VISIBLE 2-TORSION IN THE TATE-SHAFAREVICH GROUP OF AN ELLIPTIC CURVE

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Abstract. We show that every pair of 2-torsion elements in the Tate-Shafarevich group of an elliptic curve are visible in the same abelian surface. This was previously only known for a single 2-torsion element. Our result explains some of the original observations on visibility in the paper of Cremona and Mazur.

1. Introduction

Let E be an elliptic curve defined over a number field k. A torsor (or principal homogeneous space) under E is a pair (C, μ) where C is a smooth curve of genus one, and $\mu : E \times C \to C$ is morphism (both defined over k) inducing a simply transitive action on $\overline{k}$-points. The Weil-Châtelet group $H^1(k, E)$ parametrises the torsors under E up to k-isomorphism. The Tate-Shafarevich group

$$Ш(E/k) = \ker \left(H^1(k, E) \to \prod_v H^1(k_v, E) \right)$$

is the subgroup of torsors that are everywhere locally soluble. This group is conjecturally finite, and when finite, Cassels [7] showed it has order a square. Its non-zero elements correspond to genus one curves with Jacobian E that are counterexamples to the Hasse principle. Mazur introduced the notion of visibility as a way to study elements of $Ш(E/k)$.

Definition 1.1. An element $\xi \in H^1(k, E)$ is *visible* in an abelian variety A if there is an inclusion of abelian varieties $\iota : E \to A$ such that ξ is in the kernel of the induced map

$$\iota_* : H^1(k, E) \to H^1(k, A).$$

Equivalently, ξ is visible in A if the corresponding torsor may be realised as a coset of E inside A.

Cremona and Mazur [13, 2] gave examples, for each $n \in \{2, 3, 4, 5\}$, of elliptic curves $E/\mathbb{Q}$ where all of $Ш(E/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^2$ is visible in the same abelian surface. Mazur [20] showed that every 3-torsion element in $Ш(E/k)$ is visible in an abelian surface. Klenke [19] showed that every 2-torsion element in $H^1(k, E)$ is visible in

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an abelian surface. An argument using restriction of scalars, first recorded by Agashe and Stein [1], shows that every n-torsion element in $Ш(E/k)$ is visible in an abelian variety of dimension n. Fisher [15, 16] gave examples of 6 and 7-torsion elements in $Ш(E/\mathbb{Q})$ that are not visible in an abelian surface, and also of 7-torsion elements that are visible in an abelian threefold.

It follows from the restriction of scalars argument that $Ш(E/k)$ is finite if and only if all its elements are visible in the same abelian variety.

We prove the following.

Theorem 1.2. *Every pair of 2-torsion elements in* $Ш(E/k)$ *are visible in the same abelian surface.*

The proof works by showing that certain complete intersections of two quadrics in $\mathbb{P}^5$ satisfy the smooth Hasse principle.

In Section 2 we rephrase Theorem 1.2 as a statement about 2-Selmer groups. In Section 3 we introduce the relevant quadric intersections and describe some of their geometry. In Sections 4 and 5 we give two different proofs that these satisfy the smooth Hasse principle. We complete the proof of Theorem 1.2 in Section 6. In the final two sections we give some examples and discuss a variant of the proof.

We thank Shiva Chidambaram for sharing some data in connection with [4] which prompted us to revisit this problem. A Magma [6] file checking some of the calculations and examples in this paper is available on the author's webpage.

2. The 2-Selmer group

Let E be an elliptic curve defined over a number field k, and let $n \geqslant 2$ be an integer. We write $E[n]$ for the n-torsion subgroup. The n-Selmer group

$$S^{(n)}(E/k) = \ker\left(H^1(k, E[n]) \to \prod_v H^1(k_v, E) \right)$$

parametrises the n-coverings of E that are everywhere locally soluble. It is the middle term in the Kummer exact sequence

$$0 \longrightarrow E(k)/nE(k) \longrightarrow S^{(n)}(E/k) \longrightarrow Ш(E/k)[n] \longrightarrow 0.$$

Lemma 2.1. *Let F/k be a second elliptic curve and $\pi : E[n] \to F[n]$ an isomorphism of* $\mathrm{Gal}(\overline{k}/k)$*-modules. If $\xi, \eta \in S^{(n)}(E/k)$ are in the kernel of the composition*

$$H^1(k, E[n]) \xrightarrow{\pi_*} H^1(k, F[n]) \longrightarrow H^1(k, F)$$

then their images in $Ш(E/k)$ *are visible in the same abelian surface.*

Proof. We identify $E[n]$ and $F[n]$ via π and write $\Delta = E[n] = F[n]$. We embed Δ diagonally in $E \times F$ and let $A = (E \times F)/\Delta$. There is a commutative diagram

$$\begin{array}{ccc} H^1(k,\Delta) & \longrightarrow & H^1(k,F) \\ \downarrow & & \downarrow \\ H^1(k,E) & \longrightarrow & H^1(k,A). \end{array}$$

We are given that ξ and η are in the kernel of the top horizontal map. By the commutativity of the diagram, their images in Ш(E/k) are in the kernel of the bottom horizontal map, and so visible in A. □

We now take $n = 2$. Elements of the 2-Selmer group $S^{(2)}(E/k)$ may be represented by binary quartics. See for example [3, 5, 10, 12, 26].

Definition 2.2. A *binary quartic* is a homogeneous polynomial $f \in k[x,z]$ of degree 4. The invariants of the binary quartic

$$f(x,z) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4$$

are

$$\begin{aligned} I &= 12ae - 3bd + c^2, \\ J &= 72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3. \end{aligned} \tag{1}$$

A binary quartic is *non-singular* if it has no repeated roots (i.e. linear factors) over $\overline{k}$, equivalently if its discriminant $4I^3 - J^2$ is non-zero.

A non-singular binary quartic f defines a smooth projective curve of genus one with affine equation $y^2 = f(x,1)$. The projective curve is defined either by gluing affine pieces, or directly as the curve with equation $y^2 = f(x,z)$ in the weighted projective plane $\mathbb{P}(1,2,1)$. This curve is a 2-covering of its Jacobian

$$E_{I,J}: \quad y^2 = x^3 - 27Ix - 27J. \tag{2}$$

A binary quartic is *soluble* if the corresponding genus one curve has a k-point, and *everywhere locally soluble* if it has a k_v-point for all places v.

