J_\infty)(\exists\sigma \in J_\infty)(\sigma \cdot \tau \cdot \sigma^{-1} = j\tau)$$

This is very pretty: we know the cycle types of all members of J_∞ and we know that any two elements of J_∞ with the same cycle type are conjugated by something in J_∞. Of course what we are really after is showing if possible that J_∞ contains a fixed point for j. So what we really want is to swap the quantifiers around in the above to get:

$$\exists\sigma \in J_\infty\ \forall\tau \in J_\infty\ \ \sigma \cdot \tau \cdot \sigma^{-1} = j\tau$$

for then this σ must be an $\in$-automorphism of V.

Proof:

Suppose there were such a σ. Then for any $\tau \in J_\infty$

$$\sigma \cdot \tau \cdot \sigma^{-1} = j\tau$$

so in particular

$$\sigma \cdot \sigma \cdot \sigma^{-1} = j\sigma$$

so σ is an $\in$-automorphism of V.

In fact it will be sufficient for our purposes that j have a non-trivial fixed point, because the fixed point would also be an $\in$-automorphism of V.

then we would have a theorem:

THEOREM 55 *NF+ AxCount +$GC \vdash \exists \in$-automorphism of V*

The proof would go like this: Work inside a saturated model of NF+ AxCount +GC. J_∞ is a proper class of this model. j is an automorphism of it, and J_∞ is such that all automorphisms are inner. Then throw away the model.

Presumably this won't work beco's J_∞ is a saturated group and any saturated group has too many automorphisms for them all to be inner. We could try the other extreme: add axioms to make J_∞ (when nontrivial) into a group for which all automorphisms are inner. Then there will be an $\in$-automorphism of V as before.

It will be sufficient for J_∞ to have non-trivial centre. For then let τ belong to the centre. Let σ conjugate τ and $j\tau$. But $\tau^\sigma = \tau$ since τ is in the centre, so τ is an automorphism. We can show that in V^σ everything in J_∞ is an automorphism. For

$$(x \in J_\infty)^\tau$$

iff

$$(x \in J_0, x \in J_1 \ldots x \in J_n)^\tau$$

Now $(x \in J_n)^\tau$ is just $(\tau_{n+k}`x \in J_n)$ is $x \in J_n$.

But presumably it is obvious that J_∞ has trivial centre, by some compactness argument ...

9 An Old file of observations on centralisers

PROPOSITION 56 *The symmetric group on $\mathcal{P}^2(x)/\sim_1$ is a subgroup of $C_{n,2}$*

Proof:

If $\pi \in J_2$ it must fix every i-equivalence class (every J_1 orbit), for let x be a $\sim_1$-equivalence class, and let π be $j^2`\sigma$. $\pi`x$ is then $(j^2`\sigma)`x = \{(j\sigma)`y : y \in x\}$ which is $\{\sigma``y : y \in x\}$ which is easily seen to be x.

This shows that the centralisers $C_{n,k}$ start off small but become unmanageably large very quickly as k gets bigger.

■

PROPOSITION 57 *$j : \Sigma_x \to \Sigma_{\mathcal{P}(x)}$ is not elementary.*

Proof:
In the finite case (x finite) this is obvious beco's the groups are all of different sizes. In the infinite case it needs weak AC in the form "In any symmetric group elements of the same cycle type are conjugate".

Consider $\sigma = (0,1)$ and $\tau = (2,3)(4,5)$. These are not conjugate, but $j\sigma$ and $j\tau$ both have only 2-cycles and 1-cycles, and (with the help of a bit of AC) have the same number of each so they have the same cycle type and therefore *are* conjugate. ■

I think we can probably do this without using any AC. Probably worth nailing down.

