# REPRESENTATION THEORY

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## Lecture 1

## 1. Introduction

Representation Theory is the study of how symmetries occur in nature; that is the study of how groups act by linear transformations on vector spaces.

### 1.1. **Linear algebra revision.**

By *vector space* we will always mean a finite dimensional vector space over a field $k$. For this course $k$ will usually be algebraically closed and of characteristic zero, for example $\mathbb{C}$. However there are rich theories for more general fields.

Given a vector space $V$, we define

$$GL(V) = \mathrm{Aut}(V) = \{f\colon V \to V \mid f \text{ linear and invertible}\}$$

the *general linear group* of $V$; $GL(V)$ is a group under composition of linear maps.

Because all our vector spaces are finite dimensional, $V \cong k^d$ for some $d \geqslant 0$. Such an isomorphism determines a basis $e_1, \ldots, e_d$ for $V$. Then

$$GL(V) \cong \{A \in \mathrm{Mat}_d(k) \mid \det(A) \neq 0\}.$$

This isomorphism is given by the map that sends the linear map $f$ to the matrix $A$ such that $f(e_i) = A_{ji}e_j$.

*Exercise.* Check that this does indeed define an isomorphism of groups. ie check that $f$ is an isomorphism if and only if $\det A \neq 0$; and that the given map is a bijective group homomorphism.

### 1.2. **Group representations.**

Recall the definition of the action of a group $G$ on a set $X$. An *action* of $G$ on $X$ is a map $\cdot\colon G \times X \to X; (g,x) \mapsto g \cdot x$ such that

(i) $e \cdot x = x$ for all $x \in X$;
(ii) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

Recall also that to define such an action is equivalent to defining a group homomorphism $\rho\colon G \to S(X)$ where $S(X)$ denotes the symmetric group on the set $X$.

**Definition.** A *representation* $\rho$ of a group $G$ on a vector space $V$ is a group homomorphism $\rho\colon G \to GL(V)$.

By abuse of notation we will sometimes refer to the representation by $\rho$, sometimes by the pair $(\rho, V)$ and sometimes just by $V$ with the $\rho$ implied. This can sometimes be confusing but we have to live with it.

Thus defining a representation of $G$ on $V$ corresponds to assigning a linear map $\rho(g)\colon V \to V$ to each $g \in G$ such that

(i) $\rho(e) = \mathrm{id}_V$;

(ii) $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$;
(iii) $\rho(g^{-1}) = \rho(g)^{-1}$ for all $g \in G$.

*Exercise.* Show that (iii) is redundant in the above.

Given a basis for $V$ a representation $\rho$ is an assignment of a matrix $\rho(g)$ to each $g \in G$ such that (i),(ii) and (iii) hold.

**Definition.** The *degree* of $\rho$ or *dimension* of $\rho$ is $\dim V$.

**Definition.** We say a representation $\rho$ is *faithful* if $\ker \rho = \{e\}$.

*Examples.*

(1) Let $G$ be any group and $V = k$. Then $\rho\colon G \to \mathrm{Aut}(V); g \mapsto \mathrm{id}$ is called the *trivial representation.*

(2) Let $G = \mathbb{Z}/2 = \{\pm 1\}$, $V = \mathbb{R}^2$, then
$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
is a group rep of $G$ on $V$.

(3) Let $G = (\mathbb{Z}, +)$, $V$ a vector space, and $\rho$ a representation of $G$ on $V$. Then necessarily $\rho(0) = \mathrm{id}_V$, and $\rho(1)$ is some invertible linear map $f$ on $V$. Now $\rho(2) = \rho(1+1) = \rho(1)^2 = f^2$. Inductively we see $\rho(n) = f^n$ for all $n > 0$. Finally $\rho(-n) = (f^n)^{-1} = (f^{-1})^n$. So $\rho(n) = f^n$ for all $n \in \mathbb{Z}$.

Notice that conversely given any invertible linear map $f\colon V \to V$ we may define a representation of $G$ on $V$ by $\rho(n) = f^n$.

Thus we see that there is a 1-1 correspondance between representations of $\mathbb{Z}$ and invertible linear transformations given by $\rho \mapsto \rho(1)$.

(4) Let $G = (\mathbb{Z}/N, +)$, and $\rho\colon G \to GL(V)$ a rep. As before we see $\rho(n) = \rho(1)^n$ for all $n \in \mathbb{Z}$ but now we have the additional constraint that $\rho(N) = \rho(0) = \mathrm{id}_V$.

Thus representations of $\mathbb{Z}/N$ correspond to invertible linear maps $f$ such that $f^N = \mathrm{id}_V$. Of course any linear map such that $f^N = \mathrm{id}_V$ is invertible so we may drop the word invertible from this correspondance.

*Exercise.* Check the details

(5) If $G$ is a group with generated by $x_1, \ldots, x_n$ and with relations (words in $x_i, x_i^{-1}$ equal to the identity in $G$) $r_1(x_1, \ldots, x_n), \ldots, r_m(x_1, \ldots, x_n)$ , then there is a 1-1 correspondence between representations of $G$ on $V$ and $n$-tuples of invertible linear maps $(A_1, \ldots, A_n)$ on $V$ such that $r_i(A_1, \ldots, A_n) = \mathrm{id}_V$.

(6) Let $G = S_3$, the symmetric group of $\{1, 2, 3\}$, and $V = \mathbb{R}^2$. Take an equilateral triangle in $V$ centred on 0; then $G$ acts on the triangle by permuting the vertices. Each such symmetry induces a linear transformation of $V$. For example $g = (12)$ induces the reflection through the vertex three and the midpoint of the opposite side, and $g = (123)$ corresponds to a rotation by $2\pi/3$.

*Exercise.* Choose a basis for $\mathbb{R}^2$. Write the coordinates of the vertices of the triangle in this basis. For each $g \in S_3$ write down the matrix of the corresponding linear map. Check that this does define a representation of $S_3$ on $V$. Would the calculations be easier in a different basis?

(7) Given a finite set $X$ we may form the vector space $kX$ of functions $X$ to $k$ with basis $\langle \delta_x \mid x \in X \rangle$ where $\delta_x(y) = \delta_{xy}$.

Then an action of $G$ on $X$ induces a representation $\rho\colon G \to \mathrm{Aut}(kX)$ by $(\rho(g)f)(x) = f(g^{-1}\cdot x)$ called the *permutation representation* of $G$ on $X$.

To check this is a representation we must check that each $\rho(g)$ is linear, that $\rho(e) = \mathrm{id}$ and $\rho(gh) = \rho(g)\rho(h)$ for each $g, h \in G$.

For the last observe that for each $x \in X$,

$$\rho(g)(\rho(h)f)(x) = (\rho(h)f)(g^{-1}x) = f(h^{-1}g^{-1}x) = \rho(gh)f(x).$$

Notice that $\rho(g)\delta_x(y) = \delta_{x,g^{-1}\cdot y} = \delta_{g\cdot x,y}$ so $\rho(g)\delta_x = \delta_{g\cdot x}$. So by linearity $\rho(g)(\sum_{x\in X}\lambda_x\delta_x) = \sum\lambda_x\delta_{g\cdot x}$.

(8) In particular if $G$ is finite then the action of $G$ on itself induces the *regular representation* $kG$ of $G$. The regular representation is always faithful because $ge_h = e_h$ for all $h \in G$ implies that $gh = h$ for all $h \in G$ and so $g = e$.

## Lecture 2

### 1.3. **The category of representations.**

We want to classify all representations of a group $G$ but first we need a good notion of when two representations are the same.

Notice that if $\rho\colon G \to GL(V)$ is a representation and $\varphi\colon V \to V'$ is a vector space isomorphism then we may define $\rho'\colon G \to GL(V')$ by $\rho'(g) = \varphi \circ \rho(g) \circ \varphi^{-1}$. Then $\rho'$ is also a representation.

**Definition.** We say that $\rho\colon G \to GL(V)$ and $\rho'\colon G \to GL(V')$ are *isomorphic* representations if there is a linear isomorphism $\varphi\colon V \to V'$ such that

$$\rho'(g) = \varphi \circ \rho(g) \circ \varphi^{-1} \text{ for all } g \in G$$

i.e. if $\rho'(g) \circ \varphi = \varphi \circ \rho(g)$. We say that $\varphi$ *intertwines* $\rho$ and $\rho'$.

Notice that if $\varphi$ intertwines $\rho$ and $\rho'$ and $\varphi'$ intertwines $\rho'$ and $\rho''$ then $\varphi'\varphi$ intertwines $\rho$ and $\rho''$ and $\varphi^{-1}$ intertwines $\rho'$ and $\rho$. Thus isomorphism is an equivalence relation.

If $\rho\colon G \to GL_d(k)$ is a matrix representation then an intertwining map $k^d \to k^d$ is an invertible matrix $P$ and the matrices of the reps it intertwines are related by $\rho'(g) = P\rho(\text{`}g)P^{-1}$. Thus matrix representations are equivalent precisely if they correspond to the same representation with respect to different bases.

*Examples.*

(1) If $G = \{e\}$ then a representation of $G$ is just a vector space and two vector spaces are isomorphic as representations if and only if they have the same dimension.

(2) If $G = \mathbb{Z}$ then $\rho\colon G \to GL(V)$ and $\rho'\colon G \to GL(V')$ are isomorphic reps if and only if there are bases of $V$ and $V'$ such that $\rho(1)$ and $\rho'(1)$ are the same matrix. In other words isomorphism classes of representations of $\mathbb{Z}$ correspond to conjugacy classes of invertible matrices. Over $\mathbb{C}$ the latter is classified by Jordan Normal Form (more generally by rational canonical form).

(3) If $G = \mathbb{Z}/2$ then isomorphism classes of representations of $G$ correspond to conjugacy classes of matrices that square to the identity. Since the minimal polynomial of such a matrix is $X^2 - 1 = (X-1)(X+1)$ provided the field does not have characteristic 2 every such matrix is conjugate to a diagonal matrix with diagonal entries all $\pm 1$.

*Exercise.* Show that there are precisely $n+1$ isomorphism classes of representations of $\mathbb{Z}/2$ of dimension $n$.

Note that two isomorphic representations must have the same dimension but that the converse is not true.

**Definition.** Suppose that $\rho\colon G \to GL(V)$ is a rep. We say that a $k$-linear subspace $W$ of $V$ is *$G$-invariant* if $\rho(g)(W) \subset W$ for all $g \in G$ (ie $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$).

In that case we call $W$ a *subrepresentation* of $V$; we may define a representation $\rho_W : G \to GL(W)$ by $\rho_W(g)(w) = \rho(g)(w)$ for $w \in W$.

We call a subrepresentation $W$ of $V$ *proper* if $W \neq V$ and $W \neq 0$. We say that $V \neq 0$ is *irreducible* or *simple* if it has no proper subreps.

*Examples.*

(1) Any one-dimensional representation of a group is irreducible.

(2) Suppose that $\rho\colon \mathbb{Z}/2 \to GL(k^2)$ is given by $-1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (char $k \neq 2$). Then there are precisely two non-trivial subreps spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively.

*Proof.* It is easy to see that these two subspaces are $G$-invariant. Any non-trivial subspace must be one dimensional and so by spanned by an eigenvector of $\rho(-1)$. But of $\rho(-1)$ are precisely those already described. $\square$

(3) If $G$ is $\mathbb{Z}/2\mathbb{Z}$ then the only irreducible representations are one-dimensional.

*Proof.* Suppose $\rho\colon G \to GL(V)$ is an irreducible rep. The minimal polynomial of $\rho(1)$ divides $X^2 - 1 = (X-1)(X+1)$. Thus $\rho(-1)$ has an eigenvector $v$. Now $0 \neq \langle v \rangle$ is a subrep. of $V$. Thus $V = \langle v \rangle$. $\square$

Notice we've shown along the way that there are precisely two simple reps of $G$ if $k$ doesn't have characteristic 2 and only one if it does.

(4) If $G = D_6$ then every irreducible complex representation has dimension at most 2.

*Proof.* Suppose $\rho\colon G \to GL(V)$ is an irred. $G$-rep. Let $r$ be a non-trivial rotation and $s$ a reflection in $G$. Then $\rho(r)$ has a eigenvector $v$, say. So $\rho(r)v = \lambda v$ for some $\lambda \neq 0$. Consider $W := \langle v, \rho(s)v \rangle \subset V$. Since $\rho(s)\rho(s)v = v$ and $\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v$, $W$ is $G$-invariant. Since $V$ is irred, $W = V$. $\square$

*Exercise.* Classify all irred reps of $D_6$ up to iso (Hint: $\lambda^3 = 1$ above). Note in particular that $D_6$ has an irred. rep. of degree 2.

(5) If $G = \mathbb{Z}$ and $(\rho, V)$ is a representation over $\mathbb{C}$ then when is $V$ irreducible?

We can choose a basis for $V$ so that $\rho(1)$ is in Jordan Normal Form. It is easy to see that the Jordan blocks determine invariant subspaces; so if $V$ is irreducible then there is only one Jordan block. Say $\rho(1) = A$ then $Ae_i = \lambda e_i + e_{i-1}$ for some non-zero $\lambda$ and $i = 1, \ldots d$ (where by convention $e_0 = 0$).

*Exercise.* Show that the invariant subspaces are precisely the subspaces of the form $\langle e_1, \ldots, e_k \rangle$ for $k \leqslant d$.

It follows that the only irreducible representations of $\mathbb{Z}$ are one-dimensional. $\rho\colon \mathbb{Z} \to \mathbb{C}^\times$; $1 \mapsto \lambda$.

**Proposition.** *Suppose $\rho\colon G \to GL(V)$ is a rep. and $W \subset V$. Then the following are equivalent:*

*(i) $W$ is a subrep;*
*(ii) there is a basis $v_1, \ldots, v_d$ of $V$ such that $v_1, \ldots, v_r$ is a basis of $W$ and the matrices $\rho(g)$ are all block upper triangular;*
*(iii) for every basis $v_1, \ldots, v_d$ of $V$ such that $v_1, \ldots, v_r$ is a basis of $W$ the matrices $\rho(g)$ are all block upper triangular.*

*Proof.* Think about it! □

**Definition.** If $W$ is a subrep of a rep $(\rho, V)$ of $G$ then we may define a *quotient representation* by $\rho_{V/W}\colon G \to GL(V/W)$ by $\rho(g)(v + W) = \rho(g)(v) + W$. Since $\rho(g)W \subset W$ for all $g \in G$ this is well-defined.

**Definition.** If $(\rho, V)$ and $(\rho', W)$ are reps of $G$ we say a linear map $\varphi\colon V \to W$ is a $G$-linear map if $\varphi \circ \rho(g) = \rho(g) \circ \varphi$ for all $g \in G$. We write $\mathrm{Hom}_G(V, W) = \{\varphi \mid \varphi \text{ is } G \text{ linear}\}$, a $k$-vector space.

Note if $\varphi \in \mathrm{Hom}_G(V, W)$ is a vector space isomorphism then $\phi$ intertwines the isomorphic reps $V$ and $W$.

**Lemma.** *Suppose $(\rho, V)$ and $(\rho', W)$ are representations of $G$ and $\varphi \in \mathrm{Hom}_G(V, W)$ then*

*(i) $\ker\varphi$ is a subrep of $V$.*
*(ii) $\mathrm{Im}\varphi$ is a subrep of $W$.*
*(iii) $V/\ker\varphi$ is isomorphic to $\mathrm{Im}\varphi$ as reps of $G$.*

*Proof.*
(i) if $v \in \ker\varphi$ and $g \in G$ then $\varphi(\rho(g)v) = \rho'(g)\varphi(v) = 0$
(ii) if $w = \varphi(v) \in \mathrm{Im}$ and $g \in G$ then $\varphi$ then $\rho'(g)w = \varphi\rho(g)v \in \mathrm{Im}\varphi$.
(iii) We know that the linear map $\varphi$ factors through $\overline{\varphi}\colon V/\ker\varphi \to \mathrm{Im}\varphi$ and it is straightforward to check that for each $g \in G$, $\overline{\varphi}\rho_{V/\ker\varphi}(g) = \rho'_{\mathrm{Im}\varphi}(g)\overline{\varphi}$ □

## Lecture 3

## 2. Complete reducibility and Maschke's Theorem

*Question.* When can we choose a basis for a representation $V$ so that all the matrices $\rho(g)$ are block diagonal of the same size blocks?

**Definition.** We say a representation $V$ is a *direct sum* of $U$ and $W$ if $U$ and $W$ are subreps of $V$ such that $V = U \oplus W$ as vector spaces (ie $V = U + W$ and $U \cap W = 0$).

Given two representations $(\rho_1, U)$ and $(\rho_2, W)$ we may define a representation of $G$ on $U \oplus W$ by $\rho(g)(u, w) = (\rho_1(g)u, \rho_2(g)w)$.

*Examples.*

(1) If $G$ acts on a finite set $X$ so that $X$ may be written as the disjoint union of two $G$-invariant subsets $X_1$ and $X_2$. Then $kX \cong kX_1 \oplus kX_2$ under $f \mapsto (f|_{X_1}, f|_{X_2})$.

That is $kX = \{f \mid f(x) = 0\ \forall x \in X_2\} \oplus \{f \mid f(x) = 0\ \forall x \in X_1\}$.

More generally if the $G$-action on $X$ decomposes into orbits as a disjoint union $X = \bigcup \mathcal{O}_i$ then $kX \cong \bigoplus k\mathcal{O}_i$.

(2) If $G$ acts transitively on a finite set $X$ then $U := \{f \in kX \mid \sum_{x\in X} f(x) = 0\}$ and $W := \{f \in kX \mid f \text{ is constant}\}$ are subreps of $kX$. If $k$ is charactersitic 0 then $kX = U \oplus W$. What happens if $k$ has characteristic $p > 0$?

**Proposition.** *Suppose $\rho\colon G \to GL(V)$ is a rep. and $V = U \oplus W$ as vector spaces. Then the following are equivalent:*

*(i) $V = U \oplus W$ as reps;*
*(ii) there is a basis $v_1, \dots, v_d$ of $V$ such that $v_1, \dots, v_r$ is a basis of $U$ and $v_{r+1}, \dots v_d$ is a basis for $W$ and the matrices $\rho(g)$ are all block diagonal;*
*(iii) for every basis $v_1, \dots, v_d$ of $V$ such that $v_1, \dots, v_r$ is a basis of $U$ and $v_{r+1}, \dots, v_d$ is a basis for $W$ and the matrices $\rho(g)$ are all block diagonal.*

*Proof.* Think about it! □

But warning:

*Example.* $\rho\colon \mathbb{Z}/2 \to GL_2(\mathbb{R})$; $1 \mapsto \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$ defines a representation (check). The representation $\mathbb{R}^2$ breaks up as $\langle e_1\rangle \oplus \langle e_1 - e_2\rangle$ as subreps even though the matrix is upper triangular but not diagonal.

So we may rephrase our question:

*Question.* When does a representation $V$ break up as a direct sum of subreps?

Not always: clearly $V$ cannot be irreducible. But we see already from the example of $\mathbb{Z}$ that having proper invariant subspaces doesn't always suffice to be able to do this either. However there is an amazing theorem.

**Theorem** (Maschke's Theorem)**.** *Let $G$ be a finite group and $(\rho, V)$ a representation of $G$ over a field $k$ of characteristic zero. Suppose $W \subset V$ is an invariant subspace. Then there is a $G$-invariant complement to $W$ ie a $G$-invariant subspace $U$ of $V$ such that $V = U \oplus W$.*

**Corollary** (Complete reducibility)**.** *If $G$ is a finite group, $(\rho, V)$ a representation over a field of characteristic zero. Then $V \cong W_1 \oplus \cdots W_r$ is a direct sum of representations with each $W_i$ irreducible.*

*Proof.* By induction on $\dim V$. If $\dim V = 0$ or $V$ is irreducible then the result is clear. Otherwise $V$ has a non-trivial $G$-invariant subspace $W$.

By the theorem there is a $G$-invariant complement $U$ and $V \cong U \oplus W$ as $G$-reps. But $\dim U, \dim W < \dim V$, so by induction they each break up as a direct sum of irreducibles subreps. Thus $V$ does also. □

*Example.* We saw before that every representation of $\mathbb{Z}/2$ over $\mathbb{C}$ is a direct sum of 1-dimensional subreps as we may diagonalise $\rho(-1)$. Let's think about how this might generalise:

Suppose that $G$ is a finite abelian group, and $(\rho, V)$ is a complex representation of $G$. Each element $g \in G$ has finite order so has a minimal polynomial dividing $X^n - 1$ for $n = o(g)$. In particular it has distinct roots. Thus there is a basis for $V$ such that $\rho(g)$ is diagonal. But because $G$ is abelian $\rho(g)$ and $\rho(h)$ commute for each pair $g, h \in G$ and so the $\rho(g)$ may be simultaneously diagonalised (Sketch proof: if each $\rho(g)$ is a scalar matrix the result is clear. Otherwise pick $g \in G$ such that $\rho(g)$ is not a scalar matrix. Each eigenspace $E(\lambda)$ of $\rho(g)$ will be $G$-invariant since $G$ is abelian. By induction on $\dim V$ we may solve the problem for each subrep $E(\lambda)$ and then put these subreps back together). Thus $V$ decomposes as a direct sum of one-dimensional reps. Of course, this technique can't work in general because (a) $\rho(g)$ and $\rho(h)$ won't commute in general; (b) not every irreducible rep is one-dimensional in general. Thus we'll need a new idea.

*Example.* Let $G$ act on a finite set $X$, and consider the real permutation representation $\mathbb{R}X = \{f\colon X \to \mathbb{R}\}$ with $(\rho(g)f)(x) = f(g^{-1}x)$.

Idea: with respect to the given basis $\delta_x$ all the matrices $\rho(g)$ are orthogonal; that is they preserve distance. This is because the standard inner product with respect to the basis is $\langle f_1, f_2\rangle = \sum_{x\in X} f_1(x)f_2(x)$ and so for each $g \in G$

$$\langle \rho(g)f_1, \rho(g)f_2\rangle = \sum_{x\in X} f_1(g^{-1}x)f_2(g^{-1}x) = \langle f_1, f_2\rangle$$

since $g$ permutes the elements of $X$.

In particular if $W$ is a subrep of $\mathbb{R}X$ and $W^\perp := \{v \in \mathbb{R}X \mid \langle v, W\rangle = 0\}$ then if $g \in G$ and $v \in W^\perp$ and $w \in W$ we have (suppressing the $\rho$) $\langle w, gv\rangle = \langle g^{-1}w, v\rangle = 0$ since $g^{-1}w \in W$. Thus $G$ preserves $W^\perp$ which is thus a $G$-invariant complement to $W$.

We will first prove our result over $\mathbb{C}$ by showing that every complex representation over $\mathbb{C}$ is equivalent to one whose image is a subgroup of the unitary group $U_n(\mathbb{C})$ and using the idea in this example. Then we will adapt the same idea to work over an arbitrary field of characteristic zero.

Recall: if $V$ is a complex vector space then a Hermitian inner product is a positive definite Hermitian sesquilinear map $(-,-)\colon V \times V \to \mathbb{C}$ that is a map satisfying

(i) $(ax+by, z) = \overline{a}(x,z) + \overline{b}(y,z)$ and $(x, ay+bz) = a(x,y) + b(x,z)$ for $a, b \in \mathbb{C}$, $x, y, z \in V$ (sesquilinear);
(ii) $(x,y) = \overline{(y,x)}$ (Hermitian);
(iii) $(x,x) > 0$ for all $x \in V\backslash\{0\}$ (positive definite).

If $W \subset V$ is a linear subspace of a complex vector space with a Hermitian inner product and $W^\perp = \{v \in V \mid (v,w) = 0\ \forall w \in W\}$ then $W^\perp$ is a vector space complement to $W$ in $V$.

**Definition.** A Hermitian inner product on a $G$-rep $V$ is $G$-invariant if $(gx, gy) = (x,y)$ for all $g \in G$ and $x, y \in V$; equivalently if $(gx, gx) = (x,x)$ for all $g \in G$ and $x \in V$.

**Lemma.** *If $(-,-)$ is a $G$-invariant Hermitian inner product on a $G$-rep $V$ and $W \subset V$ is a subrep then $W^\perp = \{v \in V \mid (v,w) = 0$ for all $w \in W\}$ is a $G$-invariant complement to $W$.*

*Proof.* It suffices to prove that $W^\perp$ is $G$-invariant since $W^\perp$ is a complement to $W$.