Definition 2.3. Let K/k be any field extension. Binary quartics f and g are *properly K-equivalent* if

$$f(x,z) = \frac{1}{(ps-qr)^2} g(px+rz, qx+sz)$$

for some $p,q,r,s \in K$ with $ps - qr \neq 0$.

Binary quartics that are properly k-equivalent have the same invariants. The converse is true for non-singular binary quartics over an algebraically closed field. We may write any elliptic curve E defined over k in the form $E_{I,J}$ for some $I, J \in k$

with $4I^3 - J^2 \neq 0$. The 2-Selmer group $S^{(2)}(E_{I,J}/k)$ may then be identified with the set of proper k-equivalence classes of everywhere locally soluble binary quartics with invariants I and J.

The genus one curve defined by a non-singular binary quartic f is a double cover of $\mathbb{P}^1$ ramified over 4 points. These 4 points are the fibre of the 2-covering above the identity on E, and form a torsor under $E[2]$. In Section 6 we will use the fact that the class of f in $H^1(k, E[2])$ is the class of this torsor.

3. Quadric intersections

We work over a field k with $\mathrm{char}(k) \neq 2, 3$, with algebraic closure $\overline{k}$. Let $F(x_1, \ldots, x_n)$ and $G(x_1, \ldots, x_n)$ be a pair of quadratic forms in n variables. Let A and B be the $n \times n$ symmetric matrices of second partial derivatives of F and G. Then $\det(\lambda A + \mu B)$ is a binary form of degree n. By abuse of notation we write this binary form as $\det(\lambda F + \mu G)$. The next two lemmas are well known. The first is [23, Proposition 2.1].

Lemma 3.1. *The quadric intersection $X = \{F = G = 0\} \subset \mathbb{P}^{n-1}$ is smooth (and of codimension 2) if and only if $\det(\lambda F + \mu G)$ has no repeated factors.*

Proof. By the Jacobian criterion for smoothness, any singular point on X must be a singular point on some quadric in the pencil spanned by F and G. It therefore gives a root of $\det(\lambda F + \mu G)$. This is a repeated root, as may be seen by reducing to the case where F is a quadratic form in $x_2, \ldots, x_n$ only, and G has no term x_1^2. Conversely, a root of the binary form is a repeated root if and only if the corresponding quadric has rank less than $n - 1$, or it has rank exactly $n - 1$ and its singular point lies on X. In both these cases X has a singular point. $\square$

Lemma 3.2. *Let X be as in Lemma 3.1 with $n = 4$. Then X is a smooth curve of genus one. If $\mathrm{rank}\, F = 3$ then X is a 2-covering of its Jacobian elliptic curve $y^2 = \det(xF + G)$.*

Proof. This is a special case of a standard construction used in 4-descent. See for example [3] or [21]. $\square$

Given a binary quartic form

$$f(x, z) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4,$$

we define the *associated quadratic form*

$$Q_f(x_0, x_1, x_2) = ax_0^2 + bx_0x_1 + \frac{c}{3}(x_0x_2 + 2x_1^2) + dx_1x_2 + ex_2^2.$$

This choice is made so that

$$Q_f(x^2, xz, z^2) = f(x, z)$$

and in addition the following two properties hold.

Lemma 3.3. *Let $Q(x_0, x_1, x_2) = x_0 x_2 - x_1^2$. Let f be a binary quartic form, with invariants I and J as defined by* (1)*.*

(i) *We have* $\det(xQ - 9Q_f) = 2(x^3 - 27Ix - 27J)$*.*

(ii) *If $\widetilde{f}(x,z) = f(px + rz, qx + sz)$ then $Q_{\widetilde{f}}(x_0, x_1, x_2) = Q_f(\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2)$ where*

$$\begin{aligned}\widetilde{x}_0 &= p^2 x_0 + 2prx_1 + r^2 x_2,\\ \widetilde{x}_1 &= pqx_0 + (ps + qr)x_1 + rsx_2,\\ \widetilde{x}_2 &= q^2 x_0 + 2qsx_1 + s^2 x_2.\end{aligned}$$

Moreover $Q(\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2) = (ps - qr)^2 Q(x_0, x_1, x_2)$.

Proof. A direct calculation. □

Definition 3.4. Let f and g be a pair of non-singular binary quartics with the same invariants I and J. The *associated quadric intersection* is

$$X_{f,g} = \left\{ \begin{array}{c} u_0 u_2 - u_1^2 = v_0 v_2 - v_1^2 \\ Q_f(u_0, u_1, u_2) = Q_g(v_0, v_1, v_2) \end{array} \right\} \subset \mathbb{P}^5.$$

In Section 4 we show that quadric intersections of the form $X_{f,g}$ satisfy the smooth Hasse principle. First however we consider their geometry.

By Lemma 3.3(i) the quadratic forms F and G defining $X_{f,g}$ satisfy

$$\det(xF - 9G) = -4(x^3 - 27Ix - 27J)^2. \tag{3}$$

In particular the pencil of quadrics defining $X_{f,g}$ has 3 singular fibres. Each is singular along a line of the form

$$\{(\lambda u_0 : \lambda u_1 : \lambda u_2 : \mu v_0 : \mu v_1 : \mu v_2) \mid (\lambda : \mu) \in \mathbb{P}^1\}$$

for some $(u_0 : u_1 : u_2), (v_0 : v_1 : v_2) \in \mathbb{P}^2$ with $u_0 u_2 - u_1^2 \neq 0$ and $v_0 v_2 - v_1^2 \neq 0$. Such a line meets $X_{f,g}$ in 2 distinct points swapped by the involution

$$(u_0 : u_1 : u_2 : v_0 : v_1 : v_2) \mapsto (u_0 : u_1 : u_2 : -v_0 : -v_1 : -v_2). \tag{4}$$

Therefore $X_{f,g}$ has exactly 6 singular points.

Remark 3.5. We may change coordinates over $\overline{k}$ so that $X_{f,g}$ has equations $\{x_1 x_2 = x_3 x_4 = x_5 x_6\} \subset \mathbb{P}^5$. It follows that $X_{f,g}$ contains exactly 8 planes.