PROPOSITION 58 *If x is infinite, J_{n+1} is not a normal subgroup of J_n.*

Proof:
It will suffice to prove the case $n = 0$, as the isomorphism under j will do the rest. Think of permutations in J_1 as continuous over $\subseteq$ so that their actions are determined entirely by what they do to singletons. Let $\pi \in J_1$ now be conjugated by a transposition of two infinite sets. The result agrees with π on singletons and so would be identical to π if it were in J_1. ■

The following observation makes sense only if we are conducting these constructions in the context of type theory without the axiom of infinity. A_n is the alternating subgroup of $\Sigma(V_n)$. (N.B. aberrant notation but no real alternative). Then

PROPOSITION 59 $j``\Sigma_{V_n} \subseteq A_{n+1}$

Proof:
As long as V_n is finite every element of Σ_{V_n} is a product of either an even or an odd number of transpositions If σ is $a \cdot b \cdot c \cdots$ then $j\sigma$ is $(ja) \cdot (jb) \cdot (jc) \cdots$ since j is a homomorphism. It will now suffice to show that ja is an even permutation whenever a is a transposition. If a is (x, y) then ja is the product of $(u \cup \{x\}, u \cup \{y\})$ for all u disjoint from $\{x, y\}$. If k is the cardinality of V_n then there are 2^{k-2} such u. Thus everything in the range of j is an even permutation. ■

By analogy with the discussion after proposition 1 we may note that $x \sim_2 y$ defined as

$$(\exists\sigma)(x \sim (j^2`\sigma)`y)$$

is an equivalence relation. We note similarly that if π is in the centraliser $C(J_2)$ then once we know what $\pi`x$ is we know what $\pi`y$ is for any y such that $y \sim_2 x$. This is because if y is $(j^2`\sigma)`x$ then $\pi`y$ is $\pi^{??}(j^2`\sigma)`x = (j^2`\sigma)^{??}\pi`x$. Therefore each permutation in the centraliser $C(J_2)$ corresponds to (at most) one choice of value for each element of the quotient by $\sim_2$. Unfortunately the size of this object is quite hard to compute.

At this stage we can summarise the group-theoretic project: Consider a set x of size n, and its k-times power set $\mathcal{P}^k`x$. There is a natural embedding from $\Sigma_x$ into $\Sigma_{\mathcal{P}^k`x}$, namely j^k. Let $C(n,k)$ be the centraliser of $j^n``(\Sigma_x)$ in $\Sigma_{\mathcal{P}^k`x}$. What do we know about $C(n,k)$?

****

The *Theory of negative types* (TNT) is like the simple theory of types except that we have a type for each member of **Z** rather than just each member of **N**. A simple compactness argument shows that this theory is consistent & does not prove the axiom of infinity. A model of TNT is a **Z**-sequence of sets $\langle V_i : i \in Z\rangle$ where V_{i+1} "looks like" the power set of V_i. Unfortunately it can be shown very simply in all non-trivial set theories that there cannot be a sequence $\langle x_i : i \in \mathbf{Z}\rangle$ where $x_{i+1} = \mathcal{P}(x_i)$. (Curiously this does not depend in any way on the axiom of foundation)

Let us now imagine such a model displayed before us, and concentrate our attention on one level of it, say type 0. Let us consider $C(1)$, $C(2)\ldots$etc. at this type. $C(1)$ is the centraliser (in Σ_{V_0}) of $j``\Sigma_{V_{-1}}$, and $C(n)$ will be the centraliser (in $\Sigma_{V_0}$) of $j^n``\Sigma_{V_{-n}}$. Is it possible for the union of all these centralisers to be actually equal to Σ_{V_0}? This question is important for set theory because this will happen to any model in which every element is the interpretation of a set

10 A message from Peter Neumann

The story is this.

A G-space (for a given group G) is a set Ω equipped with an action of G. Formally an action is a map $\mu : \Omega \times G \to \Omega$. For brevity I'll write $\mu : (\omega, g) \mapsto \omega^g$. Then the two conditions required for an action are

(1) $\omega^1 = \omega$ for all $\omega \in \Omega$;

(2) $(\omega^g)^h = \omega^{gh}$ for all $\omega \in \Omega$ and all $g, h \in G$.

Sometimes it is convenient to think of G-sets as a species of algebraic structure with a collection of unary operators, one for each element of G, satisfying the above identical relations.

