Polyhedrality of the Brookes–Groves invariant for the non-commutative torus

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1 Introduction

In [3] Bieri and Strebel defined a geometric invariant Σ for finitely generated modules over the group algebras of finitely generated abelian groups. They used this to define a criterion for when metabelian groups are finitely presented. This invariant was further developed by Bieri, Strebel and Groves and has many interesting applications. In [2] Bieri and Groves showed that when the group algebra is defined over a Dedekind domain the complement of Σ must be a closed rational spherical polyhedral cone.

In [5] and [6] Brookes and Groves defined a similar invariant Δ for modules over the crossed product of a division ring by a free finitely generated abelian group. Such a crossed product is often known as the (coordinate ring of) the non-commutative torus since in the special case where it is commutative it is the coordinate ring of an algebraic torus. If in the commutative case we take the complement of Δ and identify points that differ by a positive scalar multiple we obtain Σ. Brookes and Groves were unable to prove that their invariant must be a rational polyhedral cone, although using the methods of [2] they do prove a weaker version of the result; they show that for any finitely generated module M, $\Delta(M)$ must contain a rational polyhedral cone $\Delta^*(M)$ of dimension equal to the Gelfand–Kirillov dimension of M and moreover that the complement $\Delta(M)\backslash\Delta^*(M)$ must be contained inside a rational polyhedral cone of strictly smaller dimension.

In this paper we use Gröbner basis methods to prove the following theorem:

Theorem A. *If DA is a crossed product of a division ring D by a free finitely generated abelian group A, then, for all finitely generated DA-modules M, $\Delta(M)$ is a closed rational polyhedral cone in $Hom(A, \mathbb{R})$.*

To make the Gröbner basis methods work we first construct a skew polynomial ring with a ring homomorphism onto our crossed product. We then construct Gröbner bases for left ideals in these skew polynomial rings and use

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them to show that Δ must always be polyhedral as required. We may view the skew polynomial rings as deformations of the crossed products analogous to those deformations made when studying quantum groups or quantum enveloping algebras.

Given a subset S of $\mathbb{R}^n$ and a point x of S, the *local cone* of S at x is defined as

$$LC_x(S) = \{y \in \mathbb{R}^n \,|\, \exists \epsilon_0 > 0 \text{ s.t. } \forall \epsilon \in (0, \epsilon_0]\ x + \epsilon y \in S\}.$$

In [8] Brookes and Groves show that given a module M and $\chi \in \Delta^*(M)$,

$$LC_\chi(\Delta^*(M)) = \Delta^*(\mathrm{gr}^\chi(M))$$

where $\mathrm{gr}^\chi(M)$ denotes the associated graded module of M with respect to a non-trival χ-filtration. We now prove:

Theorem B. *If DA is a crossed product of a division ring D by a free finitely generated abelian group A, then, for all finitely generated DA-modules M and for each $\chi \in \Delta(M)$*

$$LC_\chi(\Delta(M)) = \Delta(\mathrm{gr}^\chi(M)).$$

This means that if M is a module such that the Gelfand–Kirillov dimension of M is equal to the Gelfand–Kirillov dimension of $\mathrm{gr}^\chi(M)$ for each $\chi \in \Delta(M)$ then $\Delta(M)$ is a homogeneous polyhedron and so $\Delta(M) = \Delta^*(M)$.

In [12] we are able to show that $\mathrm{gr}^\chi(M)$ does always have the right dimension for cyclic DA-modules of codimension 1 over DA. We then use the techniques of Bieri and Groves in [2] to show that $\Delta(M)$ is a homogeneous polyhedron for all finitely generated pure modules M; that is those modules with the property that every non-zero submodule has the same dimension. This leads to a simplification of many of the results of Brookes and Groves in [6], [7] and [8].

2 Preliminaries

2.1 Filtrations and Gradings

Suppose that D is a division ring. By a *D-algebra* we will mean a ring R equipped with a ring homomorphism from D to R giving R a natural left D-module structure.

By an *$\mathbb{R}$-filtration* of a D-algebra R, we will mean a set

$$\{F_\mu R | \mu \in \mathbb{R}\}$$

such that $D \subseteq F_0 R$, $F_\mu R \subseteq F_\nu R$ whenever $\nu \leq \mu$,

$$R = \bigcup_{\mu \in \mathbb{R}} F_\mu R$$

and

$$F_\mu R.F_\nu R \subseteq F_{\mu+\nu} R$$

for each $\mu, \nu \in \mathbb{R}$. We will write $F_\mu^+ R$ for $\bigcup_{\nu > \mu} F_\nu R$.

Given a filtered D-algebra R and a left R-module M, an *$\mathbb{R}$-filtration* of M is a set

$$\{F_\mu M | \mu \in \mathbb{R}\}$$

of D-submodules of M such that $F_\mu M \subseteq F_\nu M$ whenever $\nu \leq \mu$,

$$M = \bigcup_{\mu \in \mathbb{R}} F_\mu M$$

and

$$F_\mu R . F_\nu M \subseteq F_{\mu+\nu} M$$

for each $\mu, \nu \in \mathbb{R}$. Again we write $F_\mu^+ M$ for $\bigcup_{\nu > \mu} F_\nu M$.