We recall from Section 2 that binary quartics f and g are properly $\overline{k}$-equivalent if there exist $p, q, r, s \in \overline{k}$ with $ps - qr \neq 0$ and

$$f(x,z) = \frac{1}{(ps - qr)^2} g(px + rz, qx + sz).$$

This relation is unchanged if we multiply p, q, r, s through by the same non-zero scalar. It is natural to regard such equivalences as the same. With this convention, there are exactly 4 proper $\overline{k}$-equivalences relating f and g, and these may be computed by factoring a $(4,4)$-form as a product of $(1,1)$-forms, as described in [12, Theorem 12]. Corresponding to each proper equivalence there is a plane $\Lambda \subset X_{f,g}$ with equations

$$\begin{aligned}(ps-qr)v_0 &= p^2u_0 + 2pru_1 + r^2u_2,\\ (ps-qr)v_1 &= pqu_0 + (ps+qr)u_1 + rsu_2,\\ (ps-qr)v_2 &= q^2u_0 + 2qsu_1 + s^2u_2.\end{aligned}$$

Lemma 3.6. *The planes corresponding to distinct proper equivalences intersect only at singular points of $X_{f,g}$.*

Proof. By Lemma 3.3(ii) we may reduce to the case where $f = g$. The proper equivalences then give the action of the 2-torsion of the Jacobian. We may take one to be given by the identity matrix, and the other by a matrix $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Since the latter has order 2 in PGL_2, it has trace 0. A calculation then shows that the planes have unique point of intersection

$$(u_0 : u_1 : u_2 : v_0 : v_1 : v_2) = (-r : p : q : -r : p : q) \tag{5}$$

and that this is a singular point on $X_{f,f} = X_{f,g}$. $\square$

The next lemma shows that it is easy to find a k-rational conic on $X_{f,g}$. By a conic, we shall always mean a smooth conic.

Lemma 3.7. *Let $F = u_0u_2 - u_1^2 - v_0v_2 + v_1^2$ be the first of the two quadrics defining $X_{f,g}$. Then there are two algebraic families of planes contained in $\{F = 0\}$, each parametrised by $\mathbb{P}^3$. The planes in each family that meet $X_{f,g}$ in a conic are parametrised by the complement of 4 planes in $\mathbb{P}^3$.*

Proof. We change coordinates over k so that $F = z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}$, this being the formula for the Pfaffian of a 4×4 alternating matrix Z. The two algebraic families are the orbits of the planes $\{z_{12} = z_{13} = z_{14} = 0\}$ and $\{z_{23} = z_{24} = z_{34} = 0\}$ under the action of GL_4 via $A : Z \mapsto AZA^T$. We may describe one of these families as follows. Let $\Phi : \mathbb{P}^3 \times \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ be the rational map given by

$$\begin{aligned}&((x_1 : x_2 : x_3 : x_4), (y_1 : y_2 : y_3 : y_4))\\ &\qquad\qquad \mapsto (z_{12} : \ldots : z_{34}) = (x_1y_2 - x_2y_1 : \ldots : x_3y_4 - x_4y_3).\end{aligned}$$

Then for each point $x \in \mathbb{P}^3$ the image of $y \mapsto \Phi(x, y)$ is a plane $\Pi_x \subset \{F = 0\}$.

As suggested by Remark 3.5 we now change coordinates over $\overline{k}$ so that $X_{f,g}$ is defined by $F = z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{23}$ and $G = \lambda z_{12}z_{34} + \mu z_{13}z_{24}$ where $\lambda, \mu \in \overline{k}$

with $\lambda, \mu, \lambda + \mu \neq 0$. After substituting $z_{ij} = x_i y_j - x_j y_i$ into G, its matrix of second partial derivatives (with respect to $y_1, \ldots, y_4$) is

$$\begin{pmatrix} 0 & \mu x_3 x_4 & \lambda x_2 x_4 & -(\lambda+\mu) x_2 x_3 \\ \mu x_3 x_4 & 0 & -(\lambda+\mu) x_1 x_4 & \lambda x_1 x_3 \\ \lambda x_2 x_4 & -(\lambda+\mu) x_1 x_4 & 0 & \mu x_1 x_2 \\ -(\lambda+\mu) x_2 x_3 & \lambda x_1 x_3 & \mu x_1 x_2 & 0 \end{pmatrix}.$$

The plane Π_x meets $X_{f,g}$ in a conic if and only if this matrix has rank 3. This is the case for all $x \in \mathbb{P}^3 \setminus \{x_1x_2x_3x_4 = 0\}$. □

4. The Hasse principle

We take k a number field, and show that the quadric intersections $X_{f,g}$ in Definition 3.4 satisfy the smooth Hasse principle. (The Hasse principle itself is not true. See Example 7.2 for a counterexample.)

Theorem 4.1. *If $X_{f,g}$ has a smooth k_v-point for all places v then it has a smooth k-point.*

We know by Lemma 3.7 that $X_{f,g}$ contains a k-rational conic. So if $X_{f,g}$ were smooth then it would satisfy the Hasse principle by a theorem of Salberger [24]. We initially tried to adapt his proof, to allow for the fact that $X_{f,g}$ is singular. However, we found that a simpler argument is possible in our situation. In particular, we appeal to standard facts about del Pezzo surfaces, instead of Salberger's results on conic bundle surfaces.

Remark 4.2. If $E_{I,J}$ has no k-rational 2-torsion points then (3) shows that $X_{f,g}$ is a quadric intersection of type (F), as defined by Colliot Thélène, Sansuc and Swinnerton-Dyer. This case is specifically excluded from [8, Theorem 3.20]. According to [8, Remark 3.20.1] it is not known in general whether quadric intersections of this type have trivial Brauer-Manin obstruction.

We have found two different proofs of Theorem 4.1. We give details of both, since the first motivates the second. Readers uninterested in this motivation may skip to the start of the next section.

We start our first proof of Theorem 4.1 by showing that every smooth hyperplane section of $X_{f,g}$ satisfies the Hasse principle.

