# INVARIANT THEORY FOR
# THE ELLIPTIC NORMAL QUINTIC,
# II. THE COVERING MAP

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ABSTRACT. A genus one curve $C$ of degree 5 is defined by the $4 \times 4$ Pfaffians of a $5 \times 5$ alternating matrix of linear forms on $\mathbb{P}^4$. We prove a result characterising the covariants for these models in terms of their restrictions to the family of curves parametrised by the modular curve $X(5)$. We then construct covariants describing the covering map of degree 25 from $C$ to its Jacobian and give a practical algorithm for evaluating them.

## 1. INTRODUCTION

**Definition 1.1.** Let $n \geq 3$ be an integer.

   (i) An *elliptic normal curve* $C \subset \mathbb{P}^{n-1}$ is a smooth curve of genus one and degree $n$ that spans $\mathbb{P}^{n-1}$.
  (ii) A *rational nodal curve* $C \subset \mathbb{P}^{n-1}$ is a rational curve of degree $n$ that spans $\mathbb{P}^{n-1}$ and has a single node.

If $C \subset \mathbb{P}^{n-1}$ is an elliptic normal curve then there is a covering map $\pi$ of degree $n^2$ from $C$ to its Jacobian $E$ given by $P \mapsto [nP - H] \in \operatorname{Pic}^0(C) \cong E$ where $H$ is the hyperplane section. We may also describe $\pi : C \to E$ as the map that quotients out by the action of $E[n]$ on $C$ by translation (assuming we are not in characteristic dividing $n$). The subgroup of $\operatorname{SL}_n$ consisting of matrices that describe this action is called the *Heisenberg group* of $C$. If $n$ is odd then over an algebraically closed field we may change co-ordinates so that this group is generated by

$$
(1) \qquad
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \zeta_n & 0 & \cdots & 0 \\
0 & 0 & \zeta_n^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & \zeta_n^{n-1}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
$$

where $\zeta_n$ is a primitive $n$th root of unity. (If $n$ is even then one must take scalar multiples of these matrices with determinant 1.)

In the cases $n = 2, 3, 4$ classical invariant theory gives formulae for the Jacobian $E$ and for the covering map $\pi : C \to E$. See [13], [14] for the cases $n = 2, 3$, and

[2] for a survey of the cases $n = 2, 3, 4$. In [8] we gave a practical algorithm for evaluating the invariants in the case $n = 5$ and showed that they give a formula for the Jacobian. We now extend this invariant theoretic approach to give a formula for the covering map.

We work throughout over a field $K$ of characteristic not dividing $6n$, where in due course we take $n = 5$. Except in the following paragraph, and at the end of Section 8, we assume for simplicity that $K$ is algebraically closed.

To explain the motivation for our work, let $E$ be an elliptic curve over a number field $K$. For any integer $n \geq 2$ the quotient group $E(K)/nE(K)$ injects into the $n$-Selmer group $S^{(n)}(E/K)$, which is finite and effectively computable. In an explicit $n$-descent calculation one represents each element of the $n$-Selmer group by (equations for) an elliptic normal curve $C \subset \mathbb{P}^{n-1}$ with Jacobian $E$. It is perhaps better to call $C$ a "genus one normal curve" as it need not have any $K$-rational points. The Selmer group elements with $C(K) \neq \emptyset$ make up the image of $E(K)/nE(K)$ in $S^{(n)}(E/K)$. Moreover if $P \in C(K)$ then a coset representative for the corresponding element of $E(K)/nE(K)$ is given by the image of $P$ under the covering map. Having explicit formulae for the covering map can therefore help in finding generators for the Mordell-Weil group $E(K)$.

In the case $n = 5$ the curves of Definition 1.1 are called *elliptic normal quintics* and *rational nodal quintics*. By the Buchsbaum-Eisenbud structure theorem [4], [5] they are defined by the $4 \times 4$ Pfaffians of a $5 \times 5$ alternating matrix of linear forms on $\mathbb{P}^4$. We call such a matrix $\phi$ a *genus one model* and write $C_\phi \subset \mathbb{P}^4$ for the subvariety defined by the $4 \times 4$ Pfaffians. It is shown in [8, Proposition 5.10] that $C_\phi$ is a smooth curve of genus one if and only if it is an elliptic normal quintic. In this case we say that $\phi$ is *non-singular*.

There is a natural action of $\mathrm{GL}_5 \times \mathrm{GL}_5$ on the space of genus one models. The first factor acts as $M : \phi \mapsto M\phi M^T$ and the second factor acts by changing co-ordinates on $\mathbb{P}^4$. We adopt the following notation. Let $V$ and $W$ be 5-dimensional vector spaces with bases $v_0, \ldots, v_4$ and $w_0, \ldots, w_4$. We identify the space of genus one models with $\wedge^2 V \otimes W$ via

$$\phi = (\phi_{ij}) \longleftrightarrow \sum_{i<j}(v_i \wedge v_j) \otimes \phi_{ij}(w_0, \ldots, w_4).$$

With this identification the action of $\mathrm{GL}_5 \times \mathrm{GL}_5$ becomes the natural action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ on $\wedge^2 V \otimes W$. By squaring and then identifying $\wedge^4 V \cong V^*$ there is a natural map

$$(2) \qquad\qquad P_2 : \wedge^2 V \otimes W \to V^* \otimes S^2 W = \mathrm{Hom}(V, S^2 W).$$

Explicitly $P_2(\phi) = (v_i \mapsto p_i(w_0, \ldots, w_4))$ where $p_0, \ldots, p_4$ are the $4 \times 4$ Pfaffians of $\phi$. Thus $V$ may be thought of as the space of quadrics defining $C_\phi$ and $W$ as the space of linear forms on $\mathbb{P}^4$.

**Lemma 1.2.** *The action of* $\mathrm{GL}(V) \times \mathrm{GL}(W)$ *is transitive on the the genus one models* $\phi$ *for which* $C_\phi$ *is a rational nodal quintic, and on the genus one models* $\phi$ *for which* $C_\phi$ *is an elliptic normal quintic with given j-invariant.*

*Proof.* See [8, Proposition 4.6]. $\qquad\square$

The co-ordinate ring $K[\wedge^2 V \otimes W]$ is a polynomial ring in 50 variables.

**Theorem 1.3.** *The ring of invariants for* $\mathrm{SL}(V) \times \mathrm{SL}(W)$ *acting on* $K[\wedge^2 V \otimes W]$ *is generated by invariants* $c_4$ *and* $c_6$ *of degrees* 20 *and* 30*. Moreover if we scale them as specified in* [8] *and put* $\Delta = (c_4^3 - c_6^2)/1728$ *then*

    (i) *a genus one model* $\phi$ *is non-singular if and only if* $\Delta(\phi) \neq 0$,

    (ii) *if* $\phi$ *is non-singular then* $C_\phi$ *has j-invariant* $c_4(\phi)^3/\Delta(\phi)$.

*Proof.* See [8, Theorem 4.4]. $\qquad\square$

**Lemma 1.4.** *Let* $\phi \in \wedge^2 V \otimes W$ *be a genus one model with* $C_\phi$ *either an elliptic normal quintic or a rational nodal quintic. Then the Zariski closure of the* $\mathrm{GL}(V) \times \mathrm{GL}(W)$-*orbit of* $\phi$ *is the zero locus of an irreducible homogeneous invariant* $I$. *Moreover we can take*

$$I = \begin{cases} c_4 & \text{if } j(C_\phi) = 0 \\ c_6 & \text{if } j(C_\phi) = 1728 \\ \Delta & \text{if } C_\phi \text{ is a rational nodal quintic} \\ c_4^3 - j(C_\phi)\Delta & \text{otherwise.} \end{cases}$$

*Proof.* The existence of $I$ is proved in [8, Lemma 4.10]. The invariants listed vanish at $\phi$ by Theorem 1.3 and are irreducible in $K[c_4, c_6]$. They are therefore irreducible in $K[\wedge^2 V \otimes W]$ since any factors would themselves have to be invariants. We use here that $\mathrm{SL}(V) \times \mathrm{SL}(W)$ is connected and has no 1-dimensional rational representations. Alternatively we can prove irreducibility by restricting to the Weierstrass models in [8, Section 6]. $\qquad\square$

**Lemma 1.5.** *Let* $I$ *be a non-constant homogeneous invariant. Then there exists* $\phi \in \wedge^2 V \otimes W$ *with* $I(\phi) = 0$ *and* $C_\phi$ *either an elliptic normal quintic or a rational nodal quintic.*

*Proof.* We may assume that $I$ is irreducible in $K[c_4, c_6]$. So up to scalar multiples we have $I = c_4, c_6, \Delta$ or $c_4^3 - j\Delta$ with $j \neq 0, 1728$. We take $C_\phi$ to be an elliptic normal quintic with the appropriate $j$-invariant, or in the case $I = \Delta$ a rational nodal quintic. $\qquad\square$

The covariants we need to describe the covering map are $\mathrm{SL}(V) \times \mathrm{SL}(W)$-equivariant polynomial maps $\wedge^2 V \otimes W \to S^{5d}W$ for $d = 1, 2, 3$. More generally we defined a *covariant* to be an $\mathrm{SL}(V) \times \mathrm{SL}(W)$-equivariant polynomial map $\wedge^2 V \otimes W \to Y$ where $Y$ is a rational representation of $\mathrm{GL}(V) \times \mathrm{GL}(W)$. In all our examples $Y$ will be *homogeneous* by which we mean there exist integers $r$ and $s$

such that the morphism $\rho_Y : \mathrm{GL}(V) \times \mathrm{GL}(W) \to \mathrm{GL}(Y)$ satisfies $\rho_Y(\lambda I_V, \mu I_W) = \lambda^r \mu^s I_Y$ for all $\lambda, \mu \in K^\times$.

