Duality*

Ernst Specker (translated by Thomas Forster)

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Man kann ganz gleich tun
und doch ein andrer sein
Grillparzer: *Friedrich der Streitbare*

This isn't quite the same as the taylor and Francis version: a few typos have been corrected....

The duality principle in projective geometry seems to have been stated for the first time in full generality by J.D. Gergonne. In 1826 he published in the 16th volume of his *Annales*, which was entitled *Mathematical Philosophy*, an essay "Philosophical considerations on the elements of the science of the extended." (In the list of contents at the end of the volume it is made clearer as "Philosophical Considerations on the properties of the extended which do not depend on metric relations".) In it Gergonne first explains the concept of the metric property and of the property of position and then continues as follows:

Mais un caractère extrêmement frappant de cette partie de la a géometrie qui ne dépend aucunement des relations métriques entre les parties des figures; c'est qu'à l'exception de quelques théorèmes symétriques d'eux-mêmes, tels, par exemple, que le théorème d'Euler sur les polyèdres, et son analogue sur les polygones, tous les théorèmes y sont doubles; c'est à dire que, dans la géometrie plane, à chaque théorème il en répond nécessairement un autre qui s'en déduit en y échangeant simplement entre eux les deux mots points et droites; tandis que, dans la géométrie de l'éspace, ce sont des mots points et plans qu'il faut échanger entre eux pour passer d'un théorème à son corrélatif.

Parmi un grand nombre d'exemples que nous pourrions puiser, dans le présnt recueil, de cette sorte de dualité des théorèmes qui constituent la géométrie de situation, nous nous bornerons à indiquer, comme les plus remarquables, les deux élégans théorèmes de M. Coriolis, dḿontrés d'abord à la page 326 du XI.e volume, puis à la page 69 du XII.e, et l'article que nous avons nous-même publié à la page 157 du présent volume, sur les lois générales qui régissent les polyèdres. (p 210)

"But an extremely striking character of that part of geometry which does not in any way depend on the metric relations between the parts of the figures is that—with the exception of some theorems which by themselves are symmetric, such as, for example, Euler's theorem on the polyhedra, and its analogue concerning polygons—all the theorems there are double; that is to say that in plane

*The editor is grateful to Dr. Thomas Forster whose lucid translation and diagrams have made Specker's important article more accessible *geometry there necessarily always corresponds to each theorem another which is derived from it simply by exchanging the two words* points *and* lines; *whilst in geometry of space it is the words* points *and* planes *which we have to exchange one for the other in order to pass from one proposition to its correlative.*

Among the great number of examples from which we could choose, in the present collection, of this sort of duality of the thorems which form the geometry of situation, we shall restrict ourselves to pointing out—as the most remarkable—the two elegant theorems of M. Coriolis first proved on page 326 of volume XI, then on page 69 of volume XII, and the article which we ourselves published on page 157 of the present volume, on the general laws which govern polyhedra" (p. 210).

In what follows, Gergonne exhibits a number of pairs of dual theorems; he puts them—in a manner which has often been imitated—on one page in two columns. In so doing he does not conceive of space consistently as projective; the exceptions that the duality principle then suffers are concealed by artful formulations. For example, his first pair is as follows (p. 212):

Two points, distinct from one another, given in space, determine an indefinite straight line, which when these two points are called A and B, can itself be called AB.	Two non parallel planes, given in space, determine an indefinite straight line which, when these two planes are called A and B, can itself be called AB.

Here the two concepts *two distinct points* and *two non parallel planes* are presented as dual concepts of which nothing was explicitly said above and which cannot be consistently followed through. It is assumed that a plane and a straight line will always have a point in common, and this is true in general only in projective space. (pp 212-213)

A plane in space may be determined by a line and a point not on that line, or again by two lines which meet at a point.	A point in space may be determined by a plane and a line not lying within it, or again by two lines lying within one plane.