We define the *associated graded ring* of an $\mathbb{R}$-filtered ring R by

$$\mathrm{gr}^F(R) = \bigoplus_{\mu \in \mathbb{R}} F_\mu R / F_\mu^+ R.$$

The multiplication in $\mathrm{gr}^F(R)$ is given on homogeneous elements by

$$(x_1 + F_{\mu_1}^+ R)(x_2 + F_{\mu_2}^+ R) = x_1 x_2 + F_{\mu_1+\mu_2}^+ R$$

and extended linearly. Given $x \in R$ we write $\sigma^F(x) = x + F_\mu^+ R \in \mathrm{gr}^F(R)$, the symbol of x, when $x \in F_\mu R$ but $x \notin F_\mu^+ R$.

Similarly we define the *associated graded module* of an $\mathbb{R}$-filtered R-module M

$$\mathrm{gr}^F(M) = \bigoplus_{\mu \in \mathbb{R}} F_\mu M / F_\mu^+ M,$$

and $\sigma^F(m) = m + F_\mu^+ M \in \mathrm{gr}^F(M)$, the symbol of m, for $m \in F_\mu M \backslash F_\mu^+ M$. This is naturally a $\mathrm{gr}^F(R)$ module with action on homogeneous elements given by

$$(x + F_{\mu_1}^+ R)(m + F_{\mu_2}^+ M) = x.m + F_{\mu_1+\mu_2}^+ M$$

for $x \in F_{\mu_1} R$ and $m \in F_{\mu_2} M$.

Given a monoid G and a ring R we say that R is *G-graded* if R decomposes as a direct sum of additive subgroups

$$R = \bigoplus_{x \in G} R_x$$

with $R_x R_y \subseteq R_{xy}$ for all $x, y \in G$.

Notice that the associated graded ring of an $\mathbb{R}$-filtered D-algebra is $\mathbb{R}$-graded when we think of $\mathbb{R}$ as a monoid with its usual addition.

2.2 Gelfand–Kirillov dimension

Suppose that R is a finitely generated D-algebra with finite generating set X such that $Dx = xD$ for each $x \in X$, we set $V \subset R$ to be the D-vector space spanned by X. Then we may define

$$d_X(n) = \dim_D \left(\sum_{i=0}^{n} V^i \right)$$

We then define the *GK-dimension* of R over D by

$$GK_D(R) = \overline{\lim} \log_n d_X(n).$$

The proof of Lemma 1.1 of [9] tells us that this definition is independent of the choice of generating set X.

Similarly given a finitely generated left R-module M with finite generating set F, we may define

$$d_{X,F}(n) = \dim_D \left(\sum_{i=0}^{n} V^i F \right)$$

and the *GK-dimension* of M over D by

$$GK_D(M) = \overline{\lim} \log_n d_{X,F}(n)$$

again this is independent of the choice of F and X by the proof of Lemma 1.1 of [9].

It is possible for the GK-dimension of a finitely generated algebra to be infinite; consider the free associative algebra on two generators for example. However in the rings we consider it always will be finite. For commutative algebras it agrees with the usual dimension function.

2.3 Polyhedral cones

We say a subset of $\mathbb{R}^n$ is a *convex polyhedral cone* if it can be written as the intersection of finitely many closed or open linear half spaces in $\mathbb{R}^n$. The *dimension* of a convex polyhedral cone S is defined to be the dimension of the subspace spanned by S and written $\dim(S)$. A convex polyhedral cone is said to be *rational* if each of the defining half spaces have boundaries induced from a subspace of $\mathbb{Q}^n$.

A subset Δ of $\mathbb{R}^n$ is said to be a *rational polyhedral cone* if it can be written as a finite union

$$\Delta = S_1 \cup \cdots \cup S_k$$

of rational convex polyhedral cones. The dimension of Δ, $\dim(\Delta)$ is defined to be $\max(\dim(S_i))$. A polyhedral cone Δ is said to be *homogeneous* if it may be written as a finite union of rational convex polyhedral cones of the same dimension.

2.4 Crossed products

We say that a G-graded ring R is *strongly G-graded* if $R_x R_y = R_{xy}$ for all $x, y \in G$.

If G is a group with identity element e, then we say that a G-graded ring is a *crossed product of R_e by G*, written $R_e G$, if R_x contains a unit $\bar{x}$ for each $x \in G$.

Given a crossed product of a ring R by a group G, a typical element α of RG may be written uniquely as a finite sum

$$\alpha = \sum_i \bar{g}_i r_i$$

with r_i non-zero elements of R, and g_i distinct elements of G. The set $\{g_i\}$ is called the *support* of α, and is written $\mathrm{supp}(\alpha)$.

Given a subgroup H of G, $RH = \{\alpha \in RG | \text{supp}(\alpha) \subseteq H\}$ is a crossed product of R by H. If H is normal in G then we may consider RG as a crossed product of RH by G/H.

We now recall the definition by Brookes and Groves of an invariant for modules over rings of the form DA and some of their results.

Given a group homomorphism χ from A to $\mathbb{R}$ we may define $F_\mu^\chi DA$ to be the D-linear span of $\{a \in A | \chi(a) \geq \mu\}$. This defines an $\mathbb{R}$-filtration of DA called the *χ-filtration* of DA.

We say that an $\mathbb{R}$-filtration $\{F_\mu M\}$ of a left DA-module M with respect to the χ-filtration of DA is a *χ-filtration* of M.

A χ-filtration $\{F_\mu M\}$ of a DA-module M is said to be *trivial* if $M = F_\mu M$ for some $\mu \in \mathbb{R}$.

