

The proportion of genus one curves over \mathbb{Q} defined by a binary quartic that everywhere locally have a point

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Abstract

We consider the proportion of genus one curves over \mathbb{Q} of the form $z^2 = f(x, y)$ where $f(x, y) \in \mathbb{Z}[x, y]$ is a binary quartic form (or more generally of the form $z^2 + h(x, y)z = f(x, y)$ where also $h(x, y) \in \mathbb{Z}[x, y]$ is a binary quadratic form) that have points everywhere locally. We show that the proportion of these curves that are locally soluble, computed as a product of local densities, is approximately 75.96%. We prove that the local density at a prime p is given by a fixed degree-9 rational function of p for all odd p (and for the generalized equation, the same rational function gives the local density at every prime). An additional analysis is carried out to estimate rigorously the local density at the real place.

1 Introduction

In this paper we show that most genus one curves over \mathbb{Q} of the form $z^2 = f(x, y)$, where $f \in \mathbb{Z}[x, y]$ is a binary quartic form, have a point everywhere locally.

Consider the family of equations

$$z^2 = f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4, \quad (1)$$

where $a, b, c, d, e \in \mathbb{Z}$. Provided that f is squarefree, such an equation defines a genus one curve over \mathbb{Q} . We define the *height* of the equation (1) by

$$H(f) := \max\{|a|, |b|, |c|, |d|, |e|\}. \quad (2)$$

Let ρ denote the density of equations (1), when ordered by height, that are everywhere locally soluble; that is,

$$\rho = \lim_{X \rightarrow \infty} \frac{\#\{f \mid H(f) \leq X, (1) \text{ everywhere locally soluble}\}}{\#\{f \mid H(f) \leq X\}}. \quad (3)$$

Since 100% of binary quartics over \mathbb{Z} , when ordered by height, are squarefree, ρ also represents the density of genus one curves C of the form $z^2 = f(x, y)$ that have a point everywhere locally. In this paper we describe how to compute ρ , and in particular show that $\rho \approx 75.96\%$.

We define local densities for solubility as follows. For p a prime, we identify the space of equations (1) having coefficients in \mathbb{Z}_p with the space \mathbb{Z}_p^5 equipped with its natural additive Haar measure. For measurable subsets S of \mathbb{Z}_p^5 , or \mathbb{Z}_p^n for any $n \geq 1$, we will also refer to the measure of S as its *density*, or as the *probability* that a random element of \mathbb{Z}_p^n lies in S . We then write $\rho(p)$ for the density of equations (1) over \mathbb{Z}_p that have a solution over \mathbb{Q}_p . We also write $\rho(\infty)$ for the probability that an equation of the form (1), with real coefficients independently and uniformly distributed in $[-1, 1]$, has a real solution.

It is then a theorem of Poonen and Stoll [11] (which in turn relies on the sieve of Ekedahl [8]) that the density ρ exists and is given by

$$\rho = \rho(\infty) \prod_{p \text{ prime}} \rho(p). \quad (4)$$

We prove the following theorem.

Theorem 1. *Let $\rho(p)$ denote the density of binary quartic forms f over \mathbb{Z}_p such that $z^2 = f(x, y)$ is soluble over \mathbb{Q}_p . Then*

$$\rho(p) = \begin{cases} \frac{23087}{24528} & \text{if } p = 2; \\ R(p) & \text{if } p \geq 3, \end{cases}$$

where R is the rational function

$$R(t) = 1 - \frac{4t^7 + 4t^6 + 2t^5 + t^4 + 3t^3 + 2t^2 + 3t + 3}{8(t+1)(t^2+t+1)(t^6+t^3+1)}. \quad (5)$$

We also carry out a rigorous numerical integration to prove the following.

Proposition 2. *Let $\rho(\infty)$ denote the probability that an equation $z^2 = f(x, y)$, where f is a binary quartic form with real coefficients independently and uniformly distributed in $[-1, 1]$, has a real solution. Then*

$$0.873954 \leq \rho(\infty) \leq 0.874124.$$

A Monte Carlo simulation (see §3) suggests that $\rho(\infty)$ is equal to 0.87411 to five decimal places.

Theorem 1 and Proposition 2, together with equation (4), imply the following.

Theorem 3. *When equations of the form (1) with coefficients in \mathbb{Z} are ordered by height, a proportion of*

$$\rho = \rho(\infty) \frac{23087}{24528} \prod_{p>2} \left(1 - \frac{4p^7 + 4p^6 + 2p^5 + p^4 + 3p^3 + 2p^2 + 3p + 3}{8(p+1)(p^2+p+1)(p^6+p^3+1)} \right)$$

have points everywhere locally. We have

$$0.759515 \leq \rho \leq 0.759663.$$

The Monte Carlo simulation described above suggests that the value of ρ is equal to 0.75965 to five decimal places.

One striking feature of Theorem 1 is that, for $p > 2$, the quantities $\rho(p)$ are given by a fixed rational function R evaluated at p . In a sequel to this article we will show that this phenomenon continues for higher genus hyperelliptic curves, provided that p is sufficiently large compared to the genus.

Our strategy for proving Theorem 1 is a refinement of that for testing solubility of a binary quartic form over \mathbb{Z}_p (equivalently, \mathbb{Q}_p) as described, for example, in the work of Birch and Swinnerton-Dyer [4]; the arguments are also related to those in our earlier work on determining the density of locally soluble quadratic forms (with Keating and Jones) in [3] and ternary cubic forms in [2]. We consider the reductions modulo p ; equations (1) whose reductions have smooth \mathbb{F}_p -points are soluble by Hensel's lemma, while those that have no \mathbb{F}_p -points are insoluble. Finally, to determine the probabilities of solubility in the much more difficult remaining cases, we develop certain recursive formulae, involving these and other suitable related probabilities, that allow us to solve for and obtain exact algebraic expressions for the desired probabilities.

We may instead consider equations of the form

$$z^2 + h(x, y)z = f(x, y), \quad (6)$$

with $f(x, y)$ as in (1) and $h(x, y) = lx^2 + mxy + ny^2$. For these equations, as in [6], we get a more uniform result for all primes p , including $p = 2$. For each p , we identify the space of generalized binary quartics (6) over \mathbb{Z}_p with the space \mathbb{Z}_p^8 equipped with its natural additive Haar measure. Then we prove the following theorem.

Theorem 4. *For every prime p , the density $\rho'(p)$ of generalized binary quartics (6) over \mathbb{Z}_p that are soluble over \mathbb{Q}_p is $R(p)$, where R is the rational function (5).*

Note that $\rho(2) = 23087/24528 \approx 0.94125$ while $\rho'(2) = 1625/1752 \approx 0.92751$.

We define the *height* $H(h, f)$ of a generalized binary quartic (6) given by a pair of binary forms (h, f) by

$$H(h, f) := \max\{|l|^2, |m|^2, |n|^2, |a|, |b|, |c|, |d|, |e|\}, \quad (7)$$

which generalizes the height $H(f)$ defined for binary quartics. As before, we define the probability $\rho'(\infty)$ that an equation of the form (6), with real coefficients independently and uniformly distributed in $[-1, 1]$, has a real solution. The value of $\rho'(\infty)$ is different from $\rho(\infty)$, and is approximately 0.873743: see Section 3.