Lemma 4.3. *Let $H \subset \mathbb{P}^5$ be a k-rational hyperplane. If $X_{f,g} \cap H$ is a smooth surface then it satisfies the Hasse principle.*

Proof. We are given that $Y = X_{f,g} \cap H$ is a smooth intersection of two quadrics in $\mathbb{P}^4$. It is therefore a del Pezzo surface of degree 4. As explained in Section 3 the proper equivalences between f and g determine planes $\Lambda_1, \dots, \Lambda_4 \subset X_{f,g}$. We cannot have $\Lambda_i \subset H$ as this would contradict that Y is smooth. Intersecting with H we therefore get lines $\ell_1, \dots, \ell_4 \subset Y$. These lines are jointly defined over k, by which we mean that they are permuted by the action of $\mathrm{Gal}(\overline{k}/k)$. By Lemma 3.6 they are disjoint. We may therefore blow them down to give a del Pezzo surface of degree 8. It is well known that del Pezzo surfaces of degree 8 satisfy the Hasse principle. This completes the proof. $\square$

Remark 4.4. Applying the involution (4) to the planes $\Lambda_1, \dots, \Lambda_4$ gives a further 4 planes on $X_{f,g}$ and hence a further 4 lines $\ell'_1, \dots, \ell'_4$ on the del Pezzo surface Y. It may be checked using Remark 3.5 that ℓ_i meets ℓ'_j if and only if $i \neq j$. The two sets of four lines therefore constitute a "double-four" in the terminology of Swinnerton-Dyer [25, Section 7]. It follows that the del Pezzo surface of degree 8 (considered at the end of the last proof) has no -1-curves, and so is isomorphic over $\overline{k}$ to $\mathbb{P}^1 \times \mathbb{P}^1$ (instead of $\mathbb{P}^2$ blown up at one point).

We could use Bertini's theorem to choose the hyperplane H in Lemma 4.3. However the following more explicit lemma simplifies the next steps.

Lemma 4.5. *Let $\Pi \subset \{u_0u_2 - u_1^2 = v_0v_2 - v_2^2\} \subset \mathbb{P}^5$ be a k-rational plane. Let $H \subset \mathbb{P}^5$ be a k-rational hyperplane that contains Π, but does not contain any of the singular points on $X_{f,g}$. Then the surface $X_{f,g} \cap H$ is either smooth or has a k-rational singular point.*

Proof. The pencil of quadrics defining $X_{f,g}$ contains $F = u_0u_2 - u_1^2 - v_0v_2 + v_2^2$ of rank 6, and three singular quadrics G_1, G_2, G_3 each of rank 4. We claim that on restricting to the hyperplane H, each of these quadrics has rank 4. Indeed the restriction of F has rank at most 4 since $\Pi \subset \{F = 0\}$, and rank at least 4 since the rank can drop by at most 2. If the rank of G_i dropped then H would have to contain the singular locus of G_i. However this singular locus is a line meeting $X_{f,g}$ in two singular points. The claim follows since we assumed that H does not contain any of the singular points on $X_{f,g}$.

The singular quadrics in the pencil defining $X_{f,g} \cap H$ correspond to the roots of a binary quintic form. We may account for 4 of the roots as corresponding to the restrictions of F, G_1, G_2, G_3. If any of these are repeated roots then the corresponding rank 4 quadric is defined over k, and its singular point is a k-rational singular point on $X_{f,g} \cap H$. The same argument applies when the binary quintic form is identically zero (a case also covered by [8, Lemma 1.14]), provided we select the restriction of F as our rank 4 quadric, the problem with G_1, G_2, G_3 being that these might not be defined over k. In the remaining case the binary quintic form has distinct roots, in which case $X_{f,g} \cap H$ is smooth by Lemma 3.1. $\square$

The following lemma will help us choose the hyperplane H in Lemma 4.5 so that $X_{f,g} \cap H$ is everywhere locally soluble.

Lemma 4.6. *Let $X = \{F = G = 0\} \subset \mathbb{P}^5$ be a quadric intersection. Suppose that* $\operatorname{rank} F \geqslant 5$ *and* $\Pi \subset \{F = 0\} \subset \mathbb{P}^5$ *is a plane. Then for every $P \in X_{\mathrm{sm}} \setminus \Pi$ there is a hyperplane H containing both Π and P such that P is a smooth point on the intersection $X \cap H$.*

Proof. We change coordinates so that $\Pi = \{(* : * : * : 0 : 0 : 0)\}$ and $P = (0 : 0 : 0 : 1 : 0 : 0)$. A hyperplane of the required form is given by $H = \{\lambda x_5 + \mu x_6 = 0\}$, provided that the linear form $\lambda x_5 + \mu x_6$ does not vanish on $T_P X$. So we are done unless $T_P X = \{x_5 = x_6 = 0\}$. However in that case the first 4 partial derivatives of F vanish at P. This implies that F vanishes on the linear span of Π and P, and so $\operatorname{rank} F \leqslant 4$. This contradicts our assumption that $\operatorname{rank} F \geqslant 5$. □

The remainder of the proof is based on arguments in [24, Section 2].

Proof of Theorem 4.1. Let $X_{f,g} \subset \mathbb{P}^5$ be as given in Definition 3.4. By Lemma 3.7 we may pick a k-rational plane $\Pi \subset \{u_0u_2 - u_1^2 = v_0v_2 - v_2^2\}$ such that $C = X_{f,g} \cap \Pi$ is a conic. Let S be the finite set of places v with $C(k_v) = \emptyset$.

For each place $v \in S$, we use Lemma 4.6, and our assumption that $X_{f,g}$ has a smooth k_v-point, to pick a k_v-rational hyperplane H_v containing Π and meeting $X_{f,g}$ transversely at a k_v-point. The implicit function theorem (specifically [8, Proposition 6.2]) shows that there is a v-adic neighbourhood U_v of H_v in $(\mathbb{P}^5)^\vee(k_v)$ such that every hyperplane in U_v meets $X_{f,g}$ transversely at a k_v-point. Since C is smooth, none of the singular points of $X_{f,g}$ are contained in Π. The hyperplanes containing Π, but not containing any of the 6 singular points on $X_{f,g}$, form a Zariski open subset $W \subset \mathbb{P}^2$ (the complement of at most 6 lines). By weak approximation we may pick a hyperplane H in $W(k)$ that belongs to U_v for each $v \in S$. We claim that $(X_{f,g} \cap H)(k_v) \neq \emptyset$ for all places v. Indeed, this is true for places $v \notin S$ since $C \subset X_{f,g} \cap H$ and it is true for places $v \in S$ since $H \in U_v$.