Definition. *Given a DA-module M, $\Delta(M)$ is the subset of* $\text{Hom}(A, \mathbb{R})$ *such that $\chi \in \Delta(M)$ precisely if there is a non-trivial χ-filtration of M or $\chi = 0$.*

Proposition 2.1 (Proposition 2.1 of [5]). *Suppose that M is a left DA-module with finite generating set X. The following are equivalent for $\chi \in$* $\text{Hom}(A, \mathbb{R}) \backslash 0$.

1. $\chi \notin \Delta(M)$;
2. *the χ-filtration of M given by $F_\mu M = F_\mu^\chi DA.X$ is trivial;*
3. *M is generated by X over a Noetherian subring of $F_0^\chi DA$;*
4. *M is generated by X over $F_0^\chi DA$;*
5. *for each $x \in X$, there exists $\alpha \in DA$ such that $\alpha.x = 0$ and $\sigma^{F^\chi}(\alpha) = 1$.*

Lemma 2.2 (Corollary 2.2 of [5]). *Suppose that*

$$0 \to L \to M \to N \to 0$$

is a short exact sequence of finitely generated DA-modules. Then

$$\Delta(M) = \Delta(L) \cup \Delta(N)$$

2.5 Strongly graded skew polynomial rings

A *strongly graded skew polynomial ring in n variables* over a division ring D is a strongly $\mathbb{N}^n$-graded ring, R, such that each component has dimension 1 as a D-module, i.e.

$$R = \bigoplus_{\alpha \in \mathbb{N}^n} Dr_\alpha$$

where the r_α's are non-zero elements of R_α such that we have $Dr_\alpha = r_\alpha D$ and $Dr_\alpha.Dr_\beta = Dr_{\alpha+\beta}$ for each $\alpha, \beta \in \mathbb{N}^n$.

Given an element r of R we will define the *support* of r to be the set of α in $\mathbb{N}^n$ such that the component of r in R_α is non-zero.

Lemma 2.3. *If R is a strongly graded skew polynomial ring in n variables over D then there is a sequence $D = R_0, R_1, \dots, R_n = R$ of subrings of R such that each R_i is a strongly graded skew polynomial ring in i variables over D with elements $x_{i+1} \in R_{i+1}$ and automorphisms $\sigma_i \in \mathrm{Aut}(R_i)$ with the property that*

$$rx_{i+1} = x_{i+1}\sigma_i(r) \in R$$

for each $r \in R_i$ and such that the ring generated by R_i and x_{i+1} is R_{i+1}.

Note this lemma permits us to use the following evocative notation:

$$R = \bigoplus_{\alpha\in\mathbb{N}^n} D\mathbf{x}^\alpha$$

We will call the elements $1.\mathbf{x}^\alpha$ *monomials* in R, where $\mathbf{x}^\alpha$ denotes the product $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ in that order.

Proof. We will write e_i to denote the element of $\mathbb{N}^n$ whose i-th entry is 1 and whose other entries are all zero; so $\mathbb{N}^n$ is generated as a monoid by $e_1, \dots, e_n$. For notational convenience in this proof we will write $\mathbb{N}^i$ only to mean the submonoid generated by $e_1, \dots, e_i$. We set

$$R_i = \bigoplus_{\alpha\in\mathbb{N}^i} Dr_\alpha.$$

It is easy to see that R_i is a subring of R that is a strongly graded skew polynomial ring in i variables over D and that R_{i+1} is generated by R_i and $x_{i+1} = r_{e_{i+1}}$ for each i.

It remains to prove that there is an automorphism σ_i of R_i such that $rx_{i+1} = x_{i+1}\sigma_i(r)$ for each $r \in R_i$. If $r \in R_i$ then it may be written as $\sum_{\alpha\in\mathbb{N}^i} d_\alpha r_\alpha$ with $d_\alpha \in D$ for each α. As R_{i+1} is strongly graded, for each $\alpha \in \mathbb{N}^i$ there is a $d'_\alpha \in D$ such that

$$d_\alpha r_\alpha x_{i+1} = x_{i+1} d'_\alpha r_\alpha.$$

Now $rx_{i+1} = x_{i+1}\sum_{\alpha\in\mathbb{N}^i} d'_\alpha r_\alpha$, so there is certainly a map σ_i from R_i to itself with the required property. It now just remains to prove that σ_i is an automorphism. This is an easy check. $\square$

Corollary 2.4. *Any strongly graded skew polynomial ring in n variables over a division ring is a Noetherian domain.*

Proof. Just iteratively apply Theorem 1.2.9 of [10]. $\square$

3 Gröbner bases for skew polynomial rings

In this section we develop a theory of Gröbner bases for strongly graded skew polynomial rings. From now on we assume R is a strongly graded skew polynomial ring in n variables over D. As observed in section 2.5, Lemma 2.3 enables us to choose $x_1, \dots, x_n \in R$ and then adopt the notation $R = \bigoplus_{\alpha\in\mathbb{N}^n} D\mathbf{x}^\alpha$.

Given $\chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$, the set of monoid homomorphisms from $\mathbb{N}^n$ to $\mathbb{R}$, we define an $\mathbb{R}$-filtration of R: $F^\chi_\mu R$ is the D-module spanned by those monomials $\mathbf{x}^\alpha$ such that $\chi(\alpha) \geq \mu$. We will often just write χ to denote this filtration instead of F^χ, so for example $\mathrm{gr}^\chi(M) = \mathrm{gr}^{F^\chi}(M)$. Notice that $\mathrm{gr}^\chi(R) \cong R$ for all such χ.