Nowhere does Gergonne attempt to base the duality principle on anything. He seems to be of the view that it should be an axiom of a theory of space and as such not in need of any justification. Specifically he nowhere refers to the theory of the incidence-respecting polar correspondence between points and lines developed by Poncelet.

Poncelet was irritated by this omission of Gergonne's, and subsequent volumes of *Annales de Mathématiques* contain a dispute between the two of them. The first essay appears under "Mathematical Philosophy," but later essays appear under "Mathematical Polemics." The quarrel was occasioned not so much by a dispute over priority as by a difference of views. The differences are admittedly not clearly spelled out, but this would in any case not have been possible without projective geometry being given a clearer form.

It was only much later—with the help of the axiomatic approach—that this clarification could be achieved. Accordingly, we shall now turn to the setting up of an axiom system for plane projective geometry. We begin by noting that all relations between points and straight lines can be expressed in terms of incidence and identity: the point p lies on the line g, the line g goes through the point p; p coincides with g (written (pIg)). The lines g and h intersect in $p : gIp$ and hIp. The points p, q, r do not lie on a line: there is no line g with gIp and gIq and gIr. Thus the axiom system contains the primitive concepts *point, line* and *incidence*—a two place relation. The logic on which it is based is first-order predicate logic with equality. Axiom A_0 collects the properties that we have tacitly assumed.

A_0 Every element is a point or a line but not both. Incidence is a symmetrical relation and holds between a point and a line.

A_1 Any two points are incident on precisely one line.

$A_1{}'$ Any two lines meet at precisely one point.

A_2 There are four points no three of which lie on any one line.

$A_2{}'$ There are four lines no three of which meet at any one point.

Note that the concepts *point* and *line* in A_o occur symmetrically: A is *self-dual.* The axioms A_1 and $A_1{}'$ are dual: It is easy to describe structures satisfying A_0, A_1 and $A_1{}'$, but not the whole system of axioms $A_0, A_1, A_1{}'A_2$ and $A_2{}'$ of plane projective geometry. The simplest projective plane contains 7 points and 7 lines, between which, given appropriate numbering, the following incidences hold. The points p_2, p_3, and p_4 lie on g_1; p_1, p_3, and p_5 lie on g_2; p_1, p_2, and p_6 lie on g_3; p_1, p_4, and p_7 lie on g_4; p_2, p_5, and p_7 lie on g_5; p_3, p_6 and p_7 lie on g_6; p_4, p_5, and p_6 lie on g_7 (Figure 1).

From this description it follows that the lines g_2, g_3, and g_4 meet at p_1; it can easily be verified that all the axioms are true in this structure.

It is easy to check that in this projective plane, with the points and lines numbered as above, if the point p_i lies on the line g_k then the line g_i lies on the point p_k. So the bijection π defined by by swapping g_i to p_i has the following properties. It maps the set of points and lines one-one onto itself; it preserves incidence—a lies on b iff $\pi(a)$ lies on $\pi(b)$; the image of a point is a line and the image of a line is a point; π^2 is the identity, that is, $\pi(\pi(a)) = a$ for all a. Let us call a map with these characteristics a *polarity.*

It follows immediately from the existence of a polarity that the principle of duality holds in the projective plane under consideration in full generality: for any expression S of the language of point, line, incidence, *etc.*, if S holds in this structure, so does S', the sentence obtained from S by exchanging 'point' and 'line'. All we need in order to prove this is the fact that isomorphic structures satisfy the same sentences. If given a plane E we define a plane E' whose points are the lines of E and whose lines are the points of E then the planes E amd E' are isomorphic, and the isomorphism is given by π. Given our definition of E', a formula S holds in E' precisely when S' holds in E. Since S holds in E' precisely when S holds in E, S' holds in E precisely when S holds in E. The axiom system is dual: that is to say, if A is an axiom, so is A'.