**Lemma 1.6.** *Let $Y$ be a homogeneous rational representation of $\mathrm{GL}(V) \times \mathrm{GL}(W)$ with degrees $(r,s)$. If $F : \wedge^2 V \otimes W \to Y$ is a homogeneous covariant then there exist integers $p$ and $q$ called the* weights *of $F$ such that*

$$\begin{aligned} 2 \deg F &= 5p + r \\ \deg F &= 5q + s. \end{aligned}$$

(3)

*Proof.* See [10, Lemma 2.2]. $\qquad\square$

For example the Pfaffian map (2) is a covariant of degree 2 with weights $(p,q) = (1,0)$. The covariants in the case $Y$ is the trivial representation are the invariants as described in Theorem 1.3. For general $Y$ the covariants form a module over the ring of invariants $K[c_4, c_6]$.

In Section 2 we recall our method [10] for studying the covariants via their restrictions to the Hesse family, i.e. the universal family over $X(5)$. These restrictions are nearly characterised by their invariance properties under an appropriate action of $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$. In Sections 3 and 4 we make this relationship precise. Thus our work resolves, albeit in one particular case, what is described in [1, Chapter V,§22] as the "mysterious role of invariant theory". We give examples for a range of different $Y$ in Section 5. In Section 6 we show how a free basis for the $K[c_4, c_6]$-module of covariants for $Y$ may be characterised in terms of its specialisations to the genus one models $\phi$ of the form considered in Lemma 1.4. In Section 7 we relate the covariants in the case $Y = S^5 W$ to work of Hulek [11] and finally in Section 8 we give our formula for the covering map.

## 2. DISCRETE COVARIANTS

In this section we recall some of the theory from [10]. We then state our main result on the relationship between covariants and discrete covariants.

We take $n \geq 5$ an odd integer. The *Heisenberg group* of level $n$ is

$$H_n = \langle \sigma, \tau | \sigma^n = \tau^n = [\sigma, [\sigma, \tau]] = [\tau, [\sigma, \tau]] = 1 \rangle.$$

It is a non-abelian group of order $n^3$ and its centre is a cyclic group of order $n$ generated by $\zeta = [\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1}$. In [10, Section 3] we defined a group homomorphism $s_\beta : \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{Aut}(H_n)$ by

$$s_\beta(( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} )) : \sigma \mapsto \zeta^{-ac/2} \sigma^a \tau^c ; \ \tau \mapsto \zeta^{-bd/2} \sigma^b \tau^d.$$

where the exponents are read as integers mod $n$.

**Definition 2.1.** The *extended Heisenberg group* is the semi-direct product

$$H_n^+ = H_n \ltimes \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}),$$

with group law $(h, \gamma)(h', \gamma') = (h \, s_\beta(\gamma) h', \gamma\gamma')$.

The *Schrödinger representation* $\theta : H_n \to \mathrm{SL}_n(K)$ maps $\sigma$ and $\tau$ to the matrices (1). These matrices have commutator $\theta(\zeta) = \zeta_n I_n$.

**Theorem 2.2.** *(i) The Schrödinger representation* $\theta : H_n \to \mathrm{SL}_n(K)$ *extends uniquely to a representation* $\theta^+ : H_n^+ \to \mathrm{SL}_n(K)$.
*(ii) The normaliser of* $\theta(H_n)$ *in* $\mathrm{SL}_n(K)$ *is* $\theta^+(H_n^+)$.

*Proof.* See [10, Theorem 3.6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

**Remark 2.3.** (i) The representation $\theta^+$ of Theorem 2.2 is given on the generators $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ for $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ by suitable scalar multiples of

$$
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \zeta_n & \zeta_n^2 & \cdots & \zeta_n^{-1} \\
1 & \zeta_n^2 & \zeta_n^4 & \cdots & \zeta_n^{-2} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & \zeta_n^{-1} & \zeta_n^{-2} & \cdots & \zeta_n
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \zeta_n^{1/2} & 0 & \cdots & 0 \\
0 & 0 & \zeta_n^{2^2/2} & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & \zeta_n^{1/2}
\end{pmatrix}.
$$

(ii) The Schrödinger representation has $\phi(n)$ conjugates obtained by either changing our choice of $\zeta_n$ or precomposing with an automorphism of $H_n$. We may apply Theorem 2.2 to any one of these representations.

The Hesse family of elliptic normal quintics (studied for example in [9], [11]) is given by

$$
(4) \qquad \begin{array}{rcl}
u : & \mathbb{A}^2 & \to \quad \wedge^2 V \otimes W \\
& (a, b) & \mapsto \quad a \sum (v_1 \wedge v_4) w_0 + b \sum (v_2 \wedge v_3) w_0
\end{array}
$$

where the sums are taken over all cyclic permutations of the subscripts mod 5. We define actions of the Heisenberg group $H_5$ on $V$ and $W$ so that the Hesse models $u(a, b)$ are $H_5$-invariant.

$$
(5) \qquad \begin{array}{l}
\theta_V : H_5 \to \mathrm{SL}(V) ; \quad \sigma : v_i \mapsto \zeta_5^{2i} v_i ; \quad \tau : v_i \mapsto v_{i+1} \\
\theta_W : H_5 \to \mathrm{SL}(W) ; \quad \sigma : w_i \mapsto \zeta_5^{i} w_i ; \quad \tau : w_i \mapsto w_{i+1}.
\end{array}
$$

Since $\theta_V$ and $\theta_W$ are conjugates of the Schrödinger representation they extend by Theorem 2.2 to representations of $H_5^+$. By abuse of notation we continue to write these representations as $\theta_V$ and $\theta_W$.

Let $Y$ be a homogeneous rational representation of $\mathrm{GL}(V) \times \mathrm{GL}(W)$. Then $\theta_V$ and $\theta_W$ define an action of $H_5^+$ on $Y$ and so an action of $\Gamma = \mathrm{SL}_5(\mathbb{Z}/5\mathbb{Z})$ on $Y^{H_5}$. Taking $Y = \wedge^2 V \otimes W$ the action of $\Gamma$ on $(\wedge^2 V \otimes W)^{H_5} = \mathrm{Im}(u)$ is described by a representation $\chi_1 : \Gamma \to \mathrm{GL}_2(K)$.

**Definition 2.4.** Let $\pi : \Gamma \to \mathrm{GL}(Z)$ be a representation. A *discrete covariant* for $Z$ is a polynomial map $f : \mathbb{A}^2 \to Z$ satisfying $f \circ \chi_1(\gamma) = \pi(\gamma) \circ f$ for all $\gamma \in \Gamma$.

**Theorem 2.5.** *Let* $F : \wedge^2 V \otimes W \to Y$ *be a covariant. Then* $f = F \circ u : \mathbb{A}^2 \to Y^{H_5}$ *is a discrete covariant. Moreover* $F$ *is uniquely determined by* $f$.

*Proof.* See [10, Theorem 4.3]. □

For any given $Y$ the discrete covariants may be computed using invariant theory for the finite groups $H_5$ and $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$. We say that a discrete covariant $f : \mathbb{A}^2 \to Y^{H_5}$ *is a covariant* if it arises from a covariant $F : \wedge^2 V \otimes W \to Y$ as described in Theorem 2.5. It is important to note that not every discrete covariant is a covariant. For example, taking $Y$ to be the trivial representation, the ring of invariants is $K[c_4, c_6]$ as described in Theorem 1.3 whereas the ring of discrete invariants is generated by

$$D = ab(a^{10} - 11a^5b^5 - b^{10})$$

(6)    $$c_4 = a^{20} + 228a^{15}b^5 + 494a^{10}b^{10} - 228a^5b^{15} + b^{20}$$

$$c_6 = -a^{30} + 522a^{25}b^5 + 10005a^{20}b^{10} + 10005a^{10}b^{20} - 522a^5b^{25} - b^{30}$$

subject only to the relation $c_4^3 - c_6^2 = 1728D^5$. We use the same notation for both a covariant and its restriction to the Hesse family. By the uniqueness part of Theorem 2.5 this should not cause any confusion.

There are essentially two ways in which a discrete covariant might fail to be a covariant. The first is that the weights computed using (3) might not be integers. For example $D$ has weights $(p, q) = (24/5, 12/5)$ and so cannot be an invariant. The second is that denominators might be introduced. More precisely we prove the following theorem in Section 3.

**Theorem 2.6.** *Let* $f : \mathbb{A}^2 \to Y^{H_5}$ *be an integer weight discrete covariant. Then* $\Delta^k f$ *is a covariant for some* $k \geq 0$.

In Section 4 we give a practical method for computing the least such $k$.

**Remark 2.7.** If $Y$ is homogeneous of degree $(r, s)$ and $Y^{H_5} \neq 0$ then the action of the centre of $H_5$ shows that $2r + s \equiv 0 \pmod 5$. We see by (3) that $p$ is an integer if and only if $q$ is an integer. So the integer weight condition is just a congruence mod 5 on the degree of a covariant. Since $\Delta = D^5$ and $\deg D = 12$ is coprime to 5, an equivalent formulation of Theorem 2.6 is that if $f : \mathbb{A}^2 \to Y^{H_5}$ is a homogeneous discrete covariant then $D^m f$ is a covariant for some $m \geq 0$.

## 3. FRACTIONAL COVARIANTS

In this section we prove Theorem 2.6.

**Lemma 3.1.** *Let* $\phi \in \wedge^2 V \otimes W$ *be a non-singular Hesse model.*
  (i) *The stabiliser of* $\phi$ *in* $\mathrm{SL}(V) \times \mathrm{SL}(W)$ *is*

$$H = \{(\theta_V(h), \theta_W(h)) : h \in H_5\}.$$

  (ii) *The normaliser of* $H$ *in* $\mathrm{SL}(V) \times \mathrm{SL}(W)$ *is*

$$N = \{(\theta_V(h), \zeta\theta_W(h)) : (\zeta, h) \in \mu_5 \times H_5^+\}.$$

*Proof.* (i) It is clear by (4) and (5) that $H$ is contained in the stabiliser of $\phi$. Since any automorphism of $C_\phi$ of order 5 is translation by a 5-torsion point of its Jacobian, all such automorphisms are described by elements of $H$.