Finally, we let ρ' denote the density of equations (6) with integer coefficients, when ordered by their height $H(h, f)$, that are everywhere locally soluble; that is,

$$\rho' = \lim_{X \rightarrow \infty} \frac{\#\{(h, f) \mid H(h, f) \leq X, (6) \text{ everywhere locally soluble}\}}{\#\{(h, f) \mid H(h, f) \leq X\}}. \quad (8)$$

We now obtain a version of Theorem 3 for generalized binary quartics. To give the analogue of (4) also for ρ' , we use a weighted version of [11, Lemma 1], proved by the same methods in [7, Prop. 3.4], that takes into account the weights in the definition of $H(h, f)$. We thus obtain the following theorem.

Theorem 5. *When equations of the form (6) with coefficients in \mathbb{Z} are ordered by height, a proportion of*

$$\rho' = \rho'(\infty) \prod_p \left(1 - \frac{4p^7 + 4p^6 + 2p^5 + p^4 + 3p^3 + 2p^2 + 3p + 3}{8(p+1)(p^2+p+1)(p^6+p^3+1)} \right)$$

have points everywhere locally. We have

$$\rho' \approx 0.748248.$$

The results of this paper are used by the first author in [1] to prove that a positive proportion of equations of the form (1) fail the Hasse Principle. They are also used by the third author, Ho, and Park in [9] to determine the density of bidegree $(2, 2)$ -forms over \mathbb{Z} (which correspond to genus one curves over \mathbb{Q} embedded in $\mathbb{P}^1 \times \mathbb{P}^1$) that have points everywhere locally.

The proof of Theorem 4 is given in Section 2. After some preliminaries in §2.1 and §2.2, the main proof is presented in §2.3. For odd primes p , Theorem 1 then follows immediately, since by completing the square we have $\rho(p) = \rho'(p)$. The modifications required when $p = 2$ are described in §2.4. Finally, in Section 3, we describe the methods we used to estimate the probability $\rho(\infty)$ of solubility over the reals, establishing Proposition 2 and hence Theorem 3.

2 The density of soluble generalized binary quartics over \mathbb{Z}_p

In this section, we determine the probability that a genus one curve over \mathbb{Q}_p , given by an equation in the general form (6) with coefficients in \mathbb{Z}_p , has a \mathbb{Q}_p -rational point.

A similar, slightly simpler, argument can be applied to the equation (1) over \mathbb{Z}_p for odd primes p , and yields exactly the same probability; it is easy to see that this must be the case, using a straightforward argument based on completing the square when 2 is a unit.

2.1 Notation and preliminaries

Let $h(x, y) = lx^2 + mxy + ny^2$ and $f(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$ be binary forms over \mathbb{Z}_p , and let

$$F(x, y, z) = z^2 + h(x, y)z - f(x, y).$$

We refer to the pair (h, f) or F itself as a “generalized binary quartic”. The polynomial F is weighted homogeneous, where x, y, z have weights 1, 1, 2 respectively, and defines a curve C in weighted projective space $\mathbb{P}(1, 1, 2)$ over \mathbb{Q}_p ; this has genus one provided that it is smooth. We denote reduction modulo p by a bar, so that $\bar{h}, \bar{f} \in \mathbb{F}_p[x, y]$ are (possibly zero) binary forms over \mathbb{F}_p .

For every \mathbb{Q}_p -point $(x : y : z)$, we may choose homogeneous coordinates $x, y, z \in \mathbb{Z}_p$, not all in $p\mathbb{Z}_p$; then at least one of x, y is a unit. In what follows, we will always choose such primitive integral coordinates.

Our overall strategy is based on the observation that \mathbb{Q}_p -rational points reduce modulo p to \mathbb{F}_p -points on the reduced curve \bar{C} and that smooth \mathbb{F}_p -points lift to \mathbb{Q}_p -points by Hensel’s lemma. Thus if \bar{C} has smooth \mathbb{F}_p -points, then $C(\mathbb{Q}_p) \neq \emptyset$, while if $\bar{C}(\mathbb{F}_p) = \emptyset$, then $C(\mathbb{Q}_p) = \emptyset$. If all the \mathbb{F}_p -points are singular, then we have to work harder: geometrically, we then blow up the singular points; in our exposition, we will explicitly make variable substitutions. This will lead to a recursion, from which

we will then be able to solve for the various densities or probabilities of solubility in different special configurations.

2.2 Counts of generalized binary quartics over \mathbb{F}_p

We divide into cases according to the factorization of \bar{F} over the algebraic closure $\bar{\mathbb{F}}_p$ of \mathbb{F}_p ; clearly, this occurs either over \mathbb{F}_p itself or over \mathbb{F}_{p^2} , giving the following four *factorization types*:

1. \bar{F} absolutely irreducible;
2. \bar{F} has distinct factors over \mathbb{F}_p , i.e., $\bar{F} = (z - s_1(x, y))(z - s_2(x, y))$ with $s_1, s_2 \in \mathbb{F}_p[x, y]$ distinct;
3. \bar{F} has conjugate factors over \mathbb{F}_{p^2} , i.e., $\bar{F} = (z - s_1(x, y))(z - s_2(x, y))$ with $s_1, s_2 \in \mathbb{F}_{p^2}[x, y]$ conjugate over \mathbb{F}_p ;
4. \bar{F} has a repeated factor over \mathbb{F}_p , i.e., $\bar{F} = (z - s(x, y))^2$ with $s \in \mathbb{F}_p[x, y]$.

In the following lemma we give the counts of how many of the p^8 pairs (\bar{h}, \bar{f}) fall into each of these cases, and how many there are that also satisfy the side condition

$$z^2 + lz - a \quad \text{is irreducible over } \mathbb{F}_p \quad (*)$$

which will occur later, and will be referred to as “condition $(*)$ ”.

Lemma 6. *The numbers of generalized binary quartics over \mathbb{F}_p with each factorization type are given in the following table, as well as the same counts for those satisfying condition $(*)$:*

Factorization type	All	Satisfying $(*)$
1. Absolutely irreducible	$p^6(p^2 - 1)$	$\frac{1}{2}p^5(p^2 - 1)(p - 1)$
2. Distinct factors over \mathbb{F}_p	$\frac{1}{2}p^3(p^3 - 1)$	0
3. Conjugate factors over \mathbb{F}_{p^2}	$\frac{1}{2}p^3(p^3 - 1)$	$\frac{1}{2}p^5(p - 1)$
4. Repeated factor over \mathbb{F}_p	p^3	0
Total	p^8	$\frac{1}{2}p^7(p - 1)$

Let ξ_i (respectively, ξ_i^*) denote the probability that a generalized binary quartic F over \mathbb{F}_p (respectively, a generalized binary quartic F satisfying $(*)$) has factorization type i for $i = 1, 2, 3, 4$. Then Lemma 6 implies the following.

Corollary 7. *The probabilities ξ_i, ξ_i^* are given as follows:*

i	ξ_i	ξ_i^*
1	$(p^2 - 1)/p^2$	$(p^2 - 1)/p^2$
2	$\frac{1}{2}(p^3 - 1)/p^5$	0
3	$\frac{1}{2}(p^3 - 1)/p^5$	$1/p^2$
4	$1/p^5$	0

Define a binary form $f(x, y)$ to be *monic* if $f(1, 0) = 1$. We will need the counts of binary quartics (up to scaling), and the number of monic binary quartics, over \mathbb{F}_p with certain factorization patterns over \mathbb{F}_p , distinguished by the number of roots in $\mathbb{P}^1(\mathbb{F}_p)$ with various multiplicities: (0) none, (1) at least one simple root, (2) a double and no simple roots, (3) two double roots, or (4) a quadruple root. An elementary computation yields the following.