There is a temptation (and people have probably succumbed to it) to infer from this the duality principle in the form: Given a formula S true in a plane E, the dual formula S' is also true in E. However, all that this duality of the axioms gives us is that if S can be deduced from the axioms, so can S'. A

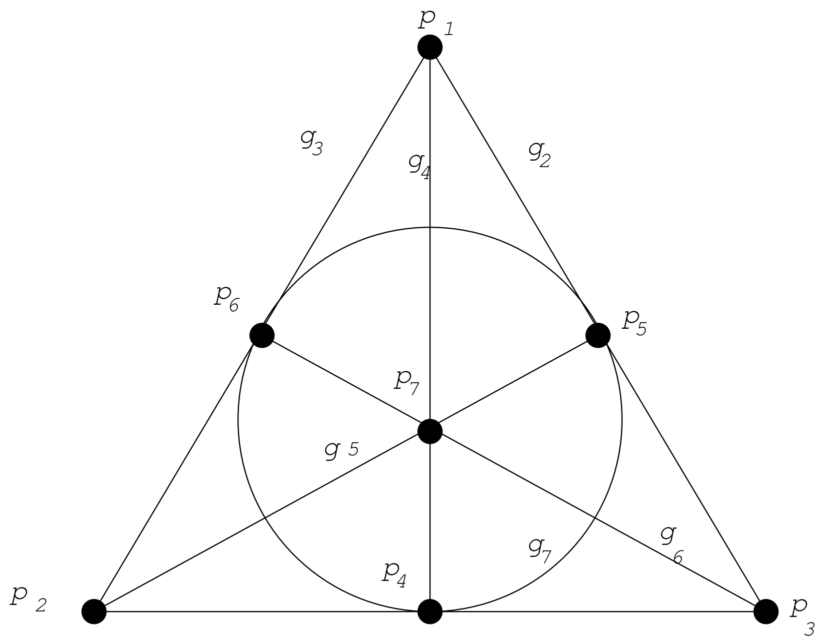

Figure 1: The seven-point projective plane

formula S can be true in a model without being deducible from the axioms. One such formula is the following: On every line lie exactly three points. This holds in the seven-point plane we have just seen, but it cannot be proved from the axioms since as is well-known there are projective planes whose lines have infinitely many points. The above construction does at least show that if E is a plane in which S holds, there is a plane E' in which S' holds.

We can give a proof that a plane satisfying a formula S satisfies the dual S' as well, even if we are merely given a **correlation**. A correlation is a map exchanging points for lines and preserving incidence. The preceding proof nowhere uses the fact that the square of the map is the identity. Chasles (1839) had already emphasised this in [1, p. 226]. On the other hand, the general duality principle does not hold in a projective plane without extra conditions. This is shown, for example, by the existence of finite projective planes that admit no correlation (Pickert 1955, p 108). A finite structure is determined up to isomorphism by the sentences it satisfies. If E always satisfies S whenever it satisfies S', then E is isomorphic to its dual plane E' (the points of which are the lines of E and the lines of which are the points of E). An isomorphism of E onto E' is nothing more or less than a correlation for E. However, this example of a finite projective geometry with a correlation but no polarity is rather cumbersome, and we shall weaken the A axioms given earlier to those that follow, the better to bring out the fundamental features of interest.

B_0 Every element is a point or a line but not both. Incidence is a symmetrical relation and holds between a point and a line.

B_1 Any two points are incident on *at most* one line.

$B_1{}'$ Any two lines meet at *at most* one point.

This system of axioms is dual: Exchanging the concepts *point* and *line* changes B_0 into itself, and swaps B_1 and B_1'. A model of this system of axioms is a *configuration*; *polarity* and *correlation* are to be defined as for projective planes. It is now a simple matter to give an example of a finite configuration that admits no correlation. If K is the configuration consisting of a line with two incident points, then there is clearly no one-one function that exchanges the points and lines of K. K satisfies "there are exactly two points", but not the dual ("there are precisely two lines"). Thus the duality principle fails in K. On the other hand there are of course configurations for which the duality principle does hold. If we add to the axioms B_0, B_1 and B_1 a further axiom B_2: "There are exactly three elements", the resulting system is still dual; but it has no dual model, for if S is the assertion "There are precisely zero or precisely two points", then in each model of the extended system the dual S' ("There are precisely zero or two lines") holds iff S does not.