You'll find a discussion of what the axioms mean in Chapter 3 (if my memory is right) of the book on Groups and Geometry by Gabrielle Stoy, Edward Thompson and me.

Let G be a group acting on a set Ω. In your case the group is $\Sigma(X)$ and Ω is the n-times power set of X but the theory is completely general. If we want to find the centraliser of G in $\Sigma(\Omega)$ then the language of groups acting on sets turns out to be just right. A permutation of Ω that commutes with every element of G is an isomorphism of Ω to itself in the category of G-spaces. Thus what we are seeking is $Aut_G(\Omega)$.

To find this, what we do is split Ω up into the disjoint union of the G-orbits. Any isomorphism (indeed, any homomorphism) in the category of G-spaces maps orbits to orbits. So collect together those orbits in Ω that are isomorphic as G-spaces: write

$$\Omega = \bigcup m_i \Gamma_i,$$

where orbits Γ_i, Γ_j are not G-isomorphic if $i \neq j$ and m_i is the number (i.e. cardinal number) of orbits G-isomorphic to Γ_i. Any G-automorphism of Ω simply permutes the m_i copies of Γ_i among themselves—for each value of i. Thus the centraliser you want, which I have expressed as $Aut_G(\Omega)$, is the cartesian product $\prod Aut_G(m_i\Gamma_i)$. This 'reduces' (a dangerous word) the problem to finding $Aut_G(\Omega)$ in case Ω is of the form $m\Gamma$.

This is where wreath products come in. The group $Aut_G(m\Gamma)$ is the wreath product $Aut_G(\Gamma) \circ \Sigma(m)$. Informally this means that any automorphism of a G-space which is the disjoint union of m copies of a transitive G-space Γ is obtained by applying individual automorphisms to each of the m separate orbits, and then permuting the set of m orbits (setwise) in any way you please.

To complete the picture we need to know what $Aut_G(\Gamma)$ is for a transitive G-space Γ. Let α be any point of Γ and let H be its stabiliser in G (i.e. the set of all members of G that do not move α). It is not very hard to prove that $Aut_G(\Gamma)$ is isomorphic to $N(H)/H$, where $N(H)$ is the normaliser of H in G.

Quite a lot of this is to be found in the undergraduate text-book 'Groups and Geometry' by Neumann, Stoy and Thompson (OUP 1993). Please forgive me for quoting my own book! I think you'll find most of it also in the pook on Permutation Groups by Dixon and Mortimer which is in the Springer Graduate Text Series a few years ago.

I have been thinking a bit what it all means for your problem and can give you some information if you'd find it useful. But there's a lot more information to be got—I haven't had time to work it out or write it all down yet. You are right that, for $n = 1$, the answer is a cartesian product of cyclic groups of order 2—the number of factors is $1+[n/2]$ if n is finite and it is the number of cardinal numbers less or equal to n if n is infinite. For $n = 2$ I'm getting something of a picture, but it is far from complete. I have half a feeling that for $n \geq 3$ the answer is a little less complicated than for $n = 2$, but I haven't thought enough about this yet.

Does this help a bit? I'll let you know if I get any more about the cases $n \geq 2$.

All best wishes, Πeter

Oberwolfach: 18.viii.98

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We consider in this note (dating from the early eighties!) a doubly-indexed family of groups, some finite, some infinite, which arise in a very natural way in the study of simple type theory. Simple type theory is a *set* theory & a model of it will consist of set X, its power set $\mathcal{P}(X)$, its power set $\mathcal{P}(\mathcal{P}(X))$ and so on, $\mathcal{P}^n(X)$ for each positive integer n. The various $\mathcal{P}^n(X)$ are the *types* that give the theory its name. $\in$ thus holds only between elements of a given type & those of the next type up. The axioms of the simple theory of types ("TST" for short) are

- (i) Extensionality at each type.
- (ii) Set existence at each type: $\{X : \Phi\}$ exists for each Φ

In TST every variable is brutally required to range over one type & one only, & can thus be conveniently thought of as having a type subscript permanently attached to it. Thus the existence of $\{X_n : X_n \in X_n\}$ cannot be an axiom at any type n—$X_n \in X_n$ is not a well-formed formula. Indeed, thus avoiding Russell's Paradox was the reason for inventing this theory in the first place.