Suppose $g \in G$, $x \in W^\perp$ and $w \in W$. Then $(gx, w) = (x, g^{-1}w) = 0$ since $g^{-1}w \in W$. Thus $gx \in W^\perp$ as required. □

**Proposition** (Weyl's unitary trick)**.** *If $V$ is a complex representation of a finite group $G$, then there is a $G$-invariant Hermitian inner product on $V$.*

*Proof.* Pick any Hermitian inner product $\langle -,-\rangle$ on $V$ (ie choose a basis and take the standard inner product obtained by declaring the basis to be orthonormal and extending sesquilinearly). Then define a new inner product $(-,-)$ on $V$ by averaging:

$$(x,y) := |G|^{-1} \sum_{g\in G} \langle gx, gy\rangle.$$

It is easy to see that $(-,-)$ is sesquilinear and Hermitian symmetric and positive definite because $\langle -,-\rangle$ is so.

But now if $h \in G$ and $x, y \in V$ then $(hx, hy) = |G|^{-1}\sum_{g\in G}\langle ghx, ghy\rangle = |G|^{-1}\sum_{g'\in G}\langle g'x, g'y\rangle$ and so $(-,-)$ is $G$-invariant. □

**Corollary** (Maschke's Theorem for complex representations)**.** *Every complex representation of a finite group $G$ splits as a direct sum of irreducible subreps.*

## Lecture 4

**Corollary** (of Weyl's unitary trick)**.** *Every finite subgroup $G$ of $GL_n(\mathbb{C})$ is conjugate to a subgroup of $U_n(\mathbb{C}) := \{A \in \mathrm{Mat}_n(\mathbb{C}) \mid A\overline{A}^T = I\}$.*

*Proof.* First notice that $A \in GL_n(\mathbb{C})$ is unitary if and only if $(Ax, Ay) = (x, y)$ for all $x, y \in \mathbb{C}^n$ (here $(-,-)$ denotes the standard inner product with respect to the standard basis of $\mathbb{C}^n$). Moreover changing basis corresponds to conjugating by an element of $GL_n(\mathbb{C})$.

So we use the unitary trick to find a $G$-invariant Hermitian inner product $\langle -,-\rangle$ and choose an orthonormal basis for $\mathbb{C}^n$ with respect to $\langle -,-\rangle$ using Gram-Schmidt, say.

Let $P$ be the change of basis matrix from the standard basis to the newly constructed basis. Then $\langle Pa, Pb\rangle = (a, b)$ for $a, b \in V$. So for each $g \in G$

$$(P^{-1}gPa, P^{-1}gPb) = \langle gPa, gPb\rangle = \langle Pa, Pb\rangle = (a, b).$$

Thus $P^{-1}gP \in U_n(\mathbb{C})$ for each $g \in G$ as required. □

Thus studying all complex representations of a finite group $G$ is equivalent to studying unitary (ie distance preserving) ones.

We now adapt our proof of complete reducibility to handle any field of characteristic $k$, even if there is no notion of inner product.

**Theorem** (Maschke's Theorem)**.** *Let $G$ be a finite group and $V$ a representation of $G$ over a field $k$ of characteristic zero. Then every subrep $W$ of $V$ has a $G$-invariant complement.*

*Proof.* Choose some projection $\pi\colon V \to W$; ie a $k$-linear map $\pi\colon V \to W$ such that $\pi(w) = w$ for all $w \in W$.

Now $\ker\pi$ is a vector space complement to $W$ since (1) if $v \in \ker\pi \cap W$ then $v = 0$ and (2) $\pi(v - \pi(v)) = 0$ for all $v \in V$ so $V = W + \ker\pi$. Moreover $\ker\pi$ is $G$-invariant if $\pi \in \mathrm{Hom}_G(V, W)$. So we try to build a $G$-linear projection $V \to W$ using $\pi$.

Let $\pi'\colon V \to W$ be defined by $\pi'(v) = \frac{1}{G}\sum_{g\in G} g\pi(g^{-1}v)$.

Notice that

- $\pi'(v) \in W$ for all $v \in V$ since $W$ is $G$-invariant so $g\pi(g^{-1}v) \in W$ for all $g \in G, v \in V$.
- $\pi'$ is $k$-linear since it is a linear combination of $k$-linear maps $g\pi g^{-1}$.
- $\pi'(w) = w$ for all $w \in W$ since $g\pi g^{-1}w = gg^{-1}w = w$.

- $\pi' \in \mathrm{Hom}_G(V, W)$ since if $v \in V$ and $h \in G$ then
$$|G|h\pi'(v) = \sum_{g \in G} hg\pi(g^{-1}v) = \sum_{g' \in G} g'\pi'(g'^{-1}hv) = |G|\pi'(hv)$$
where $g' = hg$ so $g^{-1} = g'^{-1}h$.
  Dividing by $|G|$ we see that $\pi'$ is $G$-linear as required.

Thus $\ker \pi'$ is the required $G$-invariant complement to $W$. □

*Remarks.*

(1) We can explicitly compute $\pi'$ and $\ker \pi'$ given $(\rho, V)$ and $W$.

(2) Notice that we only use char $k = 0$ when we invert $|G|$. So in fact we only need that the characteristic of $k$ does not divide $|G|$. (Exercise: What happens for $\rho\colon \mathbb{Z}/2\mathbb{Z} \to GL_2(\mathbb{F}_2); 1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $W = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \subset \mathbb{F}_2^2$.)

(3) Whenever $V, W$ are $G$-reps we can make
$$\mathrm{Hom}(V, W) = \{\varphi\colon V \to W \mid \varphi \text{ is } k \text{ linear}\}$$
into a $G$-rep by $(g\varphi)(v) = g(\varphi(g^{-1}v))$. Then
$$\mathrm{Hom}_G(V, W) = \{\varphi \in \mathrm{Hom}(V, W) \mid g\varphi = \varphi\}$$
and there is a $k$-linear projection (for char $k$ not dividing $|G|$)
$$\mathrm{Hom}(V, W) \to \mathrm{Hom}_G(V, W)$$
given by $\varphi \mapsto \frac{1}{|G|}\sum_{g \in G} g\varphi$.

(4) In fact every irreducible representation of $G$ is a submodule of the regular representation $kG$ (see Ex Sheet 1 Q10 or the section on characters for a proof in characteristic zero).

An observation that we should have made earlier: if $\theta\colon H \to G$ is a group homomorphism then every representation $\rho\colon G \to GL(V)$ of $G$ induces a representation $\rho\theta\colon H \to GL(V)$ of $H$.

If $H$ is a subgroup of $G$ and $\theta$ is inclusion we call this *restriction to $H$*.

## 3. Schur's Lemma

We've proven in characteristic zero that every representation $V$ of a finite group $G$ decomposed $V = \bigoplus V_i$ with $V_i$ irreducible. We might ask how unique this is. Three possible hopes

(1) (uniqueness of factors) If $\bigoplus_{i=1}^{k} V_i \cong \bigoplus_{i=1}^{k'} V_i'$ with $V_i, V_i'$ irreducible then $k = k'$ and there is $\sigma \in S_k$ such that $V'_{\sigma(i)} \cong V_i$.

(2) (uniqueness of isotypical decomposition) For each $V$ there exist unique subreps $W_1, \ldots, W_k$ st $V = \bigoplus W_i$ and if $V_i \leqslant W_i$ and $V_j \leqslant W_j$ are irred. subreps then $V_i \cong V_j$ if and only if $i = j$.

(3) (uniqueness of decomposition) For each $V$ there is only one way to decompose $V = \bigoplus V_i$ with $V_i$ irreducible.

Notice that (3) is clearly too strong. For example if $G$ is the trivial group and $\dim V > 1$ then every line in $V$ gives an irreducible subrep. This non-uniqueness is roughly measured in this case by $GL(V)$.

Notice also that (2) (and so (1)) is true for $\mathbb{Z}/2\mathbb{Z}$ — the $W_i$ are the eigenspaces of $\rho(1)$.

**Theorem** (Schur's Lemma)**.** *Suppose that $V$ and $W$ are irreducible reps of $G$ over $k$. Then*

*(i) every element of $Hom_G(V, W)$ is either $0$ or an isomorphism,*
*(ii) if $k$ is algebraically closed then $\dim_k \mathrm{Hom}_G(V, W)$ is either $0$ or $1$ .*

In other words irreducible representations are rigid.

*Proof.* (i) Let $\varphi$ be a non-zero $G$-linear map from $V$ to $W$. Then $\ker\varphi$ is a $G$-invariant subspace of $V$. Thus $\ker\varphi = 0$, since it cannot be the whole of $V$. Similarly $\mathrm{im}\varphi$ is a subrep of $W$ so $\mathrm{im}\varphi = W$ since it cannot be 0. Thus $\varphi$ is both injective and surjective, so an isomorphism.

(ii) Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}_G(V, W)$ are non-zero. Then by (i) they are both isomorphisms. Consider $\varphi = \varphi_1^{-1}\varphi_2 \in \mathrm{Hom}_G(V, V)$. Since $k$ is algebraically closed we may find $\lambda$ an eigenvalue of $\varphi$ then $\varphi - \lambda\,\mathrm{id}_V$ has non-trivial kernel and so is zero. Thus $\varphi_1^{-1}\varphi_2 = \lambda\,\mathrm{id}_V$ and $\varphi_2 = \lambda\varphi_1$ as required. □

## Lecture 5

**Proposition.** *If $V, V_1$ and $V_2$ are $k$-representations of $G$ then*

$$\mathrm{Hom}_G(V, V_1 \oplus V_2) \cong \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$$

*and*

$$\mathrm{Hom}_G(V_1, \oplus V_2, V) \cong \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V).$$

*Proof.* Let $\pi_i \colon V_1 \oplus V_2 \to V_i$ be the $G$-linear projection onto $V_i$ with kernel $V_{3-i}$. Then the map $\mathrm{Hom}_G(V, V_1 \oplus V_2) \to \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$ given by $\varphi \mapsto (\pi_1\varphi, \pi_2\varphi)$ has inverse $(\psi_1, \psi_2) \mapsto \psi_1 + \psi_2$.

Similarly the map $\mathrm{Hom}_G(V_1, \oplus V_2, V) \cong \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V)$ given by $\varphi \mapsto (\varphi|_{V_1}, \varphi|_{V_2})$ has inverse $(\psi_1, \psi_2) \mapsto \psi_1\pi_1 + \psi_2\pi_2$. □

Now, recall,

**Theorem** (Schur's Lemma)**.** *Suppose that $V$ and $W$ are irreducible reps of $G$ over $k$. Then*

*(i) every element of $Hom_G(V, W)$ is either $0$ or an isomorphism,*
*(ii) if $k$ is algebraically closed then $\dim_k \mathrm{Hom}_G(V, W)$ is either $0$ or $1$ .*

**Corollary.** *Suppose $k$ is algebraically closed and*

$$V \cong \bigoplus_{i=1}^{r} V_i$$

*is a decomposition of a $k$-rep. of $G$ into irreducible components.*

*Then for each irreducible representation $W$ of $G$,*

$$|\{i \mid V_i \cong W\}| = \dim \mathrm{Hom}_G(W, V).$$

*Proof.* By induction on $r$. If $r = 0, 1$ we're done.

If $r > 1$ consider $V$ as $\left(\bigoplus_{i=1}^{r-1} V_i\right) \oplus V_r$. By the Proposition

$$\dim \mathrm{Hom}_G(W, \left(\bigoplus_{i=1}^{r-1} V_i\right) \oplus V_r) = \dim \mathrm{Hom}_G(W, \bigoplus_{i=1}^{r-1} V_i) + \dim \mathrm{Hom}_G(W, V_r).$$

Now the result follows by the induction hypothesis. □

Important question: How do we actually compute these numbers $\dim \mathrm{Hom}_G(V, W)$.

**Corollary.** *(of Schur's Lemma) Every irreducible complex representation of a finite abelian group $G$ is one-dimensional.*

*Proof.* Let $(\rho, V)$ be a complex irred. rep of $G$. For each $g \in G$, $\rho(g) \in \mathrm{Hom}_G(V, V)$. So by Schur, $\rho(g) = \lambda_g \, \mathrm{id}_V$ for some $\lambda_g \in \mathbb{C}$. Thus for $v \in V$ non-zero, $\langle v \rangle$ is a subrep of $V$. □

**Corollary.** *Every finite abelian group $G$ has precisely $|G|$ complex irreducible representations.*

*Proof.* Let $\rho$ be an irred. complex rep of $G$. By the last corollary, $\dim \rho = 1$. So $\rho \colon G \to \mathbb{C}^\times$ is a group homomorphism.

Since $G$ is a finite abelian group $G \cong C_{n_1} \times \cdots \times C_{n_k}$ some $n_1, \ldots, n_k$. Now if $G = G_1 \times G_2$ is the direct product of two groups then there is a 1-1 correspondance between the set of group homomorphisms $G \to \mathbb{C}^\times$ and the of pairs $(G_1 \to \mathbb{C}^\times, G_2 \to \mathbb{C}^\times)$ given by restriction $\varphi \mapsto (\varphi|_{G_1}, \varphi|_{G_2})$. Thus we may reduce to the case $G = C_n = \langle x \rangle$ is cyclic.

Now $\rho$ is determined by $\rho(x)$ and $\rho(x)^n = 1$ so $\rho(x)$ must be an $n$th root of unity. Moreover we may choose $\rho(x)$ however we like amongst the $n$th roots of 1. □

*Examples.*

$G = C_4 = \langle x \rangle$.

| | $1$ | $x$ | $x^2$ | $x^3$ |
|---|---|---|---|---|
| $\rho_1$ | $1$ | $1$ | $1$ | $1$ |
| $\rho_2$ | $1$ | $i$ | $-1$ | $-i$ |
| $\rho_3$ | $1$ | $-1$ | $1$ | $1$ |
| $\rho_4$ | $1$ | $-i$ | $-1$ | $i$ |

$G = C_2 = \langle x, y \rangle$.

| | $1$ | $x$ | $y$ | $xy$ |
|---|---|---|---|---|
| $\rho_1$ | $1$ | $1$ | $1$ | $1$ |
| $\rho_2$ | $1$ | $-1$ | $1$ | $-1$ |
| $\rho_3$ | $1$ | $1$ | $-1$ | $-1$ |
| $\rho_4$ | $1$ | $-1$ | $-1$ | $1$ |

Note there is no natural correspondence between elements of $G$ and representations $\rho$.

Note too that the rows of these matrices are orthogonal with respect to the standard Hermitian inner product: $\langle v, w \rangle = \sum \overline{v_i} w_i$.

**Lemma.** *If $(\rho_1, V_1)$ and $(\rho_2, V_2)$ are non-isomorphic one-dimensional representations of a finite group $G$ then $\sum_{g \in G} \overline{\rho_1(g)} \rho_2(g) = 0$*

*Proof.* We've seen that $\mathrm{Hom}_G(V_1, V_2)$ is a $G$-rep under $g\varphi(v) = \rho_2(g)\varphi\rho_1(g^{-1})$ and $\sum_{g \in G} g\varphi \in Hom_G(V_1, V_2) = 0$ by Schur. Since $\rho_1(g)$ is always a root of unity, $\rho_1(g^{-1}) = \overline{\rho_1(g)}$. Pick an isomorphism $\varphi \in Hom(V_1, V_2)$. Then $0 = \sum_{g \in G} \rho_2(g)\varphi\rho_1(g^{-1}) = \sum_{g \in G} \overline{\rho_1 g} \rho_2(g)\varphi$ as required. □

**Corollary.** *Suppose $G$ is an abelian group then every complex representation $V$ of $G$ has a unique isotypical decomposition.*

*Proof.* For each homomorphism $\theta_i \colon G \to \mathbb{C}^\times$ $(i = 1, \ldots, |G|)$ we can define $W_i$ to be the subspace of $V$ defined by

$$W_i = \{v \in V \mid \rho(g)v = \theta_i(g)v \text{ for all } g \in G\}.$$

Since $V$ is completely reducible and every irreducible rep of $G$ is one dimensional $V = \sum W_i$. We need to show that for each $i$ $W_i \cap \sum_{j \neq i} W_j = 0$. It is equivalent to show that $\sum w_i = 0$ with $w_i \in W_i$ implies $w_i = 0$ for all $i$.

But $\sum w_i = 0$ with $w_i$ in $W_i$ certainly implies $0 = \rho(g)\sum w_i = \sum \theta_i(g)w_i$. By choosing an ordering $g_1, \dots, g_{|G|}$ of $G$ we see that the $|G| \times |G|$ matrix $\theta_i(g_j)$ is invertible by the lemma. Thus $w_i = 0$ for all $i$ as required. □

**Corollary.** *(of Schur's Lemma) If $G$ has a faithful complex irreducible representation then the centre of $G$, $Z(G)$ is cyclic.*

*Proof.* Let $V$ be a faithful complex irreducible rep of $G$, and let $z \in Z(G)$. Then let $\varphi_z \colon V \to V$ be defined by $\varphi_z(v) = zv$. Since $gz = zg$ for all $g \in G$, $\varphi_z \in \mathrm{Hom}_G(V, V) = \mathbb{C}\,\mathrm{id}_V$ by Schur, $\varphi_z = \lambda_z\,\mathrm{id}_V$, say.

Now $Z(G) \to \mathbb{C}; z \mapsto \lambda_z$ is a representation of $Z(G)$ that must be faithful since $V$ is faithful. In particular $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$. But every such subgroup is cyclic. □

## Lecture 6

## 4. Characters

4.1. **Definitions.** We'll now always assume $k = \mathbb{C}$ although almost always a field of characteristic zero containing all $n$th roots of unity would suffice.

We've seen that to count the number of times an irreducible representation $W$ occurs as a summand of a completely reducible representation $V$ it suffices to compute $\dim \mathrm{Hom}_G(V, W)$ but have no strategy to do this in general. It turns out that the theory of characters makes this very easy.

**Definition.** Given a representation $\rho \colon G \to GL(V)$, the *character* of $\rho$ is the function $\chi = \chi_\rho = \chi_V \colon G \to k$ given by $g \mapsto \mathrm{tr}\,\rho(g)$.

Since for matrices $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, the character does not depend on the choice of basis for $V$ $[\mathrm{tr}(X^{-1}AX) = \mathrm{tr}(AXX^{-1}) = \mathrm{tr}(A)]$. By the same argument we also see that equivalent reps have the same character.

**Proposition.** *Let $(\rho, V)$ be a complex rep of $G$ with character $\chi$*

*(i)* $\chi(e) = \dim V$;
*(ii)* $\chi(g) = \chi(hgh^{-1})$ *for all* $g, h \in G$;
*(iii)* $\chi(g^{-1}) = \overline{\chi(g)}$ *if $o(g)$ is finite.*
*(iv) If $\chi'$ is the character of $(\rho', V')$ then $\chi + \chi'$ is the character of $V \oplus V'$.*

*Proof.*
(i) $\chi(e) = \mathrm{tr}\,\mathrm{id}_V = \dim V$.

(ii) $\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h)^{-1}$. Thus $\rho(hgh^{-1})$ and $\rho(g)$ are conjugate and so have the same trace.

(iii) if $\rho(g)$ has eigenvalues $\lambda_1, \dots, \lambda_n$ (with multiplicity) then $\chi(g) = \sum \lambda_i$. But as $o(g)$ is finite each $\lambda_i$ must be a root of unity. Thus $\overline{\chi(g)} = \sum \overline{\lambda_i} = \sum \lambda_i^{-1}$ but of course the $\lambda_i^{-1}$ are the eigenvalues of $g^{-1}$.

(iv) is clear. □

The proposition tells us that the character of $\rho$ contains very little data; just a complex number for each conjugacy class in $G$. The extraordinary thing that we will see is that it contains all we need to know to reconstruct $\rho$ up to isomorphism.

*Example.* Let $G = D_{2n} = \langle s, t \mid s^2 = 1, t^n = 1, sts^{-1} = t^{-1} \rangle$, the dihedral group of order $2n$ for $n$ odd. This acts on $\mathbb{R}^2$ by symmetries of the $n$-gon; with $t$ acting by rotation by $2\pi/n$ and $s$ acting by a reflection. To compute the character of this rep we just need to know the eigenvalues of the action of each element. Each reflection (element of the form $st^i$) will act by a matrix with eigenvalues $\pm 1$. Thus $\chi(st^i) = 0$ for all $i$. The rotations $t^r$ act by matrices $\begin{pmatrix} \cos 2\pi r/n & -\sin 2\pi r/n \\ \sin 2\pi r/n & \cos 2\pi r/n \end{pmatrix}$ thus $\chi(t^r) = 2\cos 2\pi r/n$.

**Definition.** We say a function $f \colon G \to \mathbb{C}$ is a (complex-valued) *class function* if $f(hgh^{-1}) = f(g)$ for all $g, h \in G$. We'll write $\mathcal{C}_G$ for the complex vector space of class functions on $G$.

Notice that if $G$ is finite and $\mathcal{O}_1, \ldots, \mathcal{O}_r$ is a list of the conjugacy classes of $G$ then the 'delta functions' $\delta_{\mathcal{O}_i} \colon G \to \mathbb{C}$ given by $y \mapsto 1$ if $y \in \mathcal{O}_i$ and $y \mapsto 0$ otherwise form a basis for $\mathcal{C}_G$. In particular $\dim \mathcal{C}_G$ is the number of conjugacy classes in $G$.

We make $\mathcal{C}_G$ into a Hermitian inner product space by defining

$$\langle f, f' \rangle = |G|^{-1} \sum \overline{f(g)} f'(g).$$

It is easy to check that this does define an Hermitian inner product and that the functions $\delta_{\mathcal{O}_i}$ are pairwise orthogonal. Notice that $\langle \delta_{\mathcal{O}_i}, \delta_{\mathcal{O}_i} \rangle = |\mathcal{O}_i|/|G| = |C_G(x_i)|^{-1}$ for any $x_i \in \mathcal{O}_i$.

Thus if $x_1, \ldots, x_r$ are conjugacy class representatives, then we can write

$$\langle f, f' \rangle = \sum_{i=1}^{r} |C_G(x_i)|^{-1} \overline{f(x_i)} f'(x_i).$$

## 4.2. **Orthogonality.**

**Theorem** (Orthogonality of characters)**.** *If $V$ and $V'$ are complex irreducible representations of a finite group $G$ then $\langle \chi_V, \chi_{V'} \rangle$ is 1 if $V \cong V'$ and 0 otherwise.*

Notice that this theorem tells us that the characters of irreducible reps form part of an orthonormal basis for $\mathcal{C}_G$. In particular the number of irreducible representations is bounded above by the number of conjugacy classes of $G$. In fact we'll see that the characters span the space of class functions and so that the number of irreps is precisely the number of conjugacy classes in $G$. We saw this when $G$ is abelian last time.

Recall that if $V, W$ are reps of $G$, $\mathrm{Hom}(V, W) = \{f \colon V \to W \mid f \text{ is } k \text{ linear}\}$ into a $G$-rep by $(g.f)(v) = g(f(g^{-1}v))$. Let's compute the character of $\mathrm{Hom}(V, W)$.

**Lemma.** *If $V$ and $W$ are reps of a group $G$ then for $g \in G$ of finite order, $\chi_{\mathrm{Hom}_G(V,W)}(g) = \overline{\chi_V(g)} \chi_W(g)$.*

*Proof.* Given $g \in G$ we may choose bases $v_1, \ldots, v_n$ for $V$ and $w_1, \ldots, w_m$ for $W$ such that $gv_i = \lambda_i v_i$ and $gw_j = \mu_j w_j$. Then the functions $f_{ij}(v_k) = \partial_{ik} w_j$ extend to linear maps that form a basis for $\mathrm{Hom}(V, W)$ and $(g.f_{ij})(v_i) = \lambda_i^{-1} \mu_j w_j$ thus $g f_{ij} = \lambda_i^{-1} \mu_j f_{ij}$ and $\chi_{\mathrm{Hom}(V,W)}(g) = \sum_{i,j} \lambda_i^{-1} \mu_j = \chi_V(g^{-1}) \chi_W(g) = \overline{\chi_V(g)} \chi_W(g)$. □

**Lemma.** *If $U$ is a rep of $G$ then* $\dim\{u \in U \mid gu = u\} = \langle 1, \chi \rangle = |G|^{-1} \sum_{g \in G} \chi_U(g)$.

*Proof.* Let $\pi \colon U \to U$ be defined by $\pi(u) = |G|^{-1} \sum_{g \in G} gu$, and write $U^G := \{u \in U \mid gu = u\}$. Then $h\pi(u) = \pi(u)$ for all $u \in U$ so $\pi(u) \in U^G$ for all $u \in U$. Moreover $\pi_{U^G} = \mathrm{id}_{U^G}$ by direct calculation. Thus

$$\dim U^G = \mathrm{tr}\, \mathrm{id}_{U^G} = \mathrm{tr}\, \pi = |G|^{-1} \sum_{g \in G} \chi_U(g)$$

as required. □

**Proposition.** *If $V$ and $W$ are representations of $G$ then* $\dim \operatorname{Hom}_G(V,W) = \langle \chi_W, \chi_V \rangle$.