Now let $g \in \mathrm{SL}(V) \times \mathrm{SL}(W)$ with $g(\phi) = \phi$ and let $\gamma$ be the automorphism of $C_\phi$ induced by $g$. By [8, Proposition 5.19 and Lemma 2.4] $\gamma$ preserves the invariant differential and is therefore a translation map. Since $C_\phi \subset \mathbb{P}^4$ is a curve of degree 5 this translation is by a point of order 5. Composing $g$ with a suitable element of $H$ reduces us to the case $\gamma$ is the identity. Then $g = (g_V, g_W)$ is a pair of scalar matrices. Since these matrices each have determinant 1 and jointly fix $\phi$ it follows that $(g_V, g_W) = (\theta_V(h), \theta_W(h))$ for some $h$ in the centre of $H_5$.

(ii) We see by Theorem 2.2(ii) that $N$ is contained in the normaliser of $H$, and that any element of the normaliser may be composed with an element of $N$ to give an element of the form $g = (I_V, g_W)$ where $I_V$ is the identity. Since $\theta_V$ is faithful it follows that $g_W$ is in the centraliser of $\theta_W(H_5)$ in $\mathrm{SL}(W)$, which turns out to consist only of scalar matrices. $\qquad\square$

The following proposition will be used to explain the relationship between the covariants and the discrete covariants.

**Proposition 3.2.** *Let $G$ be a linear algebraic group acting on irreducible affine varieties $X$ and $Y$. Let $H \subset G$ be a subgroup whose normaliser $N \subset G$ is of finite order coprime to $\mathrm{char}\, K$. Suppose that $A \subset X^H$ is an irreducible variety acted on by $N/H$, and $U \subset A$ is a dense open subset such that*

(i) *the morphism $G \times U \to X$; $(g, \phi) \mapsto g(\phi)$ has dense image,*
(ii) *the stabiliser in $G$ of each element of $U$ is $H$,*
(iii) *either $\mathrm{char}\, K = 0$ or the derivative of the map in (i) is an isomorphism at all points of $G \times U$.*

*Then by restriction to $A$ there is a bijection between*

- *$G$-equivariant rational maps $F : X -\!\rightarrow Y$, and*
- *$N/H$-equivariant rational maps $f : A -\!\rightarrow Y^H$*

*Proof.* Let $F : X -\!\rightarrow Y$ be a $G$-equivariant rational map. Its domain of definition is a $G$-invariant open subset of $X$ and hence by (i) it meets $U$. Therefore $F$ restricts to a rational map $f$ on $A$. By hypothesis $A$ is acted on by $N$ and pointwise fixed by $H$. Since $F$ is $N$-equivariant it follows that $f(A) \subset Y^H$ and $f$ is $N/H$-equivariant.

Conversely suppose $f : A -\!\rightarrow Y^H$ is an $N/H$-equivariant rational map. We let $\delta \in N$ act on $G \times A$ via $(g, a) \mapsto (g\delta^{-1}, \delta a)$. Since $N$ is a finite group of order coprime to $\mathrm{char}\, K$ and $G \times A$ is an affine variety, the quotient $(G \times A)/N$ exists, and is an affine variety. We consider the maps

$$
\begin{aligned}
\psi_{\mathrm{id}} &: (G \times A)/N \longrightarrow X; & (g, a) &\mapsto g(a) \\
\psi_f &: (G \times A)/N -\!\rightarrow Y; & (g, a) &\mapsto g(f(a)).
\end{aligned}
$$

Shrinking $U$ if necessary, we may assume that $N/H$ acts on $U$. By (i) $\psi_{\mathrm{id}}$ has dense image, by (ii) it is injective on the dense subset $(G \times U)/N$, and by (iii) it is separable. It follows that $\psi_{\mathrm{id}}$ is birational. Then $F = \psi_f \circ \psi_{\mathrm{id}}^{-1}$ is a $G$-equivariant rational map extending $f$.                                                                              $\square$

*Proof of Theorem 2.6.* We apply Proposition 3.2 with $G = \mathrm{SL}(V) \times \mathrm{SL}(W)$, $X = \wedge^2 V \otimes W$ and $H \subset N \subset G$ as in Lemma 3.1. We also let $A = X^H$ be the space of Hesse models and $U \subset A$ the space of non-singular Hesse models.

   We check the hypotheses (i), (ii) and (iii). By [9, Proposition 4.1] every non-singular model is equivalent to a Hesse model and by Theorem 1.3 the non-singular models are Zariski dense in $\wedge^2 V \otimes W$. This proves (i). We checked (ii) in Lemma 3.1 and (iii) is checked in Lemma 3.3 below.

   By Lemma 3.1 and the definition of $H_5^+$ we have $N/H \cong \mu_5 \times \Gamma$ where $\Gamma = \mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$. Now $f$ is $\Gamma$-equivariant by definition of a discrete covariant and $\mu_5$-equivariant by the assumption it has integer weights. So by Proposition 3.2 it is the restriction of a $G$-equivariant rational map $F : \wedge^2 V \otimes W - \to Y$. (We say $F$ is a *fractional covariant*.)

   It remains to show that $\Delta^k F$ is regular for some $k \geq 0$. Let $S \in K[\wedge^2 V \otimes W]$ be a homogeneous polynomial of least degree such that $SF$ is regular. Then $F = R/S$ where $R$ is a covariant and $S$ is an invariant. Suppose $S(\phi) = 0$ for some non-singular model $\phi$. By [9, Proposition 4.1] we may suppose that $\phi$ is a Hesse model, and so by the regularity of $f$ we have $R(\phi) = S(\phi) = 0$. By Lemma 1.4 the Zariski closure of the $\mathrm{GL}(V) \times \mathrm{GL}(W)$-orbit of $\phi$ is the zero locus of a homogeneous invariant $I$. Now both $R$ and $S$ are divisible by $I$ and this contradicts the choice of $S$. Therefore $F$ is regular on all non-singular models. By Theorem 1.3(i) and the Nullstellensatz it follows that $\Delta^k F$ is regular for some $k \geq 0$.                                                                              $\square$

   The following lemma completes the proof of Theorem 2.6 in the case of positive characteristic (still assuming char $K \neq 2, 3, 5$).

**Lemma 3.3.** *The derivative of the morphism*

$$\begin{aligned}
\mathrm{SL}(V) \times \mathrm{SL}(W) \times \mathbb{A}^2 \;&\to\; \wedge^2 V \otimes W \\
(g_V, g_W, (a,b)) \;&\mapsto\; (g_V, g_W) u(a,b)
\end{aligned}$$

*is an isomorphism at all $(g_V, g_W, (a,b))$ with $D(a,b) \neq 0$.*

*Proof.* It suffices to compute the derivative at $(I_V, I_W, (a,b))$. This is a linear map

$$(7) \qquad\qquad\qquad \mathfrak{sl}(V) \times \mathfrak{sl}(W) \times \mathbb{A}^2 \to \wedge^2 V \otimes W.$$

We write $E_{ij}$ for the $n \times n$ matrix with $(i,j)$ entry 1 and all other entries 0. Then $\mathfrak{sl}_n$ has basis $\{E_{ij} : i \neq j\} \cup \{E_{00} - E_{ii} : i \neq 0\}$. Taking these bases for $\mathfrak{sl}(V)$ and $\mathfrak{sl}(W)$, the standard basis for $\mathbb{A}^2$ and the basis $\{(v_i \wedge v_j) w_k : i < j\}$ for $\wedge^2 V \otimes W$, we found by direct calculation that the derivative (7) has determinant $5^4 D(a,b)^4$.                    $\square$

## 4. Denominators

In this section we show how to find the least value of $k$ in Theorem 2.6. We consider the family of genus one models

$$(8) \qquad \begin{aligned} u_1 : \mathbb{A}^5 &\to \wedge^2 V \otimes W \\ (\lambda_0, \dots, \lambda_4) &\mapsto \sum \lambda_0 (v_1 \wedge v_4) w_0 + \sum (v_2 \wedge v_3) w_0. \end{aligned}$$

where the sums are taken over all cyclic permutations of the subscripts mod 5. These models are related to the Hesse family by

$$(9) \qquad u_1(a, \dots, a) = u(a, 1).$$

**Remark 4.1.** If $\phi = u_1(\lambda_0, \dots, \lambda_4)$ then $C_\phi \subset \mathbb{P}^4$ is defined by $\lambda_0 x_0^2 + x_1 x_4 - \lambda_2 \lambda_3 x_2 x_3 = 0$ and its cyclic permutes. These curves were studied in [7] where it is shown that $\phi = u_1(\lambda, 1, \dots, 1)$ defines the universal family of (generalised) elliptic curves parametrised by $X_1(5)$. Here $\lambda$ is a co-ordinate on $X_1(5) \cong \mathbb{P}^1$.

**Definition 4.2.** Let $D \subset \mathrm{SL}(V) \times \mathrm{SL}(W)$ be the subgroup of pairs of diagonal matrices

$$(10) \qquad \begin{pmatrix} \alpha_1 \alpha_4 & & & & \\ & \alpha_0 \alpha_2 & & & \\ & & \alpha_1 \alpha_3 & & \\ & & & \alpha_2 \alpha_4 & \\ & & & & \alpha_0 \alpha_3 \end{pmatrix}, \quad \begin{pmatrix} \alpha_0 & & & & \\ & \alpha_1 & & & \\ & & \alpha_2 & & \\ & & & \alpha_3 & \\ & & & & \alpha_4 \end{pmatrix},$$

with $\prod \alpha_i = 1$.