We know by Lemma 4.5 that $X_{f,g} \cap H$ is either smooth or has a k-point. It follows by Lemma 4.3 that $X_{f,g} \cap H$ has a k-point. By our choice of H, this is not a singular point on $X_{f,g}$. □

Remark 4.7. The conclusion of Theorem 4.1 may be strengthened to say that the k-points on $X_{f,g}$ are Zariski dense. Indeed, in Lemma 4.3 the corresponding strengthening follows from standard facts about del Pezzo surfaces, and in the proof just given there are clearly infinitely many choices for the hyperplane H that satisfy the required hypotheses.

5. An alternative proof

In the previous section we proved that the quadratic intersections $X_{f,g}$ satisfy the smooth Hasse principle. Our proof worked by considering hyperplane sections which were del Pezzo surfaces. However it turns out that a more direct approach is possible.

Proposition 5.1. *Let $X = X_{f,g}$ be as in Definition 3.4. Let $\Lambda_1, \ldots, \Lambda_4$ be the planes contained in X corresponding to the proper equivalences between f and g, and let H be the hyperplane section. Then the linear system $|3H-\Lambda_1-\Lambda_2-\Lambda_3-\Lambda_4|$ defines a birational map $\phi : X \dashrightarrow Y$ where $Y \subset \mathbb{P}^9$ is a Brauer-Severi threefold. Moreover, the birational map ϕ is regular at all smooth points of X.*

Proof. The planes $\Lambda_1, \ldots, \Lambda_4$ are jointly defined over k, by which we mean that they are permuted by the action of $\mathrm{Gal}(\overline{k}/k)$. Having made this observation, we are free for the remainder of the proof to work over $\overline{k}$. In particular, we make the change of coordinates suggested in Remark 3.5.

We label our coordinates on $\mathbb{P}^5$ as $x_I = x_{ij}$ where $I = \{i, j\}$ runs over the subsets of $\{1, 2, 3, 4\}$ of size 2. By Lemma 3.6 we may assume that

$$X = \{x_{12}x_{34} = x_{13}x_{24} = x_{14}x_{23}\} \subset \mathbb{P}^5$$

and each Λ_i is defined by the vanishing of the x_I with $i \in I$. The space of cubic forms vanishing on $\Lambda_1 \cup \ldots \cup \Lambda_4$ is spanned by the monomials $x_I x_J x_K$ with $\#(I \cup J \cup K) = 4$. Inside this space, a complement to the space of cubic forms vanishing on X is spanned by the entries on and above the leading diagonal of the following symmetric matrix, where for simplicity we put $\alpha = x_{12}x_{34}$.

$$\begin{pmatrix} x_{12}x_{13}x_{14} & \alpha x_{12} & \alpha x_{13} & \alpha x_{14} \\ \alpha x_{12} & x_{12}x_{23}x_{24} & \alpha x_{23} & \alpha x_{24} \\ \alpha x_{13} & \alpha x_{23} & x_{13}x_{23}x_{34} & \alpha x_{34} \\ \alpha x_{14} & \alpha x_{24} & \alpha x_{34} & x_{14}x_{24}x_{34} \end{pmatrix}$$

This matrix defines a rational map $\phi : X \dashrightarrow Y$ where $Y \subset \mathbb{P}^9$ is the locus of all 4×4 symmetric matrices of rank 1. Let $\psi : \mathbb{P}^3 \dashrightarrow X$ be given by

$$(x_1 : x_2 : x_3 : x_4) \mapsto (x_{12} : \ldots : x_{34}) = (x_1x_2 : \ldots : x_3x_4). \tag{6}$$

The composite $\phi \circ \psi : \mathbb{P}^3 \to Y$ is the 2-uple embedding, which is clearly an isomorphism. It follows that ϕ is birational and Y is a Brauer-Severi variety. The final statement is checked by direct calculation. $\square$

Our alternative proof of Theorem 4.1 is immediate from Proposition 5.1 and the Hasse principle for Brauer-Severi varieties. Remark 4.7 is also immediate.

6. Application to visibility

The results of Section 2 reduce the proof of Theorem 1.2 to showing that if f and g are non-singular binary quartics with the same invariants I and J, and both are everywhere locally soluble, then their classes ξ and η in $S^{(2)}(E_{I,J}/k)$ satisfy the hypotheses of Lemma 2.1.

Lemma 6.1. *Let f and g be non-singular binary quartics over k with the same invariants I and J. If both f and g are everywhere locally soluble then $X_{f,g}$ has a smooth k_v-point for all places v.*

Proof. By assumption there exist $x, y, z, X, Y, Z \in k_v$ with $y^2 = f(x, z)$ and $Y^2 = g(X, Z)$. Since any smooth curve with a k_v-point has infinitely many k_v-points we may further assume that $yY \neq 0$. Then $X_{f,g}$ has k_v-point

$$(u_0 : u_1 : u_2 : v_0 : v_1 : v_2) = (x^2Y : xzY : z^2Y : X^2y : XZy : Z^2y).$$

This is a smooth point since, as noted in Section 3, the singular points on $X_{f,g}$ all have $u_0u_2 - u_1^2 \neq 0$. □

Lemma 6.1 and Theorem 4.1 together show that $X_{f,g}(k) \neq \emptyset$. The proof of Theorem 1.2 is completed by the following proposition. In the case that E has a rational 2-torsion point, we also appeal to Remark 4.7.

Proposition 6.2. *Let f and g be non-singular binary quartics over k with the same invariants I and J. Let $E = E_{I,J}$ be given by (2). Suppose that either*

(i) *$X_{f,g}(k) \neq \emptyset$ and $E(k)[2] = 0$, or*

(ii) *the k-points on $X_{f,g}$ are Zariski dense.*

Then there exists an elliptic curve F over k and $\pi : E[2] \to F[2]$ an isomorphism of $\mathrm{Gal}(\overline{k}/k)$-modules such that the classes of f and g are in the kernel of the composition

$$H^1(k, E[2]) \xrightarrow{\pi_*} H^1(k, F[2]) \longrightarrow H^1(k, F). \tag{7}$$

Proof. Let $(u_0 : u_1 : u_2 : v_0 : v_1 : v_2)$ be a k-point on $X_{f,g}$. We put

$$\lambda = u_0u_2 - u_1^2 \quad \text{and} \quad \mu = Q_f(u_0, u_1, u_2). \tag{8}$$

If $\lambda = \mu = 0$ then either f or g has a k-rational root (i.e. linear factor) and so is trivial in $H^1(k, E[2])$. We comment on this case at the end of the proof. So for now we may suppose that $(\lambda : \mu) \in \mathbb{P}^1(k)$. We pick $\widetilde{\lambda}, \widetilde{\mu} \in k$ with $\lambda\widetilde{\mu} - \mu\widetilde{\lambda} = 1$.