We will say two monoid homomorphisms $\chi, \chi' \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ are *equivalent with respect to* a left ideal I precisely if $\mathrm{gr}^\chi(I) = \mathrm{gr}^{\chi'}(I)$.

We aim to prove the following:

Theorem. *If I is a non-zero left ideal of R then* $\mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ *has finitely many equivalence classes with respect to I and each class is a convex polyhedron in* $\mathrm{Hom}(\mathbb{N}^n, \mathbb{R}) \cong \mathbb{R}^n$.

We begin with some definitions. Suppose that $<$ is a total ordering on $\mathbb{N}^n$ and that $r = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \mathbf{x}^\alpha$.

Definition. *The* Newton diagram *of r is the set $\mathcal{N}(r) = \{\alpha | \lambda_\alpha \neq 0\}$. If $r \neq 0$ the* privileged exponent *of r with respect to $<$ is $\exp_<(r) = \max_<(\mathcal{N}(r))$. Finally given a left ideal I of R we define*

$$\mathrm{Exp}_<(I) := \{\exp_<(r) | 0 \neq r \in I\}.$$

The privileged exponent should be thought of as a generalisation of the notion of degree for a polynomial in one variable. Whenever we talk about the privileged exponent of 0 we will mean a formal symbol that is smaller than every element of $\mathbb{N}^n$.

In order to develop the theory of Gröbner bases we need to have a well ordering of the monomials in R to ensure that we have a process of reduction that stops. As the orderings that we are interested in, those coming from elements of $\mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$, are not well orderings in general, we follow Assi, Castro-Jiménez, and Granger in [1] and work inside $R[t]$ and associate our given ordering with an ordering of the terms in there. Given $\alpha \in \mathbb{N}^n$ we write $|\alpha| = \sum \alpha_i$ for the total degree of α. We say that a total ordering $<$ on $\mathbb{N}^n$ is *compatible with sums* if whenever α, β and γ are in $\mathbb{N}^n$ with $\alpha < \beta$, we have $\alpha + \gamma < \beta + \gamma$. Whenever $<$ is a total ordering on $\mathbb{N}^n$ that is compatible with sums, we define a total well-ordering $<^h$ on $\mathbb{N}^{n+1}$ that is also compatible with sums as follows:

$$(\alpha, k) <^h (\beta, l) \text{ if and only if } \begin{cases} \text{either } |\alpha| + k < |\beta| + l \\ \text{or } (|\alpha| + k = |\beta| + l \text{ and } \alpha < \beta). \end{cases}$$

Notice that $R[t]$ is a strongly $\mathbb{N}^{n+1}$-graded skew polynomial ring and so has a notion of a Newton diagram and a privileged exponent with respect to $<^h$. It also possesses an $\mathbb{N}$-grading by total degree. When we say an element p of $R[t]$ is homogeneous we mean with respect to this total degree grading and will write $\deg(p)$ to mean this degree.

Given a filtration $F_\mu R$ of R we may define a filtration of $R[t]$ by $F_\mu(R[t]) = (F_\mu R)[t]$.

The following proposition is the key to making the Gröbner basis machinery work.

Proposition 3.1 (Division algorithm). *Let $<$ be a total ordering of $\mathbb{N}^n$ compatible with sums. Suppose that $\{r_1, \ldots, r_k\}$ is a set of homogeneous elements in $R[t]$. For each $p \in R[t]$ there exists $(q_1, \ldots, q_k, r) \in R[t]^{k+1}$ such that*

1. $p = \sum_{i=1}^k q_i r_i + r$

2. $\exp_{<^h}(q_i r_i) \leq^h \exp_{<^h}(p)$ *for each* i

3. $\mathcal{N}(r) \cap \bigcup_{i=1}^k \left(\exp_{<^h}(r_i) + \mathbb{N}^{n+1}\right) = \emptyset$.

Proof. We define the q_i's and r recursively as follows: we begin by setting $q_i = 0$ for each i and setting $r = p$. If $\mathcal{N}(r) \cap \bigcup_{i=1}^k \left(\exp_{<^h}(r_i) + \mathbb{N}^{n+1}\right)$ is non-empty then we let α be its maximal element and let j be the least integer such that $\alpha \in \left(\exp_{<^h}(r_j) + \mathbb{N}^{n+1}\right)$. There exists $\beta \in \mathbb{N}^{n+1}$ and $d \in D$ such that $\alpha = \beta + \exp_{<^h}(r_j)$ and $\alpha \notin \mathcal{N}\left(r - d\mathbf{x}^\beta r_j\right)$. We now replace q_j by $q_j + d\mathbf{x}^\beta$ and then r by $p - \sum_{i=1}^k q_i r_i$ and continue.

This process terminates since at each stage the maximal element of

$$\mathcal{N}(r) \cap \bigcup_{i=1}^k \left(\exp_{<^h}(r_i) + \mathbb{N}^{n+1}\right)$$

is smaller than before and $<^h$ is a well ordering. Notice that at each stage $\exp_{<^h}(q_i r_i) \leq^h \exp_{<^h}(p)$ for each i, since on each iteration $\exp_{<^h}(d\mathbf{x}^\beta r_j) \leq \exp_{<^h}(r) \leq \exp_{<^h}(p)$. $\square$

Remark. *If we begin with a homogeneous* p *then each of the* q_i*'s and the* r *resulting from the division algorithm will also be homogeneous, and for each* i, $\deg(p) = \deg(r) = \deg(q_i) + \deg(r_i)$, *since on each iteration* $\deg(\mathbf{x}^\beta) + \deg(r_j) = \deg(r) = \deg(p)$.

