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PFAFFIAN PRESENTATIONS OF ELLIPTIC NORMAL CURVES

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Abstract. We investigate certain alternating matrices of linear forms whose Pfaffians generate the homogeneous ideal of an elliptic normal curve, or one of its higher secant varieties.

1. Introduction

We work over an algebraically closed field k of arbitrary characteristic. An elliptic normal curve $C \subset \mathbb{P}^{n-1}$ is a smooth curve of genus one and degree n that is contained in no hyperplane. The rth higher secant variety $\mathrm{Sec}^r C$ is the Zariski closure of the locus of all $(r-1)$-planes spanned by r points on C. As special cases we have $\mathrm{Sec}^0 C = \emptyset$, $\mathrm{Sec}^1 C = C$ and $\mathrm{Sec}^2 C = \mathrm{Sec}\, C$. It is shown in [L] that $\mathrm{Sec}^r C$ is an irreducible variety of codimension $\max(n-2r, 0)$. Moreover it is shown in [vBH, Proposition 8.15] that if $n \geq 2r+1$ then $\mathrm{Sec}^r C$ has singular locus $\mathrm{Sec}^{r-1} C$. Thus an elliptic normal curve is uniquely determined by any one of its higher secant varieties that is not the whole of projective space.

We write $I(X)$ for the homogeneous ideal of a projective variety X. By convention $I(\emptyset)$ is the irrelevant ideal. It is well known that if $n \geq 4$ then $I(C)$ is generated by a vector space of quadrics of dimension $n(n-3)/2$. We recall the generalisation of this result to higher secant varieties.

Proposition 1.1. *If $n \geq 2r+2$ then $I(\mathrm{Sec}^r C)$ is generated by a vector space of $(r+1)$-ics of dimension $\beta(r+1, n)$ where*

$$\beta(r,n) = \binom{n-r}{r} + \binom{n-r-1}{r-1}$$

is the number of ways of choosing r elements from $\mathbb{Z}/n\mathbb{Z}$ such that no two elements are adjacent.

Proof. This is a special case of Theorem 4.1. □

Let $\mathcal{L}(D) = H^0(C, \mathcal{O}(D))$ be the Riemann-Roch space determined by a divisor D on C. Let H be the divisor of a hyperplane section, and let D_1, D_2 be divisors on C with $D_1 + D_2 = H$. We write $\Phi(D_1, D_2)$ for the matrix of linear forms representing the multiplication map

$$\mathcal{L}(D_1) \times \mathcal{L}(D_2) \to \mathcal{L}(H).$$

It is clear that $\Phi(D_1, D_2)$ has rank at most 1 on C, and so has rank at most r on $\mathrm{Sec}^r C$.

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Definition 1.2. A matrix of linear forms is a determinantal presentation of $\mathrm{Sec}^r C$ if its $(r+1)\times(r+1)$ minors generate $I(\mathrm{Sec}^r C)$.

Determinantal presentations for curves of arbitrary genus have been studied in [EKS]. A set-theoretic generalisation to higher secant varieties is given in [Ra]. In §2 we establish the following analogue of the main result of [EKS] for higher secant varieties of elliptic normal curves.

Theorem 1.3. *Let $n \geq 2r+1$. Then $\Phi(D_1, D_2)$ is a determinantal presentation of $\mathrm{Sec}^r C$ if and only if*
(i) $\deg D_1$, $\deg D_2 \geq r+2$, and
(ii) if $\deg D_1 = \deg D_2 = r+2$ then $D_1 \not\sim D_2$.

We fit Theorem 1.3 into a bigger picture involving rank 2 vector bundles $\mathcal{E}$ on C with $\det \mathcal{E} \simeq \mathcal{O}(1)$. Here $\mathcal{O}(1)$ is the line bundle on C associated to the hyperplane section. We write $\Phi(\mathcal{E})$ for the alternating matrix of linear forms representing the determinant map

$$\wedge^2 H^0(C, \mathcal{E}) \to H^0(C, \mathcal{O}(1)) = \mathcal{L}(H).$$

It is clear that $\Phi(\mathcal{E})$ has rank at most 2 on C, and so has rank at most $2r$ on $\mathrm{Sec}^r C$. The analogue of Definition 1.2 is

Definition 1.4. An alternating matrix of linear forms is a Pfaffian presentation of $\mathrm{Sec}^r C$ if its $(2r+2)\times(2r+2)$ Pfaffians generate $I(\mathrm{Sec}^r C)$.

For $\mathcal{E}$ decomposable the situation is not new. Indeed we can write $\mathcal{E} \simeq \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$ where D_1, D_2 are divisors on C with $D_1 + D_2 = H$. Then with respect to suitable bases we have

$$\Phi(\mathcal{E}) = \begin{pmatrix} 0 & \Phi(D_1, D_2) \\ -\Phi(D_1, D_2)^T & 0 \end{pmatrix}.$$

So $\Phi(\mathcal{E})$ is a Pfaffian presentation of $\mathrm{Sec}^r C$ if and only if $\Phi(D_1, D_2)$ is a determinantal presentation of $\mathrm{Sec}^r C$.

A theorem of Atiyah (Proposition 6.1) says that for n odd there is a unique indecomposable rank 2 vector bundle $\mathcal{E}$ on C with $\det \mathcal{E} \simeq \mathcal{O}(1)$. By the vector bundle form of Riemann-Roch (recalled in §6) we have $\dim H^0(C, \mathcal{E}) = n$. Our main result is the following analogue of Theorem 1.3.

Theorem 1.5. *Let $n \geq 3$ be an odd integer. Let C be an elliptic normal curve of degree n. Let $\mathcal{E}$ be the unique indecomposable rank 2 vector bundle on C with $\det \mathcal{E} \simeq \mathcal{O}(1)$. Then*
(i) The n submaximal Pfaffians of $\Phi(\mathcal{E})$ are linearly independent.
(ii) If $n \geq 2r+3$ then $\Phi(\mathcal{E})$ is a Pfaffian presentation of $\mathrm{Sec}^r C$.

The proof of Theorem 1.5(ii) from Theorem 1.5(i) is an induction similar to the proof of Theorem 1.3. To prove Theorem 1.5(i) it is helpful to make a definition.

Definition 1.6. Let $n \geq 3$ be an odd integer. Let C be an elliptic normal curve of degree n. A Klein matrix Φ for C is an $n \times n$ alternating matrix of linear forms on $\mathbb{P}^{n-1}$ such that
(i) Φ has rank 2 at all points on C, and
(ii) the n submaximal Pfaffians of Φ are linearly independent.

We show in §6 that every Klein matrix is of the form $\Phi(\mathcal{E})$ for $\mathcal{E}$ an indecomposable rank 2 vector bundle on C with $\det \mathcal{E} \simeq \mathcal{O}(1)$. Then Theorem 1.5(i) is a consequence of Atiyah's uniqueness result and the following existence theorem.

Theorem 1.7. *Let $n \geq 3$ be an odd integer. Then every elliptic normal curve of degree n has a Klein matrix.*

In §§3,4 we give two independent proofs of Theorem 1.7. The first uses representations of the Heisenberg group and so is only valid if $\mathrm{char}\,(k) \nmid n$. The second uses the Buchsbaum-Eisenbud structure theorem for Gorenstein ideals of codimension 3, and hence applies over an arbitrary field. The latter may also be used to weaken Definition 1.6(i) to the statement that Φ has rank at most 2 on C.