Lemma 8. *The numbers of nonzero binary quartics over \mathbb{F}_p (up to scaling by \mathbb{F}_p^\times) with each factorization type are given in the following table, as well as the same for monic quartics:*

Factorization type	Binary quartics mod \mathbb{F}_p^\times	Monic quartics
0. No roots	$\frac{1}{8}p(p-1)(3p^2+p+2)$	$\frac{1}{8}p(p-1)(3p^2+p+2)$
1. Simple root	$\frac{1}{8}p(p+1)(5p^2+p+2)$	$\frac{1}{8}p(p-1)(5p^2+3p+2)$
2. One double and no simple root	$\frac{1}{2}p(p^2-1)$	$\frac{1}{2}p^2(p-1)$
3. Two double roots	$\frac{1}{2}p(p+1)$	$\frac{1}{2}p(p-1)$
4. Quadruple root	$p+1$	p
Total	$p^4 + p^3 + p^2 + p + 1$	p^4

Let η_i (respectively, η'_i) denote the probability that a nonzero binary quartic form f over \mathbb{F}_p (respectively, a monic binary quartic form f) has factorization type i for $i = 0, 1, 2, 3, 4$. Then Lemma 8 implies the following.

Corollary 9. *The probabilities η_i, η'_i are given as follows:*

i	η_i	η'_i
0	$\frac{1}{8}p(p-1)^2(3p^2+p+2)/(p^5-1)$	$\frac{1}{8}(p-1)(3p^2+p+2)/p^3$
1	$\frac{1}{8}p(p^2-1)(5p^2+p+2)/(p^5-1)$	$\frac{1}{8}(p-1)(5p^2+3p+2)/p^3$
2	$\frac{1}{2}p(p-1)(p^2-1)/(p^5-1)$	$\frac{1}{2}(p-1)/p^2$
3	$\frac{1}{2}p(p^2-1)/(p^5-1)$	$\frac{1}{2}(p-1)/p^3$
4	$(p^2-1)/(p^5-1)$	$1/p^3$

2.3 Proof of Theorem 4

Fix a prime p , and let $\rho = \rho'(p)$ be the probability that the equation (6) with coefficients in \mathbb{Z}_p is \mathbb{Q}_p -soluble. Let σ_i denote the probability of solubility of a generalized binary quartic that has factorization type i . Then

$$\rho = \sum_{i=1}^4 \xi_i \sigma_i, \tag{9}$$

where the ξ_i are as in Corollary 7.

Similarly, let ρ^* be the probability that a generalized binary quartic (6) that satisfies condition $(*)$ is \mathbb{Q}_p -soluble. For $i = 1$ and $i = 3$, let σ_i^* denote the probability of solubility of a generalized binary quartic that has factorization type i and satisfies $(*)$. (We do not define σ_2^* or σ_4^* .) Then

$$\rho^* = \xi_1^* \sigma_1^* + \xi_3^* \sigma_3^*.$$

To compute ρ , we evaluate each σ_i ; in so doing, we will also need the values of σ_1^* , σ_3^* and ρ^* . We evaluate σ_1 , σ_1^* , and σ_2 in §2.3.1, then σ_3 and σ_3^* in §2.3.3, and σ_4 in §2.3.4. The latter is expressed in terms of ρ and additional probabilities τ_i for $0 \leq i \leq 4$, defined and evaluated in §2.3.4 and §2.3.5. We then solve the resulting recursion for ρ in §2.3.6.

2.3.1 Evaluation of σ_1 , σ_1^* , and σ_2

The first two cases, where the reduction \bar{F} is either absolutely irreducible, or has distinct factors over \mathbb{F}_p , are straightforward. First, both σ_1 and σ_1^* are probabilities of \mathbb{Q}_p -solubility of a generalized binary quartic F over \mathbb{Z}_p whose reduction modulo p is absolutely irreducible. Such curves always have smooth \mathbb{F}_p -points; this would not necessarily be the case for hyperelliptic curves of genus $g \geq 2$, which will be treated in a sequel to this paper.

Proposition 10. *Every generalized binary quartic over \mathbb{Z}_p whose reduction modulo p is absolutely irreducible has a \mathbb{Q}_p -rational point; that is, $\sigma_1 = \sigma_1^* = 1$.*

Proof. The curve \bar{C} over \mathbb{F}_p defined by a generalized binary quartic over \mathbb{Z}_p whose reduction modulo p is absolutely irreducible has arithmetic genus 1. If it is smooth, then it has genus 1 and thus at least one \mathbb{F}_p -point by the Hasse bounds. Otherwise, since it is geometrically irreducible, its normalization is a smooth curve of genus zero. Since the genus drops by at least 1 for each singular point, there must be exactly one singularity, and its multiplicity must be 2. Now the normalization has $p+1$ points, of which at most two lie over the singular point of \bar{C} , so \bar{C} has at least $p-1$ smooth points. Thus, in all cases, \bar{C} has at least one smooth point over \mathbb{F}_p , which lifts to a \mathbb{Q}_p -point. \square

Proposition 11. *Every generalized binary quartic over \mathbb{Z}_p whose reduction modulo p splits into two distinct factors has a \mathbb{Q}_p -rational point, so $\sigma_2 = 1$.*

Proof. We have $\bar{F} = (z - s_1(x, y))(z - s_2(x, y))$, where $s_1, s_2 \in \mathbb{F}_p[x, y]$ are distinct binary quadratic forms. Each of the curves $z = s_i(x, y)$ has $p+1$ points over \mathbb{F}_p , and since $s_1 \neq s_2$, these two curves intersect in at most 2 points. Hence, for all p , there are smooth \mathbb{F}_p -points, which lift to \mathbb{Q}_p -points. \square

2.3.2 Some lemmas

The following lemma expresses a general principle that allows us to make a change of coordinates in the arguments that follow.

Lemma 12. *Let Φ and Φ' be subsets of $\mathbb{Z}_p[x_1, \dots, x_n]$, each defined by letting m of the coefficients range over an open subset of \mathbb{Z}_p^m and fixing the remaining coefficients. Suppose that Φ' is the image of Φ under an affine linear substitution*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + b, \quad (10)$$

where $A \in \mathrm{GL}_n(\mathbb{Z}_p)$ and $b \in \mathbb{Z}_p^n$. Then the probability that a random polynomial in Φ is soluble over \mathbb{Z}_p is the same as the probability that a random polynomial in Φ' is soluble over \mathbb{Z}_p .

Proof. The effect of the transformation (10) on the m coefficients that vary is both affine linear and invertible mod p . It therefore preserves the measure. Clearly solubility over \mathbb{Z}_p is also preserved. \square

As an example of the use of this lemma, we consider the set of monic quadratics $g(x) = x^2 + bx + c$ with coefficients in \mathbb{Z}_p , which we identify with \mathbb{Z}_p^2 in the usual way. The lemma shows that the probability g has a \mathbb{Z}_p -root, given that g has a double root mod p at $x \equiv r \pmod{p}$, is independent of r . Thus in working out the probability that g has a \mathbb{Z}_p -root, given that g has a double root mod p , we may assume without loss of generality that the double root is at $x \equiv 0 \pmod{p}$.

The following two lemmas will be used to help evaluate σ_3 and σ_3^* in the next section. The coefficients of a generalized binary quartic are labelled as described in Section 2.1.