**Lemma 4.3.** *The action of $D$ on $\mathbb{A}^5$ compatible with $u_1$ is*

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto \left( \frac{\alpha_0^2}{\alpha_1 \alpha_4} \lambda_0, \frac{\alpha_1^2}{\alpha_0 \alpha_2} \lambda_1, \frac{\alpha_2^2}{\alpha_1 \alpha_3} \lambda_2, \frac{\alpha_3^2}{\alpha_2 \alpha_4} \lambda_3, \frac{\alpha_4^2}{\alpha_0 \alpha_3} \lambda_4 \right).$$

*In particular $D$ acts transitively on the subsets of $\mathbb{A}^5$ defined by the condition that $\lambda_0, \dots, \lambda_4$ have a fixed non-zero product.*

*Proof.* Let $g_V$ and $g_W$ be the matrices (10) with $(\alpha_0, \dots, \alpha_4) = (\alpha, 1, \dots, 1)$. Then

$$(g_V, g_W) \, u_1(\lambda_0, \dots, \lambda_4) = \alpha \, u_1(\alpha^2 \lambda_0, \alpha^{-1} \lambda_1, \lambda_2, \lambda_3, \alpha^{-1} \lambda_4).$$

From this calculation and the obvious cyclic symmetry it follows that the action of $D$ on $\mathbb{A}^5$ is as stated. In the special case $(\alpha_0, \dots, \alpha_4) = (\beta^{-2}, \beta^{-1}, 1, \beta, \beta^2)$ this action is given by $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \mapsto (\beta^{-5} \lambda_0, \lambda_1, \lambda_2, \lambda_3, \beta^5 \lambda_4)$. Since we are working over an algebraically closed field the final statement is clear. $\square$

Let $Y$ be a homogeneous rational representation of $\mathrm{GL}(V) \times \mathrm{GL}(W)$.

**Theorem 4.4.** *Let $f : \mathbb{A}^2 \to Y^{H_5}$ be an integer weight discrete covariant.*

(i) *There is a unique $D$-equivariant rational map $f_1 : \mathbb{A}^5 - \to Y$ with*

$$f_1(a, \dots, a) = f(a, 1).$$

(ii) $f$ *is a covariant if and only if* $f_1$ *is regular.*

*Proof.* By Theorem 2.6 there is a fractional covariant $F : \wedge^2 V \otimes W - \to Y$ with $f = F \circ u$. It follows by (9) and Lemma 4.3 that $f_1 = F \circ u_1$ satisfies (i). Uniqueness is proved using the final part of Lemma 4.3.

It remains to show that if $f_1$ is regular then $F$ is regular. Theorem 2.6 already shows that $R = \Delta^k F$ is a covariant for some $k \geq 0$. We take the least such $k$. Let $\phi = u_1(0, 1, 1, 1, 1)$. Then $C_\phi$ is the rational nodal quintic parametrised by

$$(x_0 : \ldots : x_4) = (s^5 - t^5 : st^4 : s^2 t^3 : -s^3 t^2 : -s^4 t).$$

If $k \geq 1$ then by regularity of $f_1$ we have $R(\phi) = 0$. Then Lemma 1.4 shows that $R$ is divisible by $\Delta$ contradicting our choice of $k$. Therefore $k = 0$ and $F$ is a covariant. By the convention introduced following Theorem 2.5, we say that $f$ is a covariant. □

What makes Theorem 4.4 useful is that we can compute $f_1$ from $f$ without going via $F$. Explicitly we put

$$(11) \qquad f_1(\lambda_0, \ldots, \lambda_4) = \rho_Y(g_V, g_W) f(a, 1)$$

where $g_V$ and $g_W$ are given by (10) and satisfy $u_1(\lambda_0, \ldots, \lambda_4) = (g_V, g_W) u(a, 1)$. We then eliminate $\alpha_0, \ldots, \alpha_4$ and $a$ from the right hand side, using the relations

$$(12) \qquad \alpha_i^2 / (\alpha_{i+1} \alpha_{i+4}) \;\; = \;\; \lambda_i / a$$

$$(13) \qquad \alpha_i^5 \;\; = \;\; \lambda_i^2 / (\lambda_{i+2} \lambda_{i+3})$$

$$(14) \qquad a^5 \;\; = \;\; \lambda_0 \lambda_1 \ldots \lambda_4.$$

The first of these comes from Lemma 4.3. The other two may be deduced from the first using $\prod \alpha_i = 1$. One systematic way to proceed is by using (12) to eliminate $\alpha_0$, $\alpha_1$, $\alpha_2$, then (13) to eliminate $\alpha_3$, $\alpha_4$ and finally (14) to eliminate $a$.

**Remark 4.5.** It can be shown that Theorem 4.4(i) still holds if we weaken the condition that $f$ is $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$-equivariant and just require that it is equivariant for the action of $T = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$.

## 5. EXAMPLES

We can use Theorem 4.4 in the case $Y$ is the trivial representation to give another proof (independent of Theorem 1.3) that the discrete invariants $c_4$ and $c_6$ are in fact invariants. Indeed let $f$ be an integer weight discrete invariant. The integer weight condition is that $f$ is homogeneous of degree a multiple of 5. We construct $f_1$ from $f$ by making the substitutions $a^5 \mapsto \prod \lambda_i$ and $b \mapsto 1$. Since no denominators are introduced it follows by Theorem 4.4 that $f$ is an invariant.

In the cases $Y = \wedge^2 V \otimes W$ and $\wedge^2 V^* \otimes W^*$ the following proposition was already proved in [9] using evectants. We now have a general method. In the calculations that follow all sums and products are taken over the cyclic permutations of the

subscripts mod 5. Recall that we fixed bases $v_0, \ldots, v_4$ and $w_0, \ldots, w_4$ for $V$ and $W$. The dual bases for $V^*$ and $W^*$ are $v_0^*, \ldots, v_4^*$ and $w_0^*, \ldots, w_4^*$.

**Proposition 5.1.** *Let $Y$ be any one of*

$$\wedge^2 V \otimes W, \qquad V^* \otimes \wedge^2 W, \qquad V \otimes \wedge^2 W^*, \qquad \wedge^2 V^* \otimes W^*,$$
$$V^* \otimes S^2 W, \qquad S^2 V^* \otimes W^*, \qquad S^2 V \otimes W, \qquad V \otimes S^2 W^*.$$

*Then every integer weight discrete covariant $f : \wedge^2 V \otimes W \to Y^{H_5}$ is a covariant. In particular the covariants $F : \wedge^2 V \otimes W \to Y$ form a free $K[c_4, c_6]$-module of rank 2 or 3 and the generators have degrees as indicated in* [10, Table 4.6].

*Proof.* Let $f : \wedge^2 V \otimes W \to Y^{H_5}$ be an integer weight discrete covariant. In each case [10, Lemma 4.4] shows that $\dim Y^{H_5} = 2$ or 3 and a basis is found by inspection. We construct $f_1$ from $f$ by making the substitutions $a^5 \mapsto \prod \lambda_i$, $b \mapsto 1$, and

$$a \sum (v_1 \wedge v_4) w_0 \mapsto \sum \lambda_0 (v_1 \wedge v_4) w_0 \qquad \sum v_0^* (w_1 \wedge w_4) \mapsto \sum v_0^* (w_1 \wedge w_4)$$
$$\sum (v_2 \wedge v_3) w_0 \mapsto \sum (v_2 \wedge v_3) w_0 \qquad a^2 \sum v_0^* (w_2 \wedge w_3) \mapsto \sum \lambda_2 \lambda_3 v_0^* (w_2 \wedge w_3)$$

$$\sum v_0 (w_1^* \wedge w_4^*) \mapsto \sum v_0 (w_1^* \wedge w_4^*) \qquad a^4 \sum (v_1^* \wedge v_4^*) w_0^* \mapsto \sum \lambda_1 \lambda_2 \lambda_3 \lambda_4 (v_1^* \wedge v_4^*) w_0^*$$
$$a^3 \sum v_0 (w_2^* \wedge w_3^*) \mapsto \sum \lambda_0 \lambda_1 \lambda_4 v_0 (w_2^* \wedge w_3^*) \qquad \sum (v_2^* \wedge v_3^*) w_0^* \mapsto \sum (v_2^* \wedge v_3^*) w_0^*$$

$$a \sum v_0^* w_0^2 \mapsto \sum \lambda_0 v_0^* w_0^2 \qquad a^2 \sum v_0^{*2} w_0^* \mapsto \sum \lambda_2 \lambda_3 v_0^{*2} w_0^*$$
$$\sum v_0^* w_1 w_4 \mapsto \sum v_0^* w_1 w_4 \qquad a^4 \sum v_1^* v_4^* w_0^* \mapsto \sum \lambda_1 \lambda_2 \lambda_3 \lambda_4 v_1^* v_4^* w_0^*$$
$$a^2 \sum v_0^* w_2 w_3 \mapsto \sum \lambda_2 \lambda_3 v_0^* w_2 w_3 \qquad \sum v_2^* v_3^* w_0^* \mapsto \sum v_2^* v_3^* w_0^*$$

$$a^3 \sum v_0^2 w_0 \mapsto \sum \lambda_0 \lambda_1 \lambda_4 v_0^2 w_0 \qquad a^4 \sum v_0 w_0^{*2} \mapsto \sum \lambda_1 \lambda_2 \lambda_3 \lambda_4 v_0 w_0^{*2}$$
$$a \sum v_1 v_4 w_0 \mapsto \sum \lambda_0 v_1 v_4 w_0 \qquad \sum v_0 w_1^* w_4^* \mapsto \sum v_0 w_1^* w_4^*$$
$$\sum v_2 v_3 w_0 \mapsto \sum v_2 v_3 w_0 \qquad a^3 \sum v_0 w_2^* w_3^* \mapsto \sum \lambda_0 \lambda_1 \lambda_4 v_0 w_2^* w_3^*$$

Since these substitutions eliminate $a$ it is clear that no denominators are introduced. It follows by Theorem 4.4 that $f$ is a covariant. $\square$