Theorem 1.5 may be restated in terms of Klein matrices to give

Corollary 1.8. *Let $n \geq 3$ be an odd integer. Let C be an elliptic normal curve of degree n with Klein matrix Φ. If $n \geq 2r+3$ then Φ is a Pfaffian presentation of $\mathrm{Sec}^r C$.*

In future work we will study Klein matrices over a non-algebraically closed field. From the point of view of arithmetic, Pfaffians presentations are more interesting than determinantal presentations. Indeed it is possible for an elliptic normal curve C of degree n to have index n. This means that there are no k-rational divisors D on C with $0 < \deg D < n$. In such circumstances it can be shown that C has no determinantal presentations.

We recall some basic facts about Pfaffians. Let A be an $n \times n$ alternating matrix over a ring R. This means that A is skew-symmetric with zeros on the diagonal. If $n = 2m$ is even then the Pfaffian $\mathrm{pf}(A)$ is a polynomial of degree m in the entries of A satisfying $\mathrm{pf}(A)^2 = \det(A)$. In general the $2m \times 2m$ Pfaffians are the Pfaffians of the $2m \times 2m$ submatrices obtained by deleting the same rows and columns. They generate an ideal in R that is unchanged if we replace A by $P^T AP$ for $P \in \mathrm{GL}_n(R)$. In the case R is a field, the $(2r+2) \times (2r+2)$ Pfaffians of A vanish if and only if A has rank at most $2r$. If n is odd then we call the $(n-1) \times (n-1)$ Pfaffians of A the submaximal Pfaffians.

2. Determinantal presentations

The following lemma extends work of Knight [Kn], who considered divisors D_1 and D_2 that are multiples of a fixed point $P \in C$. The integers $\beta(r,n)$ were defined in Proposition 1.1.

Lemma 2.1. *Let C be a smooth curve of genus one. Let H be a divisor on C of degree $n \geq 2$. Let $\mathcal{L}(H)$ have basis $x_1, \ldots, x_n$. Let V_r be the vector space of r-ics in $k[x_1, \ldots, x_n]$ spanned by the $r \times r$ minors of all matrices $\Phi(D_1, D_2)$ as D_1, D_2 run over all divisors on C with $D_1 + D_2 = H$ and $\deg D_1$, $\deg D_2 \geq 1$. Then $\dim V_r \geq \beta(r,n)$.*

Proof. We begin by treating the case $r = 1$. Let P_1 and P_2 be distinct points on C with $P_1 + P_2 \not\sim H$. Then $\mathcal{L}(H) = \mathcal{L}(H - P_1) + \mathcal{L}(H - P_2)$ is spanned by the entries of $\Phi(P_1, H - P_1)$ and $\Phi(P_2, H - P_2)$. So $\dim V_r = n = \beta(1,n)$ as required.

The proof is now by induction on r and n. The case $n < 2r$ is trivial since $\beta(r,n) = 0$. We may therefore suppose that $r \geq 2$ and $n \geq 2r$, and that the result is known for all smaller values of r and n.

Let $P \in C$ be any point. We may arrange that $x_1, \ldots, x_{n-i}$ is a basis for $\mathcal{L}(H - iP)$ for $i = 0, 1, 2$. Let $g(x_1, \ldots, x_{n-2})$ be an $(r-1) \times (r-1)$ minor of $\Phi(D_1, D_2)$ where $D_1 + D_2 = H - 2P$ and $\deg D_1$, $\deg D_2 \geq 1$. It is determined by $(r-1)$-dimensional subspaces $W_1 \subset \mathcal{L}(D_1)$ and $W_2 \subset \mathcal{L}(D_2)$. Since $\deg D_1$, $\deg D_2 \geq 1$ we may pick $w_1 \in \mathcal{L}(D_1 + P) \setminus \mathcal{L}(D_1)$ and $w_2 \in \mathcal{L}(D_2 + P) \setminus \mathcal{L}(D_2)$. Then $w_1 w_2 \in \mathcal{L}(H) \setminus \mathcal{L}(H - P)$. Rescaling w_1 if necessary we may assume that

$$w_1 w_2 = x_n + \ell(x_1, \ldots, x_{n-1})$$

where ℓ is a linear form. Let f be the $r \times r$ minor of $\Phi(D_1 + P, D_2 + P)$ determined by $W_1 \oplus \langle w_1 \rangle$ and $W_2 \oplus \langle w_2 \rangle$. Then

$$f(x_1, \ldots, x_n) = x_n g(x_1, \ldots, x_{n-2}) + h(x_1, \ldots, x_{n-1})$$

for some $h \in k[x_1, \ldots, x_{n-1}]$. This construction of f from g, combined with the induction hypothesis, shows that $V_r \cap k[x_1, \ldots, x_{n-1}]$ is a subspace of V_r of codimension at least $\beta(r-1, n-2)$. But trivially, if $f(x_1, \ldots, x_{n-1})$ is an $r \times r$ minor of $\Phi(D_1, D_2)$ with $D_1 + D_2 = H - P$, then it is also an $r \times r$ minor of $\Phi(D_1, D_2 + P)$. Thus

$$\dim V_r \geq \beta(r, n-1) + \beta(r-1, n-2) = \beta(r, n).$$

□

Let $C \subset \mathbb{P}^{n-1}$ be an elliptic normal curve with hyperplane section H. We noted in the introduction that the $(r+1) \times (r+1)$ minors of $\Phi(D_1, D_2)$ vanish on $\mathrm{Sec}^r C$. So combining the last lemma with Proposition 1.1 gives

Lemma 2.2. *If $n \geq 2r+2$ then $I(\mathrm{Sec}^r C)$ is generated by the $(r+1) \times (r+1)$ minors of the matrices $\Phi(D_1, D_2)$ as D_1, D_2 run over all divisors on C with $D_1 + D_2 = H$ and* $\deg D_1$,$\deg D_2 \geq 1$.

The aim of this section is to prove Theorem 1.3. This theorem is a variant of Lemma 2.2 giving necessary and sufficient conditions for $I(\mathrm{Sec}^r C)$ to be generated by the $(r+1) \times (r+1)$ minors of a single matrix $\Phi(D_1, D_2)$. First however we record a consequence of the results obtained so far, that will be used in §4. For any $P \in C$, we write C_P and C_{2P} for the elliptic normal curves of degrees $n-1$ and $n-2$ obtained by projecting away from P and $T_P C$.

Corollary 2.3. *Let $n \geq 5$. If we choose co-ordinates on $\mathbb{P}^{n-1}$ so that $P \in C$ is the point $(x_1 : \ldots : x_n) = (0 : 0 : \ldots : 0 : 1)$ then*
(i) $I(\mathrm{Sec}^r C) \cap k[x_1, \ldots, x_{n-1}] = I(\mathrm{Sec}^r C_P)$, and
(ii) for every r-ic $g \in I(\mathrm{Sec}^{r-1} C_{2P})$ there exists an $(r+1)$-ic $h \in k[x_1, \ldots, x_{n-1}]$ with $x_n g + h \in I(\mathrm{Sec}^r C)$.