Lemma 13. *Let $l, a, b \in p\mathbb{Z}_p$ and $m, c \in \mathbb{Z}_p$ be fixed, subject to the condition that $z^2 + mz - c$ is irreducible over \mathbb{F}_p . Let α be the probability of the existence of $x, z \in \mathbb{Z}_p$ with $F(x, 1, z) = 0$ given that $n, d, e \in \mathbb{Z}_p$ (that is, α is the density of $(n, d, e) \in \mathbb{Z}_p^3$ for which such a solution exists), and β the probability of such a solution given that $n, d, e \in p\mathbb{Z}_p$. Then*

$$\alpha = \frac{p}{p+1} \quad \text{and} \quad \beta = \frac{1}{p+1}.$$

Proof. We will show that $\alpha = (1 - 1/p) + (1/p)\beta$ and $\beta = (1/p)\alpha$, from which their values follow. Note that while, *a priori*, α and β might depend on $(l, a, b) \in p\mathbb{Z}_p^3$ and $(m, c) \in \mathbb{Z}_p^2$, we will see in the proof that this is not the case.

First, assume only that $n, d, e \in \mathbb{Z}_p$. We have $F(x, 1, z) \equiv z^2 + (mx + n)z - (cx^2 + dx + e)$, which defines a conic over \mathbb{F}_p . The side condition that $z^2 + mz - c$ is irreducible over \mathbb{F}_p implies that there are no \mathbb{F}_p -rational points at infinity. If this conic is smooth then it has \mathbb{F}_p -points; the probability of this is $1 - 1/p$, since the discriminant of the conic is $e(m^2 + 4c) - (d^2 + dm - cn^2)$ and $m^2 + 4c \neq 0$, so for each pair n, d there is a unique $e \pmod{p}$ for which the discriminant vanishes. If the conic is singular, then the singular point is the only \mathbb{F}_p -rational point (by the side condition), and without loss of generality (see Lemma 12) we may suppose the singular point to be at $(0, 0) \pmod{p}$, for which the probability of solubility is β . This establishes the first equation.

Next suppose that $n, d, e \in p\mathbb{Z}_p$. Then we have $F(x, 1, z) \equiv z^2 + mxz - cx^2$, whose only zero over \mathbb{F}_p is (by the side condition) $(x, z) \equiv (0, 0)$. The equation $F(x, 1, z) = 0$ implies $p^2 \mid e$, so with probability $1 - 1/p$ we have no solutions; otherwise we may replace the variables x, z by px, pz and divide through by p^2 , leading back to the first case. This establishes the second equation.

Note that in this reduction step the values of c and m are unchanged. The coefficients l, a, b become more divisible by p but our first argument is unchanged, being independent of these values, provided only that they have positive valuation. \square

Lemma 14. *Let $l, m, c \in p\mathbb{Z}_p$ and $a, b \in p^2\mathbb{Z}_p$ be fixed, with $v(c) = 1$. Let α' be the probability of the existence of $x, z \in \mathbb{Z}_p$ with $F(x, 1, z) = 0$ given that $n, e \in \mathbb{Z}_p$ and $d \in p\mathbb{Z}_p$ (that is, α' is the density of $(n, d, e) \in \mathbb{Z}_p \times p\mathbb{Z}_p \times \mathbb{Z}_p$ for which such a solution exists), and β' the probability of such a solution given that $n, d, e \in p\mathbb{Z}_p$. Then*

$$\alpha' = \beta' = \frac{1}{2}.$$

Proof. We will show that $\alpha' = \frac{1}{2}(1 - 1/p) + (1/p)\beta'$ and $\beta' = \frac{1}{2}(1 - 1/p) + (1/p)\alpha'$, from which the result follows. Note that while, *a priori*, α' and β' might depend on $(l, m, c) \in p\mathbb{Z}_p^3$ and $(a, b) \in p^2\mathbb{Z}_p^2$, we will see in the proof that this is not the case.

First let $n, d, e \in p\mathbb{Z}_p$ and consider the quadratic $\frac{1}{p}(cx^2 + dx + e)$ over \mathbb{F}_p . If it has distinct roots, then we can lift these to obtain \mathbb{Q}_p -points with $z = 0$; this has probability $\frac{1}{2}(1 - 1/p)$. If it is irreducible (probability $\frac{1}{2}(1 - 1/p)$) then there are no solutions, while if it has repeated roots (probability $1/p$) then after shifting the root to $x \equiv 0$ (see Lemma 12), replacing x, z by px, pz and rescaling the equation we recover the original situation except that now we only have $n, e \in \mathbb{Z}_p$ with $d \in p\mathbb{Z}_p$. (The valuations of l, a, b have increased, but everything here only depends on them lying in $p\mathbb{Z}_p$ or $p^2\mathbb{Z}_p$ as specified in the statement of the lemma.) This gives $\beta' = \frac{1}{2}(1 - 1/p) + (1/p)\alpha'$.

Now with $n, e \in \mathbb{Z}_p$ and $d \in p\mathbb{Z}_p$, we have $\bar{F}(x, 1, z) \equiv z^2 + nz - e$. The equation is soluble if this quadratic has distinct roots, and insoluble if it has no roots; if it has a double root then shifting the root to $z \equiv 0$ leads us back to the first case. Hence $\alpha' = \frac{1}{2}(1 - 1/p) + (1/p)\beta'$, as required. \square

2.3.3 Evaluation of σ_3 and σ_3^*

Now we consider generalized binary quartics F whose reduction modulo p factors as $\bar{F} = (z - s_1(x, y))(z - s_2(x, y))$, where s_1 and s_2 are conjugate binary quadratics over \mathbb{F}_{p^2} . Let $\omega \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$ and denote by $\bar{\omega}$ its Galois conjugate; then we may write $s_1(x, y) = r(x, y) + \omega s(x, y)$ and $s_2 = r(x, y) + \bar{\omega}s(x, y)$, where r, s are binary quadratic forms over \mathbb{F}_p . Replacing z by $z + r(x, y)$, we may assume without loss of generality that $r = 0$, so now $\bar{F} = (z - \omega s(x, y))(z - \bar{\omega}s(x, y))$. The only \mathbb{F}_p -points are those with $z \equiv s(x, y) \equiv 0 \pmod{p}$, which are singular. The probability of solubility now depends on the factorization of s over \mathbb{F}_p . Under condition (*), the leading coefficient of s must be non-zero, and the only difference between the two cases (σ_3 and σ_3^*) arises from the different probabilities of each factorization pattern occurring, depending on whether s is an arbitrary binary quadratic form over \mathbb{F}_p , or is restricted to those whose leading coefficient is non-zero.

In the case where s has distinct roots modulo p , we must take care to show that the probabilities that the two singular \mathbb{F}_p -points lift are independent.

Lemma 15. *Suppose that F is a generalized binary quartic over \mathbb{Z}_p whose reduction modulo p factors over \mathbb{F}_{p^2} as $\bar{F} = (z - \omega s)(z - \bar{\omega}s)$, with $s \in \mathbb{F}_p[x, y]$ a binary quadratic form having distinct roots over \mathbb{F}_p . Then the curve \bar{C} over \mathbb{F}_p defined by F has two \mathbb{F}_p -points, both singular, and the probability that at least one of these points lifts to a \mathbb{Q}_p -point is $(2p + 1)/(p + 1)^2$.*

Proof. Moving the roots of s to $(0 : 1)$ and $(1 : 0)$, we may assume without loss of generality that $s = xy$. Let $l, m, n, a, b, c, d, e \in \mathbb{Z}_p$ be the coefficients of F . Then we have $l, n, a, b, d, e \in p\mathbb{Z}_p$, and $z^2 + mz - c$, being the minimal polynomial of ω , is irreducible over \mathbb{F}_p . The only \mathbb{F}_p -points are the two singular points, $(0 : 1 : 0)$ and $(1 : 0 : 0)$.

The probability that $(0 : 1 : 0)$ lifts to a \mathbb{Q}_p -point is $\beta = 1/(p + 1)$, since we are in the situation of Lemma 13 with $n, d, e \in p\mathbb{Z}_p$. By symmetry, the probability that $(1 : 0 : 0)$ lifts to a \mathbb{Q}_p -point is also β . Provided that these probabilities are independent, the probability that at least one of the points lifts is $1 - (1 - \beta)^2 = (2p + 1)/(p + 1)^2$, as stated.