It is slightly less straightforward to exhibit a configuration that admits a correlation but no polarity. Figure 2 shows one.

We can define a correlation as follows. First on the points: For all i and j, $p_i \mapsto g_i$, $q^i_j \mapsto h^i_j$, and $q_j \mapsto h_j$. Then on the lines: For all i and j, $g_i \mapsto p_{i+3}$, $h^i_j \mapsto q^i_{j+3}$ and $h_j \mapsto q_{j+3}$, with all addition mod 6.

The theory of K (the set of formulæ true in the configuration K of Figure 2) is dual and complete (every formula is provable or refutable), but it has no model with a polarity, for all its models are isomorphic to K and K has no polarity. In fact, this theory can even be axiomatized with a single axiom. The question of whether or not a dual complete system of axioms always has a model with a correlation remains open.[1] On the other hand, it is easy to show that a model of a complete dual system of axioms does not necessarily have a correlation. For this we consider the configuration that consists of a countable infinity of points and a countable infinity of lines with an empty incidence relation. The theory of this configuration is self-dual. Also, it is the same as the theory of the configuration with a countable infinity of points and continuum many lines with an empty incidence relation. Clearly there is no one-one structure-preserving map transposing points and lines in this second model. (It is not difficult to exhibit a countable configuration that satisfies the dual of every formula that it satisfies, but that allows of no correlation though of course no such configuration can be finite.)

H. Kneser (1935) discusses some of these questions in the case of projective geometry; yet many still seem not to be answered. (For example: does every finite plane with a correlation also have a polarity? Does every plane that satisfies the dual of everything it satisfies have a correlation?)

Our example of a dual axiom system without a dual model has the feature that it contains a formula S that is equivalent to the negation of its dual S'. Clearly, no system proving a theorem of the form $S \longleftrightarrow \neg S'$ can have a dual model. What we wish to show is the converse of this: namely, that a dual

[1] The author added a note at the end of the original article to the effect that a theory T with an automorphism σ such that $T \vdash \psi \longleftrightarrow \sigma(\psi)$ for all ψ has a model with a corresponding automorphism. Specifically Quine's NF is consistent iff Type Theory remains consistent if extended by taking as axioms all expressions of the form $\phi \longleftrightarrow \phi^*$. Kneser's questions about projective planes appear still to be open. (Translator's note)