**Proposition 5.2.** *Let $Y$ be any one of $S^5 W, S^5 V, S^5 V^*, S^5 W^*$. Then the covariants $F : \wedge^2 V \otimes W \to Y$ form a free $K[c_4, c_6]$-module of rank 6 with generators in degrees $10, 20, 30, 30, 40, 50$ except in the case $Y = S^5 W^*$ where the generators have degrees $30, 40, 50, 50, 60, 70$.*

*Proof.* The module of integer weight discrete covariants is computed as described in [10] and is found to be a free $K[c_4, c_6]$-module of rank 6 with generators in

degrees $10, 10, 20, 20, 30, 30$. We use Theorem 4.4 to decide which of these are covariants. We construct $f_1$ from $f$ by making the substitutions $a^5 \mapsto \prod \lambda_i$, $b \mapsto 1$ and

$$
\begin{aligned}
a^5 \sum w_0^5 &\mapsto \sum \lambda_0^3 \lambda_1 \lambda_4 w_0^5 \\
a^4 \sum w_0^3 w_1 w_4 &\mapsto \sum \lambda_0^2 \lambda_1 \lambda_4 w_0^3 w_1 w_4 \\
a^3 \sum w_0 w_1^2 w_4^2 &\mapsto \sum \lambda_0 \lambda_1 \lambda_4 w_0 w_1^2 w_4^2 \\
a^2 \sum w_0 w_2^2 w_3^2 &\mapsto \sum \lambda_2 \lambda_3 w_0 w_2^2 w_3^2 \\
a \sum w_0^3 w_2 w_3 &\mapsto \sum \lambda_0 w_0^3 w_2 w_3 \\
\prod w_0 &\mapsto \prod w_0
\end{aligned}
\qquad
\begin{aligned}
a^5 \sum v_0^5 &\mapsto \sum \lambda_0 \lambda_1^2 \lambda_4^2 v_0^5 \\
a^3 \sum v_0^3 v_1 v_4 &\mapsto \sum \lambda_0 \lambda_1 \lambda_4 v_0^3 v_1 v_4 \\
a \sum v_0 v_1^2 v_4^2 &\mapsto \sum \lambda_0 v_0 v_1^2 v_4^2 \\
a^4 \sum v_0 v_2^2 v_3^2 &\mapsto \sum \lambda_1 \lambda_2 \lambda_3 \lambda_4 v_0 v_2^2 v_3^2 \\
a^2 \sum v_0^3 v_2 v_3 &\mapsto \sum \lambda_1 \lambda_4 v_0^3 v_2 v_3 \\
\prod v_0 &\mapsto \prod v_0
\end{aligned}
$$

$$
\begin{aligned}
a^5 \sum v_0^{*5} &\mapsto \sum \lambda_0 \lambda_2^2 \lambda_3^2 v_0^{*5} \\
a^2 \sum v_0^{*3} v_1^* v_4^* &\mapsto \sum \lambda_2 \lambda_3 v_0^{*3} v_1^* v_4^* \\
a^4 \sum v_0^* v_1^{*2} v_4^{*2} &\mapsto \sum \lambda_1 \lambda_2 \lambda_3 \lambda_4 v_0^* v_1^{*2} v_4^{*2} \\
a \sum v_0^* v_2^{*2} v_3^{*2} &\mapsto \sum \lambda_0 v_0^* v_2^{*2} v_3^{*2} \\
a^3 \sum v_0^{*3} v_2^* v_3^* &\mapsto \sum \lambda_0 \lambda_2 \lambda_3 v_0^{*3} v_2^* v_3^* \\
\prod v_0^* &\mapsto \prod v_0^*
\end{aligned}
\qquad
\begin{aligned}
a^{10} \sum w_0^{*5} &\mapsto \sum \lambda_1^2 \lambda_2^3 \lambda_3^3 \lambda_4^2 w_0^{*5} \\
a^6 \sum w_0^{*3} w_1^* w_4^* &\mapsto \sum \lambda_1 \lambda_2^2 \lambda_3^2 \lambda_4 w_0^{*3} w_1^* w_4^* \\
a^2 \sum w_0^* w_1^{*2} w_4^{*2} &\mapsto \sum \lambda_2 \lambda_3 w_0^* w_1^{*2} w_4^{*2} \\
a^3 \sum w_0^* w_2^{*2} w_3^{*2} &\mapsto \sum \lambda_0 \lambda_1 \lambda_4 w_0^* w_2^{*2} w_3^{*2} \\
a^4 \sum w_0^{*3} w_2^* w_3^* &\mapsto \sum \lambda_1 \lambda_2 \lambda_3 \lambda_4 w_0^{*3} w_2^* w_3^* \\
\prod w_0^* &\mapsto \prod w_0^*
\end{aligned}
$$

In the cases $Y = S^5 W, S^5 V, S^5 V^*$ an integer weight discrete covariant is a covariant if and only if the coefficient of $\sum w_0^5, \sum v_0^5, \sum v_0^{*5}$ is divisible by $a^5$. Computing the discrete covariants we find that there is a single constraint in degree $10m$ for each $m \geq 1$. The covariants therefore have Hilbert series

$$
\frac{2(t^{10} + t^{20} + t^{30})}{(1 - t^{20})(1 - t^{30})} - \frac{t^{10}}{1 - t^{10}} = \frac{t^{10} + t^{20} + 2t^{30} + t^{40} + t^{50}}{(1 - t^{20})(1 - t^{30})}.
$$

In Section 7 we give further details of the covariants in the case $Y = S^5 W$.

In the case $Y = S^5 W^*$ an integer weight discrete covariant is a covariant if and only if the coefficient of $\sum w_0^{*5}$ is divisible by $a^{10}$ and the coefficient of $\sum w_0^{*3} w_1^* w_4^*$ is divisible by $a^5$. We find that the discrete covariants of degrees 10 and 20 are not covariants and that there are 3 constraints in degrees $10m$ for each $m \geq 3$. The covariants therefore have Hilbert series

$$
\frac{2(t^{10} + t^{20} + t^{30})}{(1 - t^{20})(1 - t^{30})} - 2t^{10} - 2t^{20} - \frac{3t^{30}}{1 - t^{10}} = \frac{t^{30} + t^{40} + 2t^{50} + t^{60} + t^{70}}{(1 - t^{20})(1 - t^{30})}.
$$

$\square$

**Example 5.3.** The degree 10 covariant for $Y = S^5 W$ is

$$
\begin{aligned}
S_{10} = {}& a^5 b^5 \sum w_0^5 - a^4 b(a^5 - 3b^5) \sum w_0^3 w_1 w_4 + a^3 b^2 (a^5 + 2b^5) \sum w_0 w_1^2 w_4^2 \\
& + a^2 b^3 (2a^5 - b^5) \sum w_0 w_2^2 w_3^2 - ab^4 (3a^5 + b^5) \sum w_0^3 w_2 w_3 + (a^{10} - 16a^5 b^5 - b^{10}) \prod w_0
\end{aligned}
$$

and the degree 30 covariant for $Y = S^5 W^*$ is

$$
\begin{aligned}
T_{30} = {}& 125 a^{10} b^{10} (3 a^{10} - 8 a^5 b^5 - 3 b^{10}) \sum w_0^{*5} \\
& - 5 a^6 b^4 (3 a^{20} + 134 a^{15} b^5 + 57 a^{10} b^{10} + 216 a^5 b^{15} - 22 b^{20}) \sum w_0^{*3} w_1^* w_4^* \\
& + a^2 b^3 (32 a^{25} - 195 a^{20} b^5 + 4110 a^{15} b^{10} + 900 a^{10} b^{15} + 480 a^5 b^{20} + 9 b^{25}) \sum w_0^* w_1^{*2} w_4^{*2} \\
& - a^3 b^2 (9 a^{25} - 480 a^{20} b^5 + 900 a^{15} b^{10} - 4110 a^{10} b^{15} - 195 a^5 b^{20} - 32 b^{25}) \sum w_0^* w_2^{*2} w_3^{*2} \\
& - 5 a^4 b^6 (22 a^{20} + 216 a^{15} b^5 - 57 a^{10} b^{10} + 134 a^5 b^{15} - 3 b^{20}) \sum w_0^{*3} w_2^* w_3^* \\
& + (a^{30} - 258 a^{25} b^5 + 3435 a^{20} b^{10} - 23040 a^{15} b^{15} - 3435 a^{10} b^{20} - 258 a^5 b^{25} - b^{30}) \prod w_0^*.
\end{aligned}
$$

The covariant $S_{10}$ is (a scalar multiple of) the determinant of the Jacobian matrix of the quadrics defining $C_\phi$. We do not know of any similar construction for $T_{30}$. The contraction of these two covariants is $\langle S_{10}, T_{30} \rangle = c_4^2$. In [8, Section 8] we used the existence of a such a covariant $T_{30}$ to justify our algorithm for computing the invariants in the case of a singular genus one model.

In [10, Section 7] we showed that the covariant $\Omega_5$ of degree 5 in the following proposition represents the invariant differential.

**Proposition 5.4.** *The covariants for $Y = \wedge^2 W^* \otimes S^2 W$ form a free $K[c_4, c_6]$-module of rank 6 with generators in degrees $5, 15, 15, 25, 25, 35$.*

*Proof.* The module of integer weight discrete covariants is computed as described in [10] and is found to be a free $K[c_4, c_6]$-module of rank 6 with generators in degrees $5, 15, 15, 25, 25, 35$. We use Theorem 4.4 to decide which of these are covariants. We construct $f_1$ from $f$ by making the substitutions $a^5 \mapsto \prod \lambda_i$, $b \mapsto 1$ and

$$
\begin{aligned}
a \sum (w_1^* \wedge w_4^*) w_0^2 &\mapsto \sum \lambda_0 (w_1^* \wedge w_4^*) w_0^2 \\
\sum (w_1^* \wedge w_4^*) w_1 w_4 &\mapsto \sum (w_1^* \wedge w_4^*) w_1 w_4 \\
a^2 \sum (w_1^* \wedge w_4^*) w_2 w_3 &\mapsto \sum \lambda_2 \lambda_3 (w_1^* \wedge w_4^*) w_2 w_3 \\
a^4 \sum (w_2^* \wedge w_3^*) w_0^2 &\mapsto \sum \lambda_0^2 \lambda_1 \lambda_4 (w_2^* \wedge w_3^*) w_0^2 \\
a^3 \sum (w_2^* \wedge w_3^*) w_1 w_4 &\mapsto \sum \lambda_0 \lambda_1 \lambda_4 (w_2^* \wedge w_3^*) w_1 w_4 \\
\sum (w_2^* \wedge w_3^*) w_2 w_3 &\mapsto \sum (w_2^* \wedge w_3^*) w_2 w_3.
\end{aligned}
$$

In this case every integer weight discrete covariant is a covariant. $\qquad \square$

## 6. Independence of covariants

Let $Y$ be a homogeneous rational representation of $\mathrm{GL}(V) \times \mathrm{GL}(W)$.

