Henkin on ω-consistency and ω-completeness JSL 1954 and 1957: A commentary by

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The titles of the two articles are "A generalisation of the concept of ω-consistency" and "A generalisation of the concept of ω-completeness". I tho'rt it might be an idea to look at them in case developments in proof theory in the 45 years since this paper was written could have any light to shed on the phenomenon he examines.

The obvious generalisation of ω-consistency is to the situation of a theory T with a set of constants, Γ say, and we want to say that:

Definition 1 *T is Γ-consistent unless—for some ϕ—$T \vdash \phi(a)$ for every constant a in Γ but nevertheless also proves $(\exists x)(\neg\phi(x))$.*

If we write this is the form $\bigvee_{a \in \Gamma}(T \nvdash \phi(a)) \vee T \nvdash (\exists x)(\neg\phi(x))$ it becomes clear that this property is infinitary horn. I'm not sure how useful this will be. In any case there are several more immediate complications to consider.

First Γ might not be the set of all constants of T. I must confess that it seems to me particularly perverse to jam this particular door open. After all T might prove $\phi(a)$ for every a in Γ and yet prove $\neg\phi(b)$ for some constant b not in Γ. It seems to me that rather than allow Γ to be a proper subset of the constants of T, one should always assume that Γ contains all constants, and if one wants some extra constants, dress them us as one-place predicates with $\exists!$ axioms *à la Quine* if desired.[1] I have taken the liberty of rewriting Henkin's examples with this in mind.

Secondly we should probably consider vectors of variables and polyadic predicate letters.

Finally (and this is what Henkin actually tackles first) is the idea that Γ-consistency is a syntactic notion and he wants to see how it ties up with the corresponding semantic (model-theoretic) notion. To this end he introduces an idea of T being Γ-**satisfied** by a model, with a view to proving results with the flavour: if T is Γ-consistent then there is a model that Γ-satisfies it.

[1] Henkin is aware of this. Indeed he makes the point that one can even use a *single* additional binary predicate letter. I can only assume he prefers a language without predicate letters.

Definition 2 *$\mathcal{M}$ is said to Γ-satisfy T if it is a model of T wherein '$=$' is interpreted by equality and everything in the domain of $\mathcal{M}$ is the denotation of a constant in Γ.*

If T is a theory without pairing and unpairing then one might fear that there could be a ϕ such that $T \vdash (\exists x)(\exists y)\phi(x, y)$ but $T \not\vdash \phi(s, t)$ for all constants s and t, *and that this can happen even if there is no ψ such that $T \vdash (\exists x)\psi(x)$ and $T \not\vdash \psi(s)$ for all constants s.* Such a theory would be pathological in that even tho' it is Γ-consistent it still won't be Γ-satisfied by any model.

Anyone with such fears would be right. There are such theories, and here is Henkin's example of one. It has constants a_n for each $n \in \mathbb{N}$, equality and two unary predicates F and G. Axioms:

1. $a_i \neq a_j$ when $i \neq j$
2. $(\forall x)(\neg(F(x) \wedge G(x)))$
3. $(\exists! x)(F(x))$
4. $(\exists! x)(G(x))$
5. $\neg F(a_i) \vee \neg G(a_j)$ for $i \neq j$.

Then $T \vdash (\exists x)(\exists y)(F(x) \wedge G(y))$ but $T \vdash \neg(F(a_i) \wedge G(a_j))$ for all i and j.

(At some point i want to think about a proof-theoretic presentation of this T and think about whether it looks funny.)

We use elimination of quantifiers to show that T is Γ-consistent, that is to say, that whenever $T \vdash \phi(a_i)$ for every i then $T \vdash (\forall x)(\phi(x))$.

Henkin's example is of course a "binary" counterexample. We need to be able to say it is Γ-consistent but not Γ^2-consistent. Let's define this new notion. We obviously want:

Definition 3 *T is Γ^k-consistent unless—for some k-ary ϕ—$T \vdash \phi(\vec{a})$ for every k-tuple $\vec{a}$ of constants from Γ but nevertheless also proves $(\exists x_1 \ldots x_k)(\neg\phi(x_1 \ldots x_k))$.*

T is Γ^∞-consistent if it is Γ^k-consistent for all k.

By complicating the construction we can come up with theories T_n such that for some ϕ $T_n \vdash (\exists x_1 \ldots x_n)(\phi(x_1 \ldots x_n))$ but $T \vdash \neg\phi(\vec{t})$ for each n-tuple $\vec{t}$ of constants. Instead of two unary predicates F and G we have $n + 1$, namely $F_1 \ldots F_{n+1}$ and analogous axioms to say that the extensions of the F_i are disjoint singletons. Axiom scheme 5 becomes the scheme

$\bigvee_{0<i\leq n+1} \neg F_i(t_i)$ for each tuple $\vec{t}$ of distinct $n{+}1$-tuples of constants.

The ϕ we want is $\bigwedge_{0<i\leq n+1} F_i(x_i)$. T_n proves that there is a tuple satisfying this but refutes each instance over the set of constants.

We presumably still use elimination of quantifiers to prove that T_n is $\Gamma^{(n-1)}$-consistent.

I think what will happen is that any theory that is Γ-satisfied by something must be Γ^k-consistent for all k.

The question now is: is T being Γ^∞-consistent sufficient for T to be Γ-satisfied by something? Sadly, the answer is 'no'. (As before, i have doctored Henkin's presentation of his theory by replacing some constants with monadic predicate letters). The theory T_∞ is as follows. There are infinitely many constants $\{a_i : i \in \mathbb{N}\}$ as before, and infinitely many one-place predicates $\{F_i : i \in \mathbb{N}\}$. As before there are axioms to say all the constants have distinct denotations, and that the extensions of the F_i are pairwise disjoint singletons.

The final suite of axioms is not obvious. It's no use taking the union of all the suites we've seen so far. The intersection of all the T_n is in fact Γ^∞ consistent by a horn-clause argument. But that doesn't help. We need an extra one-place predicate, which we will write 'G'. The extra axiom scheme is:

> $G(a_k) \to \neg(F_1(a_{i_1}) \wedge \ldots F_k(a_{i_k}))$ for each $k > 0$ and each k-tuple $\langle a_{i_1}, a_{i_2}, \ldots a_{i_k}\rangle$.

Now in a Γ^∞ model of T_∞ there will be a singleton—a_{17}, say—which is the extension of G. This means that the instances of the new suite where $k = 17$ come to life, so we can deduce $\neg(F_1(a_{i_1}) \wedge \ldots F_k(a_{i_{17}}))$ for all 17-tuples of constants. This immediately puts us back in T_{17} as it were, and ...

OK, Leon, we give up: what *is* the fancy consistency condition that implies satisfiability? He replies:

Note that Γ-consistency of T is equivalent to the condition (shades of the ϵ-calculus!) that to every fmla $A(x)$ with one free variable one can associate an $a \in \Gamma$ such that $T \nvdash (\exists x)(A(x) \wedge \neg A(a))$. Γ^∞-consistency of T is is similarly equivalent to the condition that for every n and every n-tuple of formulæ $A_1(x) \ldots A_n(x)$ all with one free variable there is an n-tuple $a_1 \ldots a_n$ all in Γ such that T does not prove the following:

$$\bigvee_{0 \leq i \leq n} ((\exists x_i)(A(x_i) \wedge \neg A(a_i)))$$