If $(1 : 0 : 0)$ lifts to a \mathbb{Q}_p -point then, by substitutions of the form $y \mapsto y + rx$ and $z \mapsto z + sx^2$ where $r, s \in p\mathbb{Z}_p$, we may assume that the \mathbb{Q}_p -point is $(1 : 0 : 0)$, and so $a = 0$. By the proof of Lemma 12, the effect of such a transformation on $(n, d, e) \in p\mathbb{Z}_p^3$ is measure preserving. Then

applying Lemma 13 with $a = 0$ shows that the probability $(0 : 1 : 0)$ lifts to a \mathbb{Q}_p -point is still β . Therefore the two probabilities of lifting are indeed independent, as claimed. \square

In the case where s has a double root modulo p , our argument expresses the probability of solubility in terms of ρ^* .

Lemma 16. *Suppose that F is a generalized binary quartic over \mathbb{Z}_p whose reduction modulo p factors over \mathbb{F}_{p^2} as a product of two conjugate factors, in the form $\bar{F} = (z - \omega s)(z - \bar{\omega}s)$, where s is a binary quadratic form over \mathbb{F}_p with a double root over \mathbb{F}_p . Then the curve \bar{C} over \mathbb{F}_p defined by \bar{F} has only one \mathbb{F}_p -point, and the probability λ that this point lifts to a \mathbb{Q}_p -point satisfies*

$$\lambda = \frac{2p^5 - p^4 + p^2 - 2p}{2p^6} + \frac{\rho^*}{p^6}.$$

Proof. Moving the repeated root of s to $(0 : 1)$, we may assume without loss of generality that $s = x^2$. Let $l, m, n, a, b, c, d, e \in \mathbb{Z}_p$ be the coefficients of F . Then we have $m, n, b, c, d, e \in p\mathbb{Z}_p$, and $z^2 + lz - a$, being the minimal polynomial of ω , is irreducible modulo p , so that condition $(*)$ holds. The only \mathbb{F}_p -point is the singular point $(0 : 1 : 0)$, and we need to determine the probability λ that this point lifts.

The situation is as shown in the first row of the following table; here λ_i denotes the probability of solubility given both condition $(*)$ and that the valuations of l, \dots, e satisfy the conditions in row i ; in particular, $\lambda = \lambda_1$. The values of l and a change from row to row but return to their original values by line 7; condition $(*)$ refers always to the original values of l and a .

	l	m	n	a	b	c	d	e
$\lambda = \lambda_1 = \frac{1}{p}\lambda_2$	≥ 0	≥ 1	≥ 1	≥ 0	≥ 1	≥ 1	≥ 1	≥ 1
$\lambda_2 = (1 - \frac{1}{p}) + \frac{1}{p}\lambda_3$	≥ 1	≥ 1	≥ 0	≥ 2	≥ 2	≥ 1	≥ 0	≥ 0
$\lambda_3 = \frac{1}{2}(1 - \frac{1}{p}) + \frac{1}{p}\lambda_4$	≥ 1	≥ 1	≥ 0	≥ 2	≥ 2	≥ 1	≥ 1	≥ 0
$\lambda_4 = \frac{1}{2}(1 - \frac{1}{p}) + \frac{1}{p}\lambda_5$	≥ 1	≥ 1	≥ 1	≥ 2	≥ 2	≥ 1	≥ 1	≥ 1
$\lambda_5 = (1 - \frac{1}{p}) + \frac{1}{p}\lambda_6$	≥ 1	≥ 1	≥ 1	≥ 2	≥ 2	≥ 2	≥ 1	≥ 1
$\lambda_6 = \frac{1}{p}\lambda_7$	≥ 1	≥ 1	≥ 1	≥ 2	≥ 2	≥ 2	≥ 2	≥ 1
$\lambda_7 = \rho^*$	≥ 0							

Assuming the correctness of this table, this gives the desired expression for λ in terms of ρ^* .

Now we justify the reduction steps and the relations between successive λ_i . At each step, we determine whether there is a smooth solution which lifts, or there is no solution, or we continue to the next line; the probability of continuing to the next line is always $1/p$.

1. In line 1, condition $(*)$ implies that every solution has $x, z \in p\mathbb{Z}_p$; so there are no solutions unless $v(e) \geq 2$, which has probability $1/p$. Replacing x by px and z by pz , and then dividing by p^2 , leads to line 2.
2. The reduced equation is now $z^2 + nz \equiv dx + e$, which is linear in x . With probability $1 - 1/p$, we have $v(d) = 0$ and a solution exists. Otherwise, with probability $1/p$, we have $v(d) \geq 1$, leading to line 3.

3. The reduced equation is now $z^2 + nz \equiv e$. If this quadratic is irreducible over \mathbb{F}_p , which happens with probability $\frac{1}{2}(1 - 1/p)$, then there are no solutions; if it splits (which happens with the same probability), then there are smooth solutions which lift to a \mathbb{Q}_p -point with $x = 0$. Finally, with probability $1/p$, the quadratic has a double root, which we shift to $z \equiv 0$, leading to line 4.
4. With probability $1 - 1/p$ we have $v(c) = 1$; by Lemma 14 the equation is soluble with probability $1/2$. Otherwise, with probability $1/p$, we have $v(c) \geq 2$, leading to line 5.
5. With probability $1 - 1/p$ we have $v(d) = 1$; then the quartic $\frac{1}{p}f$ has a simple root modulo p (at $x \equiv 0$), so we can lift it to a \mathbb{Q}_p -point with $z = 0$. Otherwise, with probability $1/p$, we have $v(d) \geq 2$, leading to line 6.
6. As in line 1, we now have no solutions unless $v(e) \geq 2$, which happens with probability $1/p$. At this point p divides l, m , and n , while p^2 divides a, b, c, d , and e , so we may scale the equation to obtain line 7.
7. We now have a generalized binary quartic satisfying no conditions other than $(*)$, because the coefficients l and a at the end are the same as at the start, so the probability of solubility is ρ^* .

□

We can now evaluate both σ_3 and σ_3^* . The rational functions recorded here, and in the remainder of Section 2, are not necessarily written in lowest terms, since we prefer to give expressions where the denominator is as simple as possible.

Proposition 17. *A generalized binary quartic F over \mathbb{Z}_p whose reduction modulo p factors into two conjugate factors over \mathbb{F}_{p^2} has a \mathbb{Q}_p -rational point with probability*

$$\sigma_3 = \frac{(p-1)^2(2p^9 + 3p^8 + 5p^7 + 3p^6 + 5p^5 + 3p^4 + 4p^3 + 5p^2 + 4p + 1)}{2(p^3-1)(p^9-1)}.$$

For such a generalized binary quartic that also satisfies condition $(*)$, the probability of having a \mathbb{Q}_p -rational point is

$$\sigma_3^* = \frac{(p-1)(2p^9 + 3p^8 + 5p^7 + 5p^6 + 5p^5 + 5p^4 + 4p^3 + 6p^2 + 6p + 2)}{2(p+1)^2(p^9-1)}.$$

Proof. Recall that we may assume without loss of generality that $\overline{F} = (z - \omega s(x, y))(z - \overline{\omega}s(x, y))$, where s is a binary quadratic form over \mathbb{F}_p , and that the only \mathbb{F}_p -points are those with $z \equiv s(x, y) \equiv 0 \pmod{p}$, which are singular. We divide into cases according to the factorization of s over \mathbb{F}_p .

