K_0 AND THE DIMENSION FILTRATION FOR p-TORSION IWASAWA MODULES

KONSTANTIN ARDAKOV AND SIMON WADSLEY

ABSTRACT. Let G be a compact p-adic analytic group. We study K-theoretic questions related to the representation theory of the completed group algebra kG of G with coefficients in a finite field k of characteristic p. We show that if M is a finitely generated kG-module whose dimension is smaller than the dimension of the centralizer of any p-regular element of G, then the Euler characteristic of M is trivial. Writing $\mathcal{F}_i$ for the abelian category consisting of all finitely generated kG-modules of dimension at most i, we provide an upper bound for the rank of the natural map from the Grothendieck group of $\mathcal{F}_i$ to that of $\mathcal{F}_d$, where d denotes the dimension of G. We show that this upper bound is attained in some special cases, but is not attained in general.

1. INTRODUCTION

1.1. **Iwasawa algebras.** In this paper we study certain aspects of the representation theory of Iwasawa algebras. These are the completed group algebras

$$\Lambda_G := \varprojlim \mathbb{Z}_p[G/U],$$

where $\mathbb{Z}_p$ denotes the ring of $p-$adic integers, G is a compact $p-$adic analytic group, and the inverse limit is taken over the open normal subgroups U of G. Closely related is the epimorphic image Ω_G of Λ_G,

$$\Omega_G := \varprojlim \mathbb{F}_p[G/U],$$

where $\mathbb{F}_p$ is the field of p elements.

This paper is a continuation of our earlier work [5], in which we investigated the relationship between the notion of characteristic element of Λ_G-modules defined in [10] and the Euler characteristic of Ω_G-modules. We focus exclusively on the p-torsion Λ_G-modules, that is, those killed by a power of p. Because we are interested in K-theoretic questions, we only need to consider those modules actually killed by p, or equivalently, the Ω_G-modules.

In this introduction we assume that the group G has no elements of order p, although all of our results hold for arbitrary G with slightly more involved formulations.

1.2. **The dimension filtration.** The category $\mathcal{M}(\Omega_G)$ of finitely generated Ω_G-modules has a canonical dimension function $M \mapsto d(M)$ defined on it which provides a filtration of $\mathcal{M}(\Omega_G)$ by admissible subcategories $\mathcal{F}_i$ whose objects are those modules of dimension at most i. If d denotes the dimension of G then $\mathcal{F}_d$ is just $\mathcal{M}(\Omega_G)$ and $\mathcal{F}_{d-1}$ is the full subcategory of torsion modules.

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Perhaps the central result of [5] was to classify those p-adic analytic groups G for which the Euler characteristic of every finitely generated torsion Ω_G-module is trivial. Following the proof of [5, Theorem 8.2] we see that the Euler characteristic of every module $M \in \mathcal{F}_i$ is trivial if and only if the natural map from $K_0(\mathcal{F}_i)$ to $K_0(\mathcal{F}_d)$ is the zero map. This raises the following

Question. *When is the natural map* $\alpha_i : K_0(\mathcal{F}_i) \to K_0(\mathcal{F}_d)$ *the zero map?*

Let Δ^+ denote the largest finite normal subgroup of G. In [5] we answered the above question in the case $i = d-1$: the map α_{d-1} is zero precisely when G is p-*nilpotent*, that is, when G/Δ^+ is a pro-p group. There is a way of rephrasing this condition in terms of G_{reg}, the set of elements of G of finite order: G is p-nilpotent if and only if the centralizer of every element $g \in G_{\text{reg}}$ is an open subgroup of G.

1.3. **Serre's work.** This question was answered by Serre in [23] in the case $i = 0$, although he did not use our language. In this case $\mathcal{F}_i$ consists of modules that are finite dimensional as k-vector spaces. He produced a formula which relates the Euler characteristic $\chi(G, M)$ of a module $M \in \mathcal{F}_0$ with its Brauer character φ_M:

$$\log_p \chi(G, M) = \int_G \varphi_M(g) \det(1 - \mathrm{Ad}(g^{-1}))dg.$$

Here $\mathrm{Ad} : G \to \mathrm{GL}(\mathcal{L}(G))$ is the adjoint representation of G. As a consequence, Serre proved that the Euler characteristic of every module $M \in \mathcal{F}_0$ is trivial precisely when the centralizer of every element $g \in G_{\text{reg}}$ is infinite.

1.4. **Trivial Euler characteristics.** The results of Serre and [5] mentioned above suggest that there might be a connection between the answer to Question 1.2 and the dimensions of the centralizers of the elements of G_{reg}. Indeed, one might wonder whether α_i is zero if and only if the $\dim C_G(g) > i$ for all elements $g \in G_{\text{reg}}$. Whilst this latter statement turns out to be not quite right — see (12.3) — such a connection indeed exists.

Theorem A. *Let M be a finitely generated Ω_G-module such that $d(M) < \dim C_G(g)$ for all $g \in G_{\text{reg}}$. Then $\chi(G, M) = 1$.*

The proof is given in (8.5).

1.5. **A related question.** In the remainder of the paper, we address the following

Question. *What is the rank of the natural map* $\alpha_i : K_0(\mathcal{F}_i) \to K_0(\mathcal{F}_d)$*?*

The group G acts on G_{reg} by conjugation and this action commutes with the action of a certain Galois group $\mathcal{G}$ on G_{reg}, which essentially acts by raising elements to powers of p; see (3.1) for details. Let $S_i = \{g \in G_{\text{reg}} : \dim C_G(g) \leqslant i\}$ — this is a union of $G \times \mathcal{G}$-orbits. For example, $S_d = G_{\text{reg}}$ and $G_{\text{reg}} - S_{d-1}$ is the set of all elements of G which have finite order and lie in a finite conjugacy class; in fact, $G_{\text{reg}} - S_{d-1}$ is just the largest finite normal subgroup Δ^+ of G mentioned in (1.2).

Our next result provides an upper bound for the rank of α_i:

Theorem B. $\mathrm{rk}\, \alpha_i$ *is bounded above by the number of $G \times \mathcal{G}$-orbits on S_i.*

The proof is given in (9.3).

1.6. **Some special cases.** We next show that the rank of α_i attains the upper bound given in Theorem B in some special cases.

Theorem C. *The rank of α_i equals the number of $G \times \mathcal{G}$-orbits on S_i if either*
(a) $i = d$, or $i = d - 1$ or $i = 0$, or if
(b) G is virtually abelian.

This follows by combining Propositions 10.1,10.6,10.7 and Theorem 11.3.

Finally, in (12.3) we give an example to show that the upper bound of Theorem B is *not* always attained. Questions 1.2 and 1.5 remain open in general.

2. Generalities

Throughout this paper, k will denote a fixed finite field of characteristic p and order q. Modules will be right modules, unless explicitly stated otherwise. We will conform with the notation of [5], with the exception of [5, §12].

2.1. **Grothendieck groups.** Let $\mathcal{A}$ be a small abelian category. A full additive subcategory $\mathcal{B}$ of $\mathcal{A}$ is *admissible* if whenever $0 \to M' \to M \to M'' \to 0$ is a short exact sequence in $\mathcal{A}$ such that M and M'' belong to $\mathcal{B}$, then M' also belongs to $\mathcal{B}$ [17, 12.4.2].

The *Grothendieck group* $K_0(\mathcal{B})$ of $\mathcal{B}$ is the abelian group with generators $[M]$ where M runs over all the objects of $\mathcal{B}$ and relations $[M] = [M'] + [M'']$ for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$ [17, 12.4.3].

We will frequently be dealing with vector space versions of these groups. To simplify the notation later on, whenever F is a field we will write

$$FK_0(\mathcal{B}) := F \otimes_{\mathbb{Z}} K_0(\mathcal{B}).$$

If A is a ring, then $\mathcal{P}(A)$, the category of all finitely generated projective modules is an admissible subcategory of $\mathcal{M}(A)$, the category of all finitely generated A-modules. The *Grothendieck groups* of A are defined as follows:

- $K_0(A) := K_0(\mathcal{P}(A))$, and
- $\mathcal{G}_0(A) := K_0(\mathcal{M}(A))$.

We also set $FK_0(A) := FK_0(\mathcal{P}(A))$ and $F\mathcal{G}_0(A) := FK_0(\mathcal{M}(A))$.

2.2. **Homology and Euler characteristics.** Let $\mathcal{A}$ be an abelian category, let Γ be an abelian group and let ψ be an additive function from the objects of $\mathcal{A}$ to Γ. This means that for every object $A \in \mathcal{A}$ there exists an element $\psi(A) \in \Gamma$ such that

- $\psi(B) = \psi(A) + \psi(C)$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$.

In this context, if $C_* = \cdots \to C_n \to \cdots \to C_0 \to \cdots$ is a bounded complex in $\mathcal{A}$ (that is, $C_i = 0$ for sufficiently large $|i|$), we define the *Euler characteristic* of C_* (with respect to ψ) to be

$$\psi(C_*) := \sum_{i \in \mathbb{Z}} (-1)^i \psi(C_i) \in \Gamma.$$

Note that if we think of any object $A \in \mathcal{A}$ as a complex A_* concentrated in degree zero, then $\psi(A_*) = \psi(A)$. The following is well-known.

Lemma. *Let C_* be a bounded complex in $\mathcal{A}$. Then the Euler characteristic of C_* equals the Euler characteristic of the homology complex $H_*(C_*)$ with zero differentials:*

$$\psi(C_*) = \psi(H_*(C_*)).$$

Thus ψ extends to a well-defined function from the objects of the derived category $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ (if this exists) to Γ.

2.3. **Spectral sequences and Euler characteristics.** We will require a version of Lemma 2.2 for spectral sequences. Let $E = E^r_{ij}$ be a homology spectral sequence in $\mathcal{A}$ starting with E^a [27, §5.2]. We say that E is *totally bounded* if E^a_{ij} is zero for all sufficiently large $|i|$ or $|j|$. This is equivalent to insisting that E is bounded in the sense of [27, 5.2.5] and that there are only finitely many nonzero diagonals on each page. It is clear that E converges.

We define the *r^{th}-total complex* $\mathrm{Tot}(E^r)_*$ of E to be the complex in $\mathcal{A}$ whose n^{th} term is

$$\mathrm{Tot}(E^r)_n = \bigoplus_{i+j=n} E^r_{ij}$$

and whose n^{th} differential is $\bigoplus_{i\in\mathbb{Z}} d^r_{i,n-i}$. Because E is totally bounded, $\mathrm{Tot}(E^r)_n \in \mathcal{A}$ for all $r \geqslant a$ and $n \in \mathbb{Z}$ and each complex $\mathrm{Tot}(E^r)$ is bounded.

The next result is folklore — see for example [18, Example 6, p.15] for a cohomological version and the proof of [16, Theorem 9.8] for a similar formulation — but we give a proof for the convenience of the reader.

Proposition. *Let E be a totally bounded homology spectral sequence in $\mathcal{A}$ starting with E^a. Then $\psi(\mathrm{Tot}(E^a)_*) = \psi(\mathrm{Tot}(E^\infty)_*)$.*

Proof. From the definition of a spectral sequence we see that

$$H_n(\mathrm{Tot}(E^r)_*) = \mathrm{Tot}(E^{r+1})_n$$

for all $r \geqslant a$ and $n \in \mathbb{Z}$. Applying Lemma 2.2 repeatedly gives

$$\psi(\mathrm{Tot}(E^a)_*) = \psi(\mathrm{Tot}(E^{a+1})_*) = \cdots = \psi(\mathrm{Tot}(E^\infty)_*)$$

as required. □

2.4. **Completed group algebras.** Let G be a profinite group. We will write $\mathbf{U}_G$ (respectively, $\mathbf{U}_{G,p}$) for the set of all open normal (respectively, open normal pro-p) subgroups U of G.