Proof. (i) Let $\pi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-2}$; $(x_1 : \ldots : x_n) \mapsto (x_1 : \ldots : x_{n-1})$ be the projection map. Since $\mathrm{Sec}^r C_P$ is the Zariski closure of $\pi(\mathrm{Sec}^r C)$ the result is clear.
(ii) Let H, $H - P$ and $H - 2P$ be the hyperplane sections of C, C_P and C_{2P}. By Lemma 2.2 we may assume that g is an $r \times r$ minor of $\Phi(D_1, D_2)$ for some D_1, D_2 with $D_1 + D_2 = H - 2P$ and $\deg D_1$, $\deg D_2 \geq 1$. The proof of Lemma 2.1 constructs $f \in I(\mathrm{Sec}^r C)$, an $(r+1) \times (r+1)$ minor of $\Phi(D_1 + P, D_2 + P)$, with $f = x_n g + h$ for some $h \in k[x_1, \ldots, x_{n-1}]$. □

Lemma 2.4. *If $n \geq 2r+3$ then $I(\mathrm{Sec}^r C)$ is generated by the ideals $I(\mathrm{Sec}^r C_P)$ as P runs over any n distinct points on C.*

Proof. Let X be a subset of C with $|X| \geq n$. Let I be the ideal in $k[x_1, \ldots, x_n]$ generated by the $I(\mathrm{Sec}^r C_P)$ for $P \in X$. By Lemma 2.2 it suffices to show that if $D_1 + D_2 = H$ then all $(r+1) \times (r+1)$ minors of $\Phi(D_1, D_2)$ belong to I. Swapping D_1 and D_2 if necessary we may assume that $\deg D_1 \geq r+2$. Let $d = \deg D_1$. We pick distinct points $P_1, \ldots, P_d \in X$ with $D_1 \not\sim P_1 + \ldots + P_d$. Then

$$\bigcap_{i=1}^{d} \mathcal{L}(D_1 - P_i) = \mathcal{L}(D_1 - (P_1 + \ldots + P_d)) = 0.$$

So there exists a basis $v_1, \ldots, v_d$ for $\mathcal{L}(D_1)$ such that $\mathcal{L}(D_1 - P_i)$ has basis $v_1, \ldots, \widehat{v_i}, \ldots, v_d$. Then each $(r+1) \times (r+1)$ minor of $\Phi(D_1, D_2)$ is an $(r+1) \times (r+1)$ minor of $\Phi(D_1 - P_i, D_2)$ for some $1 \leq i \leq d$. We are done since the latter belong to $I(\mathrm{Sec}^r C_{P_i})$. □

Remark 2.5. The proof of Lemma 2.4 shows that it suffices for P to run over any $n - r$ distinct points on C. This improvement is irrelevant for our applications.

For D an effective divisor on C we write $\overline{D}$ for the linear subspace of $\mathbb{P}^{n-1}$ cut out by $\mathcal{L}(H - D) \subset \mathcal{L}(H)$. If D is a sum of distinct points then $\overline{D}$ is simply the linear span of these points.

Lemma 2.6. *Let D, D_1, D_2 be effective divisors on C. Then*

(i) $\dim \overline{D} = \begin{cases} \deg D - 1 & \text{if } \deg D < n \\ n - 2 & \text{if } D \sim H \\ n - 1 & \text{otherwise.} \end{cases}$

(ii) The linear span of $\overline{D_1}$ and $\overline{D_2}$ is $\overline{\mathrm{lcm}(D_1, D_2)}$.

(iii) $\overline{D_1} \cap \overline{D_2} = \overline{\gcd(D_1, D_2)}$ if $\mathrm{lcm}(D_1, D_2)$ has degree at most n and is not linearly equivalent to H.

Proof. (i) This is immediate from Riemann-Roch.
(ii) We have $\mathcal{L}(H - D_1) \cap \mathcal{L}(H - D_2) = \mathcal{L}(H - \mathrm{lcm}(D_1, D_2))$.
(iii) The inclusion "$\supset$" is clear. Equality follows by counting dimensions using (i) and (ii). □

Lemma 2.7. *Let D_1, D_2 be divisors on C with $D_1 + D_2 = H$ and $\deg D_1 \leq \deg D_2$. Then*

$$\{\mathrm{rank}\, \Phi(D_1, D_2) < \deg D_1\} \;=\; \bigcup_{D \in |D_1|} \overline{D}.$$

Proof. A point $P \in \mathbb{P}^{n-1}$ corresponds to a codimension 1 subspace $V_P \subset \mathcal{L}(H)$. If $\Phi(D_1, D_2)$ evaluated at P has rank less than $\deg D_1$ then there exists $f \in \mathcal{L}(D_1)$ such that $fg \in V_P$ for all $g \in \mathcal{L}(D_2)$. Say $(f) = D - D_1$. Then $\mathcal{L}(H - D) \subset V_P$, so $P \in \overline{D}$. The converse is obtained by reversing these steps. □

We make a temporary definition.

Definition 2.8. A *divisor pair* (D_1, D_2) consists of divisors D_1, D_2 on C with $D_1 + D_2 = H$ and $\deg D_1$, $\deg D_2 \geq r + 1$. We say that divisor pairs (D_1, D_2) and (D_1', D_2') are equivalent if $D_1 \sim D_1'$ or $D_1 \sim D_2'$.

The next lemma is a reworking of [Ro, 9.22.1]. If $n = 2r+2$ then we already know by Proposition 1.1 that the space of $(r+1)$-ics vanishing on $\mathrm{Sec}^r C$ has dimension $\beta(r+1, n) = 2$.

Lemma 2.9 (Room)**.** *Let $n = 2r + 2$ with $r \geq 1$. If (D_1, D_2) and (D_1', D_2') are inequivalent divisor pairs, then*

$$\mathrm{Sec}^r C = \{\det \Phi(D_1, D_2) = \det \Phi(D_1', D_2') = 0\} \subset \mathbb{P}^{n-1}.$$

Proof. As noted in the introduction each of these $(r+1)$-ics contains $\mathrm{Sec}^r C$. Conversely, if P belongs to the right hand side, we know by Lemma 2.7 that $P \in \overline{D} \cap \overline{D'}$ for some $D \in |D_1|$ and $D' \in |D_1'|$. Since $D + D' \not\sim H$ it follows by Lemma 2.6(iii) that $P \in \overline{\gcd(D, D')}$. Since $D \neq D'$ this shows that $P \in \mathrm{Sec}^r C$. □

We strengthen Lemma 2.2.

Lemma 2.10. *If $n \geq 2r+2$ then $I(\mathrm{Sec}^r C)$ is generated by the $(r+1)\times(r+1)$ minors of $\Phi(D_1, D_2)$ and $\Phi(D_1', D_2')$ where (D_1, D_2) and (D_1', D_2') are any two inequivalent divisor pairs.*

Proof. The proof is by induction on n, the case $n = 2r + 2$ having been treated in Lemma 2.9. We may therefore suppose that $n \geq 2r+3$ and $\deg D_1$, $\deg D_1' \geq r+2$. Let P run over any n distinct points on C with $D_1 - D_2' \not\sim P$. Then $(D_1 - P, D_2)$ and $(D_1' - P, D_2')$ are inequivalent divisor pairs on $C_P \subset \mathbb{P}^{n-2}$. By induction hypothesis the $(r+1)\times(r+1)$ minors of $\Phi(D_1 - P, D_2)$ and $\Phi(D_1' - P, D_2')$ generate $I(\mathrm{Sec}^r C_P)$. Since these are submatrices of $\Phi(D_1, D_2)$ and $\Phi(D_1', D_2')$ we are done by Lemma 2.4. □

Proof of Theorem 1.3. We may assume $\deg D_1$, $\deg D_2 \geq r + 1$ since there would otherwise be no $(r + 1) \times (r + 1)$ minors to consider. The proof is divided into 4 cases.

(i) Suppose that $\deg D_1, \deg D_2 \geq r + 2$ and $D_1 \not\sim D_2$. Let P run over any n distinct points on C. Then $(D_1 - P, D_2)$ and $(D_1, D_2 - P)$ are inequivalent divisor pairs on $C_P \subset \mathbb{P}^{n-2}$. We know by Lemma 2.10 that $I(\mathrm{Sec}^r C_P)$ is generated by the $(r + 1) \times (r + 1)$ minors of $\Phi(D_1 - P, D_2)$ and $\Phi(D_1, D_2 - P)$. Since these are submatrices of $\Phi(D_1, D_2)$ we are done by Lemma 2.4.