*Proof.* This follows immediately from the two lemmas. □

**Corollary** (Orthogonality of characters)**.** *If $\chi$, $\chi'$ are characters of irreducible reps then $\langle \chi, \chi' \rangle = \delta_{\chi,\chi'}$.*

*Proof.* Apply the Proposition and Schur's Lemma □

Suppose now that $V_1, \ldots, V_k$ is the list of all irreducible complex reps of $G$ up to isomorphism and the corresponding characters are $\chi_1, \ldots, \chi_k$. Then Maschke's Theorem tells us that any representation $V$ may be written as a direct sum of copies of the $V_i$, $V \cong \bigoplus n_i V_i$. Thus $\chi = \sum n_i \chi_i$.

As the $\chi_i$ are orthonormal we may compute $\langle \chi, \chi_i \rangle = n_i$. This is another proof that the decomposition factors of $V$ are determined by their composition factors. However we get more: the composition factors of $V$ can be computed purely from its character; that is if we have a record of each of the irreducible characters, then we now have a practical way of calculating how a given representation breaks up as a direct sum of its irreducible components. Our main goal now is to investigate how we might produce such a record of the irreducible characters.

**Corollary.** *If $\rho$ and $\rho'$ are reps of $G$ then they are isomorphic if and only if they have the same character.*

*Proof.* We have already seen that isomorphic reps have the same character. Suppose that $\rho$ and $\rho'$ have the same character $\chi$. Then they are each isomorphic to $\langle \chi_1, \chi \rangle \rho_1 \oplus \cdots \oplus \langle \chi_k, \chi \rangle \rho_k$ and thus to each other. □

## Lecture 7

We begin by recalling

**Theorem** (Orthogonality of characters)**.** *If $V$ and $V'$ are complex irreducible representations of a finite group $G$ then $\langle \chi_V, \chi_{V'} \rangle$ is 1 if $V \cong V'$ and 0 otherwise.*

**Corollary.** *If $\rho$ and $\rho'$ are reps of $G$ then they are isomorphic if and only if they have the same character.*

Notice that complete irreducibility was a key part of the proof of this corollary, as well as orthogonality of characters. For example the two reps of $\mathbb{Z}$ given by $1 \mapsto id_{\mathbb{C}^2}$ and $1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are not isomorphic but have the same trace. Complete irreducibility tells us we don't need to worry about gluing.

**Corollary.** *If $\rho$ is a complex representation of $G$ with character $\chi$ then $\rho$ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

*Proof.* One direction follows immediately from the theorem on orthogonality of characters. For the other direction, assume that $\langle \chi, \chi \rangle = 1$. Then we may write $\chi = \sum n_i \chi_i$ for some non-negative integers $n_i$. By orthogonality of characters $1 = \langle \chi, \chi \rangle = \sum n_i^2$. Thus $\chi = \chi_j$ for some $j$, and $\chi$ is irreducible. □

This is a good way of calcuating whether a representation is irreducible.

*Examples.*

(1) Consider the action of $D_6$ on $\mathbb{C}^2$ by extending the symmetries of a triangle. $\chi(1) = 2$, $\chi(12) = \chi(23) = \chi(13) = 0$, and $\chi(123) = \chi(132) = -1$. Now
$$\langle \chi, \chi \rangle = 1/6(2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = 1$$
so this rep is irreducible.

(2) Consider the action of $S_4$ on $\mathbb{C}X$ for $X = \{1, 2, 3, 4\}$ induced from the natural action of $S_4$ on $X$. The conjugacy classes in $S_4$ are 1 of size 1, $(ab)$ of size $\binom{4}{2} = 6$, $(abc)$ of size $4.2 = 8$, $(ab)(cd)$ of size 3 and $(abcd)$ of size 6.

We can compute that the character of this rep is given by
$$\chi(g) = \#\{\text{fixed points of } g\}.$$
So $\chi(1) = 1$, $\chi((ab)) = 2$, $\chi((abc)) = 1$ and $\chi((ab)(cd) = \chi(abcd) = 0$. Thus $\langle \chi, \chi \rangle = 1/24(4^2+6{\cdot}2^2+8{\cdot}1^2+3{\cdot}0^2) = 2$. Thus if we decompose $\chi = \sum n_i\chi_i$ into irreducibles we know $\sum n_i^2 = 2$ then we must have $\chi = \chi' + \chi''$ with $\chi'$ and $\chi''$ non-isomorphic irreps.

Notice that $\langle 1, \chi \rangle = 1/24(4+6\cdot 2+8\cdot 1+0) = 1$ so one of the irreducible constituents is the trivial rep. The other has character $\chi - 1$.

In fact we have seen these subreps explicitly in this case. The constant functions gives a trivial subrep and the orthogonal complement with respect to the standard inner product (that is the set of functions that sum to zero) gives the other.

**Theorem** (The character table is square)**.** *The irreducible characters of a finite group $G$ form a basis for the space of class functions $\mathcal{C}_G$ on $G$.*

**Corollary.** *The number of irreducible representations is the number of conjugacy classes in the group.*

**Corollary.** *For each $g \in G$, $\chi(g)$ is real for every character $\chi$ if and only if $g$ is conjugate to $g^{-1}$.*

*Proof.* Since $\chi(g^{-1}) = \overline{\chi(g)}$, $\chi(g)$ is real for every character $\chi$ if and only if $\chi(g) = \chi(g^{-1})$ for every character $\chi$. Since the irreducible characters span the space of class functions this is equivalent to $g$ and $g^{-1}$ living in the same conjugacy class. □

*Proof of Theorem.* We already know that the irreducible characters are linearly independent (and orthonormal) we need to show that they span $\mathcal{C}_G$. Let $I = \langle \chi_1, \dots, \chi_r \rangle$ be the span of the irred. characters. We need to show that $I^\perp = 0$.

Suppose $f : G \to \mathbb{C} \in I^\perp$. For each representation $(\rho, V)$ of $G$ we may define $\varphi \in \mathrm{Hom}(V, V)$ by $\varphi = \frac{1}{|G|}\sum_{g\in G} \overline{f(g)}\rho(g)$. Now
$$\rho(h)^{-1}\varphi\rho(h) = \frac{1}{|G|}\sum_{g\in G} \overline{f(g)}\rho(h^{-1}gh) = \frac{1}{|G|}\sum_{g'\in G} \overline{f(g')}\rho(g')$$
since $f$ is a class function, and we see that in fact $\varphi \in \mathrm{Hom}_G(V, V)$. Moreover
$$\operatorname{tr}\varphi = \langle f, \operatorname{tr}\rho \rangle = 0$$
since $f \in I^\perp$.

Now if $V$ is irreducible then Schur's Lemma tells us that $\varphi = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbb{C}$. Since $\operatorname{tr}\varphi = 0$ it follows that $\lambda = 0$ and so $\varphi = 0$.

But every representation breaks up as a direct sum of irreducible representations $V = \bigoplus V_i$ and $\varphi$ breaks up as $\bigoplus \varphi_i$. So $\varphi = 0$ always.

But if $V$ is the regular representation $\mathbb{C}G$ then $\varphi \partial_e = |G|^{-1} \sum_{g \in G} \overline{f(g)} \partial_g = \overline{f}$. Thus $f = 0$. □

4.3. **Character tables.** We now want to classify all the irreducible representations of a given finite group and we know that it suffices to write down the characters of each one.

The character table of a group is defined as follows: we list the conjugacy classes of $G$, $\mathcal{O}_1, \ldots, \mathcal{O}_r$ (by convention always $\mathcal{O}_1 = \{e\}$) and choose $g_i \in \mathcal{O}_i$ we then list the irreducible characters $\chi_1, \ldots, \chi_k$ (by convention $\chi_1 = \chi_{\mathbb{C}}$ the character of the trivial rep. Then we write the matrix

| | $e$ | $x_2$ | $\cdots$ | $x_i$ | $\cdots$ | $x_k$ |
|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | $\cdots$ | 1 | $\cdots$ | 1 |
| $\vdots$ | | | | $\vdots$ | | |
| $\chi_j$ | $\cdots$ | $\cdots$ | $\cdots$ | $\chi_j(x_i)$ | $\cdots$ | $\cdots$ |
| $\vdots$ | | | | $\vdots$ | | |
| $\chi_k$ | | | | $\vdots$ | | |

*Examples.*

(1) $C_3 = \langle x \rangle$

| | $e$ | $x$ | $x^2$ |
|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | $\omega^2$ | $\omega$ |

Notice that the rows are indeed orthogonal. The columns are too in this case.

(2) $S_3$

There are three conjugacy classes: the identity is in a class on its own $\mathcal{O}_1$; the three transpositions live in a another class $\mathcal{O}_2$; and the two 3-cycles live in the third class $\mathcal{O}_3$.

There are three irreducible representations all together. We know that the trivial representation $\mathbf{1}$ has character $\mathbf{1}(g) = 1$ for all $g \in G$. We also know another 1-dimensional representation $\epsilon \colon S_3 \to \{\pm 1\}$ given by $g \mapsto 1$ if $g$ is even and $g \mapsto -1$ if $g$ is odd.

To compute the character $\chi$ of the last representation we may use orthogonality of characters. Let $\chi(e) = a$, $\chi((12)) = b$ and $\chi((123)) = c$ ($a$, $b$ and $c$ are each real since each $g$ is conjugate to its inverse). We know that $0 = \langle \nVdash, \chi \rangle = \frac{1}{6}(a + 3b + 2c)$, $0 = \langle \epsilon, \chi \rangle = \frac{1}{6}(a - 3b + 2c)$, and $1 = \langle \chi, \chi \rangle = \frac{1}{6}(a^2 + 3b^2 + 2c^2)$. Thus we see quickly that $b = 0$, $a + 2c = 0$ and $a^2 + 2c^2 = 0$. We also know that $a$ is a positive integer. Thus $a = 2$ and $c = -1$.

| | 1 | 3 | 2 |
|---|---|---|---|
| | $e$ | (12) | (123) |
| $\mathbf{1}$ | 1 | 1 | 1 |
| $\epsilon$ | 1 | $-1$ | 1 |
| $\chi$ | 2 | 0 | $-1$ |

In fact we already knew about this 2-dimensional representation; it is the one coming from the symmetries of a triangle inside $\mathbb{R}^2$.

## Lecture 8

**Proposition** (Column Orthogonality)**.** *If $G$ is a finite group and $\chi_1, \dots, \chi_r$ is a complete list of the irreducible characters of $G$ then for each $g, h \in G$,*

$$\sum_{i=1}^{r} \overline{\chi_i(g)}\chi_i(h) = 0 \text{ if } g \text{ and } h \text{ are not conjugate in } G$$
$$= |C_G(g)| \text{ otherwise.}$$

*In particular* $\sum_{i=1}^{r} \dim V_i^2 = |G|$.

*Example.* $S_3$

| | 1 | 3 | 2 |
|---|---|---|---|
| | $e$ | $(12)$ | $(123)$ |
| $\mathbf{1}$ | 1 | 1 | 1 |
| $\epsilon$ | 1 | $-1$ | 1 |
| $\chi$ | 2 | 0 | $-1$ |

$$1^2 + 1^2 + 2^2 = 6 = |S_3|$$
$$1^2 + (-1)^2 + 0^2 = 2 = |C_{S_3}((12))|$$
$$1^2 + 1^2 + (-1)^2 = 3 = |C_{S_3}((123))|$$
$$1 \cdot 1 + 1 \cdot -1 + 2 \cdot 0 = 0$$

*etc.*

*Proof of Proposition.* Let $X$ be character table thought of as a matrix; $X_{ij} = \chi_i(g_j)$ and let $D$ be the diagonal matrix whose diagonal entries are $|C_G(g_i)|$

Orthogonality of characters tell us that

$$\sum_k |C_G(g_k)|^{-1}\overline{X_{ik}}X_{jk} = \partial_{ij}$$

ie $\overline{X}D^{-1}X^t = I$.

Since $X$ is square we may write this as $D^{-1}\overline{X}^t = X^{-1}$. Thus $\overline{X}^t X = D$. This may be rewritten as $\sum_k \overline{X_{ki}}X_{kj} = D_{ij}$. ie $\sum_k \overline{\chi_k(g_i)}\chi_k(g_j) = \partial_{ij}|C_G(g_i)|$ as required. ☐

*Examples.*
$G = S_4$

| | 1 | 3 | 8 | 6 | 6 |
|---|---|---|---|---|---|
| | $e$ | $(12)(34)$ | $(123)$ | $(12)$ | $(1234)$ |
| $\mathbf{1}$ | 1 | 1 | 1 | 1 | 1 |
| $\epsilon$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 3 | $-1$ | 0 | 1 | $-1$ |
| $\epsilon\chi_3$ | 3 | $-1$ | 0 | $-1$ | 1 |
| $\chi_5$ | 2 | 2 | $-1$ | 0 | 0 |

The trivial $\mathbf{1}$ and sign $\epsilon$ characters may be constructed in the same way as for $S_3$. We calculated last time that the natural permuation character breaks up as the sum of a trivial character and a character whose values $\chi_3(g)$ are the number of fixed points of $g$ minus 1.

We saw on Example Sheet 1 (Q2) that given a 1-dimensional represntation $\theta$ and an irreducible representation $\rho$ we may form another irreducible representation

$\theta \otimes \rho$ by $\theta \otimes \rho(g) = \theta(g)\rho(g)$. It is not hard to see that $\chi_{\theta\otimes\rho}(g) = \theta(g)\chi_\rho(g)$. Thus we get another irreducible character $\epsilon\chi_3$. (Exercise: prove that $\theta(g)\rho(g)$ is always irreducible using characters)

We can then complete the character table using column orthogonality: We note that $24 = 1^2 + 1^2 + 3^2 + 3^2 + \chi_5(e)^2$ thus $\chi_5(e) = 2$. Then using $\sum_1^5 \chi_i(1)\chi_i(g) = |C_G(g)|$ we can construct the remaining values in the table.

Notice that the two dimensional representation corresponding to $\chi_5$ may be obtained by composing the surjective group homomorphism $S_4 \to S_3$ (with kernel the Klein-4-group) with the irreducible two dimension rep of $S_3$.

$G = A_4$. Each irreducible representation of $S_4$ may be restricted to $A_4$ and its character values on elements of $A_4$ will be unchanged. In this way we get three characters of $A_4$, $\mathbf{1}$, $\psi_2 = \chi_3|_{A_4}$ and $\psi_3 = \chi_5|_{A_4}$. If we compute $\langle \mathbf{1}, \mathbf{1}\rangle$ we of course get 1. If we compute $\langle \psi_2, \psi_2\rangle$ we get $\frac{1}{12}(3^2 + 3 * 1^1 + 8 * 0^2) = 1$ so $\psi_2$ remains irreducible. However $\langle \psi_3, \psi_3\rangle = \frac{1}{12}(2^2 + 3 * 2^2 + 8 * (-1)^2) = 2$ so $\psi_3$ breaks up into two non-isomorphic irreducible reps of $A_4$.

*Exercise.* Use this infomation to construct the whole character table of $A_4$.

4.4. **Permuation representations.** Suppose that $X$ is a finite set with a $G$-action. Recall that $\mathbb{C}X = \{f\colon X \to \mathbb{C}\}$ is a representation of $G$ via $gf(x) = f(g^{-1}x)$.

**Lemma.** *If $\chi$ is the character of $\mathbb{C}X$ then $\chi(g) = |\{x \in X \mid gx = x\}|$.*

*Proof.* If $X = \{x_1, \ldots, x_d\}$ and $gx_i = x_j$ then $g\partial_{x_i} = \partial_{x_j}$ so the $i$th column of $g$ has a 1 in the $j$th entry and zeros elsewhere. So it contributes 1 to the trace precisely if $x_i = x_j$. □

**Corollary.** *If $V_1, \ldots, V_k$ is a complete list of irreducible reps of a finite group $G$ then the regular representation $kG$ decomposes as $n_1V_1 \oplus \cdots \oplus n_kV_k$ with $n_i = \dim V_i = \chi_i(e)$. In particular $|G| = \sum n_i^2$.*

*Proof.* $\chi_{kG}(e) = |G|$ and $\chi_{kG}(g) = 0$ for $g \neq e$. Thus if we decompose $kG$ we obtain

$$n_i = \langle \chi_{kG}, \chi_i\rangle = |G|^{-1}|G|\chi_i(e) = \chi_i(e)$$

as required. □

**Proposition** (Burnside's Lemma)**.** *Let $G$ be a finite group and $X$ a finite set with a $G$-action and $\chi$ the character of $\mathbb{C}X$. Then $\langle \mathbf{1}, \chi\rangle$ is the number of orbits of $G$ on $X$.*

*Proof.* If we decompose $X$ into a disjoint of orbits $X_1 \cup \cdots \cup X_k$ then we've seen that $\mathbb{C}X = \bigoplus_{i=1}^k \mathbb{C}X_i$. So $\chi_X = \sum_{i=1}^k \chi_{X_i}$ and we may reduce to the case that $G$-acts transitively on $X$.

Now

$$\begin{aligned} |G|\langle \chi_X, 1\rangle &= \sum_{g\in G} \chi_X(g) = \sum_{g\in G} |\{x \in X \mid gx = x\} \\ &= |\{(g,x) \in G \times X \mid gx = x\}| = \sum_{x\in X} |\{g \in G \mid gx = x\} \\ &= \sum_{x\in X} |\operatorname{Stab}_G(x)| = |X||\operatorname{Stab}_G(X)| = |G| \end{aligned}$$

as required. □

## Lecture 9

Recall from last time,

**Lemma.** *If $\chi$ is the character of $\mathbb{C}X$ then $\chi(g) = |\{x \in X \mid gx = x\}|$.*

and

**Proposition** (Burnside's Lemma)**.** *Let $G$ be a finite group and $X$ a finite set with a $G$-action and $\chi_X$ the character of $\mathbb{C}X$. Then $\langle \mathbf{1}, \chi_X \rangle$ is the number of orbits of $G$ on $X$.*

If $X$ is a set with a $G$-action we may view $X \times X$ as a set with a $G$-action via $(g,(x,y)) \mapsto (gx, gy)$.

**Corollary.** *If $G$ is a finite group and $X$ is a finite set with a $G$-action and $\chi$ is the character of the permutation representation $\mathbb{C}X$ then $\langle \chi, \chi \rangle$ is the number of $G$-orbits on $X \times X$.*

*Proof.* Notice that $(x, y)$ is fixed by $g \in G$ if and only if both $x$ and $y$ are fixed. Thus $\chi_{X\times X}(g) = \chi_X(g)\chi_X(g)$ by the lemma.

Now $\langle \chi_X, \chi_X \rangle = \frac{1}{|G|}\sum_{g\in G} \chi_X(g)\chi_X(g) = \langle \mathbf{1}, \chi_{X\times X}\rangle$ and the result follows from Burnside's Lemma. □

*Remark.* If $X$ is any set with a $G$-action with $|X| > 1$ then $\{(x, x)|x \in X\} \subset X\times X$ is $G$-stable and so is the complement $\{(x, y) \in X \times X \mid x \neq y\}$.

We say that $G$ acts 2-*transitively* on $X$ if $G$ has only two orbits on $X\times X$. Given a 2-transitive action of $G$ on $X$ we've seen that the character $\chi$ of the permutation representation satisfies $\langle \chi, \chi\rangle = 2$ and $\langle \mathbf{1}, \chi\rangle = 1$. Thus $\mathbb{C}X$ has two irreducible summands — the constant functions and the functions $f$ such that $\sum_{x\in X} f(x) = 0$.

*Exercise.* If $G = GL_2(\mathbb{F}_p)$ then decompose the permutation rep of $G$ coming from the action of $G$ on $\mathbb{F}_p \cup \{\infty\}$ by Mobius transformations.

## 5. Normal subgroups and lifting characters

**Lemma** (cf Example Sheet 1 Q3)**.** *Suppose $N$ is a normal subgroup of $G$.*

*For every representation $\overline{\rho}\colon G/N \to GL(V)$, there is a representation $\rho\colon G \to GL(V)$ obtained by composing $\overline{\rho}$ with the natural surjection $G \to G/N$. In this way there is a 1-1 correspondance between representations of $G/N$ and representations of $G$ with kernel containing $N$.*

*The characters $\chi_\rho$ and $\chi_{\overline{\rho}}$ of $\rho$ and $\overline{\rho}$ respectively satisfy $\chi_\rho(g) = \chi_{\overline{\rho}}(gN)$ for each $g \in G$.*

*Moreover the correspondance restricts to a 1-1 correspondance between irreducible representations of $G/N$ and irreducible representations of $G$ with kernel containing $N$.*

*Proof.* The first paragraph follows from the first isomorphism theorem for groups.

For $g \in G$, $\chi_\rho(g) = \operatorname{tr}\rho(g) = \operatorname{tr}\overline{\rho}(gN) = \chi_{\overline{\rho}}(gN)$ since $\rho(g) = \overline{\rho}(gN)$.

We could do the last part directly but let's use characters:

$\langle \chi_\rho, \chi_\rho\rangle_G = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_\rho(g)}\chi_\rho(g) = \frac{|N|}{|G|}\sum_{gN\in G/N} \overline{\chi_{\overline{\rho}}(gN)}\chi_{\overline{\rho}}(gN) = \langle \chi_{\overline{\rho}}, \chi_{\overline{\rho}}\rangle_{G/N}$. □

*Remark.* We saw this when we computed the character table for $G = S_4$ and $N = V_4 = \langle (12)(34), (13)(24) \rangle$ last time.

| | 1 | 3 | 8 | 6 | 6 |
|---|---|---|---|---|---|
| | $e$ | (12)(34) | (123) | (12) | (1234) |
| **1** | 1 | 1 | 1 | 1 | 1 |
| $\epsilon$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 2 | 2 | $-1$ | 0 | 0 |
| $\chi_3$ | 3 | $-1$ | 0 | 1 | $-1$ |
| $\epsilon\chi_3$ | 3 | $-1$ | 0 | $-1$ | 1 |

$\mathbf{1}, \epsilon$ and $\chi_5$ all have kernel containing $V_4$ and we can see the character table for $S_3 \cong G/V_4$ inside the character table for $S_4$.

**Definition.** The *derived subgroup* of a group $G$ is the subgroup $G'$ generated by all elements of the form $ghg^{-1}h^{-1}$ with $g, h \in G$.

**Lemma.** *$G'$ is the unique smallest normal subgroup of $G$ such that $G/G'$ is abelian (that is if $G/N$ is abelian then $G' \subset N$).*

*$G$ has precisely $|G/G'|$ representations of dimension 1.*

*Proof.* Suppose $N$ is a normal subgroup of $G$. Then $G/N$ is abelian if and only if $gNhN = hNgN$ for all $g, h \in G$. Thus $ghg^{-1}h^{-1}N = N$ and $ghg^{-1}h^{-1} \in N$. Thus $G' \leqslant N$ and for the first part it suffices to prove that $G'$ is normal. But if $g, h, k \in G$ then

$$k(ghg^{-1}h^{-1})k^{-1} = (kgk^{-1})(khk^{-1})(kgk^{-1})^{-1}(khk^{-1})^{-1}$$

and it follows easily that $G'$ is normal.

For the last part, if $\rho \colon G \to GL_1(k)$ is a 1-dimensional rep then $\rho(ghg^{-1}h^{-1}) = \rho(g)\rho(h)\rho(g)^{-1}\rho(h)^{-1}$ so by the previous lemma, 1-dimensional reps of $G$ correspond to 1-dimensional reps of $G/G'$. We've seen already that there are $|G/G'|$ of the latter. □

## 6. The character ring

Given a group $G$, the set of class functions $\mathcal{C}_G$ comes equipped with certain algebraic structures: it is a commutative ring under pointwise addition and multiplication — ie $(f_1 + f_2)(g) = f_1(g) + f_2(g)$ and $f_1f_2(g) = f_1(g)f_2(g)$ for each $g \in G$, the additive identity is the constant function value 0 and the multiplicative identity constant value 1; there is a ring automorphism $*$ of order two given by $f^*(g) = f(g^{-1})$; and, when $G$ is finite, there is an inner product given by $\langle f_1, f_2 \rangle = \frac{1}{|G|}\sum_{g \in G} f_1^*(g)f_2(g)$.

We will see that all this structure is related to structure on the category of representations: we have already seen some of this. If $V_1$ and $V_2$ are representations with characters $\chi_1$ and $\chi_2$ then $\chi_1 + \chi_2 = \chi_{V_1 \oplus V_2}$ and $\langle \chi_1, \chi_2 \rangle = \dim \operatorname{Hom}_G(V_1, V_2)$.

**Definition.** The *character ring* $R(G)$ of a group $G$ is defined by

$$R(G) := \{\chi_1 - \chi_2 \mid \chi_1, \chi_2 \text{ are characters of reps of } G\} \subset \mathcal{C}_G.$$

We'll see that the character ring inherits all the algebraic structure of $\mathcal{C}_G$ mentioned above.