Define the *completed group algebra* of G by the formula

$$kG := k[[G]] := \varprojlim_{U\in\mathbf{U}_G} k[G/U].$$

As each group algebra $k[G/U]$ is finite, this is a compact topological k-algebra which "controls" the continuous k-representations of G in the following sense. Whenever V is a compact topological k-vector space and $\rho : G \to \mathrm{Aut}_{\mathrm{cts}}(V)$ is a continuous representation of G then ρ extends to a unique continuous homomorphism of topological k-algebras $\rho : kG \to \mathrm{End}_{\mathrm{cts}}(V)$, and V becomes a compact topological (left) kG-module. Conversely, any compact topological kG-module V gives rise to a continuous k-representation of G.

When G is a compact p-adic analytic group, kG is sometimes called an *Iwasawa algebra* — we refer the reader to [4] for more details. We note that in this case kG

is always Noetherian and has finite global dimension when G has no elements of order p — facts which we will sometimes use without further mention.

3. Brauer characters

Throughout this section, G denotes a profinite group which is virtually pro-p.

3.1. **p-regular elements and the Galois action.** An element $g \in G$ is said to be *p-regular* if its order, in the profinite sense, is coprime to p. We will denote the set of all p-regular elements of G by G_{reg} — this is a union of conjugacy classes in G. Because G is virtually pro-p, any p-regular element has finite order; moreover, if G has no elements of order p then G_{reg} is just the set of elements of finite order in G, so this definition extends the one given in (1.2).

Let m denote the p'-part of $|G|$ in the profinite sense; equivalently, m is the index of a Sylow pro-p subgroup of G. As we are assuming that G is virtually pro-p, m is finite. Let $k' = k(\omega)$, where ω is a primitive m-th root of unity over k and let $\mathcal{G}_k$ be the Galois group $\mathrm{Gal}(k(\omega)/k)$. If $\sigma \in \mathcal{G}_k$, then $\sigma(\omega) = \omega^{t_\sigma}$ for some $t_\sigma \in (\mathbb{Z}/m\mathbb{Z})^\times$. This gives an injection $\sigma \mapsto t_\sigma$ of $\mathcal{G}_k$ into $(\mathbb{Z}/m\mathbb{Z})^\times$.

We can now define a left permutation action of $\mathcal{G}_k$ on G_{reg} by setting $\sigma.g = g^{t_\sigma}$; this makes sense because t_σ is coprime to the order of any element $g \in G_{\mathrm{reg}}$ by construction. This action commutes with any automorphism of G, so $\mathcal{G}_k$ permutes the p-regular conjugacy classes of G. These constructions give a continuous action $G \times \mathcal{G}_k$ on G_{reg}; note that $G \times \mathcal{G}_k$ is also virtually pro-p because $\mathcal{G}_k$ is a finite group.

3.2. **Locally constant functions.** Let X be a compact totally disconnected space. For any commutative ring A we let $C(X; A)$ denote the ring of all locally constant functions from X to A. If G acts on X continuously on the left then it acts on $C(X; A)$ on the right as follows:

$$(f.g)(x) = f(g.x) \quad \text{for all} \quad f \in C(X; A), g \in G, x \in X.$$

We will identify the subring of invariants $C(X; A)^G$ with $C(G\backslash X; A)$, where $G\backslash X$ denotes the set of G-orbits in X.

3.3. **Brauer characters.** Our treatment is closely follows Serre [23, §2.1, §3.3].

Fix a finite unramified extension K of $\mathbb{Q}_p$ with residue field k. Let $F = K(\tilde{\omega})$, where $\tilde{\omega}$ is a primitive m-th root of 1. Then the ring of integers of F is $\mathcal{O}' = \mathcal{O}[\tilde{\omega}]$ where $\mathcal{O}$ is the ring of integers of K. Reduction modulo p gives an isomorphism of the residue field of F with k' and we may assume that $\tilde{\omega}$ maps to ω under this isomorphism. For each m-th root of unity $\xi \in k'$ there is a unique m-th root of unity $\tilde{\xi} \in F$ such that $\tilde{\xi}$ maps to ξ modulo p. This gives us an isomorphism $\tilde{\cdot} : \langle \omega \rangle \to \langle \tilde{\omega} \rangle$ between the two cyclic groups.

Let $\mathcal{F}_0 = \mathcal{F}_0(G)$ be the abelian category of all topological kG-modules which are finite dimensional over k. If $A \in \mathcal{F}_0$ and $g \in G_{\mathrm{reg}}$, the eigenvalues of the action of g on A are powers of ω — say $\xi_1, \ldots, \xi_d$ (always counted with multiplicity so that $\dim A = d$). Define

$$\varphi_A(g) = \sum_{i=1}^{d} \widetilde{\xi_i} \in F.$$

The function $\varphi_A : G_{\mathrm{reg}} \to F$ is called the *Brauer character* of A. It has the following properties:

Lemma. *Let $A, B, C \in \mathcal{F}_0$.*

(i) φ_A *is a locally constant* $G \times \mathcal{G}_k$*-invariant* F*-valued function on* G_{reg}*:*

$$\varphi_A \in C(G_{\text{reg}}; F)^{G \times \mathcal{G}_k}.$$

(ii) $\varphi_B = \varphi_A + \varphi_C$ *whenever* $0 \to A \to B \to C \to 0$ *is a short exact sequence.*
(iii) $\varphi_{A \otimes_k B} = \varphi_A \cdot \varphi_B$*, where* G *acts diagonally on* $A \otimes_k B$*.*
(iv) Let $A' = A \otimes_k k'$ *be the* $k'H$*-module obtained from* A *by extension of scalars. Then* $\varphi_{A'} = \varphi_A$*.*

Proof. As G acts continuously on the finite dimensional vector space A, some $U \in \mathbf{U}_G$ acts trivially. It follows that for any $g \in G$, φ_A is constant on the open neighbourhood gU of g, so φ_A is a locally constant function. For the remaining assertions, we may assume that G is finite. In this case, the result is well-known — see for example [12, Volume I, §17A, §21B] or [22, §18]. $\square$

3.4. **Berman–Witt Theorem.** Lemma 3.3 shows that there is an F-linear map

$$\varphi : FK_0(\mathcal{F}_0) \to C(G_{\text{reg}}; F)^{G \times \mathcal{G}_k}$$

given by $\varphi(\lambda \otimes [A]) = \lambda \varphi_A$ for all $\lambda \in F$ and $A \in \mathcal{F}_0$ — see (2.1) for the notation.

Theorem. φ *is an isomorphism.*

This is a generalization of the well-known Berman–Witt Theorem [12, Volume I, Theorem 21.25] in the case when G is finite. When $k = \mathbb{F}_p$ and G is finite, a short proof can also be found in [23, §2.3]. Our proof is given below in (3.8).

Until the end of §3, we fix $U \in \mathbf{U}_{G,p}$ and write $\overline{G}$ for the quotient group G/U.

3.5. **Lemma.** Let $\pi_U : G \twoheadrightarrow \overline{G}$ be the natural surjection. Then

$$\pi_U(G_{\text{reg}}) = \overline{G}_{\text{reg}}.$$

Proof. It is enough to show that $\pi_U(G_{\text{reg}}) \supseteq \overline{G}_{\text{reg}}$. First suppose that G is a finite group so that U is a normal p-subgroup. For any $x \in G$ we can find unique commuting $s \in G_{\text{reg}}$ and $u \in G$ such that $x = su$ and u has order a power of p. Now if $xU \in \overline{G}$ is p-regular then raising x to an appropriate sufficiently large power of p doesn't change xU and has the effect of making $x = s$ p-regular.

Now suppose that G is arbitrary. Serre [23, §1.1] has observed that G_{reg} is a compact subset of G which can be identified with the inverse limit of the various $(G/W)_{\text{reg}}$ as W runs over $\mathbf{U}_G$. Because U is open in G, we may assume that all the W's are contained in U. Now the result follows from the first part. $\square$

3.6. **Proposition.** The map π_U induces a bijection

$$\pi_U : (G \times \mathcal{G}_k)\backslash G_{\text{reg}} \to (\overline{G} \times \mathcal{G}_k)\backslash \overline{G}_{\text{reg}}.$$

Proof. In view of Lemma 3.5, it is sufficient to prove that this map is injective. So let $x, y \in G_{\text{reg}}$ be such that xU and yU lie in the same $\overline{G} \times \mathcal{G}_k$-orbit. By replacing y by a $G \times \mathcal{G}_k$-conjugate, we may assume that actually $xU = yU$. As G is virtually pro-p, it will now be sufficient to show that xW and yW are conjugate in G/W for any $W \in \mathbf{U}_{G,p}$ contained in U. Without loss of generality, we can assume that G is finite and that $W = 1$.

Now as x is p-regular and U is a p-group, $\langle x \rangle \cap U = 1$. Similarly $\langle y \rangle \cap U = 1$, so $\langle x \rangle \cong \langle xU \rangle = \langle yU \rangle \cong \langle y \rangle$.

Consider the finite solvable group $H := \langle x\rangle U$. As $xU = yU$, y lies in H and U is the unique Sylow p-subgroup of H. It follows that $\langle x\rangle$ and $\langle y\rangle$ are *Hall p'-subgroups* of H and as such are conjugate in H [15]. Hence there exists $h \in H$ such that $x^h := h^{-1}xh = y^a$ for some $a \geqslant 1$. Now as H/U is abelian, $xU = x^hU$. Hence $yU = y^aU$, but $y^i \mapsto y^iU$ is an isomorphism so $y = y^a = x^h$ as required. □

We would like to thank Jan Saxl for providing this proof.

Corollary. *The $G \times \mathcal{G}_k$-orbits in G_{reg} are closed and open in G_{reg}.*

Proof. Because $G \times \mathcal{G}_k$ is a profinite group acting continuously on the Hausdorff space G_{reg}, the orbits are closed. But they are disjoint and finite in number by the Proposition, so they must also be open. □

3.7. **Dévissage.** Any finitely generated $k\overline{G}$-module is finite dimensional over k since $\overline{G}$ is finite, so we have a natural inclusion $\mathcal{M}(k\overline{G}) \subset \mathcal{F}_0$ of abelian categories. Let $\lambda_U : \mathcal{G}_0(k\overline{G}) \to K_0(\mathcal{F}_0)$ be the map induced on Grothendieck groups.

Lemma. *λ_U is an isomorphism.*

Proof. Let $w_U := (U-1)kG$ be the kernel of the natural map $kG \twoheadrightarrow k\overline{G}$ and let $A \in \mathcal{F}_0$. As in the proof of Lemma 3.3 we can find $W \in \mathbf{U}_G$ which acts trivially on A, which we may assume to be contained in U. Now A is a $k[G/W]$-module and the image of w_U in $k[G/W]$ is nilpotent because U/W is a normal p-subgroup of the finite group G/W [19, Lemma 3.1.6]. Thus $Aw_U^t = 0$ for some $t \geqslant 0$, so A has a finite filtration

$$0 = Aw_U^t \subset Aw_U^{t-1} \subset \cdots \subset Aw_U \subset A$$

where each factor is a $k\overline{G}$-module. Hence λ_U is an isomorphism by dévissage — see, for example, [17, Theorem 12.4.7]. □

3.8. **Proof of Theorem 3.4.** Consider the commutative diagram

$$\begin{tikzcd} F\mathcal{G}_0(k\overline{G}) \arrow[r, "\varphi"] \arrow[d, "\lambda_U"'] & C(\overline{G}_{\mathrm{reg}}; F)^{\overline{G}\times\mathcal{G}_k} \arrow[d, "\pi_U^*"] \\ FK_0(\mathcal{F}_0) \arrow[r, "\varphi"] & C(G_{\mathrm{reg}}; F)^{G\times\mathcal{G}_k}, \end{tikzcd}$$

where the map π_U^* is defined by the formula $\pi_U^*(f)(g) = f(gU)$. The top horizontal map φ is an isomorphism by the usual Berman–Witt Theorem [12, Volume I, Theorem 21.25], π_U^* is an isomorphism as a consequence of Proposition 3.6 and λ_U is an isomorphism by Lemma 3.7. Hence the bottom horizontal map φ is also an isomorphism, as required. □

4. Modules over Iwasawa algebras

4.1. **Compact p-adic analytic groups.** From now on, G will denote an arbitrary compact p-adic analytic group. We will write $d := \dim G$ for the dimension of G. By the celebrated result of Lazard [13, Corollary 8.34], G has an open normal *uniform* pro-p subgroup, so G is in particular virtually pro-p and we can apply the theory developed in §3. We fix such a subgroup N in what follows, and write $\overline{G}$ for the quotient group G/N.