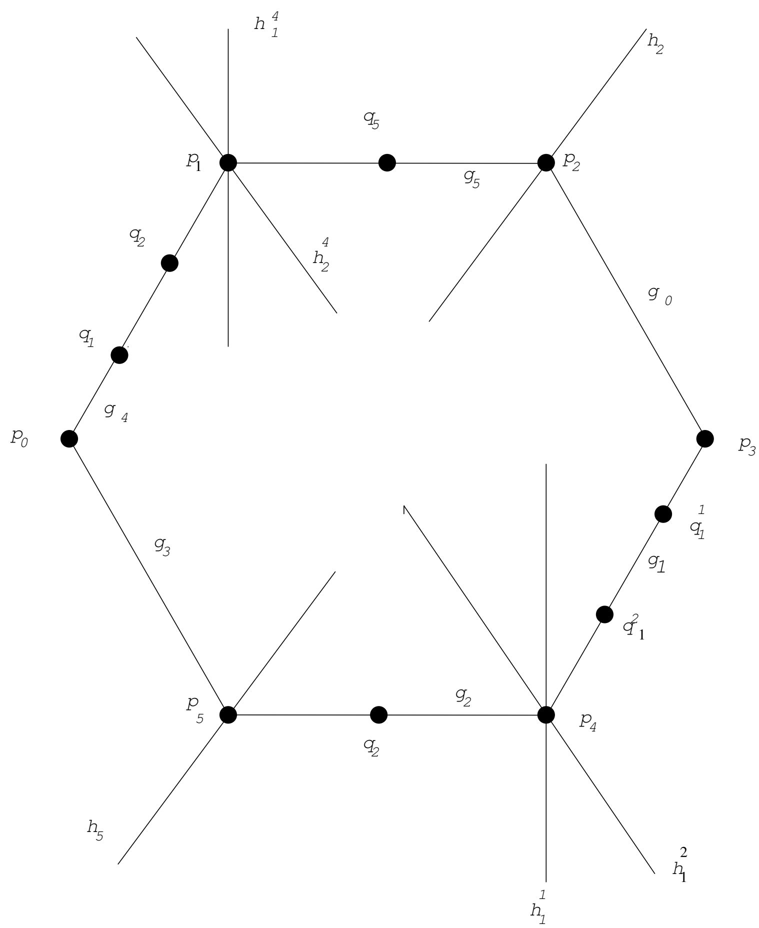

Figure 2: A configuration with a correlation but no polarity

system of axioms thatproves no theorem of the form $S \longleftrightarrow \neg S'$ has a dual model. Suppose we are given such a system. We extend it by taking as new axioms all formulæ of the form $S \longleftrightarrow S'$. If this extended system is consistent, then by the completeness theorem it has a model, and this model is obviously dual.

If on the other hand the extended system is inconsistent, then by compactness there are finitely many formulæ $S_1 \ldots S_n$ such that the conjunction of all the biconditionals $S_i \longleftrightarrow {S_i}'$ with $1 \leq i \leq n$ is refutable. We will show that in these circumstances there is a single formula S such that $S \longleftrightarrow S'$ is refutable. To do this, it will be sufficient to show that the conjunction of two biconditionals of the form $S \longleftrightarrow S'$ is equivalent to another such biconditional; induction will do the rest. We claim that $(S_1 \longleftrightarrow {S_1}') \wedge (S_2 \longleftrightarrow {S_2}')$ is equivalent to $T \longleftrightarrow T'$ where T is

$$(S_1 \longleftrightarrow {S_1}' \wedge S_2 \wedge \neg {S_2}') \vee (S_2 \longleftrightarrow {S_2}' \wedge \neg S_1 \wedge {S_1}') \vee (\neg S_1 \wedge {S_1}' \wedge S_2 \wedge \neg {S_2}')$$

In the published version there is a typo in the second disjunct

T' arises from T by swapping the indices 1 and 2. The proof procedes easily as follows: If $S_1 \longleftrightarrow S_1{}'$ and $S_2 \longleftrightarrow S_2{}'$ then T and T' are both false; if T and T' are both false, then $S_1 \longleftrightarrow S_1{}'$ and $S_2 \longleftrightarrow S_2{}'$. Notice that T and T' cannot be simultaneously true.

For this proof (and also for the earlier general considerations), it is not essential for the duality of the theory to arise from the transposition of two primitive predicates. All that is needed is a permutation of the language that commutes with the logical operations and is of order two. In this sense group theory and the theory of skew fields are dual ($a \cdot b = c$ is to be replaced by $b \cdot a = c$). Then the concepts of *antiautomorphism of order two* and *antiautomorphism* correspond to the concepts of *polarity* and *correlation*. We get a further generalization if we drop the requirement that the permutation be of order two. Thus we assume only a permutation of the language commuting with the logical operations. The most interesting example of such a theory is probably simple type theory with negative types (see Wang 1952.) We describe this theory briefly. For each type k there is a suite of variables x_i^k; primitive predicates are $x_i^k = x_j^k$ and $x_i^k \in x_j^{k+1}$. The axioms are extensionality axioms

$$(\forall x_1^{k+1})(\forall x_2^{k+1})[(\forall x_1^k)(x_1^k \in x_1^{k+1} \longleftrightarrow x_1^k \in x_2^{k+1}) \to x_1^{k+1} = x_2^{k+1}]$$

and axiom schemes of comprehension:

$$(\exists x_1^{k+1})(\forall x_1^k)[x_1^k \in x_1^{k+1} \longleftrightarrow B(x_1^k)]$$

where B is an expression formed in an admissible way from the primitive predicates.[2]