6.1. **Duality.**

**Definition.** If $G$ is group and $(\rho, V)$ is a representation of $G$ then the *dual representation* $(\rho^*, V^*)$ of $G$ is given by $\rho^*(g)(\theta)(v) = \theta(\rho(g^{-1})(v)$ for $\theta \in V^*$.

*Remark.* We've already seen the dual representation; if $k$ is the trivial representation then $V^* = \mathrm{Hom}(V, k)$.

**Lemma.** *The dual representation is a representation and* $\chi_{V^*} = \chi^*(V)$.

*Proof.* First,

$$\begin{aligned} \rho^*(gh)\theta(v) &= \theta(\rho(gh)^{-1}(v)) \\ &= \theta(\rho(h)^{-1}\rho(g)^{-1}(v)) \\ &= \rho^*(h)\theta(\rho(g)^{-1}(v)) \\ &= \rho^*(g)\rho^*(h)\theta(v) \end{aligned}$$

as required.

Suppose that $v_1, \ldots, v_m$ is a basis for $V$ and $\theta_1, \ldots, \theta_m$ is the dual basis for $V^*$. Given $g \in G$, if the diagonal entries of $\rho(g^{-1})$ wrt $v_1, \ldots, v_m$ are $\lambda_1, \ldots, \lambda_m$ then we can compute the diagonal entries of $\rho^*$ wrt $\theta_1, \ldots, \theta_m$ as $\rho^*(\theta_k)(v_k) = \theta_k(\rho(g^{-1}v_k) = \lambda_k$. In particular $\operatorname{tr} \rho^*(g) = \sum \lambda_i = \operatorname{tr} \rho(g^{-1})$ as required. □

**Definition.** We say that $V$ is *self-dual* if $V \cong V^*$ as representations of $G$.

Over $\mathbb{C}$, $V$ is self-dual if and only if $\chi_V(g) \in \mathbb{R}$ for all $g \in G$.

*Examples.*

(1) $G = C_3 = \langle x \rangle$ and $V = \mathbb{C}$. If$\rho$ is given by $\rho(x) = \omega = e^{\frac{2\pi i}{3}}$ then $\rho^*(x) = \omega^2 = \overline{\omega}$ so $V$ is not self-dual
(2) $G = S_n$: since $g$ is always conjugate to its inverse in $S_n$, $\chi^* = \chi$ always and so every representation is self-dual.

## Lecture 10

6.2. **Tensor products.** Suppose that $V$ and $W$ are vector spaces over a field $k$, with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ respectively. We may view $V \oplus W$ either as the vector space with basis $v_1, \ldots, v_m, w_1, \ldots, w_n$ (so $\dim V \oplus W = \dim V + \dim W$) or more abstractly as the vector space of pairs $(v, w)$ with $v \in V$ and $w \in W$ and pointwise operations.

**Definition.** The *tensor product* $V \otimes W$ of $V$ and $W$ is the vector space with basis given by symobls $v_i \otimes w_j$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ and so

$$\dim V \otimes W = \dim V \cdot \dim W.$$

*Example.* If $X$ and $Y$ are sets then $kX \otimes kY$ has basis $\partial_x \otimes \partial_y$ for $x \in X$ and $y \in Y$. Identifying this element with the function $\partial_{x,y}$ on $X \times Y$ given by $\partial_{x,y}(x', y') = \partial_{xx'}\partial_{yy'}$ we see that $kX \otimes kY \cong kX \times Y$.

If $v = \sum \lambda_i v_i \in V$ and $w = \sum \mu_j w_j \in W$, it is common to write $v \otimes w$ for the element $\sum_{i,j} (\lambda_i \mu_j) v_i \otimes w_j \in V \otimes W$. But note that usually not every element of $V \otimes W$ may be written in the form $v \otimes w$ (eg $v_1 \otimes w_1 + v_2 \otimes w_2$).

**Lemma.** *There is a bilinear map* $V \times W \to V \otimes W$ *given by* $(v, w) \mapsto v \otimes w$.

*Proof.* First, we should prove that if $x, x_1, x_2 \in V$ and $y, y_1, y_2 \in W$ then

$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$

and

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y.$$

We'll just do the first; the second is symmetric.

Write $x = \sum_i \lambda_i v_i$, $y_k = \sum_j \mu_j^k w_j$ for $k = 1, 2$. Then

$$x \otimes (y_1 + y_2) = \sum_{i,j} \lambda_i (\mu_j^1 + \mu_j^2) v_i \otimes w_j$$

and

$$x \otimes y_1 + x \otimes y_2 = \sum_{i,j} \lambda_i \mu_j^1 v_i \otimes w_j + \sum_{i,j} \lambda_i \mu_j^2 v_i \otimes w_j.$$

These are equal.

We should also prove that for $\lambda \in k$ and $v \in V$ and $w \in W$ then

$$(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w).$$

The proof is similar to the above. □

*Exercise.* Show that given vector spaces $U, V$ and $W$ there is a $1-1$ correspondance between

$$\{\text{linear maps } V \otimes W \to U\} \leftrightarrow \{\text{bilinear maps } V \times W \to U\}$$

given by composition with the bilinear map $(v, w) \to v \otimes w$ above.

**Lemma.** *If $x_1, \ldots, x_m$ is any basis of $V$ and $y_1, \ldots, y_m$ is any basis of $W$ then $x_i \otimes y_j$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ is a basis for $V \otimes W$. Thus the definition of $V \otimes W$ does not depend on the choice of bases.*

*Proof.* It suffices to prove that the set $\{x_i \otimes y_j\}$ spans $V \otimes W$ since it has size $mn$. But if $v_i = \sum_r A_{ri} x_r$ and $w_j = \sum_s B_{sj} y_s$ then $v_i \otimes w_j = \sum_{r,s} A_{ri} B_{sj} x_i \otimes y_j$. □

*Remark.* In fact we could have defined $V \otimes W$ in a basis independent way in the first place: let $F$ be the (infinite dimensional) vector space with basis $v \otimes w$ for every $v \in V$ and $w \in W$; and $R$ be the subspace generated by $(\lambda v) \otimes w - \lambda(v \otimes w)$, $v \otimes (\lambda w) - \lambda(v \otimes w)$ for $v \in V$, $w \in W$ and $\lambda \in k$ along with $(x_1 + x_2) \otimes y - x_1 \otimes y - x_2 \otimes y$ and $x \otimes (y_1 + y_2) - x \otimes y_1 - x \otimes y_2$ for $x, x_1, x_2 \in V$ and $y, y_1, y_2 \in W$; then $V \otimes W \cong F/R$ naturally.

*Exercise.* Show that for vector spaces $U, V$ and $W$ there is a natural (basis independent) isomorphism

$$(U \oplus V) \otimes W \to (U \otimes W) \oplus (V \otimes W).$$

**Definition.** Suppose that $V$ and $W$ are as above and $\varphi\colon V \to V$ and $\psi\colon W \to W$ are linear maps. We can define $\varphi \otimes \psi\colon V \otimes W \to V \otimes W$ as follows:

$$(\varphi \otimes \psi)(v_i \otimes w_j) = \varphi(v_i) \otimes \psi(w_j).$$

*Example.* If $\varphi$ is represented by the matrix $A_{ij}$ and $\psi$ is represented by the matrix $B_{ij}$ and we order the basis $v_i \otimes w_j$ lexicographically (ie $v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_1 \otimes w_n, v_2 \otimes w_1, \ldots, v_m \otimes w_n$) then $\varphi \otimes \psi$ is represented by the block matrix

$$\begin{pmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & A_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

**Lemma.** *The linear map $\varphi \otimes \psi$ does not depend on the choice of bases.*

*Proof.* It suffices to show that for any $v \in V$ and $w \in W$,

$$(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w).$$

Writing $v = \sum \lambda_i v_i$ and $w = \sum \mu_j w_j$ we see

$$(\varphi \otimes \psi)(v \otimes w) = \sum_{i,j} \lambda_i \mu_j \varphi(v_i) \otimes \psi(w_j) = \varphi(v) \otimes \psi(w)$$

as required. □

*Remark.* The proof really just says $V \times W \to V \otimes W$ defined by $(v, w) \mapsto \varphi(v) \otimes \psi(w)$ is bilinear and $\phi \otimes \psi$ is its correspondant in the bijection

$$\{\text{linear maps } V \otimes W \to V \otimes W\} \to \{\text{bilinear maps } V \times W \to V \otimes W\}$$

above.

**Lemma.** *Suppose that $\varphi, \varphi_1, \varphi_2 \in \mathrm{Hom}(V, V)$ and $\psi, \psi_1, \psi_2 \in \mathrm{Hom}(W, W)$*

*(i)* $(\varphi_1\varphi_2) \otimes (\psi_1\psi_2) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2) \in \mathrm{Hom}(V \otimes W, V \otimes W)$*;*
*(ii)* $\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes W}$*; and*
*(iii)* $\mathrm{tr}(\varphi \otimes \psi) = \mathrm{tr}\,\varphi \cdot \mathrm{tr}\,\psi$.

*Proof.* Given $v \in V$, $w \in W$ we can use the previous lemma to compute

$$(\varphi_1\varphi_2) \otimes (\psi_1\psi_2)(v \otimes w) = \varphi_1\varphi_2(v) \otimes \psi_1\psi_2(w) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2)(v \otimes w).$$

Since elements of the form $v \otimes w$ span $V \otimes W$ and all maps are linear it follows that

$$(\varphi_1\varphi_2) \otimes (\psi_1\psi_2) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2)$$

as required.

(ii) is clear.

For the formula relating traces it suffices to stare at the example above:

$$\mathrm{tr}\begin{pmatrix} A_{11}B & A_{12}B & \cdots \\ A_{21}B & A_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \sum_{i,j} B_{ii}A_{jj} = \mathrm{tr}\,A\,\mathrm{tr}\,B.$$

□

## Lecture 11

Recall the lemma from the end of last time

**Lemma.** *Suppose that $\varphi, \varphi_1, \varphi_2 \in \mathrm{Hom}(V, V)$ and $\psi, \psi_1, \psi_2 \in \mathrm{Hom}(W, W)$*

*(i)* $(\varphi_1\varphi_2) \otimes (\psi_1\psi_2) = (\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2) \in \mathrm{Hom}(V \otimes W, V \otimes W)$*;*
*(ii)* $\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes W}$*; and*
*(iii)* $\mathrm{tr}(\varphi \otimes \psi) = \mathrm{tr}\,\varphi \cdot \mathrm{tr}\,\psi$.

**Definition.** Given two representation $(\rho, V)$ and $(\rho', W)$ of a group $G$ we can define the representation $(\rho \otimes \rho', V \otimes W)$ by $(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g)$.

**Proposition.** *If $(\rho, V)$ and $(\rho', W)$ are representations of $G$ then $(\rho \otimes \rho', V \otimes W)$ is a representation of $G$ and $\chi_{\rho\otimes\rho'} = \chi_\rho \cdot \chi_{\rho'}$.*

*Proof.* This is an straightforward consequence of the lemma. □

*Remarks.*

(1) It follows that $R(G)$ is closed under multiplication.
(2) Tensor product of representations defined here is consistent with our earlier notion when one of the representations is one-dimensional.
(3) It follows from the lemma that if $(\rho, V), (\rho', W)$ are reps of $G$ then we may make $V \otimes W$ into a rep of $G \times G$ via
$$\rho_{V\otimes W}(g, h) = \rho(g) \otimes \rho'(h).$$
In the proposition we then restrict this representation to the diagonal subgroup $G \cong \{(g, g)\} \subset G \times G$.
(4) If $X, Y$ are finite sets with $G$-action it is easy to verify that $kX \otimes kY \cong kX \times Y$ as representations of $G$ (or even of $G \times G$).

Now return to our assumption that $k = \mathbb{C}$.

*Exercise.* Show both directly and using characters that if $U, V, W$ are representations of $G$ then $V \otimes W \cong \mathrm{Hom}(V^*, W)$ and $\mathrm{Hom}(V \otimes W, U) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, U))$ as representations of $G$.

*Question.* If $V$ and $W$ are irreducible then must $V \otimes W$ be irreducible?

We've seen the answer is yes is one of $V$ and $W$ is one-dimensional but it is not usually true.

*Example.* $G = S_3$

| | 1 | 3 | 2 |
|---|---|---|---|
| | $e$ | $(12)$ | $(123)$ |
| $\mathbf{1}$ | 1 | 1 | 1 |
| $\epsilon$ | 1 | $-1$ | 1 |
| $V$ | 2 | 0 | $-1$ |

Clearly, $\mathbf{1} \otimes W = W$ always. $\epsilon \otimes \epsilon = \mathbf{1}$, $\epsilon \otimes V = V$ and $V \otimes V$ has character $\chi^2$ given by $\chi^2(1) = 4$, $\chi^2(12) = 0$ and $\chi^2(123) = 1$. Thus $\chi^2$ decomposes as $\mathbf{1} + \epsilon + \chi$.

In fact $V \otimes V, V \otimes V \otimes V, \ldots$ are never irreducible if $\dim V > 1$.

Given a vector space $V$, define $\sigma = \sigma_V : V \otimes V \to V \otimes V$ by $\sigma(v \otimes w) \mapsto w \otimes v$ for all $v, w \in V$ (exercise: check this does uniquely define a linear map). Notice that $\sigma^2 = \mathrm{id}$ and so $\sigma$ decomposes $V \otimes V$ into two eigenspaces:
$$S^2V := \{a \in V \otimes V \mid \sigma a = a\}$$
$$\Lambda^2V := \{a \in V \otimes V \mid \sigma a = -a\}$$

**Lemma.** *Suppose $v_1, \ldots, v_m$ is a basis for $V$.*
*(i) $S^2V$ has a basis $v_iv_j := v_i \otimes v_j + v_j \otimes v_i$ for $1 \leqslant i, j \leqslant d$.*
*(ii) $\Lambda^2V$ has a basis $v_i \wedge v_j := v_i \otimes v_j - v_j \otimes v_i$ $1 \leqslant i < j \leqslant d$.*
*Thus $\dim S^2V = \frac{1}{2}m(m + 1)$ and $\dim \Lambda^2V = \frac{1}{2}m(m - 1)$.*

*Remark.* We usually write $v_i \wedge v_j =: -v_j \wedge v_i$ for $j < i$ and $v_i \wedge v_i = 0$.

*Proof.* It is easy to check that the union of the two claimed bases form a basis for $V \otimes V$, that the $v_i v_j$ do all live in $S^2V$ and that the $v_i \wedge v_j$ do all live in $\Lambda^2 V$. Everything follows. □

*Exercise.* We may view $V \otimes V$ as a representation of $C_2$ via $\rho(1) = \sigma$. What is the character $\chi$ of $\rho$? What are $\langle \mathbf{1}, \chi\rangle_{C_2}$ and $\langle \epsilon, \chi\rangle_{C_2}$? How does this relate to the lemma just proven?

**Proposition.** *Let $(\rho, V)$ be a representation of $G$.*

*(i) $S^2V$ and $\Lambda^2V$ are subreps of $V \otimes V$ and $V \otimes V = S^2V \oplus \Lambda^2 V$.*
*(ii) for $g \in G$,*

$$\chi_{S^2V}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$$
$$\chi_{\Lambda^2V}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)).$$

*Proof.* For (i) we need to show that if $a \in V \otimes V$ and $\sigma_V(a) = \lambda a$ for $\lambda = \pm 1$ then $\sigma_V \rho(g)(a) = \lambda\rho(g)(a)$ for each $g \in G$. For this it suffices to prove that $\sigma g = g\sigma$ (ie $\sigma \in \mathrm{Hom}_G(V \otimes V, V \otimes V)$). But $\sigma \circ g(v \otimes w) = gw \otimes gv = g \circ \sigma(v \otimes w)$.

To compute (ii), let $v_1, \dots, v_m$ be a basis of eigenvectors for $\rho(g)$ with eigenvalues $\lambda_1, \dots, \lambda_m$. Then $g(v_i v_j) = (\lambda_i\lambda_j)v_i v_j$ and $g(v_i \wedge v_j) = (\lambda_i\lambda_j)v_i \wedge v_j$.

Thus $\chi_{S^2V}(g) = \sum_{i\leqslant j} \lambda_i\lambda j$, whereas

$$\chi(g)^2 + \chi(g^2) = (\sum_i \lambda_i)^2 + \sum_i \lambda_i^2 = 2\sum_{i\leqslant j} \lambda_i\lambda j.$$

Similarly $\chi_{\Lambda^2V}(g) = \sum_{i<j} \lambda_i\lambda_j$, and

$$\chi(g)^2 - \chi(g^2) = (\sum_i \lambda_i)^2 - \sum_i \lambda_i^2 = \sum_{i<j} \lambda_i\lambda_j.$$

□

*Example.* $S_4$

| | 1 | 3 | 8 | 6 | 6 |
|---|---|---|---|---|---|
| | $e$ | (12)(34) | (123) | (12) | (1234) |
| $\mathbf{1}$ | 1 | 1 | 1 | 1 | 1 |
| $\epsilon$ | 1 | 1 | 1 | −1 | −1 |
| $\chi_3$ | 3 | −1 | 0 | 1 | −1 |
| $\epsilon\chi_3$ | 3 | −1 | 0 | −1 | 1 |
| $\chi_5$ | 2 | 2 | −1 | 0 | 0 |
| $\chi_3^2$ | 9 | 1 | 0 | 1 | 1 |
| $\chi_3(g^2)$ | 3 | 3 | 0 | 3 | −1 |
| $S^2\chi_3$ | 6 | 2 | 0 | 2 | 0 |
| $\Lambda^2\chi_3$ | 3 | −1 | 0 | −1 | 1 |

Thus $S^2\chi_3 = \chi_5 + \chi_3 + \mathbf{1}$ and $\Lambda^2\chi_3 = \epsilon\chi_3$.

*Exercise.* Show that if $V$ is self-dual then either $\langle \mathbf{1}, \chi_{S^2V}\rangle \neq 0$ or $\langle \mathbf{1}, \chi_{\Lambda^2V}\rangle \neq 0$.

## Lecture 12

Last time we thought about $S^2V$ and $\Lambda^2V$ as subrepresentations of $V\otimes V$. More generally, for any vector space $V$ we may consider $V^{\otimes n} = V\otimes\cdots\otimes V$. Then for any $\omega\in S_n$ we can define a linear map $\sigma_\omega\colon V^{\otimes n}\to V^{\otimes n}$ by

$$\sigma_\omega\colon v_1\otimes\cdots v_n\mapsto v_{\omega^{-1}(1)}\otimes\cdots v_{\omega^{-1}(n)}$$

for $v_1,\ldots,v_n\in V$ (exercise check this uniquely defines a linear map).

In this way we can define a representation of $S_n$ on $V^{\otimes n}$. Moreover if $V$ is a representation of $G$ then the action of $G$ on $V^{\otimes n}$ via $v_1\otimes\cdots\otimes v_n\mapsto gv_1\otimes\cdots\otimes gv_n$ commutes with the $S_n$-action. Thus we can decompose $V^{\otimes n}$ as a rep of $S_n$ and each isotypical component should be a $G$-invariant subspace of $V^{\otimes n}$. In particular we can make the following definition.

**Definition.** Suppose that $V$ is a vector space we define

(i) the $n^{th}$ *symmetric power* of $V$ to be

$$S^nV := \{a\in V^{\otimes n}\mid \sigma_\omega(a)=a \text{ for all } \omega\in S_n\}$$

and

(ii) the $n^{th}$ *exterior (or alternating) power* of $V$ to be

$$\Lambda^nV := \{a\in V^{\otimes n}\mid \sigma_\omega(a)=\epsilon(\omega)a \text{ for all } \omega\in S_n\}.$$

Note that $S^nV\oplus\Lambda^nV = \{a\in V^{\otimes n}\mid \sigma_\omega(a)=a \text{ for all } \omega\in A_n\}\subsetneq V^{\otimes n}$.

*Exercise.* Show that if $V$ is a rep of $G$ then $S^nV$ and $\Lambda^nV$ are subreps of $V^{\otimes n}$. For each $g\in G$ of finite order compute the characters of $S^nV$ and $\Lambda^nV$ in terms of the eigenvalues of $g$ on $V$.

[Hint: if $v_1,\ldots,v_r$ is a basis for $V$ then

$$\left\{\frac{1}{n!}\sum_{\sigma\in S_n} v_{\sigma(i_1)}\otimes v_{\sigma(i_n)}\mid 1\leqslant i_1\leqslant\cdots\leqslant i_n\leqslant r\right\}$$

is a basis for $S^nV$ and

$$\left\{\frac{1}{n!}\sum_{\sigma\in S_n}\epsilon(\sigma) v_{\sigma(i_1)}\otimes v_{\sigma(i_n)}\mid 1\leqslant i_1<\cdots< i_n\leqslant r\right\}$$

is a basis for $\Lambda^nV$.]

For any vector space $V$, $\Lambda^{\dim V}\cong k$ and $\Lambda^nV=0$ if $n>\dim V$.

*Exercise.* Show that if $(\rho,V)$ is a representation of $G$ then the representation $G\to GL(\Lambda^{\dim V}V)\cong k^\times$ is given by $g\mapsto\det\rho(g)$; ie the $\dim V^{th}$ exterior power of $V$ is isomorphic to $\det\rho$.

In characteristic zero, we may stick these vector spaces together to form algebras.

**Definition.** Given a vector space $V$ we may define the *tensor algebra* of $V$,

$$TV := \oplus_{n\geqslant 0}V^{\otimes n}$$

(where $V^{\otimes 0}=k$). Then $TV$ is a (non-commutative) graded ring with the product of $v_1\otimes\cdots\otimes v_r\in V^{\otimes r}$ and $w_1\otimes\cdots\otimes w_s\in V^{\otimes s}$ given by

$$v_1\otimes\cdots\otimes v_r\otimes w_1\otimes\cdots\otimes w_s\in V^{\otimes r+s}.$$

with graded quotient rings the *symmetric algebra* of $V$,

$$SV := TV/(x \otimes y - y \otimes x \mid x, y \in V),$$

and the *exterior algebra* of $V$,

$$\Lambda V := TV/(x \otimes y + y \otimes x \mid x, y \in V).$$

One can show that $SV \cong \bigoplus_{n \geqslant 0} S^n V$ under $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$ and $\Lambda V \cong \bigoplus_{n \geqslant 0} \Lambda^n V$ under $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \wedge \cdots \wedge x_n$.

Now $SV$ is a commutive ring and $\Lambda V$ is graded commutative; that is if $x \in \Lambda^r V$ and $y \in \Lambda^s V$ then $x \wedge y = (-1)^{rs} y \wedge x$.

**Proposition.** *Suppose $G$ and $H$ are finite groups.*

*Let $(\rho_1, V_1), \ldots, (\rho_r, V_r)$ be a complete list of the irreducible complex representations of $G$ and $(\rho'_1, W_1), \ldots, (\rho'_s, W_s)$ a complete list of the irreducible complex representations of $H$. For each $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, $(\rho_i \otimes \rho'_j, V_i \otimes W_j)$ is an irreducible complex representation of $G \times H$. Moreover, all the irreducible representations of $G \times H$ arise in this way.*

We have seen this before when $G$ and $H$ are abelian since then all these representations are 1-dimensional.

*Proof.* Let $\chi_1, \ldots, \chi_r$ be the characters of $V_1, \ldots, V_r$ and $\psi_1, \ldots, \psi_s$ the characters of $W_1, \ldots, W_s$.

The character of $V_i \otimes W_j$ is $\chi_i \otimes \psi_j \colon (g, h) \mapsto \chi_i(g)\psi_j(h)$. Then

$$\langle \chi_i \otimes \psi_j, \chi_k \otimes \psi_l \rangle_{G \times H} = \langle \chi_i, \chi_k \rangle_G \langle \psi_j, \psi_l \rangle_H = \partial_{ik}\partial_{jl}.$$

So the $\chi_i \otimes \psi_j$ are irreducible and pairwise distinct.

Now $\sum_{i,j} \dim(V_i \otimes W_j)^2 = (\sum_i \dim V_i^2)(\sum_j \dim W_j^2) = |G||H| = |G \times H|$ so we must have them all. □

## 7. Induction

Suppose that $H$ is a subgroup of $G$. We have a way of turning representations of $G$ into representations of $H$; we restrict the homomorphism $\rho \colon G \to GL(V)$ to $H$.

We would like a similar way of building representations of $G$ from representations of $H$. There is a good way of doing so called induction although it is a little more delicate than restriction.

If $G$ is a finite group and $W$ is a $k$-vector space we may define $\mathrm{Hom}(G, W)$ to be the vector space of all functions $G \to W$ under pointwise addition and scalar multiplication. This may be made into a representation of $G$ by defining

$$(g \cdot f)(x) := f(g^{-1}x)$$

for each $g, x \in G$. If $w_1, \ldots, w_n$ is a basis for $W$ then $\{\partial_g w_i \mid g \in G, 1 \leqslant i \leqslant n\}$ is a basis for $\mathrm{Hom}(G, W)$. So $\dim \mathrm{Hom}(G, W) = |G| \dim W$.