4.2. **The base change map.** We begin by studying $\mathcal{G}_0(kG)$ in detail. First, a preliminary result.

Lemma. *The projective dimension of the left kG-module $k\overline{G}$ is at most d.*

Proof. It is well known that kN has global dimension $\dim N = d$ [9, Theorem 4.1]. Now $k\overline{G} \cong kG \otimes_{kN} k$ as a left kG-module and kG is a free right kN-module of finite rank. The result follows. ☐

Hence the right $k\overline{G}$-modules $\mathrm{Tor}_j^{kG}(M, k\overline{G})$ are zero for all $j > d$ and all $M \in \mathcal{M}(kG)$, so we can define an element $\theta_N[M] \in \mathcal{G}_0(k\overline{G})$ by the formula

$$\theta_N[M] := \sum_{j=0}^{\infty} (-1)^j [\mathrm{Tor}_j^{kG}(M, k\overline{G})].$$

The long exact sequence for Tor shows that $M \mapsto \theta_N[M]$ is an additive function on the objects of $\mathcal{M}(kG)$, so we have a base change map [8, p. 454] on $\mathcal{G}$-theory

$$\theta_N : \mathcal{G}_0(kG) \to \mathcal{G}_0(k\overline{G})$$

that will be one of our tools for studying $\mathcal{G}_0(kG)$.

4.3. **Graded Brauer characters.** Let $\mathrm{grmod}(kG)$ denote the category of all kG-modules M which admit a direct sum decomposition

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

into a kG-submodules such that M_n is finite dimensional for all $n \in \mathbb{Z}$ and zero for all sufficiently small n, thought of as a full subcategory of the abelian category of all $\mathbb{Z}$-graded kG-modules and graded maps of degree zero.

Definition. *The* graded Brauer character *of $M \in \mathrm{grmod}(kG)$ is the function*

$$\zeta_M : G_{\mathrm{reg}} \to F[[t, t^{-1}]]$$

defined by the formula

$$\zeta_M(g) = \sum_{n \in \mathbb{Z}} \varphi_{M_n}(g) t^n.$$

This definition extends the notion of Brauer character presented in (3.3) if we think of any finite dimensional kG-module M as a graded module concentrated degree zero.

Now let $M \in \mathcal{M}(kG)$ and let $w_N := (N-1)kG$ be the kernel of the natural map $kG \twoheadrightarrow k\overline{G}$. As kG is Noetherian, w_N^n is a finitely generated right ideal in kG for all $n \geqslant 0$, so the modules Mw_N^n / Mw_N^{n+1} are finite dimensional over k for all n. Hence the associated graded module

$$\mathrm{gr}\, M := \bigoplus_{n=0}^{\infty} \frac{Mw_N^n}{Mw_N^{n+1}}$$

lies in $\mathrm{grmod}(kG)$, and as such has a graded Brauer character $\zeta_{\mathrm{gr}\, M}$.

We will see in (5.4) that $\zeta_{\mathrm{gr}\, M}(g)$ is actually a rational function in t for each $g \in G_{\mathrm{reg}}$.

4.4. **The adjoint representation.** Recall [13, §4.3] that there is an *additive structure* $(N,+)$ on our fixed uniform subgroup N. In this way N becomes a free $\mathbb{Z}_p$-module of rank d so $\mathcal{L}(G) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} N$ is a $\mathbb{Q}_p$-vector space of dimension d. There is a way of turning $\mathcal{L}(G)$ into a Lie algebra over $\mathbb{Q}_p$ [13, §9.5], but we will not need this.

The conjugation action of G on N respects the additive structure on N and gives rise to the *adjoint representation*

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathcal{L}(G))$$

given by $\mathrm{Ad}(g)(n) = gng^{-1}$ for all $g \in G$ and $n \in N$. We define a function $\Psi : G \to \mathbb{Q}_p[t]$ by setting

$$\Psi(g) := \det(1 - \mathrm{Ad}(g^{-1})t)$$

for all $g \in G$. As $\Psi(g) \cdot \det \mathrm{Ad}(g)$ is the characteristic polynomial of $\mathrm{Ad}(g)$, we can think of $\Psi(g)$ as a polynomial in $F[t]$ of degree d.

4.5. **The key result.** By Theorem 3.4 and Lemma 3.7, the composite map $\varphi \circ \lambda_N : F\mathcal{G}_0(k\overline{G}) \to C(G_{\mathrm{reg}}; F)^{G \times \mathcal{G}_k}$ is an isomorphism. We therefore do not lose any information when studying θ_N by postcomposing it with this isomorphism. Our main technical result reads as follows:

Theorem. *Let ρ_N be the composite map*

$$\rho_N := \varphi \circ \lambda_N \circ \theta_N : F\mathcal{G}_0(kG) \to C(G_{\mathrm{reg}}; F)^{G \times \mathcal{G}_k}.$$

Then for any $g \in G_{\mathrm{reg}}$ and $M \in \mathcal{F}_d$, the number

$$\rho_N[M](g) = \sum_{j=0}^{d} (-1)^j \varphi_{\mathrm{Tor}_j^{kG}(M, k\overline{G})}(g) \in F$$

equals the value at $t = 1$ of the rational function

$$\zeta_{\mathrm{gr}\, M}(g) \cdot \Psi(g) \in F(t).$$

We now begin preparing for the proof, which is given in §7.

5. Graded modules for $\mathrm{Sym}(V)\#H$

5.1. **Notation.** Let V be a finite dimensional k-vector space and let H be a finite group acting on V by k-linear automorphisms on the right. We will write v^h for the image of $v \in V$ under the action of $h \in H$. This action extends naturally to an action of H on the symmetric algebra $\mathrm{Sym}(V)$ by k-algebra automorphisms. Let

$$R := \mathrm{Sym}(V)\#H$$

denote the skew group ring [17, 1.5.4]: by definition, R is a free right $\mathrm{Sym}(V)$-module with basis H, with multiplication given by the formula

$$(hr)(gs) = (hg)(r^g s)$$

for all $g, h \in H$ and $r, s \in \mathrm{Sym}(V)$. Thus R is isomorphic to $kH \otimes_k \mathrm{Sym}(V)$ as a k-vector space and setting $R_n := kH \otimes_k \mathrm{Sym}^n V$ turns $R = \bigoplus_{n=0}^{\infty} R_n$ into a graded k-algebra.

5.2. **Dimensions.** If S is a positively graded k-algebra, let $\mathcal{M}_{\mathrm{gr}}(S)$ denote the category of all finitely generated $\mathbb{Z}$-graded right S-modules and graded maps of degree zero. Since $R = \mathrm{Sym}(V)\#H$ is a finitely generated $\mathrm{Sym}(V)$-module, we have an inclusion $\mathcal{M}_{\mathrm{gr}}(R) \subset \mathcal{M}_{\mathrm{gr}}(\mathrm{Sym}(V))$ of abelian categories.

Following [7, §11] we define the *dimension* $d(M)$ of a module $M \in \mathcal{M}_{\mathrm{gr}}(\mathrm{Sym}(V))$ to be the order of the pole of the *Poincaré series* $P_M(t)$ of M at $t = 1$, where

$$P_M(t) = \sum_{n\in\mathbb{Z}} (\dim_k M_n)t^n \in \mathbb{Z}[[t, t^{-1}]].$$

In fact [7, Theorem 11.1], there exists a polynomial $u(t) \in \mathbb{Z}[t, t^{-1}]$ such that $u(1) \neq 0$ and

$$P_M(t) = \frac{u(t)}{(1-t)^{d(M)}}.$$

Note that we have to allow Laurent polynomials and power series because our modules are $\mathbb{Z}$-graded. It is well-known that $d(M)$ also equals the Krull dimension $\mathcal{K}(M)$ of M in the sense of [17, §6.2] and the Gelfand-Kirillov dimension $\mathrm{GK}(M)$ of M [17, §8].

5.3. **Properties of graded Brauer characters.** Recall the definition of graded Brauer characters given in (4.3). As the finite group H is in particular compact p-adic analytic, we may speak of the category $\mathrm{grmod}(kH)$. The following result is a straightforward application of Lemma 3.3 and shows that graded Brauer characters behave well with respect to basic algebraic constructions.

Lemma. *Let $A, B, C \in \mathrm{grmod}(kH)$.*

(i) If $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathrm{grmod}(kH)$ then

$$\zeta_B = \zeta_A + \zeta_C.$$

(ii) Define $A \otimes_k B \in \mathrm{grmod}(kH)$ by letting H act diagonally and giving $A \otimes_k B$ the tensor product gradation

$$(A \otimes_k B)_n = \bigoplus_{i+j=n} A_i \otimes_k B_j.$$

Then $\zeta_{A\otimes_k B} = \zeta_A \cdot \zeta_B$.

(iii) For each $m \in \mathbb{Z}$ define the shifted module *$A[m] \in \mathrm{grmod}(kH)$ by setting $A[m]_n = A_{m+n}$ for all $n \in \mathbb{Z}$. Then*

$$t^m \zeta_{A[m]} = \zeta_A.$$

(iv) If k' is the finite field extension of k defined in (3.1) then $A' := A \otimes_k k'$ lies in $\mathrm{grmod}(k'H)$ and

$$\zeta_{A'} = \zeta_A.$$

5.4. **Rationality of graded Brauer characters.** By picking a homogeneous generating set we see that any $M \in \mathcal{M}_{\mathrm{gr}}(R)$ actually lies in $\mathrm{grmod}(kH)$ and as such has a graded Brauer character ζ_M. It is easy to see that $\zeta_M(1)$ is just the Poincaré series P_M and is as such a rational function (5.2). The following result shows that $\zeta_M(h)$ is also a rational function for any $h \in H_{\mathrm{reg}}$. Let m denote the p'-part of $|H|$.

Theorem. *For any $M \in \mathcal{M}_{\rm gr}(R)$ and any $h \in H_{\rm reg}$ there exists a Laurent polynomial $u_h(t) \in F[t, t^{-1}]$ such that*

$$\zeta_M(h) = \frac{u_h(t)}{(1-t^m)^{d(M)}}.$$

Proof. Without loss of generality we may assume that $H = \langle h \rangle$. Moreover, since extension of scalars doesn't change the graded Brauer character by Lemma 5.3(iv), we may also assume that $k = k'$. Hence h acts diagonalizably on V.

We now prove the result by induction on $d(M) + \dim_k V$. Suppose first of all that $d(M) = 0$. It follows that $M_n = 0$ for sufficiently large $|n|$ and so $\zeta_M(h)$ is a Laurent polynomial as required. Since $d(M) \leqslant \dim_k V$ this also deals with the case when $\dim_k V = 0$, so we assume $\dim_k V > 0$ and $d(M) > 0$.