(ii) Suppose that $\deg D_1$, $\deg D_2 \geq r + 3$ and $D_1 \sim D_2$. Let P run over any n distinct points on C. We know by case (i) that $I(\mathrm{Sec}^r C_P)$ is generated by the $(r + 1) \times (r + 1)$ minors of $\Phi(D_1, D_2 - P)$. Since this is a submatrix of $\Phi(D_1, D_2)$ we are done by Lemma 2.4.

(iii) Suppose that $\deg D_1 = r + 1$ and $\deg D_2 \geq r + 1$. Then $\Phi(D_1, D_2)$ has at most $\binom{r+t+1}{t}$ linearly independent minors where $t = \deg D_2 - \deg D_1 = n - 2r - 2$. By Proposition 1.1 the vector space of $(r+1)$-ics generating $I(\mathrm{Sec}^r C)$ has dimension

$$\beta(r + 1, n) = \binom{r + t + 1}{t} + \binom{r + t}{t} > \binom{r + t + 1}{t}.$$

Hence $\Phi(D_1, D_2)$ is not a determinantal presentation of $\mathrm{Sec}^r C$.

(iv) Suppose that $\deg D_1 = \deg D_2 = r+2$ and $D_1 \sim D_2$. Choosing suitable bases for $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$ we may arrange that $\Phi(D_1, D_2)$ is symmetric. Then $\Phi(D_1, D_2)$ has at most $(r + 2)(r + 3)/2$ linearly independent minors. By Proposition 1.1 the vector space of $(r + 1)$-ics generating $I(\mathrm{Sec}^r C)$ has dimension

$$\beta(r + 1, n) = (r + 2)^2 > (r + 2)(r + 3)/2.$$

Hence $\Phi(D_1, D_2)$ is not a determinantal presentation of $\mathrm{Sec}^r C$. □

3. The Heisenberg group

Let C be an elliptic normal curve of degree n. We prove Theorem 1.7 under the assumption $\mathrm{char}\,(k) \nmid n$. We begin by describing the action of $E[n]$ on C where E is the Jacobian of C.

Lemma 3.1. *Let $C \subset \mathbb{P}^{n-1}$ be an elliptic normal curve. Then C meets each hyperplane in at most n points. In the case of equality the points of intersection span the hyperplane.*

Proof. Since C is contained in no hyperplane, the first statement is Bézout's theorem. If the n points of intersection fail to span the hyperplane then we obtain a contradiction by considering another hyperplane through the same n points and a further point of C. □

Remark 3.2. An equally short proof of Lemma 3.1 uses Riemann-Roch instead of Bézout's theorem, cf. Lemma 2.6(i).

Lemma 3.3. *Let $C \subset \mathbb{P}^{n-1}$ be an elliptic normal curve. Let τ_P be the translation map by $P \in E$ where E is the Jacobian of C.*
(i) The automorphism τ_P lifts to PGL_n if and only if $P \in E[n]$.
(ii) If P has exact order n then τ_P has exactly n fixed hyperplanes.

Proof. (i) Let H be the divisor of a hyperplane section. Then

$$\tau_P \text{ lifts to } \mathrm{PGL}_n \iff \tau_P^* H \sim H \iff P \in E[n].$$

(ii) We lift $\tau_P \in \mathrm{PGL}_n$ to $M_P \in \mathrm{GL}_n$ with $M_P^n = I_n$. Since $\mathrm{char}\,(k) \nmid n$ we may assume that M_P is diagonal, with each eigenvalue an nth root of unity. We consider a hyperplane fixed by τ_P. The intersection of this hyperplane with C is non-empty by Bézout's theorem and contains n distinct points by the action of τ_P. So by Lemma 3.1 the points of intersection span the hyperplane. Repeating for each fixed hyperplane it follows that M_P has no repeated eigenvalues. Equivalently τ_P has exactly n fixed hyperplanes. □

We fix $\zeta_n \in k$ a primitive nth root of unity. The Heisenberg group H_n is the subgroup of GL_n generated by the matrices

$$\theta(\sigma) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta_n & 0 & \cdots & 0 \\ 0 & 0 & \zeta_n^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_n^{n-1} \end{pmatrix}, \quad \theta(\tau) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

As our notation indicates, we prefer to view H_n as an abstract group with generators σ and τ. Then $\theta : H_n \to \mathrm{GL}_n$ is a faithful representation, called the Schrödinger representation. Notice that H_n is a non-abelian group of order n^3 with centre generated by the commutator of σ and τ. Let $\overline{\theta} : H_n \to \mathrm{PGL}_n$ be the corresponding projective representation. We write $e_n : E[n] \times E[n] \to \mu_n$ for the Weil pairing.

Lemma 3.4. *Let $C \subset \mathbb{P}^{n-1}$ be an elliptic normal curve with Jacobian E. Let $S, T \in E[n]$ with $e_n(S,T) = \zeta_n$. Then we may choose co-ordinates on $\mathbb{P}^{n-1}$ such that τ_S and τ_T are given by $\overline{\theta}(\sigma)$ and $\overline{\theta}(\tau)$.*

Proof. We know by Lemma 3.3 that τ_S and τ_T lift to PGL_n and that each has exactly n fixed hyperplanes. Since τ_S and τ_T commute, the hyperplanes fixed by τ_S are permuted by τ_T. Let P belong to the intersection of C and a hyperplane fixed by τ_S. If the same hyperplane is fixed by a power of τ_T, other than the identity, then the translates of P under $E[n]$ give a contradiction to Lemma 3.1. It follows that τ_T cyclically permutes the n hyperplanes fixed by τ_S. Thus we may choose co-ordinates on $\mathbb{P}^{n-1}$ such that τ_S and τ_T are given by $\overline{\theta}(\sigma)$ and $\overline{\theta}(\tau)$, at least if we replace ζ_n by ζ_n^r for some r coprime to n. In fact τ_S and τ_T commute in PGL_n, but their commutator when lifted to GL_n is independent of these liftings and gives one possible definition of the Weil pairing. It follows that $r = 1$ as required. □

Lemma 3.5. *Let $C \subset \mathbb{P}^{n-1}$ be a Heisenberg invariant elliptic normal curve. Then C is also invariant under*

$$\iota = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Proof. Let $P_0 \in C$ with $n.P_0 \sim H$ where H is the hyperplane section. The automorphism $[-1]$ of the elliptic curve (C, P_0) extends to $\mathbb{P}^{n-1}$ to give an involution $\iota \in \mathrm{PGL}_n$ satisfying $\iota\,\overline{\theta}(h)\,\iota = \overline{\theta}(h)^{-1}$ for all $h \in H_n$. Since $\overline{\theta}(H_n)$ is its own centraliser inside PGL_n it follows that C is invariant under any involution satisfying these commutation relations. One such involution is given in the statement of the lemma. □

Remark 3.6. In the case $k = \mathbb{C}$ we refer to [H, Chapter I] for alternative proofs of Lemmas 3.4 and 3.5 using theta functions.

Following [F, Chapter 4] we derive equations for a Heisenberg invariant elliptic normal curve of odd degree. We write $(x_0 : x_1 : \ldots : x_{n-1})$ for our co-ordinates on $\mathbb{P}^{n-1}$ and agree to read all subscripts mod n.

Proposition 3.7. *Let $n \geq 5$ be an odd integer. Let C be a Heisenberg invariant elliptic normal curve of degree n. Then C contains a unique point of the form*

$$P_0 = (0 : a_1 : a_2 : \ldots : -a_2 : -a_1)$$

and has Klein matrix $\Phi = (a_{i-j}x_{i+j})$.