We consider the quadric intersection

$$C_f = \left\{ \begin{array}{l} \mu(x_0x_2 - x_1^2) - \lambda Q_f(x_0, x_1, x_2) = 0 \\ \widetilde{\mu}(x_0x_2 - x_1^2) - \widetilde{\lambda} Q_f(x_0, x_1, x_2) = y^2 \end{array} \right\} \subset \mathbb{P}^3.$$

If $E(k)[2] = 0$ then by Lemma 3.3(i) and (2) the first quadric has rank 3. If $E(k)[2] \neq 0$ then we use our additional hypothesis (ii) to adjust our choice of k-point so that the same conclusion holds. In other words, our threefold $X_{f,g}$ is fibred over the modular curve $X_E(2) \cong \mathbb{P}^1$ (which parametrises elliptic curves 2-congruent to E) and we want to make sure that our k-point is not above a cusp.

Lemmas 3.2 and 3.3(i) now show that C_f is a 2-covering of the elliptic curve

$$F : \quad y^2 = -(9\mu x + 9\widetilde{\mu})^3 + 27I(9\mu x + 9\widetilde{\mu})(\lambda x + \widetilde{\lambda})^2 + 27J(\lambda x + \widetilde{\lambda})^3. \tag{9}$$

An explicit isomorphism of $\mathrm{Gal}(\overline{k}/k)$-modules $\pi : E[2] \to F[2]$ is given, relative to the Weierstrass equations (2) and (9), by

$$(x, 0) \mapsto (-(\widetilde{\lambda} x - 9\widetilde{\mu})/(\lambda x - 9\mu), 0). \tag{10}$$

The first quadratic form in the definition of C_f defines a conic with k-rational point $(x_0 : x_1 : x_2) = (u_0 : u_1 : u_2)$. By parametrising this conic, we may rewrite C_f as a double cover of $\mathbb{P}^1$. The set of ramification points, as a set with $\mathrm{Gal}(\overline{k}/k)$-action, is isomorphic to

$$\{x_0x_2 - x_1^2 = Q_f(x_0, x_1, x_2) = 0\} \subset \mathbb{P}^2, \tag{11}$$

and hence also isomorphic to the set of ramification points for $y^2 = f(x, z)$. We upgrade this to an isomorphism of torsors, when $E[2]$ and $F[2]$ are identified via π. To do this we first note that $E[2] \setminus \{0\}$ and $F[2] \setminus \{0\}$ are naturally in bijection with the singular fibres of the pencil of quadrics defining (11). This gives a more conceptual explanation for (10). Next, since each singular fibre is a pair of lines, each element of $E[2] \setminus \{0\} = F[2] \setminus \{0\}$ naturally determines a partition of the 4 points (11) into 2 subsets of size 2. This determines the torsor structure.

Our discussion shows that the induced map $\pi_* : H^1(k, E[2]) \to H^1(k, F[2])$ sends the class of $y^2 = f(x, z)$ (a 2-covering of E) to the class of C_f (a 2-covering of F). The latter has trivial image in $H^1(k, F)$ since C_f has k-rational point $(x_0 : x_1 : x_2 : y) = (u_0 : u_1 : u_2 : 1)$. The class of f is therefore in the kernel of the composition (7).

The proof is completed by repeating the same arguments for the binary quartic g. In view of the definition of $X_{f,g}$ we may rewrite (8) as

$$\lambda = v_0v_2 - v_1^2 \quad \text{and} \quad \mu = Q_g(v_0, v_1, v_2).$$

We then obtain a 2-covering

$$C_g = \left\{ \begin{aligned} \mu(x_0x_2 - x_1^2) - \lambda Q_g(x_0, x_1, x_2) &= 0 \\ \widetilde{\mu}(x_0x_2 - x_1^2) - \widetilde{\lambda} Q_g(x_0, x_1, x_2) &= y^2 \end{aligned} \right\} \subset \mathbb{P}^3$$

of the *same* elliptic curve F. The isomorphism $\pi : E[2] \to F[2]$ is given by (10) and so is also unchanged. Since C_g has k-rational point $(x_0 : x_1 : x_2 : y) = (v_0 : v_1 : v_2 : 1)$, its class in $H^1(k, F[2])$ has trivial image in $H^1(k, F)$. The class of g is therefore in the kernel of the composition (7).

Finally we remark that if one of the binary quartics (say g) is trivial in $H^1(k, E[2])$, then the above argument is only needed for the other one (say f). For this we start (following what is essentially the proof of Klenke [19]) with any sufficiently general k-point $(u_0 : u_1 : u_2)$ on $\mathbb{P}^2$ in place of a k-point on $X_{f,g}$. This avoids the problem that the latter could have $u_0 = u_1 = u_2 = 0$. □

7. Examples

We give an example to illustrate the proof of Theorem 1.2.

Example 7.1. Let $E/\mathbb{Q}$ be the elliptic curve

$$y^2 + y = x^3 - x^2 - 929x - 10595$$

labelled $571a1$ in [9]. A 2-descent shows that $S^{(2)}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, and its non-zero elements are represented by

$$\begin{aligned} f(x,z) &= -11x^4 + 68x^3z - 52x^2z^2 - 164xz^3 - 64z^4, \\ g(x,z) &= -4x^4 - 60x^3z - 232x^2z^2 - 52xz^3 - 3z^4, \\ h(x,z) &= -31x^4 - 78x^3z + 32x^2z^2 + 102xz^3 - 53z^4. \end{aligned}$$

Each of these binary quartics has invariants $I = 44608$ and $J = 18842960$. In fact it may be shown (see [18]) that $E(\mathbb{Q}) = 0$ and $Ш(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

On the quadric intersection $X_{f,g} \subset \mathbb{P}^5$ we find the point

$$(u_0 : u_1 : u_2 : v_0 : v_1 : v_2) = (3 : 1 : 0 : -1 : 0 : 1).$$

Following the proof of Proposition 6.2, with $(\lambda, \mu, \widetilde{\lambda}, \widetilde{\mu}) = (-1, 211/3, 0, -1)$, shows that all of $Ш(E/\mathbb{Q})$ is visible in an abelian surface isogenous to $E \times F$ where $F/\mathbb{Q}$ is the elliptic curve

$$y^2 + y = x^3 + x^2 - 4x + 2$$

labelled $571b1$ in [9]. This is the first pair of elliptic curves in [13, Table 2].