Choose an h-eigenvector $v \in V$ with eigenvalue λ. As $\lambda^m = 1$ we see that $h^{-1}v^m h = (v^h)^m = (\lambda v)^m = v^m$, so $z := v^m$ is central in R.

Consider the graded submodule $T = \{\alpha \in M : \alpha.z^r = 0 \text{ for some } r \geqslant 0\}$ of M. Because R is Noetherian and M is finitely generated, T is finitely generated as an R-module and as such is killed by some power of the central element z.

Choose an h-invariant complement W for kv in V so that $\mathrm{Sym}(V) \cong \mathrm{Sym}(W)[v]$. It is now easy to see that T is a finitely generated over $\mathrm{Sym}(W)$ and in fact $T \in \mathcal{M}_{\rm gr}(\mathrm{Sym}(W)\#H)$. Note that the dimension of T viewed as a $\mathrm{Sym}(V)$-module is the same as the dimension of T viewed as an $\mathrm{Sym}(W)$-module as both depend only on the Poincaré series of T.

Since $d(T) \leqslant d(M)$ and $\dim_k W < \dim_k V$, we know by induction that

$$\zeta_T(h) \cdot (1-t^m)^{d(M)} \in F[t, t^{-1}].$$

By Lemma 5.3(i), $\zeta_M = \zeta_T + \zeta_{M/T}$, so it now suffices to prove the result for the graded module M/T. So, by replacing M by M/T we may assume M is z-torsion-free.

Now as z is a central element of R of degree m, multiplication by z induces a short exact sequence of graded R-modules

$$0 \to M \xrightarrow{z} M[m] \to L[m] \to 0,$$

where $L := M/Mz$. It follows from Lemma 5.3 that

$$t^m \zeta_M(h) = \zeta_M(h) + t^m \zeta_L(h).$$

But L is a finitely generated graded $\mathrm{Sym}(W)\#H$-module and $d(L) \leqslant d(M) - 1$ by [7, Proposition 11.3], so by induction

$$\zeta_L(h) \cdot (1-t^m)^{d(M)-1} \in F[t, t^{-1}].$$

The result follows. ◻

Inspecting the proof shows that the Theorem is still valid with m replaced by the order of h.

6. Koszul resolutions

We continue with the notation established in §5. Let $d = \dim_k V$.

6.1. **The Koszul complex for graded R-modules.** With any finitely generated graded right R-module M we associate the *Koszul complex*

$$\mathbb{K}(M)_* := 0 \to M \otimes_k (\Lambda^d V)[-d] \xrightarrow{\phi_d} \cdots \xrightarrow{\phi_2} M \otimes_k (\Lambda^1 V)[-1] \xrightarrow{\phi_1} M \to 0$$

whose maps $\phi_j : M \otimes_k (\Lambda^j V)[-j] \to M \otimes_k (\Lambda^{j-1} V)[-j+1]$ are given by the usual formula

$$\phi_j(m \otimes v_1 \wedge \cdots \wedge v_j) = \sum_{i=1}^{j} (-1)^{i+1} m.v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_j,$$

for any $m \in M$ and $v_1, \ldots, v_j \in V$. The square brackets indicate that we are thinking of $\Lambda^j V[-j]$ as a graded k-vector space concentrated in degree j.

Lemma. *$\mathbb{K}(M)_*$ is a complex inside the abelian category* $\operatorname{grmod}(kH)$.

Proof. It is well-known (and easily verified) that $\mathbb{K}(M)_*$ is a complex of k-vector spaces. Letting H act diagonally on $M \otimes_k \Lambda^j V$ makes each ϕ_j into a map of right kH-modules as $h \cdot v_i^h = v_i \cdot h$ inside the ring R. Because of the shifts, each ϕ_j is also a map of graded modules of degree zero, as required. □

6.2. **Homology of $\mathbb{K}(M)_*$.** The projection map $\epsilon : R \twoheadrightarrow R_0$ with kernel $\bigoplus_{n=1}^{\infty} R_n$ is an algebra homomorphism from R to kH which gives kH an R-R-bimodule structure. The Koszul complex is useful because it allows us to compute $\operatorname{Tor}_j^R(M, kH)$ for any $M \in \mathcal{M}_{\mathrm{gr}}(R)$.

Proposition. *(i) $\mathbb{K}(R)_*$ is a complex of R-kH-bimodules which is exact everywhere except in degree zero.*
(ii) $H_0(\mathbb{K}(R)_*) \cong kH$ as R-kH-bimodules.
(iii) For all $M \in \mathcal{M}_{\mathrm{gr}}(R)$ and all $j \geqslant 0$ there is an isomorphism of kH-modules

$$H_j(\mathbb{K}(M)_*) \cong \operatorname{Tor}_j^R(M, kH).$$

Proof. (i) Consider first the special case when $H = 1$. Thinking of V as an abelian k-Lie algebra, $R = \operatorname{Sym}(V)$ becomes the enveloping algebra of V and

$$\mathbb{K}(R)_* = \operatorname{Sym}(V) \otimes_k \Lambda^* V$$

is just the Chevalley-Eilenberg complex of left $\operatorname{Sym}(V)$-modules [27, §7.7]. By [27, Theorem 7.7.2] $\mathbb{K}(R)_*$ is exact everywhere except in degree zero.

Returning to the general case, there is a natural isomorphism of complexes of k-vector spaces

$$R \otimes_{\operatorname{Sym}(V)} (\operatorname{Sym}(V) \otimes_k \Lambda^* V) \xrightarrow{\cong} \mathbb{K}(R)_*.$$

Because R is free of finite rank as a right $\operatorname{Sym}(V)$-module, it follows that $\mathbb{K}(R)_*$ is also exact everywhere except in degree zero. We use this isomorphism to give $\mathbb{K}(R)_*$ the structure of a complex of left R-modules.

On the other hand, $\mathbb{K}(R)_*$ is a complex of right kH-modules by Lemma 6.1. It can be checked that the two structures are compatible, so $\mathbb{K}(R)_*$ is a complex of R-kH-bimodules. Explicitly, the bimodule structure is given by the following formula:

$$s.(r \otimes v_1 \wedge \cdots \wedge v_j).h = srh \otimes v_1^h \wedge \cdots \wedge v_j^h$$

for all $s, r \in R$, $v_1, \ldots, v_j \in V$ and $h \in H$.

(ii) The map $\epsilon : R \to kH$ which gives kH its R-kH-bimodule structure is the cokernel of the first map $\phi_1 : R \otimes_k V \to R$ of the complex $\mathbb{K}(R)_*$. Hence

$$H_0(\mathbb{K}(R)_*) = R/\operatorname{Im}\phi_1 \cong kH$$

as R-kH-bimodules.

(iii) Each term in $\mathbb{K}(R)_*$ is free of finite rank as a left R-module, so by parts (i) and (ii) $\mathbb{K}(R)_*$ is a free resolution of the left R-module kH. We can therefore use it to compute $\operatorname{Tor}_j^R(M, kH)$. Finally, the natural map $M \otimes_R \mathbb{K}(R)_* \to \mathbb{K}(M)_*$ is actually an isomorphism of complexes of right kH-modules, so

$$H_j(\mathbb{K}(M)_*) \cong H_j(M \otimes_R \mathbb{K}(R)_*) \cong \operatorname{Tor}_j^R(M, kH)$$

as required. □

6.3. **A formula involving graded Brauer characters.** Let $M \in \mathcal{M}_{\mathrm{gr}}(R)$. By Lemma 6.1 and Proposition 6.2(iii), $\operatorname{Tor}_j^R(M, kH)$ is an object in $\operatorname{grmod}(kH)$ and as such has a graded Brauer character $\zeta_{\operatorname{Tor}_j^R(M,kH)}$. On the other hand, since R is Noetherian M has a projective resolution consisting of finitely generated R-modules. Computing $\operatorname{Tor}_j^R(M, kH)$ using this resolution shows that these kH-modules are finite dimensional over kH, so $\zeta_{\operatorname{Tor}_j^R(M,kH)}(h)$ is actually a Laurent polynomial in $F[t, t^{-1}]$ for all $h \in H_{\mathrm{reg}}$.

Proposition. *For any $h \in H_{\mathrm{reg}}$ and $M \in \mathcal{M}_{\mathrm{gr}}(R)$,*

$$\sum_{j=0}^{d} (-1)^j \zeta_{\operatorname{Tor}_j^R(M,kH)}(h) = \zeta_M(h) \cdot \sum_{j=0}^{d} (-t)^j \varphi_{\Lambda^j V}(h).$$

Proof. In the notation of (2.2), Lemma 5.3(i) says that $\psi_h : A \mapsto \zeta_A(h)$ is an additive function from the abelian category $\operatorname{grmod}(kH)$ to $F[[t, t^{-1}]]$ thought of as an abelian group. Applying Proposition 6.2(iii) and Lemma 2.2 we obtain

$$\sum_{j=0}^{d} (-1)^j \zeta_{\operatorname{Tor}_j^R(M,kH)}(h) = \psi_h(\operatorname{Tor}_*^R(M, kH)) = \psi_h(\mathbb{K}(M)_*).$$

Now $\mathbb{K}(M)_j = M \otimes_k \Lambda^j V[-j]$ so the result follows from Lemma 5.3(ii) and (iii). □

7. Proof of Theorem 4.5

7.1. **Another expression for $\Psi(g)$.** Set $V := k \otimes_{\mathbb{F}_p} (N/N^p)$. Because N is uniform, N/N^p is an $\mathbb{F}_p$-vector space of dimension d and the right conjugation action of G on N induces a right action of G on N/N^p by linear automorphisms. In this way V becomes a right kG-module with $\dim_k V = d$.

Lemma. *For any $g \in G_{\mathrm{reg}}$, $\Psi(g) = \sum_{j=0}^{d} (-t)^j \varphi_{\Lambda^j V}(g)$.*

Proof. Let $\beta = \operatorname{Ad}(g^{-1}) \in \operatorname{GL}(\mathcal{L}(G))$ and let $j \geqslant 0$. Recall from (4.4) that $\Psi(g) = \det(1 - t\beta)$. Now, $\Lambda^j N$ is a $\Lambda^j\beta$-stable lattice inside $\Lambda^j \mathcal{L}(G)$ whose reducion modulo p is isomorphic to $\Lambda^j(N/N^p)$. Moreover, the endomorphism of $\Lambda^j V \cong \Lambda^j(N/N^p) \otimes_{\mathbb{F}_p} k$ induced by $\Lambda^j\beta$ is equal to the right action of g on $\Lambda^j V$. The result now follows from the well-known formula

$$\det(1 - t\beta) = \sum_{j=0}^{d} (-t)^j \operatorname{Tr}(\Lambda^j \beta).$$

See, for example, [23, p.487] or [14, p. 77, (6.2)]. $\square$

Corollary. *The restriction of Ψ to G_{reg} is locally constant.*

Proof. Note that V is a $k\overline{G}$-module because $[N,N] \leqslant N^p$ as N is uniform, so the Brauer characters $\varphi_{\Lambda^j V}$ are constant on the cosets of N. Now apply the Lemma. $\square$

7.2. **The associated graded ring.** We now make the connection with the theory developed on the preceeding two sections. Let $H := \overline{G} = G/N$; then V is a kH-module and we may form the skew group ring $\mathrm{Sym}(V)\#H$. Recall from (4.3) that w_N denotes the augmentation ideal $(N-1)kG$.