Proof. Since n is odd there is a point $P_0 \in C \cap \{x_0 = 0\}$ fixed by ι. It takes the form

$$\begin{aligned} & P_0 = (0 : a_1 : a_2 : \ldots : a_2 : a_1) && (+) \\ \text{or} \quad & P_0 = (0 : a_1 : a_2 : \ldots : -a_2 : -a_1). && (-) \end{aligned}$$

The n^2 translates of P_0 under $E[n]$ account for all intersections of C with the co-ordinates hyperplanes $\{x_i = 0\}$. It follows that the a_i are non-zero for $i \neq 0$. We set $a_0 = 0$. The action of $\overline{\theta}(\sigma)$ now shows that P_0 is unique.

Let $W = H^0(C, \mathcal{O}_C(1))$ be the space of linear forms on $\mathbb{P}^{n-1}$ and let $V \subset S^2W$ be the space of quadrics vanishing on C. By Proposition 1.1 we have $\dim V = n(n-3)/2$, whereas $\dim S^2W = n(n+1)/2$. The action of $\overline{\theta}(\sigma)$ allows us to write $V = \oplus V_i$ and $S^2W = \oplus (S^2W)_i$ with

$$V_i \subset (S^2W)_i = \langle x_i^2, x_{i-1}x_{i+1}, \ldots \rangle.$$

Since n is odd we deduce via the action of $\overline{\theta}(\tau)$ that $\dim V_i = (n-3)/2$ and $\dim(S^2W)_i = (n+1)/2$. Let $P_0 \in C$ satisfy either $(+)$ or $(-)$. The translates of P_0 under $\overline{\theta}(\tau)$ impose some linear conditions on the coefficients of the quadrics in V_0. Since $V_0 \subset (S^2W)_0$ has codimension 2 it follows that $\mathrm{rank}(a_{i-j}a_{i+j}) \leq 2$. If P_0 is of the form $(+)$, then the top left 3×3 minor gives $2a_1^3a_2^2a_3 = 0$. Since the a_i are non-zero it follows that P_0 must be of the form $(-)$.

Now $\Phi = (a_{i-j}x_{i+j})$ is an $n \times n$ alternating matrix of linear forms. Evaluated at P_0 it has rank 2. So the 4×4 Pfaffians of Φ are quadrics vanishing at each of the n^2 translates of P_0 under $E[n]$. It follows by Bézout's theorem that these quadrics vanish on the whole of C. Hence Φ has rank 2 on C.

Deleting the first row and column of Φ gives a submaximal Pfaffian of the form $\pm(\prod_{i=1}^{(n-1)/2} a_i)x_0^{(n-1)/2} + \ldots$. Since the a_i are non-zero this Pfaffian is non-zero. We deduce via the action of the Heisenberg group that the n submaximal Pfaffians of Φ are linearly independent. So Φ is a Klein matrix as was to be shown. □

This completes our proof of Theorem 1.7 under the assumption $\mathrm{char}\,(k) \nmid n$. (Notice that the theorem is a tautology if $n = 3$.) A little combinatorial checking shows that the Klein matrix Φ of Proposition 3.7 is a Pfaffian presentation of C. (Later this will follow as a special case of Corollary 1.8.) The details are as follows.

Lemma 3.8. *Let $n \geq 5$ be an odd integer. Let Γ be the set of subsets of $\mathbb{Z}/n\mathbb{Z}$ of size 4 and sum 0. Let Δ be the set of subsets of $(\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}$ of size 3. Then there is a surjective map*

$$f : \Gamma \to \Delta\,; \quad \{i,j,k,l\} \mapsto \{i+j, i+k, i+l\}.$$

Proof. We first check that f is well defined. Let $\gamma = \{i,j,k,l\} \in \Gamma$. If $i+j \equiv \pm(i+k) \pmod{n}$ then $j \equiv k \pmod{n}$ or $i \equiv l \pmod{n}$. Since i,j,k,l are distinct mod n it follows that $f(\gamma)$ has size 3. Moreover, since $i+j \equiv -(k+l) \pmod{n}$, the image of γ is independent of the order in which we write its elements.

Now let $\gamma = \{i,j,k,l\}$ and $\gamma' = \{i',j',k',l'\}$. If $f(\gamma) = f(\gamma')$ then reordering the elements of γ' we may suppose $i+j \equiv i'+j'$ and $i+k \equiv i'+k'$. Then $i+l \equiv \pm(i'+l')$ giving either $(i,j,k,l) = (i',j',k',l')$ or $(i,j,k,l) = (-l',-k',-j',-i')$. In an obvious notation $\gamma = \pm\gamma'$. It is readily seen that $\gamma = -\gamma$ if and only if $0 \in f(\gamma)$. So for $\delta \in \mathrm{im}\, f$, $\#f^{-1}(\delta) = 1$ or 2 according as $0 \in \delta$ or $0 \notin \delta$. Surjectivity is established by counting:

$$\frac{1}{n}\binom{n}{4} = \binom{(n-1)/2}{2} + 2\binom{(n-1)/2}{3}.$$

□

Let C be a Heisenberg invariant elliptic normal curve of odd degree $n \geq 5$. Let $\Phi = (a_{i-j}x_{i+j})$ and V as above. Then C meets the co-ordinate hyperplanes $\{x_i = 0\}$ in a total of n^2 distinct points. So every non-zero quadric in V has at least 3 non-zero terms. Since $V_0 \subset (S^2W)_0$ has codimension 2 it follows that, up to scalars, there is a unique quadric in V_0 involving any three of the monomials $x_0^2, x_1x_{n-1}, \ldots, x_{(n-1)/2}x_{(n+1)/2}$. These quadrics span V_0 and are indexed by Δ. On the other hand the 4×4 Pfaffians of Φ belonging to $(S^2W)_0$ are indexed by Γ. So by Lemma 3.8 the 4×4 Pfaffians of Φ span V, and so generate $I(C)$.

Remark 3.9. The equations we derive for an elliptic normal curve, namely the 4×4 Pfaffians of $\Phi = (a_{i-j}x_{i+j})$, appear in the work of Klein [Kl, §11] as relations among

theta functions. This is our reason for calling Φ a Klein matrix. These equations, and their relationship with the modular curve $X(n)$, have also been studied in [AR], [F], [GP], [V]. Sometimes it is useful to consider the matrices $(a_{i-j}x_{i+j})$ where $(a_0 : a_1 : \ldots : a_{n-1})$ is any point on C. These are called Moore matrices: see [ADHPR], [GP].

4. Gorenstein ideals

We give an alternative proof of Theorem 1.7, that applies over an arbitrary field k. The idea is to replace our elliptic normal curve by a higher secant variety of codimension 3 and then apply the Buchsbaum-Eisenbud structure theorem.

Let $R = k[x_1, \ldots, x_n]$ be the homogeneous co-ordinate ring of $\mathbb{P}^{n-1}$, given its usual grading by degree. For $M = \oplus M_d$ a graded R-module we write $M(c)$ for the graded R-module with $M(c)_d = M_{c+d}$.

Theorem 4.1. *Let $C \subset \mathbb{P}^{n-1}$ be an elliptic normal curve. If $m = n - 2r \geq 2$ then $I(Sec^r C)$ has a minimal graded free resolution of the form*

$$0 \to R(-n) \to R(-n+r+1)^{b_{m-1}} \to R(-n+r+2)^{b_{m-2}} \to \ldots$$
$$\ldots \to R(-r-2)^{b_2} \to R(-r-1)^{b_1} \to R \to 0.$$

In particular $\mathrm{Sec}^r C$ is projectively Gorenstein of codimension m.

Proof. This was proved independently by Graf v. Bothmer and Hulek [vBH, §8], and the author. See also [GP] for a discussion of the cases $r = 1, 2$.