**Lemma.** $\mathrm{Hom}(G, W) \cong (\dim W)kG$ *as representations of $G$.*

*Proof.* Given a basis $w_1, \ldots, w_n$ for $W$, define the linear map

$$\Theta \colon \bigoplus_{i=1}^{n} kG \to \mathrm{Hom}(G, W)$$

by

$$\Theta((f_i)_{i=1}^n)(x) = \sum_{i=1}^n f_i(x)w_i.$$

It is easy to see that $\Theta$ is injective because the $w_i$ are linearly independent so by comparing dimensions we see that $\Theta$ is a vector-space isomorphism.

It remains to prove that $\Theta$ is $G$=linear. If $g, x \in G$ then

$$g \cdot (\Theta((f_i)_{i=1}^n))(x) = \sum_{i=1}^n f_i(g^{-1}x)w_i = \Theta(g \cdot (f_i)_{i=1}^n)(x)$$

as required. □

*Exercise.* Use the basis of $\operatorname{Hom}(G, W)$ given above to find a character-theoretic proof of the lemmma.

Now, if $H$ is a subgroup of $G$ and $W$ is a representation of $H$ then we can define

$$\operatorname{Hom}_H(G, W) := \{f \in \operatorname{Hom}(G, W) \mid f(xh) = h^{-1}f(x)\ \forall x \in G, h \in H\},$$

a $k$-linear subspace of $\operatorname{Hom}(G, W)$.

*Example.* If $W = k$ is the trivial representation of $H$ then $f \in \operatorname{Hom}_H(G, k)$ if and only if $f(xh) = f(x)$ for $h \in H$ and $x \in G$. That is $\operatorname{Hom}_H(G, k)$ consists of the functions that are constant on each left coset in $G/H$. Thus $\operatorname{Hom}_H(G, k)$ can be identified with the permutation module $kG/H$ where $G$ acts on the left cosets $G/H$ in the usual way.

**Lemma.** *$\operatorname{Hom}_H(G, W)$ is a $G$-invariant subspace of $\operatorname{Hom}(G, W)$.*

*Proof.* Let $f \in \operatorname{Hom}_H(G, W)$, $g, x \in G$ and $h \in H$ we must show that

$$(g \cdot f)(xh) = h^{-1}(g \cdot f)(x).$$

But $(g \cdot f)(xh) = f(g^{-1}xh) = h^{-1}f(g^{-1}x) = h^{-1}(g \cdot f)(x)$ as required. □

**Definition.** Suppose that $H$ is a subgroup of $G$ of finite index and $W$ is a representation of $H$. We define the *induced representation* to be $\operatorname{Ind}_H^G W := \operatorname{Hom}_H(G, W)$

## Lecture 13

Recall from last time:

**Definition.** Suppose that $H$ is a subgroup of $G$ of finite index and $W$ is a representation of $H$. We define the *induced representation* by

$$\operatorname{Ind}_H^G W := \operatorname{Hom}_H(G, W) = \{f : G \to W \mid f(xh) = h^{-1}f(x) \text{ for all } x \in G, h \in H\}$$

*Remark.* Since $\operatorname{Ind}_H^G \mathbf{1} = kG/H$, $\operatorname{Ind}_H^G$ does not send irreducibles to irreducibles in general.

**Proposition.** *Suppose $W$ is a representation of $H$ then*

*(i)* $\dim \operatorname{Ind}_H^G W = \frac{|G|}{|H|} \dim W$;

*(ii) for $g \in G$,*

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \chi_W(x^{-1}gx).$$

*Remarks.*

(1) $x^{-1}gx \in H$ if and only if $gxH = xH$ so if $W$ is the trivial representation the rhs of formula in (ii) becomes $|\{xH \in G/H \mid gxH = xH\}|$ and we get the permutation character of $kG/H$ as required.

(2) If we write $\chi_W^\circ$ for the function on $G$ such that $\chi_W^\circ(g) = \chi_W(g)$ if $x \in H$ and $\chi_W^\circ(g) = 0$ if $g \notin H$, then the formula in (ii) becomes

$$\chi_{\mathrm{Ind}_H^G W}(g) = \frac{1}{|H|}\sum_{x\in G}\chi_W^\circ(x^{-1}gx);$$

this is clearly a class function.

(3) If $[h_1], \ldots, [h_m]$ is a list of the $H$-conjugacy classes such that $x^{-1}gx \in [h_i]$ some $x \in G$ then we can write this as

$$\chi_{\mathrm{Ind}_H^G W}(g) = \sum_{i=1}^{m}\frac{|C_G(g)|}{|C_H(h_i)|}\chi_W(h_i).$$

This is the most useful formula for computation.

*Proof of Proposition.* Let $x_1, \ldots, x_r$ be left coset representatives in $G/H$. Then $f \in \mathrm{Hom}_H(G, W)$ is determined by the values of $f(x_1), \ldots, f(x_r) \in W$.

Moreover, given $w_1, \ldots, w_r \in W$ we can define $f \in \mathrm{Hom}_H(G, W)$ via $f(x_ih) = h^{-1}w_i$ for $i = 1, \ldots, r$ and $h \in H$. Thus

$$\Theta\colon \mathrm{Hom}_H(G, W) \to \bigoplus_{i=1}^{r} W$$

defined by $f \mapsto (f(x_i))_{i=1}^r$ is an isomorphism of vector spaces and part (i) is done.

Following this argument, we see that given $w \in W$, and $1 \leqslant i \leqslant r$, we can define $\varphi_{i,w} \in \mathrm{Hom}_H(G, W)$ by

$$\varphi_{i,w}(x_jh) = \partial_{ij}h^{-1}w$$

for each $h \in H$ and $1 \leqslant j \leqslant r$.

Now given $g \in G$, let's consider how $g$ acts on a $\varphi_{i,w}$. For each coset representative $x_i$ there is a unique $\sigma(i)$ and $h_i \in H$ such that $g^{-1}x_i = x_{\sigma(i)}h_i \in x_{\sigma(i)}H$, and

$$(g\cdot\varphi_{i,w})(x_j) = \varphi_{i,w}(g^{-1}x_j) = \varphi_{i,w}(x_{\sigma(j)}h_j) = \partial_{i\sigma(j)}h_j^{-1}w.$$

Thus $g\cdot\varphi_{i,w} = \varphi_{\sigma^{-1}(i),h^{-1}_{\sigma^{-1}(i)}w}$.

Thus $g$ acts on $\bigoplus_{i=1}^r W$ via a block permutation matrix and we only get contributions to the trace from the non-zero diagonal blocks which correspond to the fixed points of $\sigma$. Moreover if $\sigma(i) = i$ then $g$ acts on $W_i$ via $h_i^{-1} = x_i^{-1}gx_i$

Thus

$$\mathrm{tr}\, g_{\mathrm{Ind}_H^G W} = \sum_i \chi_W^\circ(x_i^{-1}gx_i).$$

Since $G = \{x_ih \mid h \in H\}$ and $\chi_W^\circ(h^{-1}gh) = \chi_W^\circ(g)$ for all $g \in G$ and $h \in H$ we may rewrite this as

$$\mathrm{tr}\, g_{\mathrm{Ind}_H^G W} = \frac{1}{|H|}\sum_{x\in G}\chi_W^\circ(xgx^{-1})$$

as required. □

*Example.* $G = S_3$ and $H = C_3 = \{1, (123), (132)\}$.

If $W$ is any rep of $H$ then

$$\chi_{\mathrm{Ind}_H^G W}(e) = 2\chi_W(e),$$
$$\chi_{\mathrm{Ind}_H^G W}((12)) = 0, \text{ and}$$
$$\chi_{\mathrm{Ind}_H^G W}((123)) = \chi_W((123)) + \chi_W((132)).$$

So

| $C_3$ | 1 | (123) | (132) |
|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | $w$ | $w^2$ |
| $\chi_3$ | 1 | $w^2$ | $w$ |

| $s_3$ | 1 | (12) | (123) |
|---|---|---|---|
| $\mathrm{Ind}\,\chi_1$ | 2 | 0 | 2 |
| $\mathrm{Ind}\,\chi_2$ | 2 | 0 | $-1$ |
| $\mathrm{Ind}\,\chi_3$ | 2 | 0 | $-1$ |

So $\mathrm{Ind}_H^G \chi_2 = \mathrm{Ind}_H^G \chi_3$ is the 2-dimensional irreducible character of $S_3$ and $\mathrm{Ind}_H^G \chi_1 = \mathbf{1} + \epsilon$ as expected.

If $V$ is a representation of $G$, we'll write $\mathrm{Res}_H^G V$ for the representation of $H$ obtained by restriction.

**Proposition** (Frobenius reciprocity). *Let $V$ be a representation of $G$, and $W$ a representation of $H$, then*

*(i)* $\langle \chi_V, \mathrm{Ind}_H^G \chi_W \rangle_G = \langle \mathrm{Res}_H^G \chi_V, \chi_W \rangle_H$*;*
*(ii)* $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$*.*

*Proof.* We've already seen that (i) implies (ii).

Now

$$\begin{aligned}
\langle \chi_V, \mathrm{Ind}_H^G \chi_W \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_{\mathrm{Ind}_H^G W}(g) \\
&= \frac{1}{|G||H|} \sum_{g \in G} \sum_{x \in G} \overline{\chi_V(g)} \chi_W^\circ(x^{-1}gx) \\
&= \frac{1}{|G|} \sum_{x \in G} \sum_{g' \in G} \overline{\chi_V(xg'x^{-1})} \chi_W^\circ(g') \qquad (g' = x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{g' \in H} \overline{\chi_V(g')} \chi_W(g') \\
&= \langle \mathrm{Res}_H^G \chi_V, \chi_W \rangle_H
\end{aligned}$$

as required. □

*Exercise.* Prove (ii) directly by considering

$$\Theta \colon \mathrm{Hom}_G(V, \mathrm{Hom}_H(G, W)) \to \mathrm{Hom}_H(V, W)$$

defined by $\Theta(f)(v) = f(v)(e)$.

7.1. **Mackey Theory.** This is the study of representations like $\mathrm{Res}_K^G \mathrm{Ind}_H^G W$ for $H, K$ subgroups of $G$ and $W$ a representation of $H$. We can (and will) use it to chracterise when $\mathrm{Ind}_H^G W$ is irreducible.

Recall that if $G$ acts transitively on a set $X$ then for $x \in X$ there is a bijection $G/\mathrm{Stab}_G(x) \tilde{\to} X$ given by $g\,\mathrm{Stab}_G(x) \mapsto gx$ that commutes with the $G$-action (ie $g'(g\,\mathrm{Stab}_G(x)) = (g'g)\,\mathrm{Stab}_G(x) \mapsto g'gx = g'(gx)$).

If $H, K$ are subgroups of $G$ we can restrict the action of $G$ on $G/H$ to $K$

$$K \times G/H \to G/H; (k, gh) \mapsto kgH.$$

The the union of an orbit of this action is called a *double coset*. The union of the $K$-orbit of $gH$ is written $KgH := \{kgh \mid k \in K, h \in H\}$.

**Definition.** $K\backslash G/H := \{KgH \mid g \in G\}$ is the set of double cosets.

The double cosets $K\backslash G/H$ partition $G$.

Notice that $kgH = gH$ if and only if $k \in gHg^{-1}$. Thus as a set with a $K$-action, $KgH \tilde{\to} K/(K \cap gHg^{-1})$.

**Proposition.** *If $G, H, K$ as above then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \mathbf{1} \cong \bigoplus_{g \in K\backslash G/H} \operatorname{Ind}_{gHg^{-1}\cap K}^K \mathbf{1}.$$

*Proof.* This follows from the discussion above, together with the general facts that $\operatorname{Ind}_H^G \mathbf{1} = kG/H$ and that if $X = \bigcup X_i$ is a decomposition of $X$ into orbits then $kX \cong \bigoplus kX_i$. □

## Lecture 14

Recall from last time,

**Proposition.** *If $G$ is a finite group and $H$,$K$ are subgroups of $G$, then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \mathbf{1} \cong \bigoplus_{g \in K\backslash G/H} \operatorname{Ind}_{gHg^{-1}\cap K}^K \mathbf{1}.$$

Given any representation $(\rho, W)$ of $H$ and $g \in G$, we can define $({}^g\rho, {}^gW)$ to be the representation of ${}^gH := gHg^{-1} \leqslant G$ on the underlying vector space $W$ given by $({}^g\rho)(ghg^{-1}) = \rho(h)$ for $h \in H$.

**Theorem** (Mackey's Restriction Formula)**.** *If $G$ is a finite group with subgroups $H$ and $K$, and $W$ is a representation of $H$ then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{g \in K\backslash G/H} \operatorname{Ind}_{K\cap {}^gH}^K \operatorname{Res}_{{}^gH\cap K}^{{}^gH} {}^gW.$$

*Proof.* For each double coset $KgH$ we can define

$$V_g = \{f \in \operatorname{Ind}_H^G W \mid f(x) = 0 \text{ for all } x \notin KgH\}.$$

Then $V_g$ is a $K$-invariant subspace of $\operatorname{Ind}_H^G W$ since we always have $(kf)(x) = f(k^{-1}x)$. Thus there is a decomposition

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{g \in K\backslash G/H} V_g$$

and it suffices to show that for each $g$,

$$V_g \cong \operatorname{Ind}_{K\cap {}^gH}^K \operatorname{Res}_{{}^gH\cap K}^{{}^gH} {}^gW$$

as representations of $K$.

Define an injective linear map $\Theta\colon V_g \to \operatorname{Hom}(K, {}^gW)$ by $\Theta(f)(k) = f(kg)$. If $k' \in K$ then

$$(k'\Theta(f))(k) = f(k'^{-1}kg) = (k'f)(kg) = \Theta(k'f)(k)$$

and so $\Theta \in \mathrm{Hom}_K(V_g, \mathrm{Hom}(K, {}^g W))$.

Next, if $ghg^{-1} \in K$ for some $h \in H$,

$$\begin{aligned}\Theta(f)(kghg^{-1}) &= f(kgh)\\ &= \rho(h^{-1})f(kg)\\ &= ({}^g\rho)(ghg^{-1})^{-1}\Theta(f)(k)\end{aligned}$$

Thus $\mathrm{Im}\,\Theta \leqslant \mathrm{Ind}_{K\cap {}^gH}^K \mathrm{Res}_{K\cap {}^gH}^{{}^gH} {}^gW$. It remains to prove that this inclusion is an equality. We can do this by comparing dimensions:

$$\begin{aligned}\sum_{g\in K\backslash G/H} \dim V_g &= \dim W \frac{|G|}{|H|}\\ &= \dim W \sum_{g\in K\backslash G/H} \frac{|K|}{|{}^gH\cap K|} \text{ (by the proposition)}\\ &= \sum_{g\in K\backslash G/H} \dim \mathrm{Ind}_{K\cap {}^gH}^K \mathrm{Res}_{K\cap {}^gH}^{{}^gH} {}^gW\end{aligned}$$

Thus $\dim V_g = \dim \mathrm{Ind}_{K\cap {}^gH}^K \mathrm{Res}_{K\cap {}^gH}^{{}^gH} {}^gW$ as required. □

**Corollary** (Character version of Mackey's Restriction Formula)**.** *If $\chi$ is a character of a representation of $H$ then*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G \chi = \sum_{g\in K\backslash G/H} \mathrm{Ind}_{{}^gH\cap K}^K {}^g\chi.$$

*where ${}^g\chi$ is the class function on ${}^gH\cap K$ given by ${}^g\chi(x) = \chi(g^{-1}xg)$.*

*Exercise.* Prove this corollary directly with characters

**Corollary** (Mackey's irreducibility criterion)**.** *If $H$ is a subgroup of $G$ and $W$ is a representation of $H$, then $\mathrm{Ind}_H^G$ is irreducible if and only if*

*(i) $W$ is irreducible and*
*(ii) for each $g \in G\backslash H$, the two representations $\mathrm{Res}_{H\cap {}^gH}^{{}^gH} {}^gW$ and $\mathrm{Res}_{{}^gH\cap H}^H W$ of $H \cap {}^g H$ have no irreducible factors in common.*

*Proof.*

$$\begin{aligned}\mathrm{Hom}_G(\mathrm{Ind}_H^G W, \mathrm{Ind}_H^G W) &\overset{\text{Frob. recip.}}{\cong} \mathrm{Hom}_H(W, \mathrm{Res}_H^G \mathrm{Ind}_H^G W)\\ &\overset{\text{Mackey}}{\cong} \bigoplus_{g\in H\backslash G/H} \mathrm{Hom}_H(W, \mathrm{Ind}_{H\cap {}^gH}^H \mathrm{Res}_{H\cap {}^gH}^{{}^gH} {}^gW)\\ &\overset{\text{Frob. recip.}}{\cong} \bigoplus_{g\in H\backslash G/H} \mathrm{Hom}_{H\cap {}^gH}(\mathrm{Res}_{H\cap {}^gH}^H W, \mathrm{Res}_{H\cap {}^gH}^{{}^gH} {}^gW)\end{aligned}$$

We know that $\mathrm{Ind}_H^G W$ is irreducible precisely if this space has dimension 1. The summand corresponding to the coset $HeH = H$ is $\mathrm{Hom}_H(W, W)$ which has dimension 1 precisely if $W$ is irreducible and the other summands are all zero precisely if condition (ii) of the statement holds. □

**Corollary.** *If $H$ is a normal subgroup of $G$ ,and $W$ is an irreducible rep of $H$ then $\operatorname{Ind}_H^G W$ is irreducible if and only if ${}^g\chi_W \neq \chi_W$ for all $g \in G \backslash H$.*

*Proof.* Since $H$ is normal, $gHg^{-1} = H$ for all $g \in G$. Moreover ${}^gW$ is irreducible since $W$ is irreducible.

So by Mackey's irreducibility criterion, $\operatorname{Ind}_H^G W$ irreducible precisely if $W \not\cong {}^gW$ for all $g \in G \backslash H$. This last is equivalent to $\chi_W \neq {}^g\chi_W$ as required. □

*Example.*
$G = D_8 \geqslant H = C_4$, the rotations.

| $C_4$ | $1$ | $r$ | $r^2$ | $r^3$ |
|---|---|---|---|---|
| $\chi_1$ | $1$ | $1$ | $1$ | $1$ |
| $\chi_2$ | $1$ | $i$ | $-1$ | $-i$ |
| $\chi_3$ | $1$ | $-1$ | $1$ | $-1$ |
| $\chi_4$ | $1$ | $-i$ | $-1$ | $i$ |

| $D_8$ | $1$ | $r$ | $r^2$ | $s$ | $sr$ |
|---|---|---|---|---|---|
| $\operatorname{Ind}\chi_1$ | $2$ | $2$ | $2$ | $0$ | $0$ |
| $\operatorname{Ind}\chi_2$ | $2$ | $0$ | $-2$ | $0$ | $0$ |
| $\operatorname{Ind}\chi_3$ | $2$ | $-2$ | $2$ | $0$ | $0$ |
| $\operatorname{Ind}\chi_4$ | $2$ | $0$ | $-2$ | $0$ | $0$ |

We see that ${}^s\chi_1 = \chi_1$, ${}^s\chi_2 = \chi_4$, ${}^s\chi_3 = \chi_3$, ${}^s\chi_4 = \chi_2$.

We can see directly that $\operatorname{Ind}\chi_1$ and $\operatorname{Ind}\chi_3$ are reducible and $\operatorname{Ind}\chi_2 = \operatorname{Ind}\chi_4$ is irreducible.

## Lecture 15

### 7.2. **Frobenius groups.**

**Definition.** A *Frobenius group* is a finite group $G$ having a subgroup $H$ such that $H \cap gHg^{-1} = \{e\}$ for all $g \in G \backslash H$.

**Theorem.** *(Frobenius) Let $G$ be a finite group acting faithfully and transitively on a set $X$. If each $g \in G \backslash \{e\}$ fixes at most one element of $X$ then*

$$K = \{1\} \cup \{g \in G \mid gx \neq x \text{ for all } x \in X\}$$

*is a normal subgroup of $G$ of order $X$.*

*Proof.* For $x \in X$, let $H = \operatorname{Stab}_G(x)$.

We know that $\operatorname{Stab}_G(gx) = gHg^{-1}$. But by the hypothesis on the action

$$\operatorname{Stab}_G(gx) \cap \operatorname{Stab}_G(x) = \{e\}$$

whenever $gx \neq x$. Thus $H$ has $|X|$ conjugates and $G$ has $(|H| - 1)|X|$ elements that fix precisely one element of $X$.

But $|G| = |H||X|$ by the orbit-stabiliser theorem, and so

$$|K| = |H||X| - (|H| - 1)|X| = |X|$$

as required. We must show that it is a normal subgroup of $G$.

Our strategy will be to prove that it is the kernel of some representation of $G$.

Suppose $e \neq h \in H$ and that $h = gh'g^{-1}$ for some $g \in G$ and $h' \in H$ then $h \in \operatorname{Stab}_G(x) \cap \operatorname{Stab}_G(gx)$, so $gx = x$ and $g \in H$. Thus

- $h$ and $h'$ in $H$ are conjugate in $G$ if and only if they are conjugate in $H$.
- $|C_G(h)| = |C_H(h)|$ for $e \neq h \in H$

Now if $\chi$ is a character of $H$ we can compute $\operatorname{Ind}_H^G \chi$:

$$\operatorname{Ind}_H^G \chi(g) = \begin{cases} |X|\chi(e) & \text{if } g = e \\ \chi(h) & \text{if } g = h \in H\backslash\{e\} \\ 0 & \text{if } g \in K\backslash\{e\} \end{cases}$$

Suppose now that $\chi_1, \ldots, \chi_r$ is a list of the irreducible characters of $H$ and let $\theta_i = \operatorname{Ind}_H^G \chi_i + \chi_i(e)\mathbf{1}_G - \chi_i(e) \operatorname{Ind}_H^G \mathbf{1}_H \in R(G)$ for $i = 1, \ldots, r$ and so

$$\theta_i(g) = \begin{cases} \chi_i(e) & \text{if } g = e \\ \chi_i(h) & \text{if } g = h \in H \\ \chi_i(e) & \text{if } g \in K \end{cases}$$

If $\theta_i$ were a character then the corresponding representation would have kernel containing $K$. Since $\theta_i \in R(G)$ we can write it as a $\mathbb{Z}$-linear combination of irreducible characters $\theta_i = \sum n_i \psi_i$, say.

On the one hand, we can compute

$$\begin{aligned} \langle \theta_i, \theta_i \rangle_G &= \frac{1}{|G|} \sum_{g \in G} |\theta_i(g)|^2 \\ &= \frac{1}{|G|} \left( \sum_{h \in H \setminus \{e\}} |X||\chi_i(h)|^2 + \sum_{k \in K} \chi_i(e)^2 \right) \\ &= \frac{|X|}{|G|} \left( \sum_{h \in H} \chi_i(h)|^2 \right) \\ &= \langle \chi_i, \chi_i \rangle_H = 1 \end{aligned}$$

But on the other hand it must be $\sum n_i^2$. Thus $\theta_i$ is $\pm\psi$ for some character $\psi$ of $G$. Since $\theta_i(e) > 0$ it must actually be an irreducible character.

To finish we write $\theta = \sum \chi_i(e)\theta_i$ and so $\theta(h) = \sum \chi_i(e)\chi_i(h) = 0$ for $h \in H\setminus\{e\}$ by column orthogonality, and $\theta(k) = \sum \chi_i(e)^2 = |H|$ for $k \in K$. Thus $K = \ker \theta$ is a normal subgroup of $G$. □

*Remarks.*

(1) Any Frobenius group satisfies the conditions of the theorem. The normal subgroup $K$ is called the Frobenius kernel and the group $H$ is called the Frobenius complement.
(2) No non-character theoretic proof of the theorem is known.
(3) In his thesis Thompson proved, amongst other things, that the Frobenius kernel must be a direct product of its Sylow subgroups.

## 8. Arithmetic properties of characters

In this section we'll investigate how arithmetic properties of characters produce a suprising interplay between the structure of the group and properties of the character table. The highlight of this will be the proof of Burnside's famous $p^aq^b$-theorem that says that the order of a simple group cannot have precisely two distinct prime factors.

We'll need to quote some results about arithmetic without proof; proofs should have been provided in the Number Fields course. We'll continue with our assumption that $k = \mathbb{C}$ and also assume that our groups are finite.

8.1. **Arithmetic results.**

**Definition.** $x \in \mathbb{C}$ is an *algebraic integer* if it is a root of a monic polynomial with integer coefficients.