**Theorem 6.1.** *Assume* char $K = 0$.

(i) *The module of covariants $\wedge^2 V \otimes W \to Y$ is a free $K[c_4, c_6]$-module of rank $m = \dim Y^{H_5}$.*

(ii) *Let $F_1, \ldots, F_m$ be homogeneous covariants for $Y$. Then $F_1, \ldots, F_m$ are a free basis for the module in (i) if and only if for each $\phi$ with $C_\phi$ either an elliptic normal quintic or rational nodal quintic, $F_1(\phi), \ldots, F_m(\phi)$ are linearly independent over $K$.*

*Proof.* (i) The fact we obtain a free module is a standard result in invariant theory. We have assumed char $K = 0$ so that $\mathrm{SL}(V) \times \mathrm{SL}(W)$ is linearly reductive, i.e. it has a Reynolds operator. Applying the Reynolds operator to the free $K[\wedge^2 V \otimes W]$-module of polynomial maps $\wedge^2 V \otimes W \to Y$ shows that the covariants form a projective $K[c_4, c_6]$-module and hence a free $K[c_4, c_6]$-module. By Theorem 2.6 the rank is the same as for the integer weight discrete covariants. We proved in [10, Lemma 4.5] that this rank is $m$.

(ii) Let $F_1, \ldots, F_m$ be homogeneous covariants that are a basis for the module in (i) and let $\phi$ be a genus one model with $C_\phi$ either an elliptic normal quintic or rational nodal quintic. Suppose for a contradiction that there is a dependence relation
$$\lambda_1 F_1(\phi) + \ldots + \lambda_m F_m(\phi) = 0$$
for some $\lambda_1, \ldots, \lambda_m \in K$ not all zero. Let

(15)
$$d = \begin{cases} 6 & \text{if } c_4(\phi) = 0 \\ 4 & \text{if } c_6(\phi) = 0 \\ 2 & \text{otherwise.} \end{cases}$$

Since $\phi$ is equivalent to a Weierstrass model we see by [8, Proposition 4.7] that for every $\zeta \in \mu_d$ there exists $g = (g_V, g_W) \in \mathrm{SL}(V) \times \mathrm{GL}(W)$ with $g\phi = \phi$ and $\det g_W = \zeta$. Let $F_i$ have weights $(p_i, q_i)$. Applying $g$ to the above dependence relation we obtain
$$\zeta^{q_1} \lambda_1 F_1(\phi) + \ldots + \zeta^{q_m} \lambda_m F_m(\phi) = 0.$$

We may therefore reduce to the case where all the $q_i$ are congruent mod $d$. This implies by (3) that the degrees of the $F_i$ are congruent mod $5d$. We recall that $c_4$ and $c_6$ have degrees 20 and 30. It follows by (15) that there is a homogeneous covariant
$$F = I_1 F_1 + \ldots + I_m F_m$$
with $F(\phi) = 0$ where each $I_i$ is a monomial in $c_4$ and $c_6$ and $I_i(\phi) \neq 0$ for some $i$. Then $F$ is divisible by the invariant $I$ constructed in Lemma 1.4. Since we are assuming $F_1, \ldots, F_m$ are a basis for the module of covariants it follows that $I$ divides $I_i$ and so $I_i(\phi) = 0$ for all $i$. This is the required contradiction.

Conversely suppose $F_1, \ldots, F_m$ are covariants whose specialisations at $\phi$ are linearly independent over $K$ whenever $C_\phi$ is an elliptic normal quintic or rational nodal quintic. If there is a relation
$$I_1 F_1 + \ldots + I_m F_m = 0$$

for some invariants $I_1, \ldots, I_m$ then these invariants vanish on all non-singular models and so are identically zero by Theorem 1.3. Thus $F_1, \ldots, F_m$ generate a free submodule of rank $m$. By (i) it remains to show that if

$$I_1 F_1 + \ldots + I_m F_m = IF$$

for some covariant $F$ and invariants $I, I_1, \ldots, I_m$ then $I$ divides $I_i$ for all $i$. We prove this by specialising to the genus one model in Lemma 1.5. $\qquad\square$

**Remark 6.2.** (i) The result that $F_1(\phi), \ldots, F_m(\phi)$ are linearly independent for $\phi$ non-singular could equally be proved using discrete covariants. However this proof does not generalise to the case $C_\phi$ is a rational nodal quintic.
(ii) We can remove the hypothesis char $K = 0$ from Theorem 6.1 (but still of course requiring char $K \neq 2, 3, 5$) by applying the Reynolds operator for $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ to the free $K[a^5, b^5]$-module of $\langle T \rangle$-equivariant maps $\mathbb{A}^2 \to Y^{H_5}$ that pass the test of Theorem 4.4. See also Remark 4.5.

## 7. QUINTIC COVARIANTS

We give further details of the covariants in the case $Y = S^5 W$. We already noted in the proof of Proposition 5.2 that the integer weight discrete covariants form a free $K[c_4, c_6]$-module of rank 6 generated in degrees $10, 10, 20, 20, 30, 30$. A basis for $Y^{H_5}$ is

$$\mathcal{F}_1 = \sum w_0^5 - 30 \prod w_0, \qquad \mathcal{F}_2 = 10 \sum w_0^3 w_1 w_4, \qquad \mathcal{F}_3 = 10 \sum w_0^3 w_2 w_3,$$
$$\mathcal{G}_1 = \sum w_0^5 + 20 \prod w_0, \qquad \mathcal{G}_2 = 10 \sum w_0 w_1^2 w_4^2, \qquad \mathcal{G}_3 = 10 \sum w_0 w_2^2 w_3^2.$$

In terms of this basis we have generators

$$F_{10} = (a^{10} - 36 a^5 b^5 - b^{10}) \mathcal{F}_1 + 5 a^4 b(a^5 - 3 b^5) \mathcal{F}_2 + 5 a b^4 (3 a^5 + b^5) \mathcal{F}_3,$$
$$\begin{aligned}
F_{20} = {}& (a^{20} + 114 a^{15} b^5 + 114 a^5 b^{15} - b^{20}) \mathcal{F}_1 \\
& - a^4 b(a^{15} + 171 a^{10} b^5 + 247 a^5 b^{10} - 57 b^{15}) \mathcal{F}_2 \\
& - a b^4 (57 a^{15} + 247 a^{10} b^5 - 171 a^5 b^{10} + b^{15}) \mathcal{F}_3,
\end{aligned}$$
$$\begin{aligned}
F_{30} = {}& D\big(10 a^4 b^4 (9 a^{10} + 26 a^5 b^5 - 9 b^{10}) \mathcal{F}_1 \\
& + a^3 (a^{15} + 126 a^{10} b^5 + 117 a^5 b^{10} - 12 b^{15}) \mathcal{F}_2 \\
& - b^3 (12 a^{15} + 117 a^{10} b^5 - 126 a^5 b^{10} + b^{15}) \mathcal{F}_3 \big),
\end{aligned}$$
$$G_{10} = (a^{10} + 14 a^5 b^5 - b^{10}) \mathcal{G}_1 + 5 a^3 b^2 (a^5 + 2 b^5) \mathcal{G}_2 + 5 a^2 b^3 (2 a^5 - b^5) \mathcal{G}_3,$$
$$\begin{aligned}
G_{20} = {}& (a^{20} - 136 a^{15} b^5 - 136 a^5 b^{15} - b^{20}) \mathcal{G}_1 \\
& - a^3 b^2 (7 a^{15} + 272 a^{10} b^5 - 221 a^5 b^{10} + 26 b^{15}) \mathcal{G}_2 \\
& - a^2 b^3 (26 a^{15} + 221 a^{10} b^5 + 272 a^5 b^{10} - 7 b^{15}) \mathcal{G}_3,
\end{aligned}$$
$$G_{30} = 2 D^2 \big(10 a^3 b^3 \mathcal{G}_1 + a(a^5 - 3 b^5) \mathcal{G}_2 - b(3 a^5 + b^5) \mathcal{G}_3 \big).$$

We recall from Section 5 that a discrete covariant is a covariant if and only if the coefficient of $\sum w_0^5$ is divisible by $a^5$. Therefore the $K[c_4, c_6]$-module of covariants

for $Y = S^5 W$ has basis

(16)
$$S_{10} = F_{10} - G_{10}, \quad S_{30} = F_{30} - G_{30}, \quad S_{40} = c_6 F_{10} + c_4 F_{20},$$
$$S_{20} = F_{20} - G_{20}, \quad S'_{30} = F_{30} + G_{30}, \quad S_{50} = c_4^2 F_{10} + c_6 F_{20}.$$

If we evaluate these covariants at a non-singular model $\phi$ then by Theorem 6.1 we obtain a basis for the space of Heisenberg invariant quintics. The space of Heisenberg invariant quintics relative to a *fixed* elliptic normal quintic was studied by Hulek [11]. We show that our basis obtained by specialising the covariants picks out some of the quintic hypersurfaces to which Hulek was able to attach a geometric meaning.