The final statement follows from the form of the minimal free resolution and the fact, already noted in the introduction, that $\mathrm{Sec}^r C \subset \mathbb{P}^{n-1}$ has codimension m. For this one uses the graded analogue of [E, Corollary 21.16]. One consequence (which also follows directly from the proofs cited above) is that $b_{m-i} = b_i$ for all i. □

By [BH, Theorem 4.1.15(a)] the Betti numbers are

$$b_i(r, m) = \frac{m+2r}{m+r-i} \prod_{j=r+1, j\neq r+i}^{m+r-1} \frac{j}{|r+i-j|}.$$

Since $b_1(r, m) = \beta(r+1, m+2r)$ we recognise Proposition 1.1 as a special case of Theorem 4.1. We are mainly interested in Theorem 4.1 in the cases $m = 2, 3$.

Proposition 4.2. *Let $C \subset \mathbb{P}^{n-1}$ be an elliptic normal curve.*
(i) If $n = 2r + 2$ then $I(\mathrm{Sec}^r C)$ has a minimal graded free resolution of the form

$$0 \longrightarrow R(-n) \longrightarrow R(-r-1)^2 \longrightarrow R \longrightarrow 0.$$

(ii) If $n = 2r + 3$ then $I(\mathrm{Sec}^r C)$ has a minimal graded free resolution of the form

$$0 \longrightarrow R(-n) \xrightarrow{p^T} R(-r-2)^n \xrightarrow{\Phi} R(-r-1)^n \xrightarrow{p} R \longrightarrow 0$$

where Φ is an $n \times n$ alternating matrix of linear forms, and p is the row vector of submaximal Pfaffians of Φ.

Proof. This is a special case of Theorem 4.1. In case (ii) we use the Buchsbaum-Eisenbud structure theorem [BE1], [BE2]. □

The next lemma shows how the theory of unprojection, first studied by Kustin and Miller [KM], can be used to deduce Proposition 4.2(ii) from Proposition 4.2(i). More importantly for us, it shows that the matrix Φ in Proposition 4.2(ii) is a Klein matrix for C.

Lemma 4.3. *Let $n = 2r+3$ with $r \geq 1$. Let I and J be ideals in $R = k[x_1, \dots, x_{n-1}]$ with minimal free resolutions*

$$F_\bullet : \quad 0 \longrightarrow R(-n+1) \xrightarrow{\left(\begin{smallmatrix} -f_2 \\ f_1 \end{smallmatrix}\right)} R(-r-1)^2 \xrightarrow{(\,f_1\ f_2\,)} R \longrightarrow 0$$

and

$$G_\bullet : \quad 0 \longrightarrow R(-n+2) \xrightarrow{g^T} R(-r-1)^{n-2} \xrightarrow{\Psi} R(-r)^{n-2} \xrightarrow{g} R \longrightarrow 0$$

where Ψ is an $(n-2) \times (n-2)$ alternating matrix of linear forms. Suppose that $\mathcal{I}$ is a prime ideal in $\mathcal{R} = R[x_n]$, with $\mathcal{I} \cap R = I$, generated by the entries of

$$p = (f_1, f_2, x_n g_1 + h_1, \dots, x_n g_{n-2} + h_{n-2})$$

for some $h_1, \dots h_{n-2} \in R$ homogeneous of degree $r+1$. Then $\mathcal{I}$ has a minimal graded free resolution of the form

$$\mathcal{F}_\bullet : \quad 0 \longrightarrow \mathcal{R}(-n) \xrightarrow{p^T} \mathcal{R}(-r-2)^n \xrightarrow{\Psi'} \mathcal{R}(-r-1)^n \xrightarrow{p} \mathcal{R} \longrightarrow 0$$

where

$$\Psi' = \begin{pmatrix} \begin{smallmatrix} 0 & \lambda x_n + b \\ -\lambda x_n - b & 0 \end{smallmatrix} & A \\ -A^T & -\Psi \end{pmatrix}$$

for some $0 \neq \lambda \in k$, $b \in R$ and $A \in \mathrm{Mat}_{2,n-2}(R)$.

Proof. Let f, g and h be the row vectors with entries f_i, g_i and h_i. Since the entries of $h\Psi = (x_n g + h)\Psi$ belong to $\mathcal{I} \cap R = I$ there exists $A \in \mathrm{Mat}_{2,n-2}(R)$ with

$$fA = h\Psi = (x_n g + h)\Psi. \tag{4.1}$$

Post-multiplying by g^T and using the exactness of $F_\bullet$ it follows that

$$Ag^T = \lambda \left(\begin{smallmatrix} -f_2 \\ f_1 \end{smallmatrix}\right) \tag{4.2}$$

for some $\lambda \in k$. We suppose for a contradiction that $\lambda = 0$. Then (4.2) and the exactness of $G_\bullet$ give $A = D\Psi$ for some $D \in \mathrm{Mat}_{2,n-2}(k)$. Substituting in (4.1) and using the exactness of $G_\bullet$ once more gives $fD - h = \ell g$ for some linear form $\ell \in R$. Then $(x_n - \ell)g = (x_n g + h) - (\ell g + h)$ has entries in $\mathcal{I}$. Since $\mathcal{I}$ is a prime ideal generated in degree $r+1$, this forces $\ell = x_n$, contradicting that $\ell \in R$.

By (4.2) we have

$$A(x_n g + h)^T = \lambda x_n \left(\begin{smallmatrix} -f_2 \\ f_1 \end{smallmatrix}\right) + Ah^T. \tag{4.3}$$

So the entries of Ah^T belong to $\mathcal{I} \cap R = I$, and there exists $B \in \mathrm{Mat}_{2,2}(R)$ with $Ah^T = Bf^T$. So by (4.1) and our hypothesis that Ψ is alternating

$$fBf^T = fAh^T = h\Psi h^T = 0.$$

Since f_1, f_2 are coprime it follows that B is alternating. So (4.3) becomes

$$A(x_n g + h)^T = (\lambda x_n + b)\left(\begin{smallmatrix} -f_2 \\ f_1 \end{smallmatrix}\right) \tag{4.4}$$

for some linear form $b \in R$.

By (4.1) and (4.4) our resolution $\mathcal{F}_\bullet$ is a complex. It remains to prove it is exact at the terms $\mathcal{R}(-r-1)^n$ and $\mathcal{R}(-r-2)^n$. For the first of these we must show that if $(u, v) \in \mathcal{R}^2 \times \mathcal{R}^{n-2}$ with

$$uf^T + v(x_n g + h)^T = 0 \tag{4.5}$$

then (u, v) is an $\mathcal{R}$-linear combination of the rows of Ψ'. We expand u and v in powers of x_n as

$$u = u_p x_n^p + \ldots + u_1 x_n + u_0$$
$$v = v_q x_n^q + \ldots + v_1 x_n + v_0$$

where $u_0, \ldots, u_p \in R^2$ and $v_0, \ldots, v_q \in R^{n-2}$. The proof is by induction on $\max(2p, 2q+1)$ with induction step divided into the following two cases. If $p > q$ then we subtract an $\mathcal{R}$-linear combination of the first two rows of Ψ' to decrease the value of p. If $p \leq q$ then we subtract an $\mathcal{R}$-linear combination of the last $n-2$ rows of Ψ' to decrease the value of q. (The latter is possible by comparing coefficients of x_n^{q+1} in (4.5) and using the exactness of $G_\bullet$.) To start the induction we must treat the case $u \in R^2$ and $v = 0$. But then exactness of $F_\bullet$ gives $u = \lambda t(-f_2, f_1)$ for some $t \in R$, and by (4.2) we have $u = tgA^T$ and $v = tg\Psi = 0$ as required.