Lemma. *The associated graded ring of kG with respect to the w_N-adic filtration is isomorphic to $R = \mathrm{Sym}(V)\#H$.*

Proof. When $N = G$ this is follows from [13, Theorem 7.24]; see also [1, Lemma 3.11]. Letting $\mathfrak{m}$ denote the augmentation ideal of kN we see that

$$\mathrm{gr}_{w_N} kG \cong kG \otimes_{kN} \mathrm{gr}_{\mathfrak{m}} kN \cong kH \otimes_k \mathrm{Sym}(V)$$

as a right $\mathrm{Sym}(V)$-module, because kN acts trivially on its graded ring $\mathrm{Sym}(V)$. Moreover, the zeroth graded part of $\mathrm{gr}_{w_N} kG$ is isomorphic to kH as a k-algebra, so H embeds into the group of units of $\mathrm{gr}_{w_N} kG$. It is now easy to verify that the multiplication works as needed. $\square$

We will identify R with $\mathrm{gr}_{w_N} kG$ in what follows.

7.3. **A spectral sequence.** The last step in the proof involves relating Tor groups over kG with Tor groups over the associated graded ring R. There is a standard spectral sequence originally due to Serre which does the job.

Proposition. *For any finitely generated kG-module M there exists a homological spectral sequence in $\mathcal{M}(kH)$*

$$E^1_{ij} = \mathrm{Tor}^R_{i+j}(\mathrm{gr}\, M, kH)_{\mathrm{degree}\,-i} \Longrightarrow \mathrm{Tor}^{kG}_{i+j}(M, kH).$$

Proof. As in (4.3), we only consider the *deduced filtration* on M, given by $M^n = Mw_N^n$ for $n \geqslant 0$. As M is finitely generated over kG, the associated graded module $\mathrm{gr}\, M$ is finitely generated over the Noetherian ring $\mathrm{gr}\, kG \cong R$. By the proof of [3, Proposition 3.4], the w_N-adic filtration on kG is complete.

We claim that M is complete with respect to the deduced filtration. Because kG is Noetherian we can find an exact sequence $(kG)^a \stackrel{\alpha}{\to} (kG)^b \stackrel{\beta}{\to} M \to 0$ in $\mathcal{M}(kG)$. Giving all the modules involved deduced filtrations, $(kG)^a$ and $(kG)^b$ are compact and the maps α, β are continuous. Hence $\mathrm{Im}\,\alpha$ and M are compact, so $\mathrm{Im}\,\alpha$ is closed in $(kG)^b$ and M is complete as claimed.

We can now apply [25, Proposition 8.1] to the modules $A = M$ and $B = kH$ over the complete filtered k-algebra kG, where we equip B with the trivial filtration $B^0 = B$ and $B^1 = 0$. This gives us the required spectral sequence of k-vector spaces. Examining the construction shows that it is actually a spectral sequence in $\mathcal{M}(kH)$. $\square$

7.4. **Proof of Theorem 4.5.** The spectral sequence E of Proposition 7.3 gives us suitable filtrations on the kH-modules $\operatorname{Tor}_n^{kG}(M, kH)$. In the notation of (2.3) we can rewrite the information we gain from the spectral sequence as follows:

$$\operatorname{Tot}(E^1)_n = \operatorname{Tor}_n^R(\operatorname{gr} M, kH) \quad \text{and} \quad \operatorname{Tot}(E^\infty)_n = \operatorname{gr} \operatorname{Tor}_n^{kG}(M, kH)$$

for each $n \in \mathbb{Z}$. We have already observed in (6.3) that $\operatorname{Tor}_n^R(\operatorname{gr} M, kH)$ is a finite dimensional kH-module, which is moreover zero whenever $n > d$ by Proposition 6.2(iii). Thus E is totally bounded. Now, Lemma 3.3(ii) shows that $A \mapsto \varphi_A(g)$ is an additive F-valued function on the objects of $\mathcal{M}(kH)$. By Proposition 2.3,

$$\rho_N[M](g) = \sum_{j=0}^{d} (-1)^j \varphi_{\operatorname{Tor}_j^{kG}(M,kH)}(g) = \sum_{j=0}^{d} (-1)^j \varphi_{\operatorname{Tor}_j^R(\operatorname{gr} M,kH)}(g).$$

We may now apply Proposition 6.3 and Lemma 7.1 to obtain

$$\sum_{j=0}^{d} (-1)^j \varphi_{\operatorname{Tor}_j^R(\operatorname{gr} M,kH)}(g) = \sum_{j=0}^{d} (-1)^j \zeta_{\operatorname{Tor}_j^R(\operatorname{gr} M,kH)}(g)|_{t=1} = (\zeta_{\operatorname{gr} M}(g) \cdot \Psi(g))\,|_{t=1},$$

as required. $\square$

8. Euler characteristics

8.1. **Twisted μ-invariants.** Because we only deal with Iwasawa modules which are killed by p in this paper, it is easy to see that the definition of the *Euler characteristic* [5, §1.5] of a finitely generated kG-module M of finite projective dimension can be given as follows:

$$\chi(G, M) := \prod_{j \geqslant 0} |\operatorname{Tor}_j^{kG}(M, k)|^{(-1)^j}.$$

Let $\{V_1, \ldots, V_s\}$ be a complete list of representatives for the isomorphism classes of simple kG-modules. The *i-th twisted μ-invariant* of M [5, §1.5] for $i = 1, \ldots, s$ is defined by the formula

$$\mu_i(M) = \frac{\log_q \chi(G, M \otimes_k V_i^*)}{\dim_k \operatorname{End}_{kG}(V_i)},$$

where V_i^* is the dual module to V_i. We assume that V_1 is the trivial kG-module k, so that

$$\mu_1(M) = \log_q \chi(G, M).$$

We proved in [5] that these twisted μ-invariants completely determine the characteristic element of M viewed as an $\mathcal{O}G$-module [5, Theorem 1.5]. This adds to the motivation of the problem of computing the Euler characteristic $\chi(G, M)$.

8.2. **The base change map.** Before we can proceed, we need to record some information about the base change map $\theta_N : \mathcal{G}_0(kG) \to \mathcal{G}_0(k\overline{G})$.

We say that a map f between two abelian groups is an $\mathbb{Q}$-*isomorphism* if it becomes an isomorphism after tensoring with $\mathbb{Q}$. Equivalently, f has torsion kernel and cokernel.

Proposition. *There is a commutative diagram of Grothendieck groups*

$$\begin{tikzcd} K_0(kG) \arrow[r, "\pi_N"] \arrow[d, "c"'] & K_0(k\overline{G}) \arrow[d, "c_N"] \\ \mathcal{G}_0(kG) \arrow[r, "\theta_N"'] & \mathcal{G}_0(k\overline{G}). \end{tikzcd}$$

The map π_N is an isomorphism and the other maps are $\mathbb{Q}$-isomorphisms.

Proof. The vertical maps c and c_N are called *Cartan maps* and are defined by inclusions between admissible subcategories. The base change map π_N is defined by $\pi_N[P] = [P \otimes_{kG} k\overline{G}]$ for all $P \in \mathcal{P}(kG)$, and we have already discussed θ_N in (4.2). This well-known diagram appears in [8, p. 454] and expresses the fact that the Cartan maps form a natural transformation from K-theory to $\mathcal{G}$-theory.

Now, kG is a *crossed product* of kN with the finite group $\overline{G}$:

$$kG \cong kN * \overline{G},$$

see, for example [4, §2.3] for more details. Because kN is a Noetherian k-algebra of finite global dimension, the map c is a $\mathbb{Q}$-isomorphism by a general result on the K-theory of crossed products [6]. Considering the case when G is finite shows that c_N is a $\mathbb{Q}$-isomorphism as well – this also follows from a well-known result of Brauer: see, for example, [22, Corollary 1 to Theorem 35].

Finally π_N is an isomorphism because kG is a complete semilocal ring – see [5, Lemma 2.6 and Proposition 3.3(a)]. It follows that θ_N must also be a $\mathbb{Q}$-isomorphism, as required. □

Recalling Theorem 3.4 and Lemma 3.7, we obtain

Corollary. *The map ρ_N featuring in Theorem 4.5 is a $\mathbb{Q}$-isomorphism.*

If the group G has no elements of order p, then kG has finite global dimension and the Cartan map c is actually an isomorphism by Quillen's Resolution Theorem [17, Theorem 12.4.8]. In this case, therefore, we do not have to rely on [6].

8.3. **Computing Euler characteristics using θ_N.** Let $P \in \mathcal{P}(k\overline{G})$ and let $V \in \mathcal{M}(k\overline{G})$. The rule

$$(P, V) \mapsto \dim_k \operatorname{Hom}_{k\overline{G}}(P, V)$$

defines an additive function from $\mathcal{P}(k\overline{G}) \times \mathcal{M}(k\overline{G})$ to $\mathbb{Z}$ and hence a pairing

$$\langle -, - \rangle_N : K_0(k\overline{G}) \times \mathcal{G}_0(k\overline{G}) \to \mathbb{Z}.$$

This pairing appears in [22, p. 121]. Now, by extending scalars, we can define a bilinear form

$$\langle -, - \rangle_N : \mathbb{Q}K_0(k\overline{G}) \times \mathbb{Q}\mathcal{G}_0(k\overline{G}) \to \mathbb{Q}$$

which is in fact non-degenerate. We saw in Proposition 8.2 that the Cartan map $c_N : \mathbb{Q}K_0(k\overline{G}) \to \mathbb{Q}\mathcal{G}_0(k\overline{G})$ is an isomorphism. This allows us to define an non-degenerate bilinear form

$$(-, -)_N : \mathbb{Q}\mathcal{G}_0(k\overline{G}) \times \mathbb{Q}\mathcal{G}_0(k\overline{G}) \to \mathbb{Q}$$

by setting $(x, y)_N = \langle c_N^{-1}(x), y \rangle_N$ for $x, y \in \mathbb{Q}\mathcal{G}_0(k\overline{G})$.

Proposition. *For any finitely generated kG-module M of finite projective dimension, the Euler characteristic of M can be computed as follows:*

$$\log_q \chi(G, M) = (\theta_N[M], [k])_N.$$

Proof. Suppose first that M is projective. The usual adjunction between $\otimes$ and Hom gives isomorphisms

$$\operatorname{Hom}_k(M \otimes_{kG} k, k) \cong \operatorname{Hom}_{kG}(M, k) \cong \operatorname{Hom}_{k\overline{G}}(M \otimes_{kG} k\overline{G}, k).$$

As these Hom spaces are finite dimensional over k, we obtain

$$\dim_k(M \otimes_{kG} k) = \langle [M \otimes_{kG} k\overline{G}], [k] \rangle_N.$$

Now because M is projective, $\operatorname{Tor}_i^{kG}(M, k) = 0 = \operatorname{Tor}_i^{kG}(M, k\overline{G})$ for $i > 0$, and $\theta_N[M] = c_N(\pi_N[M])$ by Proposition 8.2. Hence

$$\log_q \chi(G, M) = \dim_k(M \otimes_{kG} k) = \langle \pi_N[M], [k] \rangle_N = (\theta_N[M], [k])_N$$

as required. Returning to the general case, if $0 \to P_n \to \cdots \to P_0 \to M \to 0$ is a resolution of M in $\mathcal{P}(kG)$ then Lemma 2.2 gives

$$\theta_N[M] = \sum_{i=0}^{n} (-1)^i \theta_N[P_i] \quad \text{and} \quad \log_q \chi(G, M) = \sum_{i=0}^{n} (-1)^i \log_q \chi(G, P_i).$$

The result now follows from the first part. $\square$

8.4. **Euler characteristics for modules of infinite projective dimension.** The definition of $\chi(G, M)$ given in (8.1) only makes sense when the module M has finite projective dimension. However, the expression $(\theta_N[M], [k])_N$ makes sense for arbitrary $M \in \mathcal{M}(kG)$. We include the subscripts, because *a priori* this depends on the choice of the open normal uniform subgroup N.