*Facts.*

Fact 1 The algebraic integers form a subring of $\mathbb{C}$

Fact 2 If $x \in \mathbb{Q}$ is an algebraic integer then $x \in \mathbb{Z}$ (cf Numbers and Sets 2009 Example Sheet 3 Q12)

Fact 3 Any subring of $\mathbb{C}$ that is finitely generated as an abelian group consists of algebraic integers.

**Lemma.** *If $\chi$ is the character of a representation of a finite group $G$, then $\chi(g)$ is an algebraic integer for all $g \in G$.*

*Proof.* We know that $\chi(g)$ is a sum of $n^{th}$ roots of unity for $n = |G|$. Since each $n^{th}$ root of unity is by defintion a root of $X^n - 1$ this follows from Fact 1. □

8.2. **The group algebra.** Before we go further we need to explain how to make the vector space $kG$ into a ring. There are in fact two sensible ways to do this. The first of these is by pointwise multiplication: $f_1 f_2(g) = f_1(g) f_2(g)$ for all $g \in G$ will make $kG$ into a commutative ring. But more usefully for our immediate purposes we have the convolution product

$$f_1 f_2(g) := \sum_{x \in G} f_1(gx) f_2(x^{-1})$$

that makes $kG$ into a (possibly) non-commutative ring. Notice in particular that with this product $\partial_{g_1} \partial_{g_2} = \partial_{g_1 g_2}$ and so we may rephrase the multiplication as

$$(\sum_{g \in G} \lambda_g \partial_g)(\sum_{h \in G} \mu_h \partial_h) = \sum_{k \in G} (\sum_{gh=k} \lambda_g \mu_h) \partial_k.$$

From now on this will be the product we have in mind when we think of $kG$ as a ring.

We notice in passing that a $kG$-module is the 'same' as a representation of $G$: given a representation $(\rho, V)$ of $G$ we can make it into a $kG$-module via

$$fv = \sum_{g \in G} f(g) \rho(g)(v).$$

for $f \in kG$ and $v \in V$. Conversely, given a finitely generated $kG$-module $M$ we can view $M$ as a representation of $G$ via $\rho(g)(m) = \partial_g m$.

*Exercise.* Suppose that $kX$ is a permutation representation of $G$. Calculate the action of $f \in kG$ on $kX$ under this correspondance.

## Lecture 16

For the sake of the rest of the section, we need to understand the *centre* $Z(kG)$ of $kG$; that is the set of $f \in kG$ such that $fh = hf$ for all $h \in kG$.

**Lemma.** *Suppose that $f \in kG$. Then $f$ is in $Z(kG)$ if and only if $f \in \mathcal{C}_G$, the set of class functions on $G$. In particular $\dim_k Z(kG)$ is the number of conjugacy classes in $G$.*

*Proof.* Suppose $f \in kG$. Notice that $fh = hf$ for all $h \in kG$ if and only if $f\partial_g = \partial_g f$ for all $g \in G$, since then

$$fh = \sum_{g \in G} fh(g)\partial_g = \sum_{g \in G} h(g)\partial_g f = hf.$$

But $\partial_g f = f\partial_g$ if and only if $\partial_g f \partial_{g^{-1}} = f$ and

$$(\partial_g f \partial_{g^{-1}})(x) = (\partial_g f)(xg) = f(g^{-1}xg).$$

So if $f \in Z(kG)$ if and only if $f \in \mathcal{C}_G$ as required. □

*Remark.* The multiplication on $Z(kG)$ is not the same as the multiplication on $\mathcal{C}_G$ that we have seen before even though both have the same additive groups and both are commutative rings.

**Definition.** Suppose $\mathcal{O}_1 = \{e\}, \ldots, \mathcal{O}_r$ are the conjugacy classes of $G$, define the *class sums* $C_1, \ldots, C_r$ to be the class functions on $G$ so that

$$C_i = \begin{cases} 1 & g \in \mathcal{O}_i \\ 0 & g \notin \mathcal{O}_i. \end{cases}$$

We called these $\partial_{\mathcal{O}_i}$ before. Also we'll fix $g_i \in \mathcal{O}_i$ for simplicity.

We've seen that the class sums form a basis for $Z(kG)$.

**Proposition.** *There are non-negative integers $a_{ijk}$ such that $C_iC_j = \sum_k a_{ijk}C_k$ for $i, j, k \in \{1, \ldots, r\}$.*

The $a_{ijk}$ are called the *structure constants* for $Z(kG)$.

*Proof.* Since $Z(kG)$ is a ring, we can certainly write $C_iC_j = \sum a_{ijk}C_k$ for some $a_{ijk} \in k$.

However, we can explicitly compute for $g_k \in \mathcal{O}_k$,

$$(C_iC_j)(g_k) = \sum_{x \in G} C_i(g_kx)C_j(x^{-1}) = |\{(x, y) \in \mathcal{O}_i \times \mathcal{O}_j \mid xy = g_k\}|,$$

a non-negative integer. □

Suppose now that $(\rho, V)$ is an irreducible representation of $G$. Then if $z \in Z(kG)$ we see that $z \colon V \to V$ given by $zv = \sum_{g \in G} z(g)\rho(g)v \in \mathrm{Hom}_G(V, V)$.

By Schur's Lemma it follows that $z$ acts by a scalar $\lambda_z \in k$ on $V$. In this way we get an algebra homomorphism $w_\rho \colon Z(kG) \to k; z \mapsto \lambda_z$.

Taking traces we see that

$$\dim V \cdot \lambda_z = \sum_{g \in G} z(g)\chi_V(g).$$

So

$$w_\rho(C_i) = \frac{\chi(g_i)}{\chi(e)}|\mathcal{O}_i| \text{ for } g_i \in \mathcal{O}_i.$$

We now see that $w_\rho$ only depends on $\chi_\rho$ (and so on the isomorphism class of $\rho$) and we write $w_\chi = w_\rho$.

**Lemma.** *The values $w_\chi(C_i)$ are algebraic integers.*

Note this isn't *a priori* obvious since $\frac{1}{\chi(e)}$ will not be an algebraic integer for $\chi(e) \neq 1$.

*Proof.* Since $w_\chi$ is an algebra homomorphism $Z(kG) \to k$,

$$w_\chi(C_i)w_\chi(C_j) = \sum_k a_{ijk} w_\chi(C_k).$$

So the subring of $\mathbb{C}$ generated by $w_\chi(C_i)$ for $i = 1, \ldots, r$ is a finitely generated abelian group. The result follows from Fact 3 above. □

*Exercise.* Show that

$$a_{ijk} = \frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_\chi \frac{\chi(g_i)\chi(g_j)\chi(g_k^{-1})}{\chi(1)}.$$

(Hint: use column orthogonality, the last lemma and its proof.)

8.3. **Degrees of irreducibles.**

**Theorem.** *If $V$ is an irreducible representation of a group $G$ then* $\dim V$ *divides* $|G|$.

*Proof.* Let $\chi$ be the character of $V$. We'll show that $\frac{|G|}{\chi(e)}$ is an algebraic integer and so (since it is rational) an actual integer by Fact 2 above.

$$\begin{aligned}\frac{|G|}{\chi(e)} &= \frac{1}{\chi(e)} \sum_{g \in G} \chi(g)\chi(g^{-1}) \\ &= \sum_{i=1}^{r} \frac{1}{\chi(e)} |\mathcal{O}_i| \chi(g_i)\chi(g_i^{-1}) \\ &= \sum_{i=1}^{r} w_\chi(C_i)\chi(g_i^{-1})\end{aligned}$$

But the set of algebraic integers form a ring (by Fact 1 above) and each $w_\chi(C_i)$ and $\chi(g_i^{-1})$ is an algebraic integer so $\frac{|G|}{\chi(e)}$ is an algebraic integer as required. □

*Examples.*

(1) If $G$ is a $p$-group and $\chi$ is an irreducible character then $\chi(e)$ is always a power of $p$. In particular if $|G| = p^2$ then, since $\sum_\chi \chi(e)^2 = p^2$, every irreducible rep is 1-dimensional and so $G$ is abelian.
(2) If $G = A_n$ or $S_n$ and $p > n$ is a prime, then $p$ cannot divide the dimension of an irreducible rep.

In fact a stronger result is true:

**Theorem** (Burnside (1904))**.** *If $(\rho, V)$ is an irreducible representation then* $\dim V$ *divides* $|G/Z(G)|$.

You should compare this with $|\mathcal{O}_i| = |G|/|C_G(g_i)|$ divides $|G/Z(G)|$.

*Proof.* If $z \in Z = Z(G)$ then by Schur's Lemma $z$ acts on $V$ by $\lambda_z I$ for some $\lambda_z \in k$.

For each $m \geqslant 2$, consisder the irreducible representation of $G^m$ given by

$$\rho^{\otimes m} \colon G^m \to GL(V^{\otimes m}).$$

If $z = (z_1, \ldots, z_m) \in Z^m$ then $z$ acts on $V^{\otimes m}$ via $\prod_{i=1}^m \lambda_{z_i} I$. Thus if $\prod_1^m z_i = 1$ then $z \in \ker \rho^{\otimes m}$.

Let $Z' = \{(z_1, \ldots, z_m \in Z^m \mid \prod_{i=1}^m z_i = 1\}$ so $|Z'| = |Z|^{m-1}$. We may view $\rho^{\otimes m}$ as a degree $(\dim V)^m)$ irreducible representation of $G^m/Z'$.

Since $|G^m/Z'| = |G|^m/|Z|^{m-1}$ we can use the previous theorem to deduce that $(\dim V)^m$ divides $|G|^m/|Z|^{m-1}$.

By choosing $m$ very large and considering prime factors we can deduce the result: if $p^r$ divides $\dim V$ then $p^{rm}$ divides $|G/Z|^m|Z|$ for all $m$ and so $p^r$ divides $|G/Z|$. □

**Proposition.** *If $G$ is a simple group then $G$ has no irreducible representations of degree 2.*

*Proof.* If $G$ is cyclic then $G$ has no irreducible representations of degree bigger than 1, so we may assume $G$ is non-abelian.

If $|G|$ is odd then we may apply the theorem above.

If $|G|$ is even then $G$ has an element $x$ of order 2. By example sheet 2 Q2, for every irreducible $\chi$, $\chi(x) \equiv \chi(e) \mod 2$. So if $\chi(e) = 2$ then $\chi(x) = \pm 2$, and $\rho(x) = \pm I$. Thus $\rho(x) \in Z(\rho(G))$, a contradiction since $G$ is non-abelian simple. □

## Lecture 17

### 8.4. Burnside's $p^a q^b$ Theorem.

**Theorem** (Burnside (1904))**.** *Let $p, q$ be primes and $G$ a group of order $p^a q^b$ with $a, b$ non-negative integers such that $a + b \geqslant 2$, then $G$ is not simple.*

*Remarks.*

(1) It follows that every group of order $p^a q^b$ is soluble. That is, there is a chain of subgroups $G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_r = \{e\}$ with $G_{i+1}$ normal in $G_i$ and $G_i/G_{i+1}$ abelian for all $i$.
(2) Note that $|A_5| = 2^2 \cdot 3 \cdot 5$ so the order of a simple group can have precisely 3 prime factors.
(3) If $b = 0$ then we've seen this before; $Z(G)$ has an element of order $p$ which generates a proper normal subgroup.
(4) The first purely group theoretic proof of the $p^a q^b$-theorem appeared in 1972.
(5) In 1963 Feit and Thompson published a 255 page paper proving that every group of odd order in soluble.

The key step in the proof of the $p^a q^b$-theorem is the following:

**Proposition.** *If $G$ is a non-cyclic finite group with a conjugacy class $\mathcal{O}_i \neq \{e\}$ such that $|\mathcal{O}_i|$ has prime power order then $|G|$ is not simple.*

Granting the Proposition we can prove the theorem as follows: if $a, b > 0$, then let $Q$ be a Sylow-$q$-subgroup of $G$. Since $Z(Q) \neq 1$ we can find $g \in Z(Q)$. Then $q^b$ divides $|C_G(g)|$, so the conjugacy class containing $g$ has order $p^r$ for some $0 \leqslant r \leqslant a$. The theorem now follows immediately from the Proposition.

To prove the Proposition we need some Lemmas

**Lemma.** *Suppose $0 \neq \alpha = \frac{1}{m}\sum_{i=1}^m \lambda_i$ with all $\lambda_i$ $n^{th}$ roots of 1 is an algebraic integer. Then $|\alpha| = 1$.*

*Sketch proof (non-examinable).* By assumption $\alpha \in \mathbb{Q}(\epsilon)$ where $\epsilon = e^{2\pi i/n}$.

Let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$. It is known that $\{\beta \in \mathbb{Q}(\epsilon) \mid \sigma(\beta) = \beta \text{ for all } \sigma \in \mathcal{G}\} = \mathbb{Q}$.

Consider $N(\alpha) := \prod_{\sigma\in\mathcal{G}} \sigma(\alpha)$. Since $N(\alpha)$ is fixed by every element of $\mathcal{G}$, $N(\alpha) \in \mathbb{Q}$. Moreover $N(\alpha)$ is an algebraic integer since Galois conjugates of algebraic integers are algebraic integers — they satisfy the same integer polynomials. Thus $N(\alpha) \in \mathbb{Z}$.

But for each $\sigma \in \mathcal{G}$, $|\sigma(\alpha)| = |\frac{1}{m}\sum \sigma(\lambda_i)| \leqslant 1$. Thus $N(\alpha) = \pm 1$, and $|\alpha| = 1$ as required. □

**Lemma.** *Suppose $\chi$ is an irreducible character of $G$, and $\mathcal{O}$ is a conjugacy class in $G$ such that $\chi(e)$ and $|\mathcal{O}|$ are coprime. For $g \in \mathcal{O}$, $|\chi(g)| = \chi(e)$ or $0$.*

*Proof.* By Bezout, we can find $x, y \in \mathbb{Z}$ such that $a\chi(e) + b|\mathcal{O}| = 1$. Define

$$\alpha := \frac{\chi(g)}{\chi(e)} = a\chi(g) + b\frac{\chi(g)}{\chi(e)}|\mathcal{O}|$$

Then $\alpha$ satisfies the conditions of the previous lemma and so this lemma follows. □

*Proof of Proposition.* Suppose for contradication that $G$ is simple and has an element $g \in G$ that lives in a conjugacy class $\mathcal{O}$ of order $p^r$.

If $\chi$ is a non-trivial irreducible character of $G$ then $|\chi(g)| < \chi(1)$ since otherwise $\rho(g)$ is a scalar matrix and so lies in $Z(\rho(G)) \cong Z(G)$.

Thus by the last lemma, for every non-trivial irreducible character, either $p$ divides $\chi(e)$ or $|\chi(g)| = 0$ . By column orthogonality,

$$0 = \sum_{\chi} \chi(e)\chi(g).$$

Thus $\frac{-1}{p} = \sum_{\chi\neq\mathbf{1}} \frac{\chi(e)}{p}\chi(g)$ in an algebraic integer in $\mathbb{Q}$. Thus $\frac{1}{p}$ in $\mathbb{Z}$ the desired contradiction. □

## 9. Topological groups

Consider $S^1 = U_1(\mathbb{C}) = \{g \in \mathbb{C}^\times \mid |g| = 1\} \cong \mathbb{R}/\mathbb{Z}$.

By considering $\mathbb{R}$ as a $\mathbb{Q}$-vector space we see that as a group

$$S^1 \cong \mathbb{Q}/\mathbb{Z} \oplus \bigoplus_{x\in X} \mathbb{Q}$$

for an an uncountable set $X$.

Thus we see that as an abstract group $S^1$ has uncountably many irreducible representations: for each $\lambda \in \mathbb{R}$ we can define a one-dimensional representation by

$$\rho_\lambda(e^{2\pi i\mu}) = \begin{cases} 1 & \mu \notin \mathbb{Q}\lambda \\ e^{2\pi i\mu} & \mu \in \mathbb{Q}\lambda \end{cases}$$

Then $\rho_\lambda = \rho_{\lambda'}$ if and only if $\mathbb{Q}\lambda = \mathbb{Q}\lambda'$. In this way we get uncountably many irreducible representations of $S^1$ (we haven't listed them all). We don't really have any control over the situation.

However, $S^1$ is not just a group; it comes with a topology as a subset of $\mathbb{C}$. Moreover $S^1$ acts naturally on complex vector spaces in a continuous way.

**Definition.** A *topological group* $G$ is a group $G$ which is also a topological space such that the multiplication map $G \times G \to G; (g,h) \mapsto gh$ and the inverse map $G \to G; g \mapsto g^{-1}$ are continuous maps.

*Examples.*

(1) $GL_n(\mathbb{C})$ with topology from $\mathbb{C}^{n^2}$.
(2) $G$ finite — with the discrete topology.
(3) $O(n) = \{A \in GL_n(\mathbb{R}) \mid A^T A = I\}$; $SO(n) = \{A \in O(n) \mid \det A = 1\}$.
(4) $U(n) = \{A \in GL_n(\mathbb{C}) \mid \overline{A^T} A = I\}$; $SU(n) = \{A \in U(n) \mid \det A = 1\}$.
(5) *$G$ profinite such as $\mathbb{Z}_p$, the completion of $\mathbb{Z}$ with respect to the $p$-adic metric.

**Definition.** A *representation* of a topological group $G$ on a vector space $V$ is a continuous group homomorphism $G \to GL(V)$.

*Remarks.*

(1) If $X$ is a topological space then $\alpha\colon X \to GL_n(\mathbb{C})$ is continuous if and only if the maps $x \mapsto \alpha_{ij}(x) = \alpha(x)_{ij}$ are continuous for all $i, j$.
(2) If $G$ is a finite group with the discrete topology. Then continous function $G \to X$ just means function $G \to X$.

**Theorem.** *Every one dimensional (cts) representation of $S^1$ is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.*

It is easy to see that the given maps are representations, we must show that they are the only ones.

**Lemma.** *If $\psi\colon (\mathbb{R}, +) \to \mathbb{R}$ is a continous group homomorphism then there is some $\lambda \in \mathbb{R}$ such that $\psi(x) = \lambda x$ for all $x \in \mathbb{R}$.*

*Proof.* Let $\lambda = \psi(1)$. Since $\psi$ is a group homomorphism, $\psi(n) = \lambda n$ for all $n \in \mathbb{Z}$. Then m$\psi(n/m) = \psi(n) = \lambda n$ and so $\psi(n/m) = \lambda n/m$. That is $\psi(x) = \lambda x$ for all $x \in \mathbb{Q}$. But $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\psi$ is continuous so $\psi(x) = \lambda x$ for all $x \in \mathbb{R}$. □

**Lemma.** *If $\psi\colon (\mathbb{R}, +) \to S^1$ is a continuous group homomorphism then $\psi(x) = e^{2\pi i \lambda x}$ for some $\lambda \in \mathbb{R}$.*

*Proof.* Claim: if $\psi\colon \mathbb{R} \to S^1$ is any continuous function with $\psi(0) = 1$ then there is a unique continuous function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\psi(x) = e^{2\pi i \alpha(x)}$. (Sketch proof of claim: locally $\alpha(x) = \frac{1}{2\pi i} \log \psi(x)$ we can choose the branches of log to make the pieces glue together continuously).

Now given the claim, if $\psi$ is a group homomorphism and $\alpha$ is the map defined by the claim we can define a continuous function $\mathbb{R}^2 \to \mathbb{R}$ by

$$\Delta(a, b) := \alpha(a + b) - \alpha(a) - \alpha(b).$$

Since $e^{2\pi i \Delta(a,b)} = \psi(a + b)\psi(a)^{-1}\psi(b)^{-1} = 1$, $\Delta$ only takes values in $\mathbb{Z}$. Thus $\Delta$ is constant. Since $\Delta(a, 0) = 0$ for all $a$ we see that $\Delta \equiv 0$ and so $\alpha$ is a group homomorphism. By the previous lemma we see $\alpha(x) = \lambda x$ for some $\lambda \in \mathbb{R}$ and so $\psi(x) = e^{2\pi i \lambda x}$ as required. □

## Lecture 18

Last time we proved

**Lemma.** *If $\psi\colon (\mathbb{R}, +) \to S^1$ is a continuous group homomorphism then $\psi(x) = e^{2\pi i \lambda x}$ for some $\lambda \in \mathbb{R}$.*

We'll now use this to prove

**Theorem.** *Every one dimensional (cts) representation of $S^1$ is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.*

*Proof.* Let $\rho\colon S^1 \to GL_1(\mathbb{C})$ be a continuous representation. Since $S^1$ is compact, $\rho(S^1)$ has closed and bounded image. Since $\rho(z^n) = \rho(z)^n$ for $n \in \mathbb{Z}$, it follows that $\rho(S^1) \subset S^1$.

Now let $\psi\colon \mathbb{R} \to S^1$ be defined by $\psi(x) = \rho(e^{2\pi i x})$, a continuous homomorphism. By the most recent Lemma, $\rho(e^{2\pi i x}) = \psi(x) = e^{2\pi i \lambda x}$ for some $\lambda \in \mathbb{R}$.

Since also $\rho(e^{2\pi i x}) = 1$ we see $\lambda \in \mathbb{Z}$. □

Our most powerful idea for studying representations of finite groups has been averaging over the group; that is the operation $\frac{1}{|G|}\sum_{g\in G}$. When considering more general topological groups we should replace $\sum$ by $\int$.

**Definition.** Let $G$ be a topological group. Let $C(G) = \{f\colon G \to \mathbb{C} \mid f \text{ is continuous}\}$. Then a linear map $\int_G\colon C(G) \to \mathbb{C}$ (write $\int_G f = \int_G f(g)\,\mathrm{d}g$) is called a *Haar measure* if

(i) $\int_G 1 = 1$ (so $\int_G$ is normalised so total volume is 1;
(ii) $\int_G f(xg)dg = \int_G f(g)dg$ for all $x \in G$ (so $\int_G$ is translation invariant).

*Examples.*
(1) If $G$ finite, then $\int_G f = \frac{1}{|G|}\sum_{g\in G} f(g)$.
(2) If $G = S^1$, $\int_G f = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\mathrm{d}\theta$.

**Theorem.** *If $G$ is a compact Hausdorff group, then there is a unique Haar measure on $G$.*

*Proof.* Omitted □

We've seen a Haar measure on $S^1$ and will compute one on $SU(2)$ later.

**Corollary** (Weyl's Unitary Trick)**.** *If $G$ is a compact Hausdorff group then every continuous representation $(\rho, V)$ has a $G$-invariant invariant Hermitian inner product*

*Proof.* Same as for finite groups: let $(-,-)$ be any inner product on $V$, then

$$\langle v, w\rangle = \int_G (\rho(g)v, \rho(g)w)\,\mathrm{d}g$$

is the required $G$-invariant inner product. □

**Corollary** (Maschke's Theorem)**.** *If $G$ is a compact Hausdorff group then every continuous representation of $G$ is completely reducible.*

*Proof.* Same as for finite groups: Given a rep $(\rho, V)$ choose a $G$-invariant inner product. If $W$ is a subrep of $V$ then $W^\perp$ is a $G$-invariant complement. □

We can use the Haar measure to put an inner product on the space $\mathcal{C}_G$ of (continuous) class functions:

$$\langle f, f'\rangle := \int_G \overline{f(g)} f'(g)\,\mathrm{d}g.$$

If $\rho\colon G \to GL(V)$ is a continuous representation then $\chi_\rho := \operatorname{tr}\rho$ is a continuous class function since each $\rho(g)_{ii}$ is continuous.

**Corollary** (Orthogonality of Characters)**.** *If $G$ is a compact Hausdorff group and $(\rho, V)$ and $(\rho', W)$ are continuous irreducible reps of $G$ then*

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } \chi_V \neq \chi_W. \end{cases}$$

*Proof.* Same as for finite groups:

$$\begin{aligned} \langle \chi_V, \chi_W \rangle &= \int_G \overline{\chi_V(g)} \chi_W(g) \, \mathrm{d}g \\ &= \dim \operatorname{Hom}_G(\mathbf{1}, \operatorname{Hom}(V, W)) \\ &= \dim \operatorname{Hom}_G(V, W). \end{aligned}$$

Then apply Schur's Lemma. □

It is also possible to make sense of 'the characters span the space of class functions' but this requires a little more analysis in the form of the Peter–Weyl theorem.

*Example.* $G = S^1$.

We've already seen that the one-dimensional reps of $S^1$ are all of the form $z \mapsto z^n$ for $n \in \mathbb{Z}$. Since $S^1$ is abelian we can use our usual argument to see that these are all irreducible reps — given any rep $\rho$ we can find a simultaneous eigenvector for each $\rho(g)$. Thus the 'character table' of $S^1$ has rows $\chi_n$ indexed by $\mathbb{Z}$ with $\chi_n(e^{i\theta}) = e^{in\theta}$.