If we remember that $\in$ holds only between things of type n and things of type $n+1$, it becomes evident that any permutation π of X can be naturally extended to an ($\in$-) automorphism of the whole model. That is to say the theory has no way of telling apart elements of type 0. We will need a notation for the restriction of this automorphism to any given $\mathcal{P}^n(X)$, where it will of course be a permutation, and this will be $j^n\pi$. If we wish we may think of j as an operation that takes a permutation of a set and returns the obvious permutation of the power set. j is a natural injection from $\Sigma_X \to \Sigma_{\mathcal{P}(X)}$ and indeed j^n will give us an injection from $\Sigma_{\mathcal{P}(X)}$ into $\Sigma_{\mathcal{P}^n(X)}$.

Let us now consider what relations we can define between members of $\mathcal{P}^n$‘x and which other levels we have to talk about in order to make those definitions. Suppose we ask ourselves what relations we can assert between x and y, (elements of $\mathcal{P}^n(x)$) without talking about their members, then it is clear that all we can say is $x = y$ or $x \neq y$. This is why any permutation of x extends to an automorphism of the whole structure. If we allow ourseves to quantify over members of x and y, then we then we can say $x \subseteq y$ for example. If we then allow ourselves to talk about members of members of x, y etc, then we can say even more.

One thing we will want to say about this topic is that if we have a predicate $\Phi(x)$ on members of x which can be specified without talking about members of members of x, then the set of things in x which are Φ is fixed by all permutations of x. There are more complicated versions of this: in fact if we consider a relation $\Phi(x, y)$ between members of $\mathcal{P}^n`x$ which is such that deciding whether or not $\Phi(x, y)$ for any given $x, y \in \mathcal{P}^n`x)$ involves looking only at members of $\mathcal{P}^k`x$ for some $k$ tuple $\vec{x}$, then we know that the relation $\Phi$ is fixed by all permutations of $\mathcal{P}^k`x$, or to be precise we can show

$(\forall x)(\forall y)(\Phi(x, y) \rightarrow \Phi((j^{n-k}`\tau)`x, (j^{n-k}`\tau)`y))$ where x, y are in $\mathcal{P}^n`x$ and $\tau$ is a permutation of $\mathcal{P}^k`X$.

In set theory we have a concept of *definability*, (the unique thing which so-and-so) and in algebra we have a notion of *invariance under permutation*. The reason for interest in the groups I mention is that their definable members consist of (among others) *definable* permutations.

Let us consider a set abstract $\{x : \Phi\}$. The variable x will belong to some type (i.e., have subscript) $n - 1$ say, and thus the set abstract will belong to type n. Of some interest however, is the numerical difference between the type of x and the type of the variable of *lowest* type in Φ. This is because of the following theorem (which I am not proposing to prove)

Proposition 60 *Let $t = \{x : \Phi\}$ be a set abstract of type n, and let the variable of lowest type in Φ be of type n-k. Then t is fixed by every permutation in j^k "$\Sigma_{V_{n-k}}$.*

(What is j^k "$\Sigma_{V_{n-k}}$? V_{n-k} is of course the universe of things at type $n - k$. We take the symmetric group on it, and then lift k times to obtain a subgroup of Σ_{V_n}. t is now fixed by everything in this group). This is simply the way in which simple type theory expresses the fact that definable objects tend to be fixed by automorphisms. A certain amount of reflection should persuade the reader that altho' elements of Σ_x, (x an arbitrary set) can do anthing at all, elements of j "Σ_x must send singletons to singletons, pairs to pairs, in fact must respect cardinality & commute with complementation in $\mathcal{P}(x)$.