- If s irreducible over \mathbb{F}_p , then there are no \mathbb{F}_p -points and the probability of solubility is 0. This occurs with probability $\frac{1}{2}p(p-1)^2/(p^3-1)$ in general and with probability $\frac{1}{2}(p-1)/p$ under condition $(*)$.
- The binary quadratic form s splits into distinct factors over \mathbb{F}_p with probability $\frac{1}{2}p(p^2-1)/(p^3-1)$ in general, or $\frac{1}{2}(p-1)/p$ under $(*)$. In this case there are two \mathbb{F}_p -points, one for each root of s , and by Lemma 15, the probability that at least one of these two \mathbb{F}_p -points lifts to \mathbb{Q}_p is $(2p+1)/(p+1)^2$.

- The binary quadratic form s has a repeated factor over \mathbb{F}_p with probability $(p^2 - 1)/(p^3 - 1)$ in general, or $1/p$ under (*). In this case there is only one \mathbb{F}_p -point, corresponding to the unique root of s , and Lemma 16 expresses the probability λ that it lifts to a \mathbb{Q}_p -point in terms of ρ^* .

Combining the cases, we find that

$$\sigma_3 = \frac{p(p^2 - 1)}{2(p^3 - 1)} \cdot \frac{(2p + 1)}{(p + 1)^2} + \frac{p^2 - 1}{p^3 - 1} \cdot \lambda \quad (11)$$

while

$$\sigma_3^* = \frac{(p - 1)}{2p} \cdot \frac{(2p + 1)}{(p + 1)^2} + \frac{1}{p} \cdot \lambda. \quad (12)$$

Recall also that

$$\rho^* = \xi_1^* \sigma_1^* + \xi_3^* \sigma_3^* = \xi_1^* + \xi_3^* \sigma_3^*, \quad (13)$$

(since $\sigma_1^* = 1$ by Proposition 10), and the values of ξ_1^* and ξ_3^* are given in Corollary 7. The linear equations (12) and (13), together with Lemma 16, can now be solved for λ , σ_3^* and ρ^* , and finally equation (11) gives σ_3 . \square

Corollary 18. *The probability that a generalized binary quartic (6) that satisfies condition (*) is \mathbb{Q}_p -soluble is*

$$\rho^* = \frac{p(p - 1)(2p^9 + 6p^8 + 6p^7 + 4p^6 + 3p^5 + 5p^4 + 5p^3 + 5p^2 + 5p + 2)}{2(p + 1)^2(p^9 - 1)}.$$

For reference, we also record here the value of λ (defined in Lemma 16):

$$\lambda = \frac{2p^{10} + 3p^9 - p^5 + 2p^4 - 2p^2 - 3p - 1}{2(p + 1)^2(p^9 - 1)}.$$

2.3.4 Evaluation of σ_4

We now evaluate σ_4 , the probability of solubility given that the generalized binary quartic F has repeated factors over \mathbb{F}_p , i.e., $F \equiv (z - s)^2 \pmod{p}$ for some binary quadratic form s . Replacing z by $z + s$ does not change densities, so we may assume that $s \equiv 0 \pmod{p}$, so that $F \equiv z^2 \pmod{p}$ or, equivalently, that all eight coefficients of F lie in $p\mathbb{Z}_p$.

Now all solutions $(x : y : z)$ must have $z \in p\mathbb{Z}_p$ and $f(x, y) \equiv 0 \pmod{p^2}$. Writing $f_1 = \frac{1}{p}f$, we see that each solution satisfies $f_1(x, y) \equiv z \equiv 0 \pmod{p}$. We divide into cases according to the factorization of $\overline{f_1}$, the reduction of f_1 modulo p .

If $\overline{f_1} = 0$, all coefficients of f are divisible by p^2 , and we may replace z by pz and divide through by p^2 to obtain an arbitrary generalized binary quartic over \mathbb{Z}_p . The probability of this is $1/p^5$, and the probability of solubility in this case is just ρ . Otherwise, for $0 \leq i \leq 4$, let τ_i be the probability of solubility given each possible factorization pattern for $\overline{f_1}$, with the cases numbered as in Lemma 8. Then

$$\sigma_4 = \frac{1}{p^5} \rho + \left(1 - \frac{1}{p^5}\right) \sum_{i=0}^4 \eta_i \tau_i, \quad (14)$$

where the η_i are as in Corollary 9.

We also let σ'_4 be the probability that (6) is \mathbb{Q}_p -soluble assuming again that all coefficients lie in $p\mathbb{Z}_p$ and also that $v_p(a) = 1$. Then

$$\sigma'_4 = \sum_{i=0}^4 \eta'_i \tau_i,$$

where the η'_i are also as in Corollary 9.

We will evaluate each τ_i ; then (9) and (14) will give two linear equations relating ρ and σ_4 , from which their values will then be uniquely determined.

The first two cases are easy.

Proposition 19. *We have $\tau_0 = 0$ and $\tau_1 = 1$.*

Proof. If $\overline{f_1}$ has no roots in \mathbb{F}_p , then $f(x, y) \equiv 0 \pmod{p^2}$ has no solutions, giving $\tau_0 = 0$.

If $\overline{f_1}$ has a simple root in \mathbb{F}_p , it lifts to a root in \mathbb{Q}_p , giving a \mathbb{Q}_p -point with $z = 0$, so $\tau_1 = 1$. \square

The next two cases are handled using Lemma 14.

Proposition 20. *We have $\tau_2 = 1/2$ and $\tau_3 = 3/4$.*

Proof. Each double root of $\overline{f_1}$ over \mathbb{F}_p lifts to a \mathbb{Q}_p -point with probability $\alpha' = 1/2$; indeed, without loss of generality, the double root is at $(x : y) = (0 : 1)$, so after replacing x by px and z by pz , and rescaling the equation, we may apply Lemma 14.

If $\overline{f_1}$ has one double root over \mathbb{F}_p , and no simple roots, then every \mathbb{Q}_p -point must arise from lifting the double root, giving $\tau_2 = 1/2$.

If $\overline{f_1}$ has two double roots over \mathbb{F}_p , we may assume that they are $(x : y) = (0 : 1)$ and $(1 : 0)$. Each lifts to a \mathbb{Q}_p -point with probability $1/2$ by Lemma 14, and examination of the proof of Lemma 14 shows that (just as in Lemma 15) the two probabilities are independent. Hence $\tau_3 = 1 - (1 - 1/2)^2 = 3/4$. \square

2.3.5 Evaluation of τ_4

Finally, we consider the case where $\overline{f_1}$ has a quadruple root.

Proposition 21. *We have $\tau_4 = \frac{4p^{10} + 8p^9 - 4p^8 + 4p^6 - 3p^4 + p^3 - 5p - 5}{8(p+1)(p^9 - 1)}$.*

Proof. Moving the quadruple root of $\overline{f_1}$ to 0, without loss of generality we have $\frac{1}{p}f \equiv a_1x^4 \pmod{p}$ where $a_1 = a/p \not\equiv 0 \pmod{p}$.