Finally we prove that $\mathcal{F}_\bullet$ is exact at the term $\mathcal{R}(-r-2)^n$. To do this we must show that if $(u, v) \in \mathcal{R}^2 \times \mathcal{R}^{n-2}$ with $(u, v)\Psi' = 0$ then (u, v) is a multiple of $p = (f, x_n g + h)$. We have $uA = v\Psi$ and so $uAg^T = 0$. It follows by (4.2) and the exactness of $F_\bullet$ that u is a multiple of f. This reduces us to the case $u = 0$. Our hypothesis on v is now that (i) $vA^T = 0$ and (ii) $v\Psi = 0$. By (ii) and the exactness of $G_\bullet$ we see that v is a multiple of g. Then (i) combined with (4.2) shows that $v = 0$. □

Proposition 4.4. *Let $n \geq 3$ be an odd integer. Let C be an elliptic normal curve of degree n. Let Φ be an $n \times n$ alternating matrix of linear forms. Then Φ is a Klein matrix for C if and only if the submaximal Pfaffians of Φ generate $I(\mathrm{Sec}^r C)$ where $r = (n-3)/2$.*

Proof. Let Φ be a Klein matrix for C. Then Φ has rank 2 on C and so has rank at most $2r$ on $\mathrm{Sec}^r C$. So the submaximal Pfaffians of Φ belong to $I(\mathrm{Sec}^r C)$. Comparing dimensions in Definition 1.6 and Proposition 1.1 it follows that they generate $I(\mathrm{Sec}^r C)$.

Conversely suppose that the submaximal Pfaffians of Φ generate $I(\mathrm{Sec}^r C)$. By Proposition 1.1 they are linearly independent. It remains to show that Φ has rank 2 on C. The case $n = 3$ being a tautology, we suppose $n \geq 5$. Let $P \in C$ be any point. We choose co-ordinates $(x_1 : \ldots : x_n)$ on $\mathbb{P}^{n-1}$ so that $P = (0 : 0 : \ldots : 0 : 1)$. Let C_P and C_{2P} be the elliptic normal curves of degrees $n-1$ and $n-2$ obtained by projecting C away from P and $T_P C$. By Proposition 4.2, the ideals $I = I(\mathrm{Sec}^r C_P)$ and $J = I(\mathrm{Sec}^{r-1} C_{2P})R$ in $R = k[x_1, \ldots, x_{n-1}]$ have minimal free resolutions of the form specified in Lemma 4.3. We noted in the introduction that $\mathrm{Sec}^r C$ is an irreducible variety. So $\mathcal{I} = I(\mathrm{Sec}^r C)$ is a prime ideal in $\mathcal{R} = R[x_n]$. The remaining hypotheses of Lemma 4.3 follow by Proposition 1.1 and Corollary 2.3.

The matrix Ψ' constructed in Lemma 4.3 has rank 2 when evaluated at $P = (0 : 0 : \ldots : 0 : 1)$. By the Buchsbaum-Eisenbud structure theorem [BE1], [BE2], and the uniqueness of minimal free resolutions the same is true for Φ. □

Combining Propositions 4.2 and 4.4 gives our second proof of Theorem 1.7. This not only establishes the existence of Klein matrices over an arbitrary field k, but also gives a practical algorithm for computing them.

Remark 4.5. We may weaken Definition 1.6(i) to the statement that Φ has rank at most 2 on C. This is equivalent to the original definition by Proposition 4.4.

5. Linear sections of Grassmannians

The Grassmannian $\mathrm{Gr}(2,m) = \mathrm{Gr}(2,V)$ is the set of 2-dimensional vector subspaces of an m-dimensional vector space V. It is an irreducible projective variety of dimension $2(m-2)$. The Plücker embedding is

$$p : \mathrm{Gr}(2,V) \to \mathbb{P}(\wedge^2 V); \quad \langle v_1, v_2\rangle \mapsto v_1 \wedge v_2. \tag{5.1}$$

There is an exact sequence of vector bundles on $\mathrm{Gr}(2,V)$

$$0 \longrightarrow S \longrightarrow V^* \xrightarrow{\alpha} Q \longrightarrow 0 \tag{5.2}$$

where V^* is the trivial bundle of rank m, S the universal sub-bundle of rank $m-2$, and Q the universal quotient bundle of rank 2. The relationship between (5.1) and (5.2) is given by

Lemma 5.1. *There is an isomorphism of line bundles $p^*\mathcal{O}(1) \simeq \det Q$ on $\mathrm{Gr}(2,V)$ and a commutative diagram*

$$\begin{array}{ccc} H^0(\mathbb{P}(\wedge^2 V), \mathcal{O}(1)) & \xrightarrow{p^*} & H^0(\mathrm{Gr}(2,V), p^*\mathcal{O}(1)) \\ \| & & \| \\ \wedge^2 V^* & \xrightarrow{H^0(\wedge^2\alpha)} & H^0(\mathrm{Gr}(2,V), \det Q). \end{array}$$

Now let C be a smooth projective curve embedded by a complete linear system. As in the introduction we write $\Phi(\mathcal{E})$ for the matrix of linear forms representing the determinant map on global sections for a rank 2 vector bundle $\mathcal{E}$ on C with $\det \mathcal{E} \simeq \mathcal{O}(1)$.

Lemma 5.2. *Let Φ be an alternating matrix of linear forms. If Φ has rank 2 on C then there exists a rank 2 vector bundle $\mathcal{E}$ on C with $\det \mathcal{E} \simeq \mathcal{O}(1)$ and a matrix B with entries in k such that $\Phi = B^T \Phi(\mathcal{E}) B$.*

Proof. Let $\phi : C \to \mathbb{P}(\wedge^2 V)$ be the morphism determined by Φ. Since Φ has rank 2 on C, this morphism factors via the Plücker embedding, say $\phi = p\,\pi$ for some $\pi : C \to \mathrm{Gr}(2,V)$. Let $\mathcal{E} = \pi^* Q$. Then $\mathcal{E}$ is a rank 2 vector bundle on C. Since the entries of Φ are linear forms, we have $\phi^*\mathcal{O}(1) = \mathcal{O}(1)$. So by Lemma 5.1,

$$\det \mathcal{E} \simeq \pi^*(\det Q) \simeq \pi^* p^* \mathcal{O}(1) \simeq \mathcal{O}(1).$$

We obtain a commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}(\wedge^2 V), \mathcal{O}(1)) & \xrightarrow{\phi^*} & H^0(C, \mathcal{O}(1)) \\ \| & & \| \\ \wedge^2 V^* & \xrightarrow{H^0(\wedge^2\beta)} & H^0(C, \det \mathcal{E}) \end{array}$$

where $\beta = \pi^*(\alpha) : V^* \to \mathcal{E}$. We are done since ϕ^* is represented by Φ and $H^0(\wedge^2\beta)$ is represented by $B^T\Phi(\mathcal{E})B$ where B is the matrix representing the linear map $H^0(\beta) : V^* \to H^0(C, \mathcal{E})$. □

We describe the vector bundles that can arise in the case C is an elliptic normal curve.

Lemma 5.3. *Let C be an elliptic normal curve of degree n. Suppose that C is the rank 2 locus of an alternating matrix of linear forms Φ. Then the rank 2 vector bundle of Lemma 5.2 is either indecomposable or of the form $\mathcal{L}_1 \oplus \mathcal{L}_2$ with* $\deg \mathcal{L}_1, \deg \mathcal{L}_2 \geq 3$.