We are now ready to define a Gröbner basis

Definition. *Given a left ideal* I *in* R *we say that a finite subset* $\{r_1, \ldots, r_k\}$ *of* I *is a* Gröbner basis *with respect to* $<$ *if* $\operatorname{Exp}_<(I) = \bigcup_{i=1}^k (\exp_<(r_i) + \mathbb{N}^n)$.

We say that a Gröbner basis $\{r_1, \ldots, r_k\}$ *is* minimal *if for all pairs of distinct* i, j, $\exp_<(r_i) \notin \exp_<(r_j) + \mathbb{N}^n$.

We say that a Gröbner basis $\{r_1, \ldots, r_k\}$ *is* reduced *if for all pairs of distinct* i, j, $\mathcal{N}(r_i) \cap (\exp_<(r_j) + \mathbb{N}^n) = \emptyset$.

We say that a Gröbner basis is homogeneous *if it consists of homogeneous elements with respect to the total degree grading.*

We continue by showing that a homogeneous Gröbner basis for an ideal of $R[t]$ is a generating set and that every graded ideal with respect to the total degree grading (we will just call such a graded ideal from now on) has a homogeneous Gröbner basis.

Lemma 3.2. *If* J *is a graded left ideal in* $R[t]$ *and* $\{r_1, \ldots, r_k\}$ *is a homogeneous Gröbner basis for* J *with respect to* $<^h$, *then*

$$J = \sum_{i=1}^k R[t] r_i.$$

Proof. Let $p \in J$. By the division algorithm there exist $q_1, \ldots, q_k$ and r in $R[t]$ such that $p = \sum_{i=1}^k q_i r_i + r$ and $\mathcal{N}(r) \cap \operatorname{Exp}_{<^h}(J) = \emptyset$. But $r \in J$, so $r = 0$. $\square$

Proposition 3.3. *Every non-zero graded left ideal,* J, *in* $R[t]$ *has a homogeneous reduced Gröbner basis.*

Proof. Let $\bar{J}$ be the left ideal in $R[t]$ generated by the set of $\mathbf{x}^\alpha$ for $\alpha \in \mathrm{Exp}_{<^h}(J)$. As $R[t]$ is Noetherian, $\bar{J}$ has a finite generating set of monomials each of which occurs as the privileged exponent of some $r \in J$. Since J is graded these elements may be chosen to be homogeneous. Thus J has a homogeneous Gröbner basis $\{r_1, \ldots, r_k\}$, say.

Now suppose that no homogeneous Gröbner basis for J has fewer than k elements. It follows that $\{r_1, \ldots, r_k\}$ is actually a minimal Gröbner basis: if $\exp_{<^h}(r_i) \in \exp_{<^h}(r_j) + \mathbb{N}^{n+1}$ for some distinct pair i, j, then $\mathrm{Exp}_{<^h}(J) = \bigcup_{l \neq i}(\exp_{<^h}(r_l) + \mathbb{N}^{n+1})$, contradicting the minimality of k.

Beginning with our minimal homogeneous Gröbner basis $\{r_1, \ldots, r_k\}$ we now construct $\{s_1, \ldots, s_k\}$ a homogeneous reduced Gröbner basis for J inductively as follows:

Using the division algorithm to divide r_i by $\{s_1, \ldots, s_{i-1}, r_{i+1}, \ldots, r_k\}$ for each $i \leq k$ in turn we may find q_{ij} and s_i such that

$$r_i = \sum_{j<i} q_{ij} s_j + \sum_{j>i} q_{ij} r_j + s_i$$

and

$$\mathcal{N}(s_i) \cap \left(\bigcup_{j<i} (\exp_{<^h}(s_j) + \mathbb{N}^{n+1}) \cup \bigcup_{j>i} (\exp_{<^h}(r_j) + \mathbb{N}^{n+1}) \right) = \emptyset$$

At each stage, $\{s_1, \ldots, s_{i-1}, r_i, \ldots, r_k\}$ is a minimal Gröbner basis for J since, as we will see, $\exp_{<^h}(s_j)$ is the same as $\exp_{<^h}(r_j)$. Thus

$$\exp_{<^h} \left(\sum_{j<i} q_{ij} s_j + \sum_{j>i} q_{ij} r_j \right) <^h \exp_{<^h}(r_i)$$

and so $\exp_{<^h}(s_i) = \exp_{<^h}(r_i)$. In particular $\{s_1, \ldots, s_k\}$ is a homogeneous Gröbner basis for J. By construction

$$\mathcal{N}(s_i) \cap \left(\bigcup_{j \neq i} \exp_{<^h}(s_j) + \mathbb{N}^{n+1} \right) = \emptyset.$$

for each i, so $\{s_1, \ldots, s_k\}$ is reduced as required.

□

Lemma 3.4. *Let J be a non-zero graded left ideal of $R[t]$. Write J_m for the m^{th} component of J, a subset of $R[t]_m$, the m^{th} component of $R[t]$ with respect to the total degree grading. For any total ordering $<$ of $\mathbb{N}^n$ that is compatible with sums*

$$H_J(m) := \dim_D(R[t]_m / J_m) = |\{\alpha \in \mathbb{N}^{n+1} \backslash \mathrm{Exp}_{<^h}(J) | |\alpha| = m\}|.$$

Proof. Using Proposition 3.3, pick a Gröbner basis $\{r_1, \ldots, r_k\}$ for J. Using the division algorithm we see that for all $p \in R[t]_m$ there exist $q_1, \ldots, q_k, r$ such that for each i, $q_i r_i \in J_m$, $p = \sum q_i r_i + r$ and $\mathcal{N}(r) \subseteq \mathbb{N}^{n+1} \backslash \mathrm{Exp}_{<^h}(J)$.