Every primitive \mathbb{Q}_p -point $(x : y : z)$ must satisfy $x \equiv z \equiv 0 \pmod{p}$ and $y \not\equiv 0 \pmod{p}$, so we replace x, z by px, pz and divide through by p^2 ; also, without loss of generality, we may assume that $y = 1$. This leads to the situation indicated in the first row of the following table, where ν_i is the probability of solubility given that the valuations of l, \dots, e satisfy the conditions in row i of the table,

with $\nu_1 = \tau_4$.

	l	m	n	a	b	c	d	e
$\nu_4 = \nu_1 = \frac{1}{2}(1 - \frac{1}{p}) + \frac{1}{p}\nu_2,$	≥ 2	≥ 1	≥ 0	$= 3$	≥ 3	≥ 2	≥ 1	≥ 0
$\nu_2 = (1 - \frac{1}{p}) + \frac{1}{p}\nu_3,$	≥ 2	≥ 1	≥ 1	$= 3$	≥ 3	≥ 2	≥ 1	≥ 1
$\nu_3 = \frac{1}{p}\nu_4,$	≥ 2	≥ 1	≥ 1	$= 3$	≥ 3	≥ 2	≥ 2	≥ 1
$\nu_4 = \frac{1}{2}(1 - \frac{1}{p})\frac{p}{p+1} + \frac{1}{2}(1 - \frac{1}{p}) + \frac{1}{p}\nu_5,$	≥ 1	≥ 0	≥ 0	$= 1$	≥ 1	≥ 0	≥ 0	≥ 0
$\nu_5 = (1 - \frac{1}{p}) + \frac{1}{p}\nu_6,$	≥ 1	≥ 1	≥ 0	$= 1$	≥ 1	≥ 1	≥ 0	≥ 0
$\nu_6 = \frac{1}{2}(1 - \frac{1}{p}) + \frac{1}{p}\nu_7,$	≥ 1	≥ 1	≥ 0	$= 1$	≥ 1	≥ 1	≥ 1	≥ 0
$\nu_7 = \sigma'_4$	≥ 1	≥ 1	≥ 1	$= 1$	≥ 1	≥ 1	≥ 1	≥ 1

Assuming the correctness of this table, we can express τ_4 in terms of σ'_4 . But we also have

$$\sigma'_4 = \sum_{i=0}^4 \eta'_i \tau_i = \eta'_1 + \eta'_2 \frac{1}{2} + \eta'_3 \frac{3}{4} + \eta'_4 \tau_4,$$

using the previously established values of τ_i for $0 \leq i \leq 3$ and the weights η'_i instead of η_i , as given in Corollary 9. Solving the two equations gives the value of τ_4 as stated in the proposition, and also

$$\sigma'_4 = \frac{5p^{10} + 5p^9 - p^7 + 3p^6 - 4p^5 + 4p^3 - 8p - 4}{8(p+1)(p^9 - 1)}.$$

Now we justify the reduction steps and the relations between successive ν_i . At each step, we determine whether there is a smooth solution which lifts, or there is no solution, or we continue to the next line; the probability of continuing to the next line is always $1/p$.

1. In line 1, the equation reduces to $z^2 + nz - e \equiv 0 \pmod{p}$. With probability $\frac{1}{2}(1 - 1/p)$ this has no roots over \mathbb{F}_p and the equation is insoluble; with the same probability it has simple \mathbb{F}_p -roots which lift, so the equation is soluble; and with probability $1/p$ there is a double root. In the latter case we shift the root to $z \equiv 0$, leading to line 2.
2. With probability $1 - 1/p$ we have $v(d) = 1$; then $\frac{1}{p}f$ is linear modulo p , so has a simple root which lifts, and we obtain a solution (with $z = 0$). Otherwise, $v(d) \geq 2$, leading to line 3.
3. With probability $1 - 1/p$ we have $v(e) = 1$; then there are no solutions. Otherwise rescale, replacing z by pz and dividing through by p^2 , leading to line 4.
4. The reduced equation now has the form $z^2 + (mx + n)z \equiv cx^2 + dx + e$, a possibly singular conic. If $z^2 + mz - c$ is irreducible (which has probability $\frac{1}{2}(1 - 1/p)$), then by Lemma 13 we have solubility with probability $\alpha = p/(p+1)$. If $z^2 + mz - c$ splits over \mathbb{F}_p (again with probability $\frac{1}{2}(1 - 1/p)$), then the conic has two distinct \mathbb{F}_p -points at infinity, so (whether or not it is singular) certainly has at least one more smooth \mathbb{F}_p -point. Lastly, with probability $1/p$ the quadratic has a double root modulo p ; we shift it to $z \equiv 0$, leading to line 5.

5. With probability $1 - 1/p$ we have $v(d) = 0$; then the reduced equation is linear in x and we have solubility. Otherwise $v(d) \geq 1$, leading to line 6.
6. The reduced equation is just as in line 1, and with probability $1/p$ we reach line 7.
7. In line 7, the probability ν_7 is the probability of solubility of any generalized binary quartic that reduces to z^2 modulo p , given also that $v(a) = 1$; this, by definition, is σ'_4 .

□

2.3.6 Conclusion of the proof of Theorem 4

We have established two linear equations (9) and (14) relating ρ and σ_4 . Solving for these, we obtain

$$\sigma_4 = \frac{5p^{10} + 8p^9 + p^8 - p^7 + 2p^6 - 3p^5 + 4p^3 - 10p - 6}{8(p+1)(p^9-1)},$$

and finally

$$\begin{aligned} \rho &= \frac{8p^{10} + 8p^9 - 4p^8 + 2p^6 + p^5 - 2p^4 + p^3 - p^2 - 8p - 5}{8(p+1)(p^9-1)} \\ &= 1 - \frac{4p^7 + 4p^6 + 2p^5 + p^4 + 3p^3 + 2p^2 + 3p + 3}{8(p+1)(p^2 + p + 1)(p^6 + p^3 + 1)} \end{aligned}$$

as stated in Theorem 4.

2.4 The density of soluble binary quartics over \mathbb{Z}_p

To complete the proof of Theorem 1, we need to know the density $\rho(p)$ of soluble binary quartics (1) over \mathbb{Z}_p , as opposed to the density $\rho'(p)$ of generalized binary quartics (6) over \mathbb{Z}_p . For odd primes, it is clear that we may complete the square without affecting the density, and obtain the same density as given in Theorem 4, so $\rho(p) = \rho'(p) = \rho$ (in the notation used above).

Now let $p = 2$, and consider the binary quartic

$$z^2 = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4.$$

If b or d is odd, then there are smooth points on the reduction modulo 2. If instead b and d are both even, then replacing z by $z + ax^2 + cxy + ey^2$ gives a generalized binary quartic with all coefficients even. The probability of solubility in this case is $\sigma_4 = 4691/6132$, as computed in §2.3.6, giving a final answer of $\rho(2) = (3/4) + (1/4)\sigma_4 = 23087/24528$. This completes the proof of Theorem 1.

3 The density of soluble binary quartics over \mathbb{R}

In this section we use rigorous numerical computational methods to establish bounds for $\rho(\infty)$, the probability that a random binary quartic form f with real coefficients independently and uniformly distributed in $[-1, 1]$ is not negative definite. Clearly, $1 - \rho(\infty)$ is the probability that f is negative

definite, and the probability that f is positive definite is the same, so $2 - 2\rho(\infty)$ is the probability that f has no real roots.

It suffices to consider inhomogeneous polynomials $f(x) \in \mathbb{R}[x]$. Writing $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, let $\Delta = \Delta(a, b, c, d, e)$ be the discriminant of f , which is a polynomial in a, b, c, d, e of degree 6, with 16 terms. We also define the quantities

$$H = 8ac - 3b^2; \\ Q = 3b^4 - 16ab^2c + 16a^2c^2 + 16a^2bd - 64a^3e.$$

Then the condition that f has no real roots is

$$\Delta > 0, \quad \text{and} \quad H > 0 \quad \text{or} \quad Q < 0 \quad (15)$$

(see [5, Prop. 7]). Hence $\rho(\infty) = 1 - \text{vol}(\mathcal{R})/64$, where $\text{vol}(\mathcal{R})$ is the volume of the region

$$\mathcal{R} = \{(a, b, c, d, e) \in [-1, 1]^5 \mid (\Delta > 0) \text{ and } ((H > 0) \text{ or } (Q < 0))\}.$$

We have been unable to compute this value exactly by analytic means. Instead, we have computed rigorous lower and upper bounds for $\text{vol}(\mathcal{R})$, and hence for $\rho(\infty)$, numerically.