Now if $V$ is any rep of $S^1$ then by Machke's Theorem $V$ breaks up as a direct sum of one dimensional subreps and so its character $\chi_V$ is of the form

$$\chi_V(z) = \sum_{n \in \mathbb{Z}} a_n z^n$$

with $a_n$ non-negative integers and only finitely many non-zero. As usual $a_n$ is the number of copies of $\rho_n \colon z \mapsto z^n$ in the decomposition of $V$. Thus we can compute

$$a_n = \langle \chi_n, \chi_V \rangle = \frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\theta}) e^{-in\theta} \, \mathrm{d}\theta.$$

Thus

$$\chi_V(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \left( \frac{1}{2\pi} \int_0^{2\pi} \chi_V(e^{i\theta'}) e^{-in\theta'} \, \mathrm{d}\theta' \right) e^{in\theta}.$$

So Fourier decomposition gives the decomposition of $\chi_V$ into irreducible characters and the Fourier mode is the multiplicity of an irreducible character.

*Remark.* In fact by the theory of Fourier series any continuous function on $S^1$ can be uniformly approximated by a finite $\mathbb{C}$-linear combination of the $\chi_n$.

Moreover the $\chi_n$ form a complete orthonormal set in the Hilbert space of square-integrable complex-valued functions on $S^1$. That is every function $f$ on $S^1$ such that $\int_0^{2\pi} |f(e^{i\theta})|^2 \, \mathrm{d}\theta$ exists has a unique series expansion

$$f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \left( \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta'}) e^{-in\theta'} \, \mathrm{d}\theta' \right) e^{in\theta}$$

converging in the norm $||f|| = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 \, \mathrm{d}\theta$.

## Lecture 19

### 9.1. Conjugacy classes of $SU(2)$.

Recall that $SU(2) = \{A \in GL_2(\mathbb{C}) \mid \overline{A^T}A = I, \det A = 1\}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ then since $\det A = 1$, $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Thus $d = \overline{a}$ and $c = -\overline{b}$. Moreover $a\overline{a} + b\overline{b} = 1$. In this way we see that

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\}$$

which may be viewed topologically as $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$.

More precisely if

$$\mathbb{H} := \mathbb{R} \cdot SU(2) = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid w, z \in \mathbb{C} \right\} \subset M_2(\mathbb{C}).$$

Then $||A||^2 = \det A$ defines a norm on $\mathbb{H} \cong \mathbb{R}^4$ and $SU(2)$ is the unit sphere in $\mathbb{H}$. If $A \in SU(2)$ and $X \in \mathbb{H}$ then $||AX|| = ||X||$ since $||A|| = 1$. So, after normalisation, usual integration of functions on $S^3$ defines a Haar measure on $SU(2)$.

**Definition.** Let $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}, |a| = 1 \right\} \cong S^1$, a *maximal torus* in $SU(2)$.

Also define $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)$

**Lemma.**
*(i) if $t \in T$ then $sts^{-1} = t^{-1}$;*
*(ii) $s^2 = -I \in Z(SU(2))$*
*(iii) $N_{SU(2)}(T) = T \cup sT = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \mid a \in \mathbb{C}, |a| = 1 \right\}$*

*Proof.* All three parts follow from direct computation (exercise). □

**Proposition.**
*(i) Every conjugacy class $\mathcal{O}$ in $SU_2$ contains an element of $T$.*
*(ii) More precisely. if $\mathcal{O}$ is a conjugacy class then $\mathcal{O} \cap T = \{t, t^{-1}\}$ for some $t \in T$ — $t = t^{-1}$ if and only if $t = \pm I$ when $\mathcal{O} = \{t\}$.*
*(iii) There is a bijection*

$$\{\textit{conjugacy classes in } SU(2)\} \to [-1, 1]$$

*given by $A \mapsto \frac{1}{2} \operatorname{tr} A$.*

*Proof.* (i) For every unitary matrix $A$ there is an orthonormal basis of eigenvectors of $A$; that is there is a unitary matrix $P$ such that $PAP^{-1}$ is diagonal. We want to arrange that $\det P = 1$. But we can replace $P$ by $Q = \sqrt{\det P}P$. Thus every conjugacy class $\mathcal{O}$ in $SU(2)$ contains a diagonal matrix $t$. Since additionally $t \in SU(2)$, $t \in T$.

(ii) If $\pm I \in \mathcal{O}$ the result is clear.

Suppose $t \in \mathcal{O} \cap T$ for some $t \neq \pm I$. Then

$$\mathcal{O} = \{gtg^{-1} \mid g \in SU(2)\}.$$

We've seen before that $sts^{-1} = t^{-1}$ so $\mathcal{O} \cap T \supset \{t, t^{-1}\}$.

Conversely, if $t' \in \mathcal{O} \cap T$ then $t'$ and $t$ must have the same eigenvalues since they are conjugate. This suffices to see that $t' \in \{t^{\pm 1}\}$.

(iii) To see the given function is injective, suppose that $\frac{1}{2}\operatorname{tr} A = \frac{1}{2}\operatorname{tr} B$. Then since $\det A = \det B = 1$, $A$ and $B$ must have the same eigenvalues. By part (i) they are both diagonalisable and by the proof of part (ii) this suffices to see that they are conjugate.

To see that it is surjective notice that $\frac{1}{2}\operatorname{tr}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = cos\theta$. Since $\cos\colon \mathbb{R} \to \mathbb{R}$ has image $[-1,1]$ the given function is surjective. $\square$

Let's write $\mathcal{O}_x = \{A \in SU(2) \mid \frac{1}{2}\operatorname{tr} A = x\}$ for $x \in [-1,1]$. We've proven that the $\mathcal{O}_x$ are the conjugacy classes in $SU(2)$. Clearly $\mathcal{O}_1 = \{I\}$ and $\mathcal{O}_{-1} = \{-I\}$.

**Proposition.** *If $-1 < x < 1$ then $\mathcal{O}_x$ is homeomorphic to $S^2$.*

*Proof.* First we observe that $\mathcal{O}_x \cong SU(2)/T$ for each $-1 < x < 1$. To see this it suffices to show that $T = C_{SU_2}\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right)$ for $\lambda \neq \lambda^{-1}$. But

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda^{-1} c & \lambda^{-1} d \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda^{-1} b \\ \lambda c & \lambda^{-1} d \end{pmatrix}.$$

For these to be equal for $\lambda \neq \lambda^{-1}$ we require $b = c = 0$.

Next we recall that $SU(2)$ acts on $S^2 \cong \mathbb{C} \cup \{\infty\}$ by Mobius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

This action is transitive since for each $z \in \mathbb{C}$ there are $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$ and $a/b = z$ (exercise). Then $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \cdot \infty = a/b$.

But $\operatorname{Stab}_{SU(2)}(\infty) = T$ so $SU(2)/T \cong S^2$. $\square$

### 9.2. Representations of $SU(2)$.

Now we understand the conjugacy classes of $SU(2)$, we'll try to work out its representation theory.

Let $V_n$ be the complex vectorspace of homogeneous polynomials in two variables $x, y$. So $\dim V_n = n+1$. Then $GL_2(\mathbb{C})$ acts on $V_n$ via

$$\rho_n\colon GL_2(\mathbb{C}) \to GL(V_n)$$

given by

$$\rho_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f(x,y) = f(ax+cy, bx+dy).$$

*Examples.*

$V_0 = \mathbb{C}$ has the trivial action.

$V_1 = \mathbb{C}^2$ is the standard representation of $GL_2(\mathbb{C}^2)$ on $\mathbb{C}^2$ with basis $x, y$.

$V_2 = \mathbb{C}^3$ has basis $x^2, xy, y^2$ then

$$\rho_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

Since $SU(2)$ is a subgroup of $GL_2(\mathbb{C})$ we can view $V_n$ as a representation of $SU(2)$ by restriction. In fact as we'll see, the $V_n$ are all irreducible reps of $SU(2)$ and every irreducible rep of $SU(2)$ is isomorphic to one of these.

*Remark.* $-I \in Z(SU(2))$ acts on $V_n$ as $-1$ on if $n$ is odd and as 1 if $n$ is even.

**Lemma.** *A continuous class function $f\colon SU(2) \to \mathbb{C}$ is determined by its restriction to $T$ and $f|_T$ is even ie* $f\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right) = f\left(\begin{pmatrix} z^{-1} & 0 \\ 0 & z\end{pmatrix}\right)$.

*Proof.* We've seen that each conjugacy class in $SU(2)$ meets $T$ and so a class fucntion is determined by its restriction to $T$. Then evenness follows from the additional fact that $T \cap \mathcal{O} = \{t^{\pm 1}\}$ for some $t \in T$. □

Thus we can view the character of a representation $\rho$ of $SU(2)$ as an even function $\chi_\rho\colon S^1 \to \mathbb{C}$.

**Lemma.** *If $\chi$ is a character of a representation of $SU(2)$ then $\chi|_T$ is a Laurent polynomial ie a finite $\mathbb{N}$ linear combination of functions*

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix} \mapsto z^n \text{ for } n \in \mathbb{Z}.$$

*Proof.* If $V$ is a continuous representation of $SU(2)$ then $\operatorname{Res}_T^{SU(2)} V$ is a continuous representation of $T$ and $\chi_{\operatorname{Res}_T V}$ is the restriction of $\chi_V$ to $T$. But we've proven already that every continuous representation of $T$ has character of the given form. □

## Lecture 20

Write

$$\mathbb{N}[z, z^{-1}] := \left\{ \sum_{n \in \mathbb{Z}} a_n z^n \mid a_n \in \mathbb{N} \text{ and only finitely many } a_n \neq 0 \right\}$$

and

$$\mathbb{N}[z, z^{-1}]^{ev} = \{ f \in \mathbb{N}[z, z^{-1}] \mid f(z) = f(z^{-1}) \}.$$

We showed last time that for every continuous representation $V$ of $SU(2)$, the character $\chi_V \in \mathbb{N}[z, z^{-1}]^{ev}$ after identifying it with its restriction to $T$.

The next thing to do is compute the character $\chi_n$ of $(\rho_n, V_n)$, the representation consisting of degree $n$ homogeneous polynomials in $x$ and $y$.

$$\rho_n\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right)(x^i y^j) = (zx)^i (z^{-1}y)^j = z^{i-j} x^i y^j.$$

So $x^i y^j$ is an eigenvector for each $t \in T$ and $T$ acts on $V_n$ via

$$\rho_n\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right) = \begin{pmatrix} z^n & & & & & \\ & z^{n-2} & & & & \\ & & z^{n-4} & & & \\ & & & \ddots & & \\ & & & & z^{2-n} & \\ & & & & & z^{-n} \end{pmatrix}.$$

Thus

$$\chi_n\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right) = z^n + z^{n-2} + \cdots + z^{2-n} + z^{-n} = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} \in \mathbb{N}[z, z^{-1}]^{ev}.$$

**Theorem.** *$V_n$ is irreducible as a reperesentation of $SU(2)$.*

*Proof.* Let $0 \neq W \leqslant V_n$ be a $SU(2)$-invariant subspace. We want to show that $W = V_n$.

Let $0 \neq w = \sum \lambda_i(x^{n-i}y^i) \in W$. We claim that $x^{n-i}y^i \in W$ whenever $\lambda_i \neq 0$.

We prove the claim by induction on $k = |\{i \mid \lambda_i \neq 0\}|$.

If $k = 1$ then $w$ is a non-zero scalar multiple of $x^{n-i}y^i$ and we're done.

If $k > 1$ choose $i$ such that $\lambda_i \neq 0$ and $z \in S^1$ such that $\{z^n, z^{n-2}, \ldots, z^{2-n}, z^n\}$ are distict complex numbers. Then

$$\rho_n\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}\right) w - z^{n-2i}w = \sum \lambda_j(z^{n-2j} - z^{n-2i})(x^{n-j}y^j) \in W$$

since $W$ is $SU(2)$-invariant. Now $\lambda_j(z^{n-2j} - z^{n-2i}) \neq 0$ precisely if $\lambda_j \neq 0$ and $j \neq i$. Thus by the induction hypothesis $x^jy^{n-j} \in W$ for all $j \neq i$ with $\lambda_j \neq 0$. It follows that also $x^iy^{n-i} = \frac{1}{\lambda_i}(w - \sum_{j\neq i} \lambda_j x^jy^{n-j}) \in W$ as required.

Now we know that $x^iy^{n-i} \in W$ for some $i$. Since

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix} x^iy^{n-i} = \frac{1}{\sqrt{2}}((x-y)^i(x+y)^{n-i}) \in W$$

we can use the claim to deduce that $x^n \in W$. Repeating the same calculation for $i = n$, we see that $(x+y)^n \in W$ and so, by the claim again, $x^iy^{n-i} \in W$ for all $i$.

Thus $W = V_n$. □

Alternative proof:

We can identify $\mathcal{O}_{\cos\theta} = \{A \in SU(2) \mid \frac{1}{2}\operatorname{tr} A = \cos\theta\}$ with the two-sphere $\{(\operatorname{Im}(a))^2 + |b|^2 = \sin^2\theta\}$ of radius $|\sin\theta|$. Thus if $f$ is a class-function on $SU(2)$, since $f$ is constant on each $\mathcal{O}_{\cos\theta}$,

$$\int_{SU(2)} f(g)\,\mathrm{d}g = \frac{1}{2\pi^2}\int_0^{2\pi} \frac{1}{2} f\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\right) 4\pi \sin^2\theta\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})\sin^2\theta\,\mathrm{d}\theta.$$

Note this is normalised correctly, since $\frac{1}{\pi}\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta = 1$. So it suffices to prove that $\frac{1}{\pi}\int_0^{2\pi} |\chi_{V_n}(e^{i\theta})|^2 \sin^2\theta\,\mathrm{d}\theta = 1$ for $z = e^{i\theta}$. (exercise: verify this).

**Theorem.** *Every irreducible representation of $SU(2)$ is isomorphic to $V_n$ for some $n \geqslant 0$.*

*Proof.* Let $V$ be an irreducible representation of $SU(2)$ so $\chi_V \in \mathbb{N}[z, z^{-1}]^{ev}$. Now $\chi_0 = 1, \chi_1 = z + z^{-1}, \chi_2 = z^2 + 1 + z^{-2}, \ldots$ form a basis of $\mathbb{Q}[z, z^{-1}]^{ev}$ as (non-f.d.) $\mathbb{Q}$-vector spaces. Thus $\chi_V = \sum a_i\chi_i$ for some $a_i \in \mathbb{Q}$, only finitely many non-zero.

Clearing denominators and moving negative terms to the left-hand-side, we get a formula

$$m\chi_V + \sum_{i\in I} m_i\chi_i = \sum_{j\in J} m_j\chi_j$$

for some disjoint finite subsets $I, J \subset \mathbb{N}$ and $m, m_i \in \mathbb{N}$. By orthogonality of characters and complete reducibility we obtain

$$mV \oplus \bigoplus_{i\in I} m_iV_i \cong \bigoplus_{j\in J} m_jV_j$$

since $V$ is irreducible and $V \cong V_j$ some $j \in J$. $\square$

9.3. **Tensor products of representations of $SU(2)$.** We've seen that if $V, W$ are representations of $SU(2)$ such that $\mathrm{Res}_T^{SU(2)} V \cong \mathrm{Res}_T^{SU(2)} W$ then $V \cong W$. We want to understand $\otimes$ for representations of $SU(2)$.

**Proposition.** *If $G \cong SU(2)$ or $S^1$ and $V, W$ are representations of $G$ then*

$$\chi_{V\otimes W} = \chi_V \cdot \chi_W.$$

*Proof.* By the discussion above we only need to consider $G \cong S^1$.

If $V$ and $W$ have eigenbases $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ such that $ze_i = z^{n_i} e_i$ and $zf_j = z^{m_j} f_j$ then $z(e_i \otimes f_j) = z^{n_i+m_j}(e_i \otimes f_j)$. So

$$\chi_{V\otimes W}(z) = \sum_{i,j} z^{n_i+m_j} = \left(\sum_i z^{n_i}\right)\left(\sum_j z^{m_j}\right) = \chi_V(z)\chi_W(z)$$

as required. $\square$

Let's compute some examples for $SU(2)$:

$$\chi_{V_1\otimes V_1}(z) = (z+z^{-1})^2 = z^2 + 1 + z^{-2} + 1 = \chi_{V_2} + \chi_{V_0}$$

and

$$\chi_{V_2\otimes V_1}(z) = (z^2+1+z^{-2})(z+z^{-1}) = z^3 + 2z + 2z^{-1} + z^{-3} = \chi_{V_3} + \chi_{V_1}.$$

**Proposition** (Clebsch–Gordan rule)**.** *For $n, m \in \mathbb{N}$,*

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|+2} \oplus V_{|n-m|}.$$

*Proof.* Without loss of generality, $n \geqslant m$. Then

$$\begin{aligned}(\chi_n \cdot \chi_m)(z) &= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} \cdot (z^m + z^{m-2} + \cdots + z^{-m}) \\ &= \sum_{j=0}^{m} \frac{z^{n+m+1-2j} - z^{-(n+m+1-2j)}}{z - z^{-1}} \\ &= \sum_{j=0}^{m} \chi_{n+m-2j}(z)\end{aligned}$$

$\square$

9.4. **Representations of $SO(3)$.**

**Proposition.** *There is an isomorphism of topological groups $SU(2)/\{\pm I\} \cong SO(3)$.*

**Corollary.** *Every irreducible representation of $SO(3)$ is of the form $V_{2n}$ for some $n \geqslant 0$.*

*Proof.* It follows from the Proposition that irreducible representations of $SO(3)$ correspond to irreducible representations of $SU(2)$ such that $-I$ acts trivially. But we saw before that $-I$ acts on $V_n$ as $-1$ when $n$ is odd and as 1 when $n$ is even. $\square$

## Lecture 21

Let's prove the proposition from the end of last time:

**Proposition.** *There is an isomorphism of topological groups $SU(2)/\{\pm I\} \cong SO(3)$.*

*Proof.* Consider $\mathbb{H}^\circ = \{A \in \mathbb{H} \mid \operatorname{tr} A = 0\} = \mathbb{R}\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \rangle$ equipped with the norm $||A|| = \det A$.

$SU(2)$ acts by isometries on $\mathbb{H}^\circ$ via $(X, A) \mapsto XAX^{-1}$ giving a group homomorphism

$$\theta \colon SU(2) \to SO(3)$$

with kernel $Z(SU(2)) = \{\pm I\}$. Since $SU(2)$ is compact and $SO(3)$ is Hausdorff the continuous group isomorphism $\bar{\theta} \colon SU(2)/\{\pm I\} \to \mathrm{Im}\theta$ is a homeomorphism so it suffices to prove that $\mathrm{Im}\theta = SO(3)$.

Now

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} ai & b \\ -\bar{b} & -ai \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} ai & e^{2i\theta} b \\ -e^{-i\theta}\bar{b} & -ai \end{pmatrix}$$

so $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ acts on $\mathbb{R}\langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ by rotation in the $\mathbf{jk}$-plane through an angle $2\theta$.

*Exercise.* Show that $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ acts by rotation through $2\theta$ in the $\mathbf{ik}$-plane, and $\begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$ acts by rotation through $2\theta$ in the $\mathbf{ij}$-plane. Deduce that $\mathrm{Im}\theta = SO(3)$.

□

## 10. Character table of $GL_2(\mathbb{F}_q)$ and related groups

**10.1. $GL_2(\mathbb{F}_q)$.** Let $p > 2$ be a prime, $q = p^a$ a power of $p$ for some $a > 0$, and $\mathbb{F}_q$ be the field with $q$-elements.

$$G := GL_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}_q \text{ and } ad - bc \neq 0 \right\}.$$

We are going to construct the character table of $G$. Our main strategy will be induction from 1-dimensional representations of large subgroups.

Let $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q \right\}$ an abelian subgroup of $G$ of order $q$ (a Sylow $p$-subgroup of $G$) and $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid b \in \mathbb{F}_q, a, d \in \mathbb{F}_q^\times \right\}$ a *Borel subgroup* of $G$. Then $N$ is normal in $B$ and $B/N \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$.

$G$ acts transitively on $\mathbb{F}_q \cup \{\infty\}$ via Mobius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d} \text{ for } z \in \mathbb{F}_q$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\infty) = a/c$$

so $B = \mathrm{Stab}_G(\infty)$. Thus $|G| = |B|(q+1) = q(q-1)^2(q+1)$.

Writing $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we see that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} s \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a + b\beta \\ d & \beta d \end{pmatrix}.$$

Hence $BsN$ contains $q|B|$ elements so must be $G \backslash B$. Thus $BsN = BsB$ and $B\backslash G/B$ has two double cosets $B$ and $BsB$ (this is called Bruhat decomposition).

By Mackey's irreduciblity criterion it follows that if $W$ is an irreducible representation of $B$, then $\operatorname{Ind}_B^G W$ is an irreducible representation of $G$ precisely if $\operatorname{Res}^B_{B\cap {}^sB} W \not\cong \operatorname{Res}^{{}^sB}_{B\cap {}^sB} {}^sW$. Since $s$ swaps $0, \infty \in \mathbb{F}_q \cup \{\infty\}$,

$${}^sB = \operatorname{Stab}_G(0) = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mid a, d \in {\mathbb{F}_q}^\times, c \in \mathbb{F}_q \right\}$$

and $B \cap {}^sB = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\} =: T$.

One final important subgroup is $K := \left\{ \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} \mid x, y \in \mathbb{F}_q \text{ not both zero} \right\}$ where $\epsilon$ is fixed non-square in $\mathbb{F}_q$ (the squaring map on $\mathbb{F}_q$ is a group homomorphism with kernel $\pm 1$ so half of the elements of $\mathbb{F}_q$ are non-squares so we may fix one).

Now $K \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a field with $q^2$ elements under usual matrix operations (exercise: check this). So $K \cong C_{q^2-1}$ — a 'non-split torus'.

If $y \neq 0$, then $\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}$ is not diagonalisable over $\mathbb{F}_q$ since its characteristic polynomial is $t^2 - 2xt + x^2 - \epsilon y^2 = (t-x)^2 - \epsilon y^2$ which has no roots in $\mathbb{F}_q$.

Next, we compute the conjugacy classes in $G$. Certainly if two elements of $G$ are conjugate they have the same minimal polynomial. In fact, we will see this is a total invariant for conjugacy classes (exercise: prove this directly).

Suppose $A \in GL_2(\mathbb{F}_q)$ has linear minimal polynomial $X - \lambda$, say, for some $\lambda \in {\mathbb{F}_q}^\times$. Then $A = \lambda I$. So $A$ lives in a conjugacy class of size 1. There are $q - 1$ such classes.

Next, if $A$ has minimal polynomial $(X - \lambda)^2$ for some $\lambda \in {\mathbb{F}_q}^\times$ then there is $w \in {\mathbb{F}_q}^2$ such that $(A - \lambda)w \neq 0$ but $(A - \lambda)^2 w = 0$. Then $v := (A - \lambda)w, w$ is a basis for ${\mathbb{F}_q}^2$ and $Av = \lambda v$, $Aw = v + \lambda w$ so $A$ is conjugate to $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Now $C_G\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{F}_q, a \neq 0 \right\}$ so $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is in a conjugacy class of order $\frac{q(q-1)(q^2-1)}{(q-1)q} = q^2 - 1$. There are $q - 1$ such classes.

If $A$ has minimal polynomial $(X - \lambda)(X - \mu)$ for some distinct $\lambda, \mu \in {\mathbb{F}_q}^\times$. Then $A$ is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ and to $\begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}$. Moreover $C_G\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) = T$. So $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ is in a conjugacy class of order $\frac{q(q-1)(q^2-1)}{(q-1)^2} = q(q+1)$. There are $\binom{q-1}{2}$ such classes.

Finally if $A$ has minimal polynomial $(X - \lambda)^2 - \epsilon\mu^2$ for $\mu \in {\mathbb{F}_q}^\times$ then $A$ could be $\begin{pmatrix} \lambda & \epsilon\mu \\ \mu & \lambda \end{pmatrix}$.

Now

$$C_G\left(\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}\right)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mid\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}=\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right\}=K$$

so $\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}$ lives in a conjugacy class of size $\frac{q(q-1)(q^2-1)}{q^2-1}=q(q-1)$. There are at least $q(q-1)/2$ such classes.

We've now covered

$$(q-1)+(q^2-1)(q-1)+q(q+1)\binom{q-1}{2}+(q^2-q)\frac{q(q-1)}{2}=|G|$$

elements so there are precisely $q(q-1)/2$ classes with irreducible quadratic minimal polynomial.

In summary

| Representative | $C_G$ | No of elts | No of such classes |
|---|---|---|---|
| $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ | $G$ | $1$ | $q-1$ |
| $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ | $\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}$ | $q^2-1$ | $q-1$ |
| $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ | $T$ | $q(q+1)$ | $\binom{q-1}{2}$ |
| $\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}$ | $K$ | $q(q-1)$ | $\binom{q}{2}$ |

## LECTURE 22

Recall our notation from last time. $G=GL_2(\mathbb{F}_q)\geqslant B=\left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\right\}$ has normal subgroup $N=\left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right\}$.