It follows that the image of $\{\mathbf{x}^\alpha | |\alpha| = m \text{ and } \alpha \notin \mathrm{Exp}_{<^h}(J)\}$ in $R[t]_m / J_m$ is a spanning set for $R[t]_m / J_m$ as a D-vector space. Since the privileged exponent of every element of J_m must lie in $\bigcup_{i=1}^k (\exp_<(r_i) + \mathbb{N}^{n+1})$ the set is also linearly independent. □

Theorem 3.5. *Let J be a non-zero graded left ideal of $R[t]$ with respect to the total degree grading. The set of* $\mathrm{Exp}_{<^h}(J)$ *is finite as $<$ ranges over all the total orderings of $\mathbb{N}^n$ that are compatible with sums.*

Proof. By Lemma 3.4 it is sufficient to prove that the set of subsets E of $\mathbb{N}^{n+1}$ such that $E + \mathbb{N}^{n+1} = E$ and $H_J(m) = |\{\alpha \in \mathbb{N}^{n+1} \backslash E || \alpha| = m\}|$ is finite. Suppose for contradiction that there is an infinite sequence $(E_i)_{i \geq 1}$ of distinct sets of this type.

Let k_1 be the minimal integer such that the k_1^{th} component J_{k_1} is non-zero. Then there is element $\alpha_1 \in \mathbb{N}^{n+1}$ of total degree k_1 such that α_1 lies in infinitely many of the E_i. Indeed, by passing to a subsequence E_{i_j} if necessary we may assume that α_1 lies in each E_i.

It follows that $S_1 := \alpha_1 + \mathbb{N}^{n+1} \subsetneq E_i$ for all i. So there is a k_2 least such that

$$H_J(k_2) \neq |\{\alpha \in \mathbb{N}^{n+1} \backslash S_1 || \alpha| = k_2\}|.$$

Now there is an element $\alpha_2 \in \mathbb{N}^{n+1}$ of total degree k_2 such that α_2 lies in infinitely many of the E_i but not in S_1. Again by passing to a subsequence if necessary we may assume that $S_2 := S_1 \cup (\alpha_2 + \mathbb{N}^{n+1}) \subsetneq E_i$ for all i. Continuing in this way we may construct an infinite strictly ascending chain of subsets S_i of $\mathbb{N}^{n+1}$ such that $S_i + \mathbb{N}^{n+1} = S_i$ for each i. This is impossible. $\square$

We now fix a total well ordering $<$ on $\mathbb{N}^n$ that is compatible with sums. Given an element $\chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$, we define a total ordering $<_\chi$ on $\mathbb{N}^n$ compatible with sums as follows:

$\alpha <_\chi \beta$ if and only if $(\chi(\alpha) > \chi(\beta))$ or $(\chi(\alpha) = \chi(\beta)$ and $\alpha < \beta)$.

Notice that the inequality sign here is the opposite of what might be expected. This is to retain consistency with other notation. Also notice that $<_\chi$ is compatible with sums because $<$ and χ are both compatible with sums.

Recall that for each left ideal J in $R[t]$, $\mathrm{gr}^\chi(J)$ denotes the associated graded ideal of J with respect to the χ-filtration on $R[t]$ that comes from the χ-filtration on R and that for each p in $R[t]$, $\sigma^\chi(p)$ denotes the symbol of p with respect to this filtration.

Definition. *Let h be the map from R to $R[t]$*

$$h\left(\sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \mathbf{x}^\alpha\right) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha t^{k-|\alpha|} \mathbf{x}^\alpha$$

where $k = \max\{|\alpha| | \lambda_\alpha \neq 0\}$.

Let H be the ring homomorphism from $R[t]$ to R given by evaluation of t at 1.

We will also, by an abuse of notation, use H to denote the map $\mathbb{N}^{n+1}$ to $\mathbb{N}^n$ given by projecting onto the first n terms.

Notice that each element of the image of h is homogeneous and $H \circ h = id_R$, and so if I is a left ideal of R then $R[t]h(I)$ is a graded left ideal of $R[t]$ and $H(R[t]h(I)) = I$.

Also notice that $H(\sigma^\chi(p)) = \sigma^\chi(H(p))$ for each homogeneous $p \in R[t]$.

Lemma 3.6. *If $\{r_1, \ldots, r_k\}$ is a homogeneous Gröbner basis for a graded left ideal J of $R[t]$ with respect to $<_\chi^h$, then $\mathrm{gr}^\chi(J) = \sum_{i=1}^k R[t]\sigma^\chi(r_i)$. Moreover*

$\{\sigma^\chi(r_1), \ldots, \sigma^\chi(r_k)\}$ *is a homogeneous Gröbner basis for* $\mathrm{gr}^\chi(J)$ *with respect to* $<^h$.