The map that sends each formula ϕ to the result of raising all superscripts on all variables in ϕ by 1 is an automorphism of the language sending axioms to axioms.

A model of the system of axioms consists of a family of sets $\langle T_k : k \in \mathrm{Z}\rangle$, of **types** and of an $\in$-relation which holds between elements of T_k and elements of T_{k+1}. T_k is to be the domain over which range the variables with superscript 'k'.

In this context we have—corresponding to the question about dual models and models with correlations—a question about models that satisfy S^* whenever they satisfy S (where S^* is the result of lifting all type superscripts by one) and models which have the obvious kind of $\in$-automorphism which (for each k) sends T_k onto T_{k+1}. Clearly a model with the second characteristic has also the first.

To find a model satisfying S^* whenever it satisfies S it is necessary and sufficient that for no S is the conjunction of the biconditionals $S^k \longleftrightarrow S^{k+1}$ refutable, where S^0 is S and S^{k+1} is $(S^k)^*$.

As in the case of dual systems of axioms it is sufficient to show the following:

Given formulæ $S_1 \ldots S_n$ such that $\vdash \neg \bigwedge_{1 \leq i \leq m}(S_i \longleftrightarrow S_1{}^*)$ there will be a formula T and a natural number n such that

$$\vdash \neg \bigwedge_{0<k<n} (T^k \longleftrightarrow T^{k+1}).$$

[2] The author presumably means this axiom scheme to be the scheme of universal closures of formulæ of this kind, with parameters allowed in B. [translator's note]

Proof: consider ordered m-tuples of truth values ordered lexicographically. Let $V(A_1, \ldots A_m; B_1, \ldots, B_m)$ be that boolean combination of $A_1, \ldots B_1, \ldots, B_m$, which says that the truth value of the m-tuple of As lexicographically precedes that of the Bs.[3]

Now let T be the formula $V(S_1, \ldots S_m; {S_i}^*, \ldots {S_m}^*)$.

Since we can refute $\bigwedge_{1 \leq i \leq m}(S_i \longleftrightarrow S_i^*)$ it follows that for each k the two m-tuples of truth values taken by ${S_1}^k \ldots {S_n}^k$ and ${S_1}^{k+1} \ldots {S_n}^{k+1}$ must be different. For the moment let n be an arbitrary natural number and suppose *per impossibile* that all the biconditionals $T^k \longleftrightarrow T^{k+1}$ with $k < n$ held. Now either the Ts are all true, in which case the n m-tuples form a strictly *increasing* sequence of elements of 2^m; or they are all false, in which case the n m-tuples form a strictly *decreasing* sequence of elements of 2^m. But there cannot be arbitrarily long strictly increasing or decreasing sequences from 2^m, and in fact if we take n to be 2^m we obtain a contradiction. We conclude that our assumption of constancy of the truth-value of the Ts was mistaken. ■

In contrast it is not the case that if T is a type theory such that, for all S, $T \cup \{S \longleftrightarrow S^*\}$ is consistent then the union of all such extensions of T is also consistent.

To prove this we consider the following type theory: The sole relation is the identity; the axioms are the two following schemes, one instance of each for each k:

1. There are precisely 1, 2, or 3 elements of each type;

2. There are not equally many elements of type k and of type $k+1$.

This axiom system obviously has no model satisfying the scheme of biconditionals $S \longleftrightarrow S^*$. However, for any one formula S there is a model in which $S \longleftrightarrow S^*$ holds. A formula S of the kind under consideration deals only with finitely many types—for example, the types $T_0, ..., T_{n-1}$—and expresses certain conditions on the number of elements in these types. So each formula S corresponds to a partition of $\{1, 2, 3\}^n$ into two pieces.