Then $Z=Z(G)=\left\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right\}$, $T=\left\{\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\right\}$, $K=\left\{\begin{pmatrix}x & \epsilon y\\ y & x\end{pmatrix}\right\}$ for some fixed non-square $\epsilon$ in $\mathbb{F}_q$.

Finally $s=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $G=B\cup BsB$.

The conjugacy classes in $GL_2(\mathbb{F}_q)$ are

| Representative | $C_G$ | No of elts | No of such classes |
|---|---|---|---|
| $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ | $G$ | $1$ | $q-1$ |
| $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ | $ZN$ | $q^2-1$ | $q-1$ |
| $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ | $T$ | $q(q+1)$ | $\binom{q-1}{2}$ |
| $\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}$ | $K$ | $q(q-1)$ | $\binom{q}{2}$ |

Let's warm ourselves up by computing the character table of $B$.

If $x, y \in B$ are conjugate in $G$ then because $G = B \cup BsB$ either $x$ is conjugate to $y$ in $B$ or $x$ is conjugate to $sys^{-1}$ (or both). So classes in $G$ split into at most two pieces when restricted to $B$.

The conjugacy classes in $B$ are

| Representative | $C_G$ | No of elts | No of such classes |
|---|---|---|---|
| $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ | $B$ | 1 | $q-1$ |
| $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ | $ZN$ | $q-1$ | $q-1$ |
| $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ | $T$ | $q$ | $(q-1)(q-2)$ |

Now $B/N \cong T \cong {\mathbb{F}_q}^\times \times {\mathbb{F}_q}^\times$. So if $\theta_0, \ldots, \theta_{q-2}$ are the characters of ${\mathbb{F}_q}^\times$ arranged so that $\theta_i\theta_j = \theta_{i+j}$ (where $+$ is understood mod $q-1$) then for every pair $i, j$ between 0 and $q-2$ we have a 1-dimensional representation of $B$ given by

$$\alpha_{ij}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \theta_i(a)\theta_j(d)$$

giving $(q-1)^2$ linear reps.

Fix $\gamma$ a non-trivial 1-dimensional representation of $\mathbb{F}_q$. Then for each $i$ between 0 and $q-2$ we can define a 1-dimensional representation of $ZN$ by

$$\rho_i\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \theta_i(a)\gamma(b).$$

Defining $\mu_i$ to be the character of $\operatorname{Ind}_{ZN}^B \rho_i$ we see that

$$\begin{aligned}
\mu_i\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) &= (q-1)\theta_i(\lambda), \\
\mu_i\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) &= \sum_{b \in {\mathbb{F}_q}^\times} \theta_i(\lambda)\gamma(b) \\
&= \theta_i(\lambda)(q\langle \mathbf{1}, \gamma\rangle_{\mathbb{F}_q} - 1) \\
&= -\theta_i(\lambda) \\
\mu_i\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) &= 0
\end{aligned}$$

So $\langle \mu_i, \mu_i \rangle = \frac{1}{q(q-1)^2}\big((q-1)(q-1)^2 + (q-1)(q-1)1\big) = 1$ and the character table of $B$ is

| | $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ | $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ | $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ |
|---|---|---|---|
| $\alpha_{ij}$ | $\theta_i(\lambda)\theta_j(\lambda)$ | $\theta_i(\lambda)\theta_j(\lambda)$ | $\theta_i(\lambda)\theta_j(\mu)$ |
| $\mu_i$ | $(q-1)\theta_i(\lambda)$ | $-\theta_i(\lambda)$ | 0 |

Let's start computing some representations of $G$.

As $\det\colon G \to {\mathbb{F}_q}^\times$ is a surjective group homomorphism, for each $i = 0, \ldots, q-2$, $\chi_i := \theta_i \circ \det$ is a 1-dimensional representation of $G$.

Next, we consider $\operatorname{Ind}_B^G \mathbf{1} = \mathbb{C}(\mathbb{F}_q \cup \{\infty\})$.

$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ acts on $\mathbb{F}_q \cup \{\infty\}$ as $z \mapsto z$ so with $q+1$ fixed points.

$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ acts on $\mathbb{F}_q \cup \{\infty\}$ as $z \mapsto z + \frac{1}{\lambda}$ so only $\infty$ is fixed.

$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ acts on $\mathbb{F}_q \cup \{\infty\}$ via $z \mapsto \frac{\lambda}{\mu}$ so 0 and $\infty$ are the fixed points.

Finally $\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \epsilon\end{pmatrix}$ acts on $\mathbb{F}_q \cup \{\infty\}$ without fixed points.

Since $G$ acts 2-transitively on $\mathbb{F}_q \cup \{\infty\}$, the representation $V_0 := \mathrm{Ind}_B^G \mathbf{1} - \mathbf{1}$ (consisting of $G$-invariant functions on $\mathbb{F}_q \cup \{\infty\}$ that sum to zero) is irreducible with character $\chi_{V_0}\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) = q$, $\chi_{V_0}\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) = 0$, $\chi_{V_0}\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) = 1$, $\chi_{V_0}\left(\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \epsilon\end{pmatrix}\right) = -1$. This is known as the Steinberg representation.

By tensoring $V_i$ with $\chi_i$ we obtain $q-1$ representations of dimension $q$, (If you prefer, $\chi_{V_i} = \chi_{\mathrm{Ind}_B^G \alpha_{ii}} - \chi_i$).

Next we can induce $\alpha_{ij}$ for $i \neq j$. Since $\mathrm{Res}_T^B \alpha_{ij} \neq \mathrm{Res}_T^{{}^sB}\, {}^s\alpha_{ji}$, $\mathrm{Ind}_B^G \alpha_{ij}$ is an irreducible character by Mackey's irreducibility criterion.

Thus we get irreducible characters $\chi_{W_{ij}}$ so that $\chi_{W_{ij}}\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) = (q+1)\theta_i(\lambda)\theta_j(\lambda)$, $\chi_{W_{ij}}\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) = \theta_i(\lambda)\theta_j(\lambda)$, $\chi_{W_{ij}}\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) = \theta_i(\lambda)\theta_j(\mu) + \theta_j(\lambda)\theta_i(\mu)$ and $\chi_{W_{ij}}\left(\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}\right) = 0$.

Notice that $W_{ij} \cong W_{ji}$ and $W_{ij} \otimes \chi_k = W_{i+k,j+k}$ so no new representations this way.

So far we have

| | $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ | $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ | $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ | $\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}$ | # of reps |
|---|---|---|---|---|---|
| $\chi_i$ | $\theta_i(\lambda)^2$ | $\theta_i(\lambda)^2$ | $\theta_i(\lambda)\theta_i(\mu)$ | $\theta_i(\lambda^2 - \epsilon\mu^2)$ | $q-1$ |
| $V_i$ | $q\theta_i(\lambda)^2$ | 0 | $\theta_i(\lambda)\theta_i(\mu)$ | $-\theta_i(\lambda^2 - \epsilon\mu^2)$ | $q-1$ |
| $W_{ij}$ | $(q+1)\theta_i(\lambda)\theta_j(\lambda)$ | $\theta_i(\lambda)\theta_j(\lambda)$ | $\theta_i(\lambda)\theta_j(\mu) + \theta_j(\lambda)\theta_i(\mu)$ | 0 | $\frac{(q-1)(q-2)}{2}$ |

The next natural thing to do is compute $\mathrm{Ind}_B^G \mu_i$. It has character given by

$$\mathrm{Ind}_B^G \mu_i\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) = (q+1)(q-1)\theta_i(\lambda),$$

$$\mathrm{Ind}_B^G \mu_i\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) = -\theta_i(\lambda),$$

$$\mathrm{Ind}_B^G \mu_i\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) = 0 \text{ and}$$

$$\mathrm{Ind}_B^G \mu_i\left(\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}\right) = 0.$$

Thus

$$\begin{aligned}\langle \operatorname{Ind}_B^G \mu_i, \operatorname{Ind}_B^G \mu_i\rangle =& \frac{1}{|G|}\left((q+1)^2(q-1)^2(q-1) + (q-1)(q^2-1)\right)\\ & \frac{1}{q}\left(q^2-1) + 1\right) = q\end{aligned}$$

so $\operatorname{Ind}_B^G \mu_i$ has many irreducible factors.

## Lecture 23

The story so far:

| # classes | $q-1$ | $q-1$ | $\binom{q-1}{2}$ | $\binom{q}{2}$ | |
|---|---|---|---|---|---|
| $\lvert\mathrm{ccl}\rvert$ | $1$ | $q^2-1$ | $q(q+1)$ | $q(q-1)$ | |
| $C_G$ | $G$ | $ZN$ | $T$ | $K$ | |
| rep | $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ | $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ | $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ | $\begin{pmatrix}\lambda & \epsilon\mu\\ \mu & \lambda\end{pmatrix}$ | # of reps |
| $\chi_i$ | $\theta_i(\lambda)^2$ | $\theta_i(\lambda)^2$ | $\theta_i(\lambda)\theta_i(\mu)$ | $\theta_i(\lambda^2-\epsilon\mu^2)$ | $q-1$ |
| $V_i$ | $q\theta_i(\lambda)^2$ | $0$ | $\theta_i(\lambda)\theta_i(\mu)$ | $-\theta_i(\lambda^2-\epsilon\mu^2)$ | $q-1$ |
| $W_{ij}$ | $(q+1)\theta_i(\lambda)\theta_j(\lambda)$ | $\theta_i(\lambda)\theta_j(\lambda)$ | $\theta_i(\lambda)\theta_j(\mu)+\theta_j(\lambda)\theta_i(\mu)$ | $0$ | $\frac{(q-1)(q-2)}{2}$ |
| $\operatorname{Ind}_B^G \mu_i$ | $(q^2-1)\theta_i(\lambda)$ | $-\theta_i(\lambda)$ | $0$ | $0$ | $(q-1)$ |

We also computed $\langle \operatorname{Ind}_B^G \mu_i, \operatorname{Ind}_B^G \mu_i\rangle = q$.

Our next strategy is to induce characters from $K$. $K \cong C_{q^2-1}$. Recall

$$K \cup \{0\} = \left\{\begin{pmatrix}x & \epsilon y\\ y & x\end{pmatrix}\right\}$$

is a field with $q^2$ elements. If we write $x + \sqrt{\epsilon}y$ for the matrix $\begin{pmatrix}x & \epsilon y\\ y & x\end{pmatrix}$, then $(x+\sqrt{\epsilon}y)(w+\sqrt{\epsilon}z) = (xw+\epsilon yz) + (xz+yw)\sqrt{\epsilon}$ as we might expect. Moreover $(x+\sqrt{\epsilon}y)^q = x^q + \sqrt{\epsilon}^q y^q = x - \sqrt{\epsilon}y$ and

$$\det(x+\sqrt{\epsilon}y) = (x+\sqrt{\epsilon}y)(x-\sqrt{\epsilon}y) = (x+\sqrt{\epsilon}y)^{q+1}.$$

Notice that $Z \leqslant K$ with $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} = \lambda$ in our new notation.

Suppose that $\varphi\colon K \to \mathbb{C}^\times$ is a 1-dimensional character of $K$. Then $\Phi := \operatorname{Ind}_K^G \varphi$ has character given by

$\Phi(\lambda) = q(q-1)\varphi(\lambda)$, $\Phi(\lambda+\sqrt{\epsilon}\mu) = \varphi(\lambda+\sqrt{\epsilon}\mu) + \varphi(\lambda-\sqrt{\epsilon}\mu)$ for $\mu \neq 0$ and $\Phi = 0$ away from these conjugacy classes.

Let's compute

$$\langle \Phi, \Phi\rangle = \frac{1}{|G|}\left((q-1)q^2(q-1)^2 + \frac{q(q-1)}{2}\sum_{\nu\in K\setminus Z}|\varphi(\nu)+\varphi(\nu^q)|^2\right)$$

But

$$\sum|\varphi(\nu)+\varphi(\nu^q)|^2 = \sum_{\nu\in K\setminus Z}(\varphi(\nu)+\varphi(\nu^q)\left(\varphi(\nu^{-1})+\varphi(\nu^{-q})\right)$$
$$= \sum_{\nu\in K\setminus Z}\left(2+\varphi(\nu^{q-1})+\varphi(\nu^{1-q})\right)$$
$$= 2(q^2-q)+2\sum_{\nu\in K}\varphi^{q-1}(\nu)-2\sum_{\lambda\in Z}\varphi(\lambda^{q-1})$$

But if $\varphi^{q-1}\neq\mathbf{1}$ then the middle term in the last sum is 0 since $\langle\varphi^{q-1},\mathbf{1}\rangle=0$. Since $\lambda^{q-1}=1$ for $\lambda\in\mathbb{F}_q$ the third term is also easy to compute. Putting this together we get $\langle\Phi,\Phi\rangle=q-1$ when $\varphi^{q-1}\neq\mathbf{1}$.

We similarly compute

$$\langle\operatorname{Ind}_B^G\mu_i,\Phi\rangle = \frac{1}{|G|}\sum_{\lambda\in Z}(q^2-1)\overline{\theta_i(\lambda)}q(q-1)\varphi(\lambda)$$
$$= (q-1)\langle\theta_i,\operatorname{Res}_Z^K\varphi\rangle_Z$$

Now, for each $\varphi$ such that $\varphi^{q-1}\neq\mathbf{1}$ (there are $q^2-q$ such choices) there is some $i$ such that $\operatorname{Res}_Z^K\varphi=\theta_i$ then our calculations tell us that if $\beta_\varphi=\operatorname{Ind}_B^G\mu_i-\Phi\in R(G)$ then

$$\langle\beta_\varphi,\beta_\varphi\rangle = q-2(q-1)+(q-1)=1.$$

Since also $\beta_\varphi(1)=q-1>0$ it follows that $\beta_\varphi$ is an irreducible character. Since $\beta_\varphi=\beta_{\varphi^q}$ (and $\varphi^{q^2}=\varphi$) we get $\binom{q}{2}$ characters in this way and the character table of $GL_2(\mathbb{F}_q)$ is complete.

| # classes | $q-1$ | $q-1$ | $\binom{q-1}{2}$ | $\binom{q}{2}$ | |
|---|---|---|---|---|---|
| rep | $\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}$ | $\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}$ | $\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix}$ | $\begin{pmatrix}\lambda&\epsilon\mu\\\mu&\lambda\end{pmatrix}$ | # of reps |
| $\chi_i$ | $\theta_i(\lambda)^2$ | $\theta_i(\lambda)^2$ | $\theta_i(\lambda)\theta_i(\mu)$ | $\theta_i(\lambda^2-\epsilon\mu^2)$ | $q-1$ |
| $V_i$ | $q\theta_i(\lambda)^2$ | $0$ | $\theta_i(\lambda)\theta_i(\mu)$ | $-\theta_i(\lambda^2-\epsilon\mu^2)$ | $q-1$ |
| $W_{ij}$ | $(q+1)\theta_i(\lambda)\theta_j(\lambda)$ | $\theta_i(\lambda)\theta_j(\lambda)$ | $\theta_i(\lambda)\theta_j(\mu)+\theta_j(\lambda)\theta_i(\mu)$ | $0$ | $\binom{q-1}{2}$ |
| $\beta_\varphi$ | $(q-1)\varphi(\lambda)$ | $-\varphi(\lambda)$ | $0$ | $-(\varphi+\varphi^q)(\lambda+\sqrt{\epsilon}\mu)$ | $\binom{q}{2}$ |

10.2. $PGL_2(\mathbb{F}_q)$.

The group $PGL_2(\mathbb{F}_q):=GL_2(\mathbb{F}_q)/Z$ may be viewed as 'the Mobius group' on $\mathbb{F}_q\cup\{\infty\}$ since $Z$ is the kernel of this action of $GL_2$ on this set.

We can write down the character table of $PGL_2(\mathbb{F}_q)$ immediately from the character table of $GL_2(\mathbb{F}_q)$: irreducible reps of $PGL_2$ correspond $1-1$ with irreducible reps of $GL_2$ with kernel containing $Z$.

$\theta_i(\lambda)^2=1$ for $\lambda\in Z$ if $i=0$ or $\frac{q-1}{2}$ so writing $\theta=\theta_{\frac{q-1}{2}}$ and naively observing that $\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\in Z$, $\begin{pmatrix}\lambda&\lambda\\0&\lambda\end{pmatrix}\in\begin{pmatrix}1&1\\0&1\end{pmatrix}Z$ , $\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix}\in\begin{pmatrix}1&0\\0&\frac{\mu}{\lambda}\end{pmatrix}Z$ and that $\begin{pmatrix}\lambda&\sqrt{\epsilon}\mu\\\mu&\lambda\end{pmatrix}\in\begin{pmatrix}\frac{\lambda}{\mu}&\sqrt{\epsilon}\\1&\frac{\mu}{\lambda}\mu\end{pmatrix}$, we see that the character table of $PGL_2(\mathbb{F}_q)$ is

| rep | $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ | $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$ | $\begin{pmatrix}1 & 0\\0 & \mu\end{pmatrix}$ | $\begin{pmatrix}\lambda & \epsilon\\1 & \lambda\end{pmatrix}$ | # of reps |
|---|---|---|---|---|---|
| $\mathbf{1}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_{\frac{q-1}{2}}$ | 1 | 1 | $\theta(\mu)$ | $\theta(\lambda^2-\epsilon)$ | 1 |
| $V_0$ | $q$ | 0 | 1 | $-1$ | 1 |
| $V_{\frac{q-1}{2}}$ | $q$ | 0 | $\theta(\mu)$ | $-\theta(\lambda^2-\epsilon)$ | 1 |
| $W_{i,(q-1-i)}$ | $(q+1)$ | 1 | $\theta_i(\mu)+\theta_i(\mu^{-1})$ | 0 | $\frac{q-3}{2}$ |
| $\beta_\varphi$ | $(q-1)$ | $-1$ | 0 | $-(\varphi+\varphi^q)(\lambda+\sqrt{\epsilon})$ | $\frac{q-1}{2}$ |

where in the last row we require $\operatorname{Res}_Z^K \varphi = \mathbf{1}$ so $\varphi^{q+1} = \mathbf{1}$ but as before $\varphi^{q-1} \neq \mathbf{1}$. These two conditions are equivalent to $\varphi^{q+1} = \mathbf{1}$ and $\varphi^2 \neq \mathbf{1}$ so there are $q+1-2 = q-1$ such choices. Since $\beta_\varphi = \beta_{\varphi^q}$ we see that there are $\frac{q-1}{2}$ such characters as claimed.

The conjugacy classes still need to be more carefully computed though. The first two columns are fine. In each case there is precisely one conjugacy class of this form.

In $PGL_2(\mathbb{F}_q)$ the elements $\begin{pmatrix}1 & 0\\0 & \mu\end{pmatrix} Z$ and $\begin{pmatrix}1 & 0\\0 & \mu^{-1}\end{pmatrix} Z$ are conjugate via $s$ (we can also see they are conjugate by staring at the character table and remembering the characters span the space of class functions). There is a special case when $\mu = -1$ since then $\mu = \mu^{-1}$. Thus we get one class $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} Z$ with centraliser $\langle T, s\rangle/Z$ and $\frac{q-3}{2}$ classes with representatives $\begin{pmatrix}1 & 0\\0 & \mu^{\pm 1}\end{pmatrix} Z$ and centralizer $T/Z$.

Similarly in $PGL_2(\mathbb{F}_q)$ the elements $\pm\begin{pmatrix}0 & \epsilon\\1 & 0\end{pmatrix}$ are conjugate via $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$ (again their conujugacy can also be established by considering the character table) so these are representatives of a single class with centraliser $\langle K, \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\rangle/Z$ and for each $\lambda \neq 0$ there is a conjugacy class with representatives $\begin{pmatrix}\pm\lambda & \epsilon\\1 & \pm\lambda\end{pmatrix}$ and centralizer $K/Z$. there are $\frac{q-1}{2}$ of these classes.

So we now have 7 types of conjugacy classes with $1, 1, 1, 1, \frac{q-3}{2}, \frac{q-1}{2}$ classes of the different types. Notice a correspondance between these numbers and the number of representations of each type.

## Lecture 24

10.3. $PSL_2(\mathbb{F}_q)$**.** We see from the character table of $PGL_2(\mathbb{F}_q)$ that it has an index 2 normal subgroup given by $\ker \chi_{q-1}2$. This subgroup is the image of $SL_2(\mathbb{F}_q) \to PGL_2(\mathbb{F}_q)$ so is isomorphic to $SL_2(\mathbb{F}_q)/\{\pm I\}$. We call it $PSL_2(\mathbb{F}_q)$. It has order $\frac{q(q^2-1)}{2}$.

What happens next depends on whether or not $-1$ is a square in $\mathbb{F}_q$. We know that $(-1)$ is a square if and only if $(1)^{\frac{q-1}{2}} = 1$ if and only if $q \equiv 1 \mod 4$.

Let's consider the case that $q \equiv 1 \mod 4$ and write $i$ for some square root of $-1$.

*Exercise.* Show that the following table describes the conjugacy classes of $PSL_2(\mathbb{F}_q)$

| rep | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&\epsilon\\0&1\end{pmatrix}$ | $\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ | $\begin{pmatrix}\mu&0\\0&\mu^{-1}\end{pmatrix}$ | $\begin{pmatrix}\lambda&\mu\epsilon\\\mu&\lambda\end{pmatrix}$ |
|---|---|---|---|---|---|---|
| # classes | 1 | 1 | 1 | 1 | $\frac{q-5}{4}$ | $\frac{q-1}{4}$ |
| size of class | 1 | $\frac{q^2-1}{2}$ | $\frac{q^2-1}{2}$ | $\frac{q(q+1)}{2}$ | $q(q+1)$ | $q(q-1)$ |

Restricting the character table of $PGL_2(\mathbb{F}_q)$ to $PSL_2(\mathbb{F}_q)$ gives

| rep | $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ | $\begin{pmatrix}1&\epsilon\\0&1\end{pmatrix}$ | $\begin{pmatrix}i&0\\0&-i\end{pmatrix}$ | $\begin{pmatrix}\mu&0\\0&\mu^{-1}\end{pmatrix}$ | $\begin{pmatrix}\lambda&\epsilon\mu\\\mu&\lambda\end{pmatrix}$ | # of reps |
|---|---|---|---|---|---|---|---|
| $\mathbf{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $V_0$ | $q$ | 0 | 0 | 1 | $-1$ | 1 | 1 |
| $W_{j,(q-1-j)}$ | $(q+1)$ | 1 | 1 | $2\theta_j(-1)$ | $\theta_j(\mu^2)+\theta_j(\mu^{-2})$ | 0 | $\frac{q-5}{4}+1$ |
| $\beta_\varphi$ | $(q-1)$ | $-1$ | $-1$ | 0 | 0 | $-(\varphi+\varphi^q)(\lambda^2-\epsilon\mu^2)$ | $\frac{q-1}{4}$ |

To see the number of reps of each type observe that when restricted to $PSL_2(\mathbb{F}_q)$, $W_{j,q-1-j} \cong W_{\frac{q-1}{2}-j,\frac{q-1}{2}+j}$ and $\beta_\varphi \cong \beta_{\varphi^{-1}}$.

If $H \leqslant G$ is an index 2 subgroup and $\chi$ is an irreducible character of $G$ then $\langle \operatorname{Res}^G_H \chi, \operatorname{Res}^G_H \chi\rangle \leqslant 2$ with equality if and only if $\chi(g) = 0$ for all $g \in G\backslash H$. (Proof: $\langle \operatorname{Res}\chi, \operatorname{Res}\chi\rangle = \frac{1}{H}\sum_{h\in H}|\chi(h)|^2 \leqslant 2\langle\chi,\chi\rangle$ with equality precisely when claimed.)

*Exercise.* Deduce that $\mathbf{1}$,$V_0$ and $\beta_\varphi$ are irreducible as reps of $PS_2(\mathbb{F}_q)$ and $W_{j,q-1-j}$ is irreducible whenever $i \neq \frac{q-1}{4}$.

Thus we have $1+1+\frac{q-1}{4}+\frac{q-5}{4}$ irreducible characters already and $W_{\frac{q-1}{4},3\frac{q-1}{4}}$ splits into the remaining two irreducible characters. Use column orthogonality to see the two characters both have degree $\frac{q+1}{2}$ and so complete the character table. Deduce that $PSL_2(\mathbb{F}_q)$ is simple for $q \equiv 1 \mod 4$.

Repeat everything for $q \equiv 3 \mod 4$ and deduce $PSL_2(\mathbb{F}_q)$ is always simple for $q \geqslant 5$.