Lemma. *Let M be a finitely generated kG-module. Then*

(i) $\psi_N : M \mapsto (\theta_N[M], [k])_N$ is an additive function on the objects of $\mathcal{M}(kG)$,
(ii) $\psi_N(M)$ does not depend on the choice of N.

Proof. It suffices to prove part (ii). Now if $U \in \mathbf{U}_{G,p}$ is another uniform subgroup of G, then $\psi_U(M) = \log_q \chi(G, M) = \psi_N(M)$ whenever M is projective, by Proposition 8.3. Rephrasing this in the language of Grothendieck groups,

$$\mathbb{Q}K_0(kG) \xrightarrow{c} \mathbb{Q}\mathcal{G}_0(kG) \xrightarrow{\psi_U - \psi_N} \mathbb{Q}$$

is a complex of $\mathbb{Q}$-vector spaces. By Proposition 8.2, the Cartan map c is an isomorphism, so $\psi_U(M) = \psi_N(M)$ for *any* finitely generated kG-module M, as required. $\square$

In view of this result, we propose to extend the definition given in (8.1) as follows.

Definition. *The* Euler characteristic *of a finitely generated kG-module M is defined to be*

$$\chi(G, M) := q^{(\theta_N[M], [k])_N} \in q^{\mathbb{Q}}$$

for any choice of open normal uniform subgroup N of G.

8.5. **Trivial Euler characteristics.** First, a preliminary

Lemma. *For any $g \in G_{\mathrm{reg}}$, the multiplicity of* 1 *as a root of the polynomial* $\Psi(g)$ *equals* $\dim C_G(g)$.

Proof. As $\Psi(g) \cdot \det \mathrm{Ad}(g)$ is the characteristic polynomial of $\mathrm{Ad}(g)$, the first number equals the dimension of the space $C := \{x \in \mathcal{L}(G) : \mathrm{Ad}(g)(x) = x\}$. The definition of $\mathrm{Ad}(g)$ shows that $C \cap N$ is just the centralizer $C_N(g)$ of g in N. Because $C_N(g) = C_G(g) \cap N$ is open in $C_G(g)$, the dimension of $C_G(g)$ as a compact p-adic analytic group equals the dimension of C as a $\mathbb{Q}_p$-vector space. The result follows. □

Recall [4, §5.4] that as kG is an Auslander-Gorenstein ring, every finitely generated kG-module M has a *canonical dimension* which we will denote by $d(M)$. By [4, §5.4(3)] this is a non-negative integer which equals the dimension $d(\mathrm{gr}\, M)$ of the associated graded module $\mathrm{gr}\, M$, defined in (5.2).

Proposition. *Let M be a finitely generated kG-module and let $g \in G_{\mathrm{reg}}$ be such that $d(M) < \dim C_G(g)$. Then $\rho_N[M](g) = 0$.*

Proof. By Theorem 4.5 and Theorem 5.4 there exists a Laurent polynomial $u_g(t) \in F[t, t^{-1}]$ such that $\rho_N[M](g)$ equals the value at $t = 1$ of the rational function

$$\frac{u_g(t) \cdot \Psi(g)}{(1 - t^m)^{d(\mathrm{gr}\, M)}}.$$

Because we are assuming that $\dim C_G(g) > d(M) = d(\mathrm{gr}\, M)$, this rational function has a zero at $t = 1$ in view of the Lemma. □

We can now give our first application of Theorem 4.5.

Proof of Theorem A. By the Proposition, $\rho_N[M] = (\varphi \circ \lambda_N)(\theta_N[M]) = 0$. As φ and λ_N are isomorphisms by Theorem 3.4 and Lemma 3.7, $\theta_N[M] = 0$. The result now follows from the new definition of $\chi(G, M)$ given in (8.4). □

9. K-THEORY

9.1. **The dimension filtration.** Recall from (3.3) that $\mathcal{F}_0$ denotes the category of all kG-modules which are finite dimensional over k.

Now, a finitely generated kG-module M is finite dimensional over k if and only if $d(M) = d(\mathrm{gr}\, M) = 0$, because both conditions are equivalent to the Poincaré series (5.2) of $\mathrm{gr}\, M$ being a polynomial in t. We can therefore unambiguously define $\mathcal{F}_i = \mathcal{F}_i(G)$ to be full subcategory of $\mathcal{M}(kG)$ consisting of all modules M with $d(M) \leqslant i$, for each $i = 0, \ldots, d$. Thus we have an ascending chain of subcategories

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{d-1} \subseteq \mathcal{F}_d = \mathcal{M}(kG).$$

Using [4, §5.3] we see that each $\mathcal{F}_i$ is an admissible (in fact, Serre) subcategory of $\mathcal{M}(kG)$, so we can form the Grothendieck groups $K_0(\mathcal{F}_i)$. The inclusions $\mathcal{F}_i \subseteq \mathcal{F}_d$ induce maps

$$\alpha_i : K_0(\mathcal{F}_i) \to K_0(\mathcal{F}_d) \qquad \text{and} \qquad \alpha_i : FK_0(\mathcal{F}_i) \to FK_0(\mathcal{F}_d)$$

which we would like to understand. Because ρ_N is a $\mathbb{Q}$-isomorphism by Corollary 8.2, we focus on the image of $\rho_N \circ \alpha_i$.

9.2. **Module structures.** Note that $K_0(\mathcal{F}_0)$ is a commutative ring with multiplication induced by the tensor product. By adapting the argument used in [5, Proposition 7.3] and using [4, §5.4(5)], we see that the twist $M \otimes_k V$ of any $M \in \mathcal{F}_d$ and any $V \in \mathcal{F}_0$ satisfies $d(M \otimes_k V) = d(M)$. In this way $K_0(\mathcal{F}_i)$ becomes a $K_0(\mathcal{F}_0)$-module and it is clear that the maps $\alpha_i : K_0(\mathcal{F}_i) \to K_0(\mathcal{F}_d)$ respect this module structure.

Lemma. *The map $\lambda_N \circ \theta_N : K_0(\mathcal{F}_d) \to K_0(\mathcal{F}_0)$ is a map of $K_0(\mathcal{F}_0)$-modules.*

Proof. Let $V \in \mathcal{M}(k\overline{G})$. For any $M \in \mathcal{F}_d$ we can define a function

$$\beta_M : (M \otimes_k V) \otimes_{kG} k\overline{G} \to (M \otimes_{kG} k\overline{G}) \otimes_k V$$

by the formula $\beta_M((m \otimes v) \otimes h) = (m \otimes h) \otimes vh$ for $m \in M$, $v \in V$ and $h \in \overline{G}$. A straightforward check shows that β_M is a homomorphism of right kG-modules with inverse γ_M, defined by the formula $\gamma_M((m \otimes h) \otimes v) = (m \otimes vh^{-1}) \otimes h$. Hence the functors $M \mapsto (M \otimes_k V) \otimes_{kG} k\overline{G}$ and $M \mapsto (M \otimes_{kG} k\overline{G}) \otimes_k V$ are isomorphic, so

$$\mathrm{Tor}_j^{kG}(M \otimes_k V, k\overline{G}) \cong \mathrm{Tor}_j^{kG}(M, k\overline{G}) \otimes_k V$$

for all $j \geqslant 0$. It follows that $\theta_N[M \otimes_k V] = \theta_N[M].[V]$ for all $M \in \mathcal{F}_d$ and $V \in \mathcal{M}(k\overline{G})$. Because $\lambda_N : \mathcal{G}_0(k\overline{G}) \to K_0(\mathcal{F}_0)$ is an isomorphism by Lemma 3.7,

$$\lambda_N(\theta_N[M \otimes_k V]) = \lambda_N(\theta_N[M]).[V]$$

for all $M \in \mathcal{F}_d$ and $V \in \mathcal{F}_0$, as required. □

Hence the image of $\lambda_N \circ \theta_N \circ \alpha_i$ is always an *ideal* of $K_0(\mathcal{F}_0)$. On the other hand, $C(G_{\mathrm{reg}}; F)^{G \times \mathcal{G}_k}$ is a commutative F-algebra via pointwise multiplication of functions, and Lemma 3.3(iii) shows that the map

$$\varphi : FK_0(\mathcal{F}_0) \to C(G_{\mathrm{reg}}; F)^{G \times \mathcal{G}_k}$$

appearing in Theorem 3.4 is an F-algebra isomorphism, so $\mathrm{Im}(\rho_N \circ \alpha_i)$ is always an ideal of $C(G_{\mathrm{reg}}; F)^{G \times \mathcal{G}_k}$.

It is easy to see that the ideals of this algebra are in bijection with the subsets of the orbit space $(G \times \mathcal{G}_k)\backslash G_{\mathrm{reg}}$; which subset does $\mathrm{Im}(\rho_N \circ \alpha_i)$ correspond to?

9.3. **An upper bound for** $\mathrm{rk}\,\alpha_i$**.** Define a subset S_i of G_{reg} by the formula

$$S_i := \{g \in G_{\mathrm{reg}} : \dim C_G(g) \leqslant i\}.$$

We record some basic facts about these subsets of G_{reg}.

Lemma. *(i) S_i is a union of conjugacy classes in G.*
(ii) S_i is stable under the action of $\mathcal{G}_k$ on G_{reg}.
(iii) S_i is a clopen subset of G_{reg} and hence a closed subset of G.

Proof. (i) This is clear.

(ii) If g is a power of h then $C_G(h) \leqslant C_G(g)$, therefore if g and h lie in the same $\mathcal{G}_k$-orbit inside G_{reg} then their centralizers are equal.

(iii) This follows from Corollary 3.6. □

We can now give our second application of Theorem 4.5.

Proof of Theorem B. By Proposition 8.5, $\rho_N[M]$ is zero on $G_{\mathrm{reg}} - S_i$ for any $M \in \mathcal{F}_i$. Hence $\mathrm{Im}(\rho_N \circ \alpha_i) \subseteq C(S_i; F)^{G \times \mathcal{G}_k}$ and the result follows from Corollary 8.2. □

10. Some special cases

10.1. **The rank of α_d.** We begin by recording the rank of α_d, or equivalently the rank of $K_0(\mathcal{F}_d) = \mathcal{G}_0(kG)$.

Proposition. *The rank of α_d equals the number of $G \times \mathcal{G}_k$-orbits on G_{reg}:*

$$\operatorname{rk} \alpha_d = \operatorname{rk} K_0(\mathcal{F}_d) = |(G \times \mathcal{G}_k)\backslash G_{\mathrm{reg}}|.$$

Proof. This follows from Corollary 8.2. □

10.2. **A localisation sequence.** Consider the localisation sequence of K-theory [20, Theorem 5.5] for the Serre subcategory $\mathcal{F}_i$ of the abelian category $\mathcal{F}_d$:

$$K_0(\mathcal{F}_i) \xrightarrow{\alpha_i} K_0(\mathcal{F}_d) \longrightarrow K_0(\mathcal{F}_d/\mathcal{F}_i) \longrightarrow 0.$$

Because we already know the rank of $K_0(\mathcal{F}_d)$, this sequence shows the problem of computing the rank of α_i is equivalent to the problem of computing the rank of the Grothendieck group of the quotient category $\mathcal{F}_d/\mathcal{F}_i$. At present, the only non-trivial case when we understand $\mathcal{F}_d/\mathcal{F}_i$ is the case $i = d-1$ — see (10.6). First, we require some results from noncommutative algebra.

10.3. **Artinian rings and minimal primes.** Recall that if R is a (not necessarily commutative) ring, then an ideal I of R is said to be *prime* if whenever A, B are ideals of R strictly containing I, their product AB also strictly contains I. A *minimal prime* of R is a prime ideal which is minimal with respect to inclusion - equivalently, it has height zero. The following result is well-known.