We now give examples to show that our proof of Theorem 1.2 for elements of $Ш(E/\mathbb{Q})[2]$ does not extend to elements of $H^1(\mathbb{Q}, E)[2]$. In other words, we really do need the hypothesis in Lemma 6.1 that both binary quartics are everywhere locally soluble.

Example 7.2. Let $E/\mathbb{Q}$ be the elliptic curve $y^2 = (x-1)(x-3)(x+3)$ labelled $192a2$ in [9]. The non-singular binary quartics

$$\begin{aligned} f(x,z) &= -x^4 - 4x^2z^2 - z^4, \\ g(x,z) &= x^4 + 2x^3z + 2x^2z^2 + 4xz^3 + 4z^4, \end{aligned}$$

have the same invariants $I = 28$ and $J = -160$, and represent 2-coverings of E. The associated quadric intersection is

$$X_{f,g} = \left\{ \begin{array}{r} (u_0+u_2)^2 + 2u_1^2 + (v_0+v_1)^2 + (v_1+2v_2)^2 = 0 \\ (u_0-u_2)^2 + 6u_1^2 + (v_0+v_1+2v_2)^2 - 3v_1^2 = 0 \end{array} \right\} \subset \mathbb{P}^5.$$

The 6 singular points on $X_{f,g}$ are $(\sqrt{2} : 0 : -\sqrt{2} : 2 : -2 : 1)$, $(\sqrt{-2} : 0 : \sqrt{-2} : -2 : 0 : 1)$, $(0 : \sqrt{-1} : 0 : 2 : -1 : 1)$ and their Galois conjugates. For $p \geqslant 5$ a prime, the genus one curves $y^2 = f(x,z)$ and $y^2 = g(x,z)$ have good reduction, and so are soluble over $\mathbb{Q}_p$. It follows by Lemma 6.1 that $X_{f,g}$ has a smooth $\mathbb{Q}_p$-point. There is also a smooth $\mathbb{Q}_2$-point given by

$$(u_0 : u_1 : u_2 : v_0 : v_1 : v_2) = (\alpha^2 - 1 : 1 : 0 : \alpha^2 - 1 : \alpha : 1)$$

where $\alpha \in \mathbb{Z}_2$ is a root of $x^4 + x^3 - x^2 + x + 4 = 0$. However, our first and second equations for $X_{f,g}$ show that $X_{f,g}(\mathbb{R})$ and $X_{f,g}(\mathbb{Q}_3)$ consist only of the singular points defined over $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$ respectively.

In conclusion $X_{f,g}$ is a counterexample to the Hasse principle, but not the smooth Hasse principle.

Finally we give some examples with $X_{f,g}(\mathbb{Q}_p) = \emptyset$.

Proposition 7.3. *Let $p \geqslant 7$ be a prime with $p \equiv 2, 3 \pmod 5$. The binary quartics*

$$\begin{aligned} f(x,z) &= 20x^2z^2 + p(x^4 + 30x^2z^2 + 5z^4), \\ g(x,z) &= (x^2 - 5z^2)^2 + p(x^4 + 4x^3z - 30x^2z^2 + 20xz^3 + 25z^4), \end{aligned}$$

are non-singular and have the same invariants. However, the associated quadric intersection $X_{f,g} \subset \mathbb{P}^5$ has no $\mathbb{Q}_p$-points.

Proof. Both f and g have invariants

$$\begin{aligned} I &= 2^4 \cdot 5 \cdot (12p^2 + 15p + 5), \\ J &= -2^6 \cdot 5^2 \cdot (3p+2)(9p^2 + 15p + 5), \end{aligned}$$

and discriminant

$$4I^3 - J^2 = 2^{12} \cdot 3^3 \cdot 5^3 \cdot p^2 (11p^2 + 15p + 5)^2 \neq 0.$$

One of the quadrics in the pencil defining $X_{f,g} \subset \mathbb{P}^5$ reduces mod p to $20u_1^2 - (v_0 - 5v_2)^2$. So if $(u_0 : u_1 : u_2 : v_0 : v_1 : v_2)$ is a $\mathbb{Q}_p$-point on $X_{f,g}$, with all coordinates in $\mathbb{Z}_p$, but not all in $p\mathbb{Z}_p$, then (since 5 is a quadratic nonresidue mod p) we must have $u_1 \equiv v_0 - 5v_2 \equiv 0 \pmod p$. We may therefore write

$$(u_0, u_1, u_2, v_0, v_1, v_2) = (x_1, px_5, x_2, 5x_4 + px_6, x_3, x_4) \tag{12}$$

for some $x_1, \ldots, x_6 \in \mathbb{Z}_p$. These then satisfy

$$\begin{aligned} x_1x_2 + x_3^2 - 5x_4^2 &\equiv 0 \pmod p, \\ x_1^2 + 10x_1x_2 + 5x_2^2 + 20x_3^2 - 40x_3x_4 &\equiv 0 \pmod p. \end{aligned}$$

Since $p \equiv 2, 3 \pmod 5$ we have $[\mathbb{F}_p(\zeta) : \mathbb{F}_p] = 4$, where ζ is a primitive 5th root of unity. We put $\sqrt{5} = 1 + 2(\zeta + \zeta^4)$, and write $\widetilde{x}_i$ for $x_i \pmod p$. Taking a linear combination of the last two equations gives

$$(\widetilde{x}_1 + \sqrt{5}\widetilde{x}_2)^2 = (\zeta - \zeta^4)^2(2\widetilde{x}_3 - (5 - \sqrt{5})\widetilde{x}_4)^2.$$

It follows that $x_1, \ldots, x_4$ all vanish mod p. Combined with (12) this gives that $u_0, u_1, u_2, v_0, v_1, v_2$ all vanish mod p, contradicting our earlier assumption. □

8. Relation to the obstruction map

The obstruction map $\mathrm{Ob}_n : H^1(k, E[n]) \to \mathrm{Br}(k)$ is defined in [11, 22]. Taking $n = 2$, the significance of this map is that a 2-covering $\xi \in H^1(k, E[2])$ may be represented by a binary quartic if and only if $\mathrm{Ob}_2(\xi) = 0$.