Proof. Suppose that $\mathcal{E}$ is decomposable, say $\mathcal{E} \simeq \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$ with $D_1 + D_2 = H$. Then C is the rank 1 locus of $\Phi(D_1, D_2)$. Since $C \neq \mathbb{P}^{n-1}$ we have $\deg D_1, \deg D_2 \geq 2$. If $\deg D_1 = 2$ then C is a $\mathbb{P}^{n-1}$-section of the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-3} \hookrightarrow \mathbb{P}^{2n-5}$. So C has dimension at least $(n-1) + (n-2) - (2n-5) = 2$. This is a contradiction. □

6. Pfaffian presentations

We recall a theorem of Atiyah.

Proposition 6.1. *Let C be a smooth curve of genus one. Let $(r, d) = 1$ and let $\mathcal{L}$ be a line bundle on C of degree d. Then there is a unique indecomposable rank r vector bundle $\mathcal{E}$ on C with determinant $\mathcal{L}$.*

Proof. See [A, Corollary to Theorem 7]. □

The degree of a vector bundle is by definition the degree of its determinant. We will use the following explicit form of Riemann-Roch.

Lemma 6.2. *Let C be a smooth curve of genus one. Let $\mathcal{E}$ be an indecomposable vector bundle on C. If $\det \mathcal{E} \not\simeq \mathcal{O}$ then $\dim H^0(C, \mathcal{E}) = \max(0, \deg \mathcal{E})$.*

Proof. See [A, Lemmas 6′, 15]. □

Proposition 6.3. *Let $n \geq 3$ be an odd integer. Let C be an elliptic normal curve of degree n. Let Φ be an $n \times n$ alternating matrix of linear forms on $\mathbb{P}^{n-1}$. Then Φ is a Klein matrix for C if and only if $\Phi = \Phi(\mathcal{E})$ for some indecomposable rank 2 vector bundle $\mathcal{E}$ on C with $\det \mathcal{E} \simeq \mathcal{O}(1)$.*

Proof. Let Φ be a Klein matrix for C. By Lemma 5.2 we have $\Phi = B^T \Phi(\mathcal{E}) B$ where B is a matrix with entries in k. By Definition 1.6 the $(n-1) \times (n-1)$ Pfaffians of Φ span a vector space of dimension at least n. The same must therefore be true of $\Phi(\mathcal{E})$.

We suppose for a contradiction that $\mathcal{E}$ is decomposable, say

$$\mathcal{E} \simeq \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$$

with $D_1 + D_2 = H$. Then the $(n-1) \times (n-1)$ Pfaffians of $\Phi(\mathcal{E})$ are the $(n-1)/2 \times (n-1)/2$ minors of $\Phi(D_1, D_2)$. If $n \geq 5$ then there are no such minors unless $\deg D_1 = (n-1)/2$ and $\deg D_2 = (n+1)/2$. But even in that case there are at most $(n+1)/2$ linearly independent minors. This is the required contradiction.

In the case $n = 3$ we must also rule out the possibility $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(1)$. It suffices to note that there are no 4×3 matrices B over k for which

$$B^T \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & 0 & 0 \\ -x_2 & 0 & 0 & 0 \\ -x_3 & 0 & 0 & 0 \end{pmatrix} B$$

has 3 linearly independent entries.

We have established that $\mathcal{E}$ is indecomposable. Lemma 6.2 gives $\dim H^0(C,\mathcal{E}) = n$. So $\Phi(\mathcal{E})$ is an $n\times n$ alternating matrix of linear forms. The linear independence of the submaximal Pfaffians of Φ implies that B is invertible. Since in the definition of $\Phi(\mathcal{E})$ we made an arbitrary choice of basis for $H^0(C,\mathcal{E})$, we are now free to suppose that B is the identity matrix. Hence $\Phi = \Phi(\mathcal{E})$.

The converse follows by Atiyah's uniqueness result (Proposition 6.1) and the existence of Klein matrices (Theorem 1.7). □

Proof of Theorem 1.5. Let C be an elliptic normal curve of degree n and let $\mathcal{E}$ be the unique indecomposable rank 2 vector bundle on C with $\det\mathcal{E}\simeq\mathcal{O}(1)$.

(i) The matrix $\Phi(\mathcal{E})$ is a Klein matrix by Proposition 6.3. So by definition its submaximal Pfaffians are linearly independent.

(ii) It is clear that $\Phi(\mathcal{E})$ has rank at most 2 on C and so has rank at most $2r$ on $\mathrm{Sec}^r C$. So the $(2r+2)\times(2r+2)$ Pfaffians of $\Phi(\mathcal{E})$ belong to $I(\mathrm{Sec}^r C)$. We must show that they generate this ideal. If $n = 2r+3$ then we are done by (i) and Proposition 1.1.

The proof is now by induction on odd values of n. Accordingly we suppose $n \geq 2r+5$ and that the result is known for $n-2$. Let D be any effective divisor on C of degree 2. Let $C_D \subset \mathbb{P}^{n-3}$ be the elliptic normal curve of degree $n-2$ obtained by projecting away from D. There is a natural inclusion $I(\mathrm{Sec}^r C_D) \subset I(\mathrm{Sec}^r C)$ where these ideals belong to different rings. By Lemma 2.4 it suffices to find generators for $I(\mathrm{Sec}^r C_D)$ among the $(2r+2)\times(2r+2)$ Pfaffians of $\Phi(\mathcal{E})$.

Let $P \in C$ with $D \sim 2P$, and let $\mathcal{E}_P = \mathcal{E}\otimes\mathcal{O}(-P)$. Then

$$\det\mathcal{E}_P \simeq (\det\mathcal{E})\otimes\mathcal{O}(-2P) \simeq \mathcal{O}(H-D)$$

where H and $H-D$ are the hyperplane sections for C and C_D. By induction hypothesis $\Phi(\mathcal{E}_P)$ is a Pfaffian presentation of $\mathrm{Sec}^r C_D$. We are done since $\Phi(\mathcal{E}_P)$ is a submatrix of $\Phi(\mathcal{E})$. □

Corollary 1.8 now follows from Proposition 6.3 and Theorem 1.5. In the case $n = 5$ we may characterise Klein matrices more simply as follows.

Corollary 6.4. *Let $C \subset \mathbb{P}^4$ be an elliptic normal quintic. Let Φ be a 5×5 alternating matrix of linear forms on $\mathbb{P}^4$. Then Φ is a Klein matrix for C if and only if C is the rank 2 locus of Φ.*

Proof. Let Φ be a Klein matrix for C. Taking $r = 1$ in Corollary 1.8 shows that C is the rank 2 locus of Φ.

Conversely, if C is the rank 2 locus of Φ we must show that the 4×4 Pfaffians of Φ are linearly independent. By Lemmas 5.2 and 5.3 we have $\Phi = B^T\Phi(\mathcal{E})B$ where $\mathcal{E}$ is an indecomposable rank 2 vector bundle on C with $\det\mathcal{E}\simeq\mathcal{O}(1)$. Then Lemma 6.2 gives $\dim H^0(C,\mathcal{E}) = 5$. So $\Phi(\mathcal{E})$ is a 5×5 alternating matrix of linear forms. If B were singular then the 4×4 Pfaffians of Φ would belong to a 1-dimensional vector space. But C cannot be defined by a single quadric. So B is invertible and Φ is a Klein matrix by Proposition 6.3. □

The analogue of Corollary 6.4 for $n \geq 7$ is false. To see this let D_1, D_2 be divisors on C with $D_1 + D_2 = H$ and $\deg D_1, \deg D_2 \geq 3$. Put $\mathcal{E} = \mathcal{O}(D_1)\oplus\mathcal{O}(D_2)$. Then C is the rank 2 locus of $\Phi(\mathcal{E})$ by Theorem 1.3, but $\Phi(\mathcal{E})$ is not a Klein matrix for C.

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