**Lemma 7.1.** *Let $\phi \in \wedge^2 V \otimes W$ be non-singular and write $S_{10}, \ldots, S_{50}$ for the quintic forms obtained by evaluating the covariants (16) at $\phi$.*

 (i) *$S_{10}$ is (a scalar multiple of) the determinant of the Jacobian matrix of the quadrics defining $C_\phi$.*
 (ii) *The Heisenberg invariant quintics vanishing on the tangent variety of $C_\phi$ are linear combinations of $S_{10}, S_{20}, S'_{30}$.*
 (iii) *The quintics $S_{10}, S_{20}, S_{30}$ are singular along $C_\phi$.*
 (iv) *The quintics $S_{10}, S_{20}, S_{30}, S'_{30}, S_{40}$ vanish on $C_\phi$.*

*Proof.* For the proof we may take $\phi = u(a, b)$ a Hesse model. Let $p_0, \ldots, p_4$ be the equations for $C_\phi$, i.e. $p_i = abw_i^2 + b^2 w_{i+1} w_{i+4} - a^2 w_{i+2} w_{i+3}$.
(i) We compute $S_{10} = 25 \det(\partial p_i / \partial w_j)$.
(ii) The tangent line to $C_\phi$ at $P = (0 : a : b : -b : -a)$ also passes through

$$Q = (5a^3 b^3 : 0 : -b(2a^5 - b^5) : -b(a^5 + 2b^5) : a(a^5 - 3b^5)).$$

Evaluating the quintic forms at $\lambda P + Q$ we find that $S_{10}, S_{20}, S'_{30}$ vanish on the tangent line whereas $S_{30}, S_{40}, S_{50}$ give polynomials in $\lambda$ of degrees $0, 2, 4$.
(iii) We may write these quintics as linear combinations of $\sum p_0^2 w_0$, $\sum p_1 p_4 w_0$ and $\sum p_2 p_3 w_0$.
(iv) We may write these quintics as linear combinations of $\sum p_0 w_0^3$, $\sum p_0 w_0 w_1 w_4$, $\sum p_0 w_0 w_2 w_3$, $\sum p_0 (w_1^2 w_3 + w_2 w_4^2)$ and $\sum p_0 (w_1 w_2^2 + w_3^2 w_4)$. $\qquad\square$

**Theorem 7.2.** *Let $C = C_\phi$ be an elliptic normal quintic. Let $\operatorname{Tan} C$ and $\operatorname{Sec} C$ be the tangent and secant varieties of $C$. Let $F$ be the locus of singular lines of the rank 3 quadrics containing $C$. Then*

 (i) *$\operatorname{Sec} C$ is the degree 5 hypersurface defined by $S_{10}$.*
 (ii) *$\operatorname{Tan} C$ and $F$ are irreducible surfaces of degrees 10 and 15 and their union is the complete intersection defined by $S_{10}$ and $S_{20}$.*
 (iii) *The space of Heisenberg invariant quintics containing $\operatorname{Tan} C$ has basis $S_{10}, S_{20}, S'_{30}$.*
 (iv) *The space of Heisenberg invariant quintics containing $F$, equivalently that are singular along $C$, has basis $S_{10}, S_{20}, S_{30}$.*

(v) *The space of Heisenberg invariant quintics containing $C$ has basis $S_{10}$, $S_{20}$, $S_{30}$, $S'_{30}$, $S_{40}$.*

*Proof.* This follows by Lemma 7.1 and work of Hulek [11]. □

## 8. THE COVERING MAP

We call the covariants $\wedge^2 V \otimes W \to S^d W$ *covariants of order $d$*. The action of the Heisenberg group shows that the order must be a multiple of 5. By Theorem 6.1 and [10, Lemma 4.4], the $K[c_4, c_6]$-modules of covariants of orders $5, 10, 15$ have ranks $6, 41, 156$. Fortunately we do not need to classify all these covariants since most of them vanish on $C_\phi$ and therefore are of no use for describing the covering map.

**Lemma 8.1.** *Let $C \subset \mathbb{P}^{n-1}$ be an elliptic normal curve. Then the space of Heisenberg invariant polynomials of degree $nd$, quotiented out by the subspace vanishing on $C$, has dimension $d$.*

*Proof.* Let $\pi : C \to E$ be the covering map of degree $n^2$ from $C$ to its Jacobian $E$. Then $\pi^*(d.0_E) \sim ndH$ where $H$ is the hyperplane section for $C$. So if $f_1, \ldots, f_d$ is a basis for the Riemann-Roch space $\mathcal{L}(d.0_E)$ then $\pi^* f_1, \ldots, \pi^* f_d$ are basis for the space of forms of degree $nd$ in $K[x_0, \ldots, x_{n-1}]/I(C)$ that are invariant under the action of $E[n]$. Applying the Reynold's operator for the Heisenberg group shows that every such form has a representative in $K[x_0, \ldots, x_{n-1}]$ that is itself Heisenberg invariant. □

**Lemma 8.2.** *Let $C \subset \mathbb{P}^{n-1}$ be either an elliptic normal curve or a rational nodal curve, and let $P \in C$ be a smooth point. Suppose $Z, X, Y$ are homogeneous polynomials in $K[x_0, \ldots, x_{n-1}]$ of degrees $n, 2n, 3n$ with $\mathrm{ord}_P(Z) = 1$, $\mathrm{ord}_P(X) = 0$, $\mathrm{ord}_P(Y) = 0$. Then for each $d \geq 1$ the forms*

$$\{X^i Y^j Z^k : i, k \geq 0, j \in \{0, 1\}, 2i + 3j + k = d\}$$

*are linearly independent in the co-ordinate ring $K[x_0, \ldots, x_{n-1}]/I(C)$.*

*Proof.* This is clear since $\mathrm{ord}_P(X^i Y^j Z^k) = k$ and the forms listed have distinct values of $k$. □

**Lemma 8.3.** *There are covariants $Z, X, Y$ of orders $5, 10, 15$ and degrees $50, 110, 165$ such that whenever $C_\phi$ is an elliptic normal quintic or rational nodal quintic there is a smooth point $P \in C_\phi$ such that the evaluations of $Z, X, Y$ at $\phi$ satisfy $\mathrm{ord}_P(Z) = 1$, $\mathrm{ord}_P(X) = 0$, $\mathrm{ord}_P(Y) = 0$.*

*Proof.* We start with the covariants $U, H : \wedge^2 V \otimes W \to \wedge^2 V \otimes W$ and $Q_6 : \wedge^2 V \otimes W \to S^2 V \otimes W$ where $U$ is the identity map and (on the Hesse family)

$$H = -(\partial D/\partial b) \sum (v_1 \wedge v_4) w_0 + (\partial D/\partial a) \sum (v_2 \wedge v_3) w_0$$
$$Q_6 = \sum (5a^3 b^3 v_0^2 + a(a^5 - 3b^5) v_1 v_4 - b(3a^5 + b^5) v_2 v_3) w_0.$$

There are covariants $P_2, P_{12}, P_{22} : \wedge^2 V \otimes W \to V^* \otimes S^2 W$ where $P_2$ is the Pfaffian map (2) and $P_{12}, P_{22}$ satisfy

$$P_2(\lambda U + \mu H) = \lambda^2 P_2 + 2\lambda\mu P_{12} + \mu^2 P_{22}.$$

We define covariants $M_{30} : \wedge^2 V \otimes W \to S^5 V$ and $N_{30} : \wedge^2 V \otimes W \to S^5 V^*$ where $M_{30} = \det Q_6$ and $N_{30}$ is the coefficient of $t$ in $\det(P_2 + t P_{22})$. We also define $T_{23}$ and $T_{28}$ taking values in $V \otimes S^3 W$ by

$$(\otimes^2 V \otimes W) \times (V^* \otimes S^2 W) \to V \otimes S^3 W$$

$$(U, P_{22}) \mapsto T_{23}$$

$$(Q_6, P_{22}) \mapsto T_{28}.$$

We then put

$$Z = (1/2) Q_6(P_{22}, P_{22})$$

$$X = (3^3/2^6) M_{30}(P_{12}, P_{12}, P_{12}, P_{22}, P_{22})$$

$$Y = (3^3/2^8) N_{30}(T_{23}, T_{28}, T_{28}, T_{28}, T_{28}).$$

As required these are covariants of orders $5, 10, 15$ and degrees $50, 110, 165$.

Suppose $C_\phi$ is a rational nodal quintic. By Lemma 1.2 we may assume that $\phi$ is as given in Section 4, i.e. $\phi = u_1(0, 1, 1, 1, 1)$. Then $C_\phi$ is parametrised by

$$(x_0 : \ldots : x_4) = (s^5 - t^5 : st^4 : s^2 t^3 : -s^3 t^2 : -s^4 t)$$

Evaluating $Z, X, Y$ at $\phi$ we find

$$Z(s^5 - t^5, st^4, s^2 t^3, -s^3 t^2, -s^4 t) = -2^8 3^4 s^{10} t^{10} (s^5 - t^5)$$

(17) $\qquad X(s^5 - t^5, st^4, s^2 t^3, -s^3 t^2, -s^4 t) = 2^{16} 3^9 s^{20} t^{20} (s^{10} + 10 s^5 t^5 + t^{10})$

$$Y(s^5 - t^5, st^4, s^2 t^3, -s^3 t^2, -s^4 t) = 2^{26} 3^{15} s^{35} t^{35} (s^5 + t^5).$$

The conclusions of the lemma are satisfied for $P = (0 : 1 : 1 : -1 : -1)$.