Proof. Let $p \in J$ be homogeneous. We define a sequence p_i as follows:

Let $p_0 = p$. For $i \geq 1$, let $p_i = p_{i-1} - (d_i \mathbf{x}^{\alpha_i} r_{j_i})$ where $1 \leq j_i \leq k$ and $\alpha_i \in \mathbb{N}^{n+1}$ are chosen such that $\exp_{<^h_\chi}(r_{j_i}) + \alpha_i = \exp_{<^h_\chi}(p_{i-1})$ and $d_i \in D$ such that $\exp_{<^h_\chi}(p_i) <^h_\chi \exp_{<^h_\chi}(p_{i-1})$. We may make these choices because $\exp_{<^h_\chi}(J) = \bigcup(\exp_{<^h_\chi}(r_i) + \mathbb{N}^{n+1})$, since $\{r_1, \ldots, r_k\}$ is a homogeneous Gröbner basis for J.

Since we chose $<$ to be a well ordering of $\mathbb{N}^n$, there is a least m such that $\chi(\exp_{<_\chi}(H(p_m))) > \chi(\exp_{<_\chi}(H(p_0)))$. Then

$$\sigma^\chi(p) = \sum_{i=1}^m d_i \mathbf{x}^{\alpha_i} \sigma^\chi(r_{j_i}) \in \sum_{i=1}^k R[t]\sigma^\chi(r_i).$$

Finally, notice that by construction

$$\exp_{<^h}(\sigma^\chi(p)) = \exp_{<^h_\chi}(p) = \exp_{<^h_\chi}(r_{j_1}) + \alpha_1 \in \exp_{<^h}(\sigma^\chi(r_{j_1})) + \mathbb{N}^{n+1}.$$

□

Lemma 3.7. *Let I be a left ideal of R. We have*

$$H(\mathrm{gr}^\chi(R[t]h(I))) = \mathrm{gr}^\chi(I)$$

Proof. Suppose that f is in $R[t]h(I)$. We may write $f = \sum_i f_i$ with each $f_i \in h(I)$ homogeneous of degree i. Now $\sigma^\chi(f) = \sum_j \sigma^\chi(f_{i_j})$ for some suitable indexing set $\{i_j\}$. So

$$H(\sigma^\chi(f)) = \sum_j H(\sigma^\chi(f_{i_j})) = \sum_j (\sigma^\chi(H(f_{i_j}))).$$

But $H(f_{i_j}) \in I$ for each j and so $H(\sigma^\chi(f)) \in \mathrm{gr}^\chi(I)$ as required.

Conversely, if $r \in I$ then $\sigma^\chi(r) = H(\sigma^\chi(h(r))) \in H(\mathrm{gr}^\chi(R[t]h(I)))$ as required. □

Proposition 3.8. *Let J be a non-zero homogeneous left ideal of $R[t]$. The set of associated graded ideals $\{\mathrm{gr}^\chi(J) | \chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})\}$ is finite. Moreover for each $\chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$, $\{\chi' \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R}) | \mathrm{gr}^{\chi'}(J) = \mathrm{gr}^\chi(J)\}$ is a convex polyhedron.*

Proof. We know from Theorem 3.5 that the set $\{\mathrm{Exp}_{<^h_\chi}(J) | \chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})\}$ is finite, so to prove the first part it is enough to show that the set $\{\mathrm{gr}^\chi(J) | \mathrm{Exp}_{<^h_\chi}(J) = E, \chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})\}$ is finite for any $E \subset \mathbb{N}^{n+1}$ with $E + \mathbb{N}^{n+1} = E$.

Fix such an E and pick $\chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ such that $\mathrm{Exp}_{<^h_\chi}(J) = E$ if any exist. By Proposition 3.3 there is a homogeneous reduced Gröbner basis $S = \{r_1, \ldots, r_k\}$ for J with respect to $<^h_\chi$ when $E = \bigcup_{i=1}^k(\exp_{<^h_\chi}(r_i) + \mathbb{N}^{n+1})$.

Suppose now that $\chi' \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ with $\mathrm{Exp}_{<^h_{\chi'}}(J) = E$. Since

$$\mathcal{N}(r_i) \cap \bigcup_{j \neq i}(\exp_{<^h_\chi}(r_j) + \mathbb{N}^{n+1}) = \emptyset$$

and $\exp_{<^h_\chi}(r_i) \in \mathrm{Exp}_{<^h_{\chi'}}(J)$, $\exp_{<^h_{\chi'}}(r_i) = \exp_{<^h_\chi}(r_i)$ for each i and so S is a reduced Gröbner basis for J with respect to $<^h_{\chi'}$.

It follows from Lemma 3.6 that

$$\mathrm{gr}^{\chi'}(J) = \sum_{i=1}^{k} R[t]\sigma^{\chi'}(r_i)$$

for each such χ'. It is clear that this just leaves finitely many possibilities.

We now fix $\chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ and let $E[\chi]$ denote the set

$$\{\chi' \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R}) | \mathrm{gr}^{\chi'}(J) = \mathrm{gr}^{\chi}(J)\}.$$

Suppose that $\{r_1, \ldots, r_k\}$ is a homogeneous reduced Gröbner basis for J with respect to $<^h_\chi$. We claim that

$$E[\chi] = \{\chi' | \sigma^{\chi'}(r_i) = \sigma^{\chi}(r_i) \text{ for each } i\}.$$

The proof of the claim will complete the proof of this Proposition since it expresses $E[\chi]$ as an intersection of hyperplanes of the form $\chi'(\alpha) = \chi'(\beta)$ for α and β in $\mathcal{N}(\sigma^{\chi}(H(r_i)))$ and open half-spaces of the form $\chi'(\alpha) < \chi'(\gamma)$ for α in $\mathcal{N}(\sigma^{\chi}(H(r_i)))$ and γ in $\mathcal{N}(H(r_i)) \backslash \mathcal{N}(\sigma^{\chi}(H(r_i)))$.