Elements of j^n "Σ_x must respect ever more sophisticated constraints as n gets larger, and look more & more like automorphisms. We can say, in general:

Proposition 61 *If, in $\Phi(x, y)$, x and y belong to the same type, n, say, and the variable of lowest type in Φ is of type k, say, then simple type theory proves: $(\forall x)(\forall y)[\tau$ an element of j^{n-k} "$\Sigma_{V_k}. \rightarrow .(\Phi(x, y) \rightarrow \Phi(\tau`x, \tau`y))]$*

(I am not proposing to prove this either!)

Now what if t (our definable thing at type n) is itself a permutation?

t will be $\{\langle x, y\rangle : \Phi\}$ for some Φ, where x and y are of type n, & the variable of lowest type in Φ is of type k. If we now apply proposition 0.0 we infer $(\forall x)(\forall y)[\tau$ an element of $j^{n-k}``\Sigma_{V_k} \rightarrow\ .(\Phi(x,y) \rightarrow\ \Phi(\tau`x, \tau`y))]$

Now $\Phi(x,y)$ says that t moves x to y, so this says that t commutes with everything in $j^{n-k}``\Sigma_{V_k}$. *Thus the centraliser of* $j^{n-k}``\Sigma_{V_k}$ *in* $\Sigma_{V_{n-1}}$ *will contain every permutation* $\{\langle x, y\rangle : \Phi\}$ *definable where* x *and* y *are of a type* n, & *the variable of lowest type in* Φ *is of type* k.

The problem I want to bring to the reader's attention is this: what do these centralisers look like? Clearly they can tell us a lot about sets definable in simple type theory. (Peter Neumann tells me they are all direct products of symmetric groups with wreath-products of symmetric groups.) I shall here follow an ancient practice of Russell in connection with simple type theory, namely omitting the type subscripts from variables, and from the constants, such as "V_n" (the universal set at type n) that the theory allows us to define. This enables us to avoid double subscripting in the following definitions: J_0 is the symmetric group on V (what I had previously notated Σ_{V_k})). J_{n+1} is to be $j``J_n$. We will be interested in the centralisers $C(J_n)$. $C(J_n)$ is, as I have emphasised, an ambiguous notation. A $C(J_n)$ exists at each type, and it is that object which we obtain by looking at the universe n types below where we are, taking its symmetric group (J_0) (we have here, by typical ambiguity, dropped a subscript which would have told us which V_n J_0 was the symmetric group of), lifting that n times by j so that we have a subgroup of the symmetric group on the (very much larger) universe at the type we first thought of.

Let us introduce the notation $C_{n,k}$ for the Centraliser (in $\Sigma_{\mathcal{P}^k`x}$) of $j^k`(\Sigma_x)$ where x is a set with n members. And let J_k be short for $j^k``(\Sigma_x)$

Consider the operation j on permutations defined by $(jf)x = f``x$ so that j is an injection from the symmetric group on X (aka Σ_X) into the symmetric group on the power set of X (aka $\Sigma_{\mathcal{P}(X)}$).

This embedding is **not** elementary even if X is infinite. Consider the following. (According to my notes Adrian Mathias pointed this out to me. It may be trivial but it is missable!). The successor operation S on the integers is a permutation of the integers which does not commute with any involution of the integers, but jS commutes with some involutions.

Proof:

The key is the fact that there is only one cycle of S over Z. Let π be a permutation of Z that commutes with S. Then

$$x < \pi(x) \longleftrightarrow (Sx < \pi(Sx))$$

(by commutativity) so $(\forall x)(x < \pi(x)) \vee (\forall x)(\pi(x) < x)$ so π is not an involution.

However, jS is a permutation of $\mathcal{P}$(Z which commutes with at least one involution, namely complementation. But in a symmetric group G on an uncountable set every element τ commutes with at least one involution. (My scribbles attribute this to Peter Kropholler, but it's presumably folklore). If G is the full symmetric group on an uncountably infinite set, then—by the pigeonhole principle—τ must have two cycles the same size. Then just let π be a bijection between these two cycles that fixes everything else.

Therefore the symmetric group on N is not elementarily equivalent to the symmetric group on $\mathbb{R}$.

Wilfrid tells me that Shelah has much stronger results than this, but i have not had the stomach to attack the paper he mentioned to me!