The assertion is proven if we can show that for each such partition there is a sequence $\langle a_0, \ldots, a_n \rangle$ such that for all $i < n-1$, $a_i \neq a_{i+1}$, and that $\langle a_0, \ldots, a_{n-1} \rangle$ and $\langle a_1, \ldots, a_n \rangle$ belong to the same piece. Once that is done we can exhibit a model of the theory satisfying $S \longleftrightarrow S^*$ as follows. For $0 \leq i \leq n$ the number of elements of type i is to be a_i; for $i < 0$ it is to be $a_0 + i \pmod 3$ and for $i > n$ it is to be $a_n + i \pmod 3$. The existence of a sequence $\langle a_0, \ldots, a_n \rangle$ with the desired property can be demonstrated as follows. For $i = 1, 2, 3$ let f_i be the $n+1$-tuple whose kth element is $i + k \pmod 3$, and let g_i be n-tuples defined similarly.

If f_1 does not have the desired property, then g_1 and g_2 belong to different pieces; if f_2 does not have the desired property then g_2 and g_3 lie in different pieces; but then g_1 and g_3 lie in the same piece and f_3 is a sequence of the desired kind.

It remains an open question whether or not simple type theory has a model satisfying the scheme $S \longleftrightarrow S^*$. If, on the other hand, the Axiom of Choice is

[3]Translator's note: thus, for example, $V(a, b)$ is $\neg a \wedge b$; $V(a_1, a_2; b_1, b_2)$ is $(\neg a_1 \wedge b_1) \vee ((a_1 \longleftrightarrow b_1) \wedge (\neg a_2 \wedge b_2))$; $V(a_1, a_2, a_3; b_1, b_2, b_3))$ is $(\neg a_1 \wedge b_1) \vee ((a_1 \longleftrightarrow b_1) \wedge (\neg a_2 \wedge b_2)) \vee ((a_1 \longleftrightarrow b_1) \wedge (a_2 \longleftrightarrow b_2)) \wedge (\neg a_3 \wedge b_3))$.

added to the type theory (perhaps in Russell's multiplicative form), then by the method of Specker (1953) a number less than 3 can be assigned to each type and it can be proved that the numbers assigned to successive types are different. If then S is the assertion "The number assigned to type 0 is smaller than the number assigned to type 1" then the conjunction $S \longleftrightarrow S^* \wedge S^* \longleftrightarrow S^{**}$ is refutable.

We now turn to the question of whether the simple type theory has a model with an $\in$-automorphism, which maps type k one-to-one onto the type $k+1$.

Corresponding to the question about dual models and models with correlations is a question about the relation between type theories satisfying $S \longleftrightarrow S^*$ and the existence of models with corresponding automorphisms.

We show that such a model exists iff Quine's "New Foundations" system (1937) is consistent.[4] In contrast to simple type theory, NF is a one sorted system of set theory. It contains an axiom of extensionality:

$$(\forall x)(\forall y)(x = y \longleftrightarrow (\forall z)(z \in x \longleftrightarrow z \in y))$$

and an axiom scheme of comprehension

$$(\forall \vec{x})(\exists y)(\forall z)(z \in y \longleftrightarrow \Phi(z, \vec{x}))$$

where 'z' is not free in Φ and Φ is **stratified**. That is to say, Φ can be turned into a formula of the language of type theory by decorating all its variables with superscripts.

If NF is consistent, then it has a model $\mathcal{M}$. We define a model $\mathcal{N}$ of simple type theory as follows: T_k is to be $M \times \{k\}$, and we set $\mathcal{N} \models \langle x, n\rangle \in \langle y, n+1\rangle$ iff $\mathcal{M} \models x \in y$. It is easy to confirm that T is a model of the simple type theory, and that the map sending $\langle x, n\rangle \mapsto \langle x, n+1\rangle$ is an $\in$-automorphism sending each type k one-one onto type $k+1$.