Let $\xi, \eta \in H^1(k, E[2])$ be represented by binary quartics f and g. We saw in Section 5 that $X_{f,g}$ is birational to a Brauer-Severi variety Y. It is natural to ask: what is the class of Y in $\mathrm{Br}(k)$? We guess it should be $\mathrm{Ob}_2(\xi + \eta)$. This suggested the following result, which gives an alternative (more computational) proof of Theorem 1.2. We work over a field k of characteristic 0.

Proposition 8.1. *Let E/k be an elliptic curve. Let $\xi, \eta \in H^1(k, E[2])$ with $\mathrm{Ob}_2(\xi) = \mathrm{Ob}_2(\eta) = 0$. The following statements are equivalent.*

(i) *There exists an elliptic curve F/k, and $\pi : E[2] \to F[2]$ an isomorphism of $\mathrm{Gal}(\overline{k}/k)$-modules, such that ξ and η are in the kernel of the composition*

$$H^1(k, E[2]) \xrightarrow{\pi_*} H^1(k, F[2]) \longrightarrow H^1(k, F).$$

(ii) $\mathrm{Ob}_2(\xi + \eta) = 0$.

Proof. "(i) ⇒ (ii)" We write $\cup$ for the Tate pairing

$$H^1(k, E[n]) \times H^1(k, E[n]) \to \mathrm{Br}(k)$$

given by the Weil pairing and cup product. It is known (see [22, 27]) that for any $\xi, \eta \in H^1(k, E[n])$ we have

$$\xi \cup \eta = \mathrm{Ob}_n(\xi + \eta) - \mathrm{Ob}_n(\xi) - \mathrm{Ob}_n(\eta).$$

Compatibility with another pairing, also due to Tate (see for example [14, Proposition 2.1] or [22, Section 5]), shows that $\xi \cup \eta = 0$ whenever ξ and η are both in the image of the connecting map $E(k)/nE(k) \to H^1(k, E[n])$.

As in the statement of the proposition, we now take $\xi, \eta \in H^1(k, E[2])$ with $\mathrm{Ob}_2(\xi) = \mathrm{Ob}_2(\eta) = 0$. Then $\mathrm{Ob}_2(\xi + \eta) = \xi \cup \eta = \pi_*(\xi) \cup \pi_*(\eta) = 0$, where the first and last equality follow from the two facts we just quoted, and the middle equality holds since $\pi : E[2] \to F[2]$ preserves the Weil pairing. This must be the case since there is only one nondegenerate alternating pairing on $(\mathbb{Z}/2\mathbb{Z})^2$.

"(ii) $\Rightarrow$ (i)" We are given that $\mathrm{Ob}_2(\xi) = \mathrm{Ob}_2(\eta) = \mathrm{Ob}_2(\xi + \eta) = 0$. We may therefore represent ξ, η, $\xi + \eta$ by binary quartics f, g, h, all with the same invariants. Our aim is to show that the k-points on $X_{f,g}$ are Zariski dense, since (i) then follows by Proposition 6.2. We are free to replace f, g, h by $\lambda f, \lambda g, \lambda h$ for any $\lambda \in k^\times$, as this does not change $X_{f,g}$. We may therefore suppose that $y^2 = h(x, z)$ has a k-point, equivalently $y^2 = f(x, z)$ and $y^2 = g(x, z)$ are k-isomorphic as curves. Let C be this genus one curve, equipped with two different maps $C \to \mathbb{P}^1$. For the same reasons as we explained at the end of the proof of Proposition 6.2, we may ignore the case where f and g are equivalent. The image of the map $C \to \mathbb{P}^1 \times \mathbb{P}^1$ is therefore defined by a $(2, 2)$-form:

$$F(x_1, x_2; y_1, y_2) = \begin{pmatrix} x_1^2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} y_1^2 \\ y_1y_2 \\ y_2^2 \end{pmatrix}.$$

We may compute this $(2, 2)$-form by specialising the $(2, 2, 2)$-form in [17, Theorem 8.2] at a k-point on $y^2 = h(x, z)$. The binary quartics $f(x_1, x_2)$ and $g(y_1, y_2)$ are the discriminant of F, when this is viewed as a binary quadratic form in the other set of variables. A short calculation reveals that

$$X_{f,g} = \left\{ \begin{matrix} u_0u_2 - u_1^2 = v_0v_2 - v_1^2 \\ l_2^2 - 4l_1l_3 = m_2^2 - 4m_1m_3 \end{matrix} \right\} \subset \mathbb{P}^5$$

where

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}.$$

The rational parametrisation (a twisted form of (6))

$$\begin{aligned}
u_0 &= -a_{21}x^2 - a_{22}xy - a_{23}y^2 - 2a_{31}xz - a_{32}(xt + yz) - 2a_{33}yt, \\
u_1 &= a_{11}x^2 + a_{12}xy + a_{13}y^2 - a_{31}z^2 - a_{32}zt - a_{33}t^2, \\
u_2 &= a_{21}z^2 + a_{22}zt + a_{23}t^2 + 2a_{11}xz + a_{12}(xt + yz) + 2a_{13}yt, \\
v_0 &= -a_{12}x^2 - a_{22}xz - a_{32}z^2 - 2a_{13}xy - a_{23}(xt + yz) - 2a_{33}zt, \\
v_1 &= a_{11}x^2 + a_{21}xz + a_{31}z^2 - a_{13}y^2 - a_{23}yt - a_{33}t^2, \\
v_2 &= a_{12}y^2 + a_{22}yt + a_{32}t^2 + 2a_{11}xy + a_{21}(xt + yz) + 2a_{31}zt,
\end{aligned}$$

then shows that the k-points on $X_{f,g}$ are Zariski dense. □

We now take k a number field. If $\xi, \eta \in S^{(2)}(E/k)$ then a standard argument, using the local-global principle for the Brauer group, shows that $\mathrm{Ob}_2(\xi) = \mathrm{Ob}_2(\eta) = \mathrm{Ob}_2(\xi + \eta) = 0$. Combining Lemma 2.1 and Proposition 8.1 therefore gives an alternative (more computational) proof of Theorem 1.2.

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