Lemma. *Let R be an Artinian ring. Then*

(i) $\mathcal{G}_0(R)$ is a free abelian group on the isomorphism classes of simple R-modules.
(ii) There is a natural bijection between the isomorphism classes of simple R-modules and the minimal primes of R, given by

$$[M] \mapsto \operatorname{Ann}_R(M).$$

10.4. **The finite radical.** Recall [3, 1.3] the important characteristic subgroup Δ^+ of G, defined by

$$\Delta^+ = \{x \in G : |G : C_G(x)| < \infty \quad \text{and} \quad o(x) < \infty\}.$$

This group is sometimes called the *finite radical* of G and consists of all elements of finite order in G whose conjugacy class is finite. It is also the largest finite normal subgroup of G. In our notation, $\Delta^+_{\mathrm{reg}} = \Delta^+ \cap G_{\mathrm{reg}}$ is just the complement of S_{d-1} in G_{reg}:

$$\Delta^+_{\mathrm{reg}} = G_{\mathrm{reg}} - S_{d-1}$$

and as such is a union of $G \times \mathcal{G}_k$-orbits in G_{reg}.

10.5. **The classical ring of quotients $Q(kG)$.** By [5, Proposition 7.2] kG has an Artinian classical ring of quotients $Q(kG)$. We have already computed the rank of $\mathcal{G}_0(Q(kG))$ under the assumption that the order of the finite group Δ^+ is coprime to p [5, Theorem 12.7(b)]. We can now present a generalization of this result, valid without any restrictions on G.

Theorem. *The rank of $\mathcal{G}_0(Q(kG))$ equals the number of $G \times \mathcal{G}_k$-orbits on Δ^+_{reg}:*

$$\operatorname{rk} \mathcal{G}_0(Q(kG)) = |(G \times \mathcal{G}_k)\backslash \Delta^+_{\mathrm{reg}}|.$$

Proof. We will show that both numbers in question are equal to the number of minimal primes of kG, r say. Let J denote the *prime radical* of kG, defined as the intersection of all prime ideals of kG. By passing to kG/J and applying [17, Proposition 3.2.2(i)], we see that minimal primes of kG are in bijection with the minimal primes of $Q(kG)$. As $Q(kG)$ is Artinian, Lemma 10.3 implies that $\operatorname{rk}\mathcal{G}_0(Q(kG)) = r$.

Next, the group G acts on Δ^+ by conjugation and therefore permutes the the minimal primes of $k\Delta^+$. It was shown in [2, Theorem 5.7] that there is a natural bijection between the minimal primes of kG and G-orbits on the minimal primes of $k\Delta^+$. The group G also permutes the simple $k\Delta^+$-modules and respects the correspondence between these and the minimal primes of $k\Delta^+$ given in Lemma 10.3(ii). Hence r is also the number of G-orbits on the simple $k\Delta^+$-modules. Finally, the G-equivariant version of the Berman–Witt Theorem [5, Corollary 12.6] shows that the latter is just $|(G \times \mathcal{G}_k)\backslash\Delta^+_{\mathrm{reg}}|$ and the result follows. □

10.6. **The rank of α_{d-1}.** Recall that a finitely generated kG-module is *torsion* if and only if $M \otimes_{kG} Q(kG) = 0$. By [11, Lemma 1.4], M is torsion if and only if $d(M) < d(kG) = d$, so $\mathcal{F}_{d-1}$ is just the category of all finitely generated torsion kG-modules, as mentioned in the introduction.

Lemma. *The quotient category $\mathcal{F}_d/\mathcal{F}_{d-1}$ is equivalent to $\mathcal{M}(Q(kG))$.*

Proof. This follows from [24, Propositions XI.3.4(a) and XI.6.4], with appropriate modifications to handle the finitely generated case. □

We can now use the localisation sequence of (10.2) to show that the upper bound for $\operatorname{rk}\alpha_i$ given in Theorem B is attained in the case when $i = d-1$.

Proposition. *The rank of α_{d-1} equals the number of $G \times \mathcal{G}_k$-orbits on S_{d-1}:*

$$\operatorname{rk}\alpha_{d-1} = |(G \times \mathcal{G}_k)\backslash S_{d-1}|.$$

Proof. In view of the Lemma, the localisation sequence becomes

$$K_0(\mathcal{F}_{d-1}) \xrightarrow{\alpha_{d-1}} K_0(\mathcal{F}_d) \longrightarrow \mathcal{G}_0(Q(kG)) \longrightarrow 0.$$

Hence $\operatorname{rk}(\alpha_{d-1}) = \operatorname{rk}K_0(\mathcal{F}_d) - \operatorname{rk}\mathcal{G}_0(Q(kG))$. Now apply Proposition 10.1 and Theorem 10.5, bearing in mind that $S_{d-1} = G_{\mathrm{reg}} - \Delta^+_{\mathrm{reg}}$. □

10.7. **The rank of α_0.** We will see in (12.3) that the rank of α_i does *not* always attain the upper bound of Theorem B. Here is another special case when $\operatorname{rk}\alpha_i$ is well-behaved.

Proposition. *The rank of α_0 equals the number of $G \times \mathcal{G}_k$-orbits on S_0:*

$$\operatorname{rk}\alpha_0 = |(G \times \mathcal{G}_k)\backslash S_0|.$$

Proof. Let $M \in \mathcal{F}_0$. Because M is finite dimensional over k, the graded Brauer character $\zeta_{\mathrm{gr}\,M}$ is a polynomial, so

$$\varphi_M = \zeta_{\mathrm{gr}\,M}|_{t=1},$$

thought of as F-valued functions on G_{reg}. Applying the explicit formula for ρ_N given in Theorem 4.5 shows that

$$\rho_N[M](g) = \varphi_M(g) \cdot \det(1 - \mathrm{Ad}(g^{-1}))$$

for any $g \in G_{\mathrm{reg}}$. We therefore have a commutative diagram

$$\begin{tikzcd} FK_0(\mathcal{F}_0) \arrow[r, "\varphi"] \arrow[d, "\alpha_0"'] & C(G_{\mathrm{reg}}; F)^{G\times \mathcal{G}_k} \arrow[d, "\eta"] \\ FK_0(\mathcal{F}_d) \arrow[r, "\rho_N"'] & C(G_{\mathrm{reg}}; F)^{G\times \mathcal{G}_k} \end{tikzcd}$$

where η is multiplication by the locally constant function

$$\Psi|_{t=1} : g \mapsto \det(1 - \mathrm{Ad}(g^{-1})).$$

Using Lemma 8.5 we see that $\Psi|_{t=1}(g) \neq 0$ if and only if $\dim C_G(g) = 0$. It follows that the image of η is precisely $C(S_0; F)^{G\times\mathcal{G}_k}$. As the maps φ and ρ_N are isomorphisms by Theorem 3.4 and Corollary 8.2,

$$\mathrm{rk}\, \alpha_0 = \mathrm{rk}\, \eta = |(G \times \mathcal{G}_k)\backslash S_0|$$

as required. □

We now start preparing for the proof of Theorem 11.3 which says that the upper bound of Theorem B is always attained if the group G is virtually abelian.

11. Induction of modules

11.1. **Dimensions.** In what follows we fix a closed subgroup H of G of dimension e. Recall [9, Lemma 4.5] that kG is a *flat* kH-module. Therefore the induction functor

$$\mathrm{Ind}_H^G : \mathcal{M}(kH) \to \mathcal{M}(kG)$$

which sends $M \in \mathcal{M}(kH)$ to $M \otimes_{kH} kG \in \mathcal{M}(kG)$ is exact and induces a map

$$\mathrm{Ind}_H^G : \mathcal{G}_0(kH) \to \mathcal{G}_0(kG).$$

We can obtain precise information about the dimension of an induced module.

Lemma. *Let $M \in \mathcal{M}(kH)$. Then*

$$d(M \otimes_{kH} kG) = d(M) + d - e.$$

Proof. Recall that the *grade* $j(X)$ of a finitely generated R-module X over an Auslander-Gorenstein ring R is defined by the formula

$$j(X) = \min\{j : \mathrm{Ext}^j_R(X, R) \neq 0\}.$$

The *canonical dimension* of X is the non-negative integer $\mathrm{id}(R) - j(X)$ where $\mathrm{id}(R)$ is the injective dimension of R.

Now, choosing a free resolution of M and using the fact that kG is a flat kH-module, we see that

$$kG \otimes_{kH} \mathrm{Ext}^j_{kH}(M, kH) \cong \mathrm{Ext}^j_{kG}(M \otimes_{kH} kG, kG)$$

as left kG-modules, for any $j \geqslant 0$. In fact, kG is a *faithfully flat* (left) kH-module by [2, Lemma 5.1], which implies that $kG \otimes_{kH} A = 0$ if and only if $A = 0$ for any finitely generated left kH-module A.

Hence $j(M) = j(M \otimes_{kH} kG)$ and the result follows because $\mathrm{id}(kG) = \dim G = d$ and $\mathrm{id}(kH) = \dim H = e$. □

11.2. **Proposition.** Suppose that the group $H \cap N$ is uniform. Then there exists a map $\iota_N : C(H_{\mathrm{reg}}; F)^{H \times \mathcal{G}_k} \to C(G_{\mathrm{reg}}; F)^{G \times \mathcal{G}_k}$ such that the following diagram commutes:

$$\begin{tikzcd} FK_0(\mathcal{F}_{d-e}(G)) \arrow[r, "\alpha_{d-e}"] & FK_0(\mathcal{F}_d(G)) \arrow[r, "\rho_N"] & C(G_{\mathrm{reg}}; F)^{G \times \mathcal{G}_k} \\ FK_0(\mathcal{F}_0(H)) \arrow[r, "\alpha_0"'] \arrow[u, "\mathrm{Ind}_H^G"] & FK_0(\mathcal{F}_e(H)) \arrow[r, "\rho_{H\cap N}"'] \arrow[u, "\mathrm{Ind}_H^G"] & C(H_{\mathrm{reg}}; F)^{H \times \mathcal{G}_k}. \arrow[u, "\iota_N"'] \end{tikzcd}$$

Proof. We assume that $H \cap N$ is uniform only to make sure that the map $\rho_{H \cap N}$ makes sense. We construct this diagram in several steps — note that the left-hand square makes sense by Lemma 11.1.

Let $\overline{H} := HN/N \cong H/(H \cap N)$, let $\overline{G} := G/N$ and define a map

$$\mathrm{Ind}_{\overline{H}}^{\overline{G}} : C(\overline{H}_{\mathrm{reg}}; F)^{\overline{H} \times \mathcal{G}_k} \to C(\overline{G}_{\mathrm{reg}}; F)^{\overline{G} \times \mathcal{G}_k}$$

as follows:

$$\mathrm{Ind}_{\overline{H}}^{\overline{G}}(f)(g) = \frac{1}{|\overline{H}|} \sum_{x \in \overline{G}} f(xgx^{-1})$$

for any $f \in C(\overline{H}_{\mathrm{reg}}; F)^{\overline{H} \times \mathcal{G}_k}$ and $g \in \overline{G}_{\mathrm{reg}}$, with the understanding that $f(u) = 0$ if $u \notin \overline{H}$. Consider the following diagram:

$$\begin{tikzcd} FK_0(\mathcal{F}_d(G)) \arrow[r, equal] & F\mathcal{G}_0(kG) \arrow[r, "\theta_N"] & F\mathcal{G}_0(k\overline{G}) \arrow[r, "\varphi"] & C(\overline{G}_{\mathrm{reg}}; F)^{\overline{G} \times \mathcal{G}_k} \\ FK_0(\mathcal{F}_e(H)) \arrow[r, equal] \arrow[u, "\mathrm{Ind}_H^G"] & F\mathcal{G}_0(kH) \arrow[r, "\theta_{H\cap N}"'] \arrow[u, "\mathrm{Ind}_H^G"] & F\mathcal{G}_0(k\overline{H}) \arrow[r, "\varphi"'] \arrow[u, "\mathrm{Ind}_{\overline{H}}^{\overline{G}}"'] & C(\overline{H}_{\mathrm{reg}}; F)^{\overline{H} \times \mathcal{G}_k}. \arrow[u, "\mathrm{Ind}_{\overline{H}}^{\overline{G}}"'] \end{tikzcd}$$

The middle square commutes by functoriality of $\mathcal{G}_0$ and the right-hand square commutes by the well-known formula [22, Theorem 12 and Exercise 18.2] for the character of an induced representation. So the whole diagram commutes.