Proposition 22 (= Proposition 2). *The probability $\rho(\infty)$ that a random real quartic with coefficients independently and uniformly distributed in $[-1, 1]$ is not negative definite satisfies*

$$0.873954 \leq \rho(\infty) \leq 0.874124.$$

The simplest way to estimate $\rho(\infty)$ non-rigorously is by Monte Carlo sampling. Taking 10^7 sampling points in $[-1, 1]^5$ and using (15) to test for being positive or negative definite gives the estimate $\rho(\infty) \approx 0.8741239$; using 10^8 sampling points gives $\rho(\infty) \approx 0.874112095$. This suggests that $\rho(\infty) \approx 0.87411$, and one expects this to be close to the actual value, but we cannot make any rigorous statement without additional work.

To obtain rigorous bounds as in Proposition 22 we have tried several methods, each implemented as a C program for efficiency, using only exact arithmetic to avoid any rounding errors. Here we only describe a basic recursive strategy and sketch one improvement, which takes advantage of homogeneity to reduce from a 5-dimensional problem to a 4-dimensional one.

The basic recursive method proceeds as follows. Identify points $(a, b, c, d, e) \in \mathbb{R}^5$ with quartics $f = f_{(a,b,c,d,e)} \in \mathbb{R}[x]$. Given two vectors $l = (l_0, l_1, l_2, l_3, l_4)$ and $u = (u_0, u_1, u_2, u_3, u_4)$ in \mathbb{R}^5 satisfying $l \leq u$ (meaning $l_i \leq u_i$ for all i), we consider the 5-dimensional box

$$B(l, u) = \{f = (a, b, c, d, e) \in \mathbb{R}^5 \mid l_0 \leq a \leq u_0, \dots, l_4 \leq e \leq u_4\} = \{f \in \mathbb{R}^5 \mid l \leq f \leq u\}.$$

Let $s = (l_0, u_1, l_2, u_3, l_4)$ and $t = (u_0, l_1, u_2, l_3, u_4)$: these are both corners of the box. Then for $x \geq 0$ we have

$$f_l(x) \leq f(x) \leq f_u(x) \quad \text{for all } f \in B(l, u),$$

while for $x < 0$ we have

$$f_s(x) \leq f(x) \leq f_t(x) \quad \text{for all } f \in B(l, u).$$

It follows that

- all $f \in B(l, u)$ are negative definite if and only if both f_u and f_t are negative definite;
- no $f \in B(l, u)$ are negative definite if either $f_l(x) \geq 0$ for some $x \geq 0$, or $f_s(x) \geq 0$ for some $x \leq 0$.

Note that the second condition is only sufficient, not necessary. To test it, we need to be able to test whether a quartic f takes only negative values on the positive or negative real half-lines. In our code we do this by using a function in the PARI/GP library [10] based on Descartes’ “rule of signs”, which gives the exact number of real roots in any interval, using only exact arithmetic for polynomials with rational coefficients.

Hence, by testing just four quartics, defined by four of the 32 corners of the box $B(l, u)$, we are able to determine whether one of three cases occurs: (i) all $f \in B(l, u)$ are negative definite; (ii) no $f \in B(l, u)$ are negative definite, or (iii) neither (undecided). In case (iii), we may then divide the box into two sub-boxes of half the volume by bisecting the longest edge (halving the maximum value of $u_i - l_i$) and recurse. We start with the box $B = [-1, 1]^5$ defined by $l = (-1, -1, -1, -1, -1)$ and $u = (1, 1, 1, 1, 1)$, and we also initialise to zero two variables v_1, v_2 , which will hold lower bounds for the total volume of sub-boxes containing all, respectively no, negative definite quartics. On testing each box, we either add its volume to one of these variables if case (i) or (ii) holds, or recurse. The stopping condition for the recursion is that we do not recurse when a sub-box is undecided and below a given volume threshold; equivalently, we bound the depth of recursion.

At the end of this process, we may conclude that the volume we require is at least v_1 and at most $32 - v_2$, and hence

$$\frac{1}{32}v_2 \leq \rho(\infty) \leq 1 - \frac{1}{32}v_1.$$

The length of this interval is $1/32$ times the total volume of the sub-boxes left undecided, which is $32 - v_1 - v_2$.

Note that all the boxes considered during this process have all their vertices (and hence their volume) rational, with a denominator which is a power of 2. Also, all the quartics we test for being positive or negative have integer coefficients scaled by a power of 2, so this test (using (15)) is also exact. Hence we may use exact arithmetic throughout, so that there are no rounding errors involved, and obtain exact rational values (with denominator a power of 2) for v_1 and v_2 , and hence for the bounds on $\rho(\infty)$. Here we express them as decimals to 6 decimal places for simplicity, rounding down the lower bound and rounding up the upper bound.

In our implementation, we use various obvious symmetries, such as reversing the coefficient sequence or changing x to $-x$, to reduce the running time. By increasing the depth of recursion, we may reduce the undecided volume and hence the length of the interval in which $\rho(\infty)$ certainly lies. In practice, however, we have found that the convergence of this process is extremely slow. In order to speed up the computation and hence obtain tighter bounds, we implemented the following improvement. The condition that the quartic with coefficients (a, b, c, d, e) is negative definite is obviously homogeneous with respect to scaling by positive constants. We subdivide the set of quartics according to which coefficient is greatest in absolute value, and whether it is positive or negative, giving ten subsets. (We may ignore the boundary regions where the maximum is attained at more than one coefficient, since these have measure zero.) Some of these subsets are trivial to deal with: for example, if either the leading coefficient or the constant coefficient are positive, then the quartic is certainly not negative definite. Each subset may be scaled so that the maximum coefficient is ± 1 , and then a 4-dimensional recursion

similar to the 5-dimensional one described above can be used to give lower and upper bounds on the volume of the negative definite forms in each subset. The final step is to add these and scale appropriately (effectively integrating with respect to the actual maximum coefficient, from 0 to 1) to obtain the lower and upper bounds.

The last scaling step introduces a factor of 5 in the denominator, since $\int_0^1 x^4 dx = 1/5$. In the table below we round the exact bounds computed to 6 decimal places.

Using this scaling method, we were able to increase the depth of recursion to 46. The following table shows the bounds obtained, and the computation time (on a single processor), for recursion depths up to 46 (the latter taking nearly 116 days):

Depth	Time	Lower bound	Upper bound
20	10s	0.863648	0.885568
25	2m 24s	0.869623	0.878944
30	39m 28s	0.872427	0.875876
35	516m 52s	0.873360	0.874896
40	6620m 35s	0.873767	0.874447
45	100988m 50s	0.873930	0.874157
46	167990m 53s	0.873954	0.874124

This justifies the bounds given in Proposition 22 for $\rho(\infty) \approx 0.874$ (to 3 significant figures).

We also used a Monte Carlo simulation to estimate $\rho'(\infty)$, the density of generalized binary quartics which are soluble over the reals. Taking 10^8 samples from $[-1, 1]^8$ we obtained the value of 0.873742745. We have not determined rigorous bounds for the actual value of $\rho'(\infty)$, but expect it to be a little smaller than $\rho(\infty)$.

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