Now suppose $C_\phi$ is an elliptic normal quintic. Then by [9, Proposition 4.1] we may assume that $\phi = u(a, b)$ is a Hesse model. There is a flex (i.e. hyperosculating point) of $C_\phi$ at $P = (0 : a : b : -b : -a)$. Evaluating $Z, X, Y$ at $\phi$ we find

$$Z(0, a, b, -b, -a) = 0$$

(18) $\qquad X(0, a, b, -b, -a) = 2^{18} 3^{10} D^{10}$

$$Y(0, a, b, -b, -a) = -2^{27} 3^{15} D^{15}$$

where $D = ab(a^{10} - 11 a^5 b^5 - b^{10})$. Since $\Delta = D^5$ it is clear that $X$ and $Y$ do not vanish at $P$. Now $C_\phi \subset \mathbb{P}^4$ is a curve of degree 5 meeting the degree 5 hypersurface defined by $Z$ at the 25 flexes of $C_\phi$. So by Bezout's theorem either $\mathrm{ord}_P(Z) = 1$ or $Z$ vanishes identically on $C_\phi$. To rule out the latter we write $Z$ in terms of the basis (16). Explicitly we find

$$Z = (39/10) c_4^2 S_{10} + 4 c_6 S_{20} - 54 c_4 S_{30} - (198/5) c_4 S'_{30} + 12 S_{50}.$$

By Theorem 6.1 the specialisations of $S_{10}, \ldots, S_{50}$ at $\phi$ are linearly independent. It follows by Theorem 7.2(v) that $Z$ does not vanish identically on $C_\phi$. $\square$

**Lemma 8.4.** *Let $\mathbb{L}/\mathbb{K}$ be a finite Galois extension with Galois group $\Gamma$. Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{L}$. Suppose there is an action of $\Gamma$ on $\mathbb{V}$ satisfying $\gamma(v + w) = \gamma(v) + \gamma(w)$ and $\gamma(\lambda v) = \gamma(\lambda)\gamma(v)$ for all $\gamma \in \Gamma$, $\lambda \in \mathbb{L}$ and $v, w \in \mathbb{V}$. Then $\dim_{\mathbb{K}} \mathbb{V}^\Gamma = \dim_{\mathbb{L}} \mathbb{V}$.*

*Proof.* A generalised form of Hilbert's Theorem 90 states that $H^1(\Gamma, \mathrm{GL}_n(\mathbb{L})) = \{1\}$. See for example [12, Chapter X, Proposition 3]. We fix a basis for $\mathbb{V}$ over $\mathbb{L}$, and then compare this basis with its Galois conjugates. By writing the resulting cocycle as a coboundary, we find a new basis for $\mathbb{V}$ over $\mathbb{L}$ consisting of vectors fixed by $\Gamma$. $\square$

**Lemma 8.5.** *Let $M_d$ be the $K[c_4, c_6]$-module of covariants for $Y = S^{5d}W$, quotiented out by the submodule of covariants that vanish on the curve. Then $M_d$ is a free $K[c_4, c_6]$-module of rank $d$ generated by*

$$\{X^i Y^j Z^k : i, k \geq 0, j \in \{0, 1\}, 2i + 3j + k = d\}$$

*where $Z, X, Y$ are the covariants in Lemma 8.3.*

*Proof.* Let $Z = (S^{5d}W)^{H_5}$ and $m = \dim Z$. We apply Lemma 8.4 with $\mathbb{K} = K(a,b)^\Gamma$, $\mathbb{L} = K(a,b)$ and $\mathbb{V}$ either $\mathbb{U} = \mathbb{L} \otimes_K Z$ or the subspace $\mathbb{U}_0$ of forms that vanish on the curve defined by the generic Hesse model $u(a,b)$. Since the action of $\Gamma$ on $\mathbb{A}^2$ (and hence on $\mathbb{L} = K(\mathbb{A}^2)$) was defined so that $u : \mathbb{A}^2 \to (\wedge^2 V \otimes W)^{H_5}$ is $\Gamma$-equivariant, we do indeed have that $\Gamma$ acts on $\mathbb{U}_0$. By Lemmas 8.1 and 8.4 we compute

$$\dim_{\mathbb{K}} \mathbb{U}^\Gamma = \dim_{\mathbb{L}} \mathbb{U} = m,$$
$$\dim_{\mathbb{K}} \mathbb{U}_0^\Gamma = \dim_{\mathbb{L}} \mathbb{U}_0 = m - d.$$

Thus the $K[a,b]^\Gamma$-module of discrete covariants $\mathbb{A}^2 \to Z$ has rank $m$, and the submodule of discrete covariants vanishing on the curve has rank $m - d$. It follows by Theorem 2.6, and the proof of [10, Lemma 4.5], that the $K[c_4, c_6]$-module of covariants $\wedge^2 V \otimes W \to S^{5d}W$ has rank $m$, and the submodule of covariants vanishing on the curve has rank $m - d$. Therefore $M_d$ has rank $d$.

Let $F_1, \ldots, F_d$ be the covariants in the statement of the lemma. Lemmas 8.2 and 8.3 show that if $C_\phi$ is an elliptic normal quintic or rational nodal quintic then $F_1(\phi), \ldots, F_d(\phi)$ are linearly independent over $K$. An argument similar to the proof of Theorem 6.1(ii) now shows that $F_1, \ldots, F_d$ are a free basis for $M_d$. $\square$

We show that the covariants $Z, X, Y$ give a formula for the covering map. The formula for the Jacobian was already proved in [8] by a different method.

**Theorem 8.6.** *Let $\phi \in \wedge^2 V \otimes W$ be non-singular. Then $C_\phi$ has Jacobian elliptic curve $E$ with Weierstrass equation*

$$(19) \qquad\qquad y^2 = x^3 - 27c_4(\phi)x - 54c_6(\phi)$$

*and the covering map $C_\phi \to E$ is given by $(x, y) = (X/Z^2, Y/Z^3)$ where $Z, X, Y$ are the evaluations at $\phi$ of the covariants in Lemma 8.3.*

*Proof.* By Lemma 8.5 the $K[c_4, c_6]$-module $M_6$ has basis

$$X^3, \ XYZ, \ X^2Z^2, \ YZ^3, \ XZ^4, \ Z^6.$$

Since $Z, X, Y$ have degrees $50, 110, 165$ and $c_4, c_6$ have degrees $20, 30$ we must therefore have

$$Y^2 = \lambda X^3 + \mu c_4 X Z^4 + \nu c_6 Z^6$$

for some $\lambda, \mu, \nu \in K$. We determine these scalars by specialising to the case $C_\phi$ is a rational nodal quintic. Using (17) we find $\lambda = 1, \mu = -27, \nu = -54$. Thus $(x, y) = (X/Z^2, Y/Z^3)$ defines a morphism $\pi : C_\phi \to E$ where $E$ is the curve defined by (19). The fibre above the point at infinity on $E$ is $C_\phi \cap \{Z = 0\}$. By (18) and Bezout's Theorem this consists of the 25 flexes on $C_\phi$. Thus $\deg \pi = 25$. Since $Z, X, Y$ are covariants it is clear that $\pi$ quotients out by the action of the Heisenberg group on $C_\phi$. Hence $E$ is the Jacobian of $C_\phi$ and $\pi$ is the covering map. $\qquad\square$

We gave algorithms for computing $Q_6$ and $H$ in [8, Section 8] and [9, Section 11]. So we can evaluate the covariants $Z, X, Y$ by following the proof of Lemma 8.3. This gives a practical algorithm for computing the covering map. Although we have been working over an algebraically closed field it is clear that Theorem 8.6 still holds without this assumption. We give an example in the case $K = \mathbb{Q}$.

**Example 8.7.** Let $C \subset \mathbb{P}^4$ be the elliptic normal quintic defined by the $4 \times 4$ Pfaffians of

$$\begin{pmatrix} 0 & 2x_2 + 3x_4 & 2x_2 + x_3 + x_4 + 4x_5 & x_1 - x_3 + 3x_4 - x_5 & -x_1 - x_2 - x_5 \\ & 0 & x_1 + 2x_2 - x_3 - 2x_4 + x_5 & 2x_1 - x_2 + x_3 + 3x_4 & -x_1 + x_2 - x_3 + x_5 \\ & & 0 & -2x_2 + x_3 + x_4 + 2x_5 & -2x_4 + x_5 \\ & - & & 0 & x_2 + x_3 + 2x_4 - x_5 \\ & & & & 0 \end{pmatrix}$$

The invariants of this model are $c_4 = 21288863488$ and $c_6 = 3106257241074688$. Our Magma function `CoveringCovariants` evaluates the covariants of Lemma 8.3 to give forms $Z, X, Y$. The first of these is

$$\begin{aligned} Z = \ & 20808951703645242324 1728 x_1^5 + 48134837542811845741 3632 x_1^4 x_2 \\ & - 10673310974337084618 09664 x_1^4 x_3 - 8615654010321956648 71424 x_1^4 x_4 \\ & - 2713065303844178403 139584 x_1^4 x_5 - 1159509369215265868 720128 x_1^3 x_2^2 \\ & + \ldots + 85118002593548552632 52480 x_5^5. \end{aligned}$$

Evaluating these forms at $(4013 : -2384 : -1616 : 1388 : 1021) \in C(\mathbb{Q})$ we obtain

$Z = 341237760995163802216399617872078722048 32,$

$X = 1214124219511158599909710742588931125361747039387286150121962457 7\backslash$
$\quad 3843080932512669892608,$

$Y = 13341702475842976696719854379608150742217144829049714776419935109\backslash$
$\quad 10164201520123953599858396067352426339710162835468918162316066816.$

The Jacobian of $C$ is the elliptic curve $E$ with Weierstrass equation $y^2 = x^3 - 27c_4 x - 54c_6$ and $P = (X/Z^2, Y/Z^3) \in E(\mathbb{Q})$ is a point of canonical height $164.90718\ldots$. In fact $E(\mathbb{Q})$ has rank 1 and is generated by $P$.

**Remark 8.8.** The elliptic curve $E$ in the above example is labelled 17472bz1 in Cremona's tables [6]. It satisfies a 5-congruence with the elliptic curve $F$ labelled $17472bx2$. In fact $F$ has Weierstrass equation $y^2 = x(x+16)(x-26)$ and the genus one model in Example 8.7 may be constructed from the point $(-2, 28) \in F(\mathbb{Q})$ using visibility as described in [10].

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