Finally, using the isomorphisms π_N^* and $\pi_{H\cap N}^*$ which feature in (3.8) we can define the required map ι_N to be the map which makes the following diagram commute:

$$\begin{tikzcd} C(\overline{G}_{\mathrm{reg}}; F)^{\overline{G} \times \mathcal{G}_k} \arrow[r, "\pi_N^*"] & C(G_{\mathrm{reg}}; F)^{G \times \mathcal{G}_k} \\ C(\overline{H}_{\mathrm{reg}}; F)^{\overline{H} \times \mathcal{G}_k} \arrow[r, "\pi_{H\cap N}^*"'] \arrow[u, "\mathrm{Ind}_{\overline{H}}^{\overline{G}}"] & C(H_{\mathrm{reg}}; F)^{H \times \mathcal{G}_k}. \arrow[u, "\iota_N"'] \end{tikzcd}$$

The commutative diagram appearing in (3.8) now shows that

$$\pi_N^* \circ \varphi = \varphi \circ \lambda_N \quad \text{and} \quad \pi_{H\cap N}^* \circ \varphi = \varphi \circ \lambda_{H \cap N}$$

and the result follows by pasting the above diagrams together. □

11.3. **The case when G is virtually abelian.** We can now apply the theory developed above. If $g \in G_{\mathrm{reg}}$ let $\delta_g : G_{\mathrm{reg}} \to F$ be the function which takes the value 1 on the $G \times \mathcal{G}_k$-orbit of g and is zero elsewhere. Note that δ_g is locally constant by Corollary 3.6.

Theorem. *Suppose that G is virtually abelian. Then the upper bound of Theorem B is always attained:*

$$\operatorname{rk}\alpha_i = |(G \times \mathcal{G}_k)\backslash S_i|$$

for all $i = 0, \ldots, d$.

Proof. Fix the integer i and fix $g \in G_{\mathrm{reg}}$ such that $\dim C_G(g) \leqslant i$. As $\operatorname{rk}\alpha_i = \operatorname{rk}(\rho_N \circ \alpha_i)$ by Corollary 8.2, it will be sufficient to show that

$$\delta_g \in \operatorname{Im}(\rho_N \circ \alpha_i).$$

Because we are assuming that G is virtually abelian, the uniform pro-p subgroup N is abelian. Thus N is a free $\mathbb{Z}_p$-module of rank $d = \dim G$. By considering the conjugation action of g on N we see that the submodule of fixed points

$$C_N(g) = \{x \in N : gx = xg\} = N \cap C_G(g)$$

has a unique g-invariant $\mathbb{Z}_p$-module complement in N which we will denote by L. Note that L is a *subgroup* of G as N is abelian. We could alternatively have defined L as the isolator of $[N, \langle g\rangle]$ in N.

Let H be the closed subgroup of G generated by L and g. Because g normalises L, H is isomorphic to a semi-direct product of L with the finite group $\langle g\rangle$:

$$H = L \rtimes \langle g\rangle.$$

Let $\epsilon_g : H_{\mathrm{reg}} \to F$ be the locally constant function which is 1 on the $H \times \mathcal{G}_k$-orbit of g inside H_{reg} and zero elsewhere. Note that ϵ_g is constant on the cosets of the open uniform subgroup $H \cap N = L$ of H.

By construction, g acts without nontrivial fixed points on L by conjugation. So if β_g denotes this action, then

$$D_g := \det(1 - \beta_g) \in F$$

is a nonzero constant. In view of the commutative diagram which appeared in the proof of Proposition 10.7 the element $X := \varphi^{-1}(\epsilon_g) \in FK_0(\mathcal{F}_0(H))$ satisfies

$$(\rho_{H\cap N} \circ \alpha_0)(X) = D_g \epsilon_g.$$

Using the definition of the map $\iota_N : C(H_{\mathrm{reg}}; F)^{H\times\mathcal{G}_k} \to C(G_{\mathrm{reg}}; F)^{G\times\mathcal{G}_k}$ of Proposition 11.2 we may calculate that there exists a nonzero constant $A_g \in F$ such that for all $y \in G_{\mathrm{reg}}$ we have

$$\iota_N(\epsilon_g)(y) = A_g \cdot \delta_g(y).$$

We can now apply Proposition 11.2 and obtain

$$(\rho_N \circ \alpha_{d-e})\left(\operatorname{Ind}_H^G X\right) = \iota_N(\rho_{H\cap N}(\alpha_0(X))) = \iota_N\left(D_g\epsilon_g\right) = D_g A_g \delta_g.$$

But $d - e = \dim G - \dim H = \dim C_G(g) \leqslant i$ and $D_g A_g \neq 0$ so

$$\delta_g \in \operatorname{Im}(\rho_N \circ \alpha_{d-e}) \subseteq \operatorname{Im}(\rho_N \circ \alpha_i)$$

as required. $\square$

12. An example

12.1. **Central torsion modules.** Suppose that z is a central element of G contained in N. Write Z for the closed central subgroup of G generated by z.

Because N is torsion-free by [13, Theorem 4.5], Z is isomorphic to $\mathbb{Z}_p$. Hence kZ is isomorphic to the power series ring $k[[z-1]]$ and its maximal ideal is generated by $z-1$. Because kG is a flat kZ-module, it follows that $z-1$ generates the kernel of the map $kG \twoheadrightarrow k[[G/Z]]$ as an ideal in kG.

Lemma. *Let $M \in \mathcal{M}(kG)$ and suppose that $M.(z-1) = 0$ so that M is also a right $k[[G/Z]]$-module. Then as right $k[[G/Z]]$-modules, we have*

$$\mathrm{Tor}_n^{kG}(M, k[[G/Z]]) \cong \begin{cases} M & \text{if} \quad n = 0 \text{ or } n = 1 \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

Proof. Because $z-1$ is not a zero-divisor in kG,

$$0 \to kG \xrightarrow{z-1} kG \to k[[G/Z]] \to 0$$

is a free resolution of $k[[G/Z]]$ as a left kG-module. Hence the complex

$$0 \to M \xrightarrow{z-1} M \to 0$$

computes the required modules $\mathrm{Tor}_n^{kG}(M, k[[G/Z]])$. The result follows because $M.(z-1) = 0$. $\square$

12.2. **Proposition.** Let $M \in \mathcal{M}(kG)$ and suppose that $M.(z-1)^a = 0$ for some $a \geqslant 1$. Then $\theta_N[M] = 0$.

Proof. Suppose first that $a = 1$ so that M is killed by $z-1$. We recall the base-change spectral sequence for Tor [27, Theorem 5.6.6] associated with a ring map $f : R \to S$:

$$E^2_{ij} = \mathrm{Tor}_i^S(\mathrm{Tor}_j^R(A, S), B) \Longrightarrow \mathrm{Tor}_{i+j}^R(A, B).$$

This is a first quadrant convergent homological spectral sequence. We apply it to the map $R := kG \twoheadrightarrow k[[G/Z]] =: S$ with $A := M$ and $B := k\overline{G} = k[G/N]$ — note that B is a left $k[[G/Z]]$-module because we are assuming that $Z \leqslant N$.

By Lemma 12.1 this spectral sequence is concentrated in rows $j = 0$ and $j = 1$, so we may apply [27, Exercise 5.2.2] and obtain a long exact sequence

$$\begin{array}{cccccccc} \cdots \to & \mathrm{Tor}_{n+1}^{kG}(M, k\overline{G}) & \to & \mathrm{Tor}_{n+1}^{k[[G/Z]]}(M, k\overline{G}) & \to & \mathrm{Tor}_{n-1}^{k[[G/Z]]}(M, k\overline{G}) & \to \\ \to & \mathrm{Tor}_n^{kG}(M, k\overline{G}) & \to & \mathrm{Tor}_n^{k[[G/Z]]}(M, k\overline{G}) & \to & \mathrm{Tor}_{n-2}^{k[[G/Z]]}(M, k\overline{G}) & \to \cdots . \end{array}$$

We can now apply Lemma 2.2 and deduce that

$$\theta_N[M] = \sum_{n=0}^{d} (-1)^n [\mathrm{Tor}_n^{kG}(M, k\overline{G})] = 0 \in \mathcal{G}_0(k\overline{G})$$

as required. The general case follows quickly by an induction on a. $\square$

Corollary. *Let M be as above. Then its Euler characteristic is trivial:*

$$\chi(G, M) = 1.$$

Proof. This follows from Definition 8.4. $\square$

Using this result we now show that the upper bound for $\mathrm{rk}\,\alpha_i$ given in Theorem B is not *always* attained.

12.3. **Example.** Let p be odd and let N be a *clean Heisenberg pro-p group* [26, Definition 4.2] of dimension $2r+1$. By definition, N has a topological generating set $\{x_1,\ldots,x_r,y_1,\ldots,y_r,z\}$ and relations $[x_i,y_i]=z^p$ for each $i=1,\ldots,r$, all other commutators being trivial. Note that N is uniform.

The presentation for N given above makes it possible to define an automorphism γ of N which fixes z and sends all the other generators to their inverses:

$$\gamma(x_i)=x_i^{-1},\gamma(y_i)=y_i^{-1},\gamma(z)=z.$$

Now let G be the semidirect product of N with a cyclic group $\langle g\rangle$ of order 2, where the conjugation action of g on N is given by the automorphism γ. Thus the p'-part of $|G/N|$ is equal to 2 so the Galois group $\mathcal{G}_k$ defined in (3.1) is trivial for any finite field k of characteristic p.

Now $(G/N)_{\mathrm{reg}}=G/N$ has two G/N-conjugacy classes, so by Proposition 3.6 there are just two $G\times\mathcal{G}_k$-orbits on G_{reg}, represented by the elements 1 and g.

By construction, the endomorphism $\gamma=\mathrm{Ad}(g)$ has eigenvalue -1 with multiplicity $2r$ and eigenvalue 1 with multiplicity 1:

$$\Psi(g)=(1+t)^{2r}(1-t),$$

so $\dim C_G(g)=1$ and of course $\dim C_G(1)=2r+1$. Hence in the notation of (9.3)

$$|(G\times\mathcal{G}_k)\backslash S_i|=\begin{cases}0 & \text{if } i=0,\\ 1 & \text{if } 1\leqslant i\leqslant 2r,\\ 2 & \text{if } i=2r+1.\end{cases}$$

Now if M is a finitely generated kG-module with $d(M)\leqslant r$ then M is killed by some power of $z-1$ by [26, Theorem B]. It follows from Proposition 12.2 that $\theta_N[M]=0$, so $\mathrm{rk}\,\alpha_i=0$ for all $i\leqslant r$ — thus the upper bound given in Theorem B is not attained for all values of i between $i=1$ and $i=r$.

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Christ's College, Cambridge; DPMMS, University of Cambridge
E-mail address: `K.Ardakov@dpmms.cam.ac.uk; S.J.Wadsley@dpmms.cam.ac.uk`