Conversely if there is a model $\mathcal{N}$ of simple type theory with such an automorphism, π, say, then a model $\mathcal{M}$ of NF can be defined as follows: let the domain of $\mathcal{M}$ be T_0, and set $\mathcal{M} \models x \; \eta \; y$ iff $\mathcal{N} \models x \in \pi(y)$. ($\eta$ is to be the membership relation of the model of NF). Now to check the validity of the axiom of existensionality and of the axiom scheme of comprehension.

Extensionality

$$(\forall x_1^0 x_2^0)[(\forall x_3^0)(x_3^0 \; \eta \; x_1^0 \longleftrightarrow x_3^0 \; \eta \; x_2^0) \to x_1^0 = x_2^0]$$

holds in the new model iff

$$(\forall x_1^0 x_2^0)[(\forall x_3^0)(x_3^0 \in f(x_1^0) \longleftrightarrow x_3^0 \in f(x_2^0)) \to x_1^0 = x_2^0]$$

Because f is a bijection we can rewrite '$x_1^0 = x_2^0$' as '$f(x_1^0) = f(x_2^0)$' and because f is onto, this becomes

$$(\forall x_1^0 x_2^0)[(\forall x_3^0)(x_3^0 \in x_1^1 \longleftrightarrow x_3^0 \in x_2^1) \to x_1^0 = x_2^0]$$

which is an extensionality axiom of simple type theory.

As an example of a comprehension axiom let us consider the following:

$$(\exists x_1^0)(\forall x_2^0)[x_2^0 \; \eta \; x_1^0 \longleftrightarrow (\exists x_3^0)(x_3^0 \; \eta \; x_2^0)]$$

[4] See addendum to the abstract.

Rewriting using the definition of η this becomes

$$(\exists x_1^0)(\forall x_2^0)[x_2^0 \in f(x_1^0) \longleftrightarrow (\exists x_3^0)(x_3^0 \in f(x_2^0))]$$

Now we can replace "$x_2^0 \in f(x_1^0)$" by "$f(x_2^0) \in f(f(x_1^0))$" and the resulting formula is equivalent to

$$(\exists x_1^2)(\forall x_2^1)[x_2^1 \in x_1^2 \longleftrightarrow (\exists x_3^0)(x_3^0 \in x_2^1)]$$

which is a comprehension axiom of simple type theory.

Projective geometry can now be founded anew in a way corresponding to the derivation of NF from the theory of types. The one-sorted theory with a single primitive two place predicate I has the following axioms.

C_0 I is symmetrical.

C_1 For two distinct elements a and b there is a unique c such that $I(a, c)$ and $I(b, c)$.

C_2 There are four elements a_1, a_2, a_3, a_4, of which no three are in the relation I with any element b.

There is a one-one correspondence between models of this new geometry and projective planes with a polarity.

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Abstract:

The axiom system of plane projective geometry is dual in the sense that it is transformed into itself by exchange of the notions "point" and "line". It follows that for every theorem the dual sentence is also a theorem. However, from the duality of the axiom system one cannot conclude that in a model the truth of a sentence implies that of the dual sentence; even less can one conclude that each model admits a one-one transformation interchanging points and lines and preserving the incidence relation. For projective geometry, models of this kind

are well known. For the simple theory of types (where duality is replaced by ambiguity of types) it is shown that the existence of such models is equivalent to the consistency of "New Foundations".

Additional remark. The following theorem answers both of the questions proposed in the paper: if it is complete, then a theory with an automorphism has a model with a corresponding automorphism. NF is therefore consistent if simple theory of type [*sic*] with the additional axioms $S \longleftrightarrow S^*$ (in the notation of the paper) is consistent.