We now prove the claim.

Firstly suppose that $\chi' \in E[\chi]$.

Consider $s_i = \sigma^{\chi'}(r_i) \in \mathrm{gr}^{\chi'}(J) = \mathrm{gr}^{\chi}(J)$. Since $\{r_1, \ldots, r_k\}$ is a reduced Gröbner basis for J with respect to $<^h_\chi$, $\exp_{<^h}(\sigma^{\chi}(r_j)) = \exp_{<^h_\chi}(r_j)$ and $\mathcal{N}(s_i) \subset \mathcal{N}(r_i)$ we have

$$\mathcal{N}(s_i) \cap \left(\exp_{<^h}(\sigma^{\chi}(r_j)) + \mathbb{N}^{n+1}\right) \subseteq \mathcal{N}(r_i) \cap \left(\exp_{<^h_\chi}(r_j) + \mathbb{N}^{n+1}\right) = \emptyset.$$

for each $j \neq i$.

But as $s_i \in \mathrm{gr}^{\chi}(J)$ it follows from Lemma 3.6 that

$$\exp_{<^h}(s_i) \in \bigcup_{j=1}^{k} (\exp_{<^h}(\sigma^{\chi}(r_j)) + \mathbb{N}^{n+1}),$$

and so $\exp_{<^h}(s_i) \in \exp_{<^h}(\sigma^{\chi}(r_i)) + \mathbb{N}^{n+1}$. As $\mathcal{N}(s_i) \subset \mathcal{N}(r_i)$ this forces $\exp_{<^h}(s_i) = \exp_{<^h}(\sigma^{\chi}(r_i))$. So $\exp_{<^h}(s_i - \sigma^{\chi}(r_i)) <^h \exp_{<^h}(\sigma^{\chi}(r_i))$. But we have $\mathcal{N}(s_i - \sigma^{\chi}(r_i)) \subset \mathcal{N}(r_i)$ and $(s_i - \sigma^{\chi}(r_i)) \in \mathrm{gr}^{\chi}(J)$. So $\exp_{<^h}(s_i - \sigma^{\chi}(r_i)) \in \exp_{<^h}(\sigma^{\chi}(r_i) + \mathbb{N}^{n+1})$. This means that $s_i - \sigma^{\chi}(r_i) = 0$ and so $\sigma^{\chi'}(r_i) = \sigma^{\chi}(r_i)$ as required.

Conversely, suppose that $\chi' \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ such that $\sigma^{\chi}(r_i) = \sigma^{\chi'}(r_i)$ for each i. Lemma 3.6 implies that $\mathrm{gr}^{\chi}(J)$ is generated by $\{\sigma^{\chi}(r_i) | 1 \leq i \leq k\}$ and so $\mathrm{gr}^{\chi}(J) \subset \mathrm{gr}^{\chi'}(J)$. If the containment here were strict it would follow that

$$\mathrm{Exp}_{<^h_\chi}(J) = \mathrm{Exp}_{<^h}(\mathrm{gr}^{\chi}(J)) \subsetneq \mathrm{Exp}_{<^h}(\mathrm{gr}^{\chi'}(J)) = \mathrm{Exp}_{<^h_{\chi'}}(J)$$

a contradiction of Lemma 3.4.

The claim is now proved.

$\square$

Recall that given a left ideal I of R we say two monoid homomorphisms $\chi, \chi' \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ are equivalent with respect to I precisely if $\mathrm{gr}^{\chi}(I) = \mathrm{gr}^{\chi'}(I)$.

Theorem 3.9. *If I is a non-zero left ideal of R then $\mathrm{Hom}(\mathbb{N}^n, \mathbb{R})\}$ has finitely many equivalence classes with respect to I and each class is a polyhedral cone in $\mathrm{Hom}(\mathbb{N}^n, \mathbb{R}) \cong \mathbb{R}^n$.*

Proof. This follows from Lemma 3.7 and Propostion 3.8 since these imply that equivalence class containing $\chi \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ is the finite union of the cones

$$\{\chi' \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R}) | \mathrm{gr}^{\chi'}(R[t]h(I)) = J\}$$

over those left ideals J in $R[t]$ with $H(J) = \mathrm{gr}^{\chi}(I)$. ☐

Lemma 3.10. *If I is a left ideal of R and $\chi, \chi' \in \mathrm{Hom}(\mathbb{N}^n, \mathbb{R})$ then for $\epsilon > 0$ sufficiently small*

$$\mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(I)) = \mathrm{gr}^{\chi+\epsilon\chi'}(I).$$

Proof. By Proposition 3.8 there exists ϵ_0 such that $\{\chi + \epsilon\chi' | \epsilon \in (0, \epsilon_0]\}$ is contained in one equivalence class with respect to $R[t]h(I)$. It also follows from the proof of Proposition 3.8 that if $\{r_1, \ldots, r_k\}$ be a reduced Grönber basis for $R[t]h(I)$ with respect to the ordering $<^h_{\chi+\epsilon_0\chi'}$ then it is a reduced Gröbner basis for $R[t]h(I)$ with respect to the ordering $<^h_{\chi+\epsilon\chi'}$ for each $\epsilon \in (0, \epsilon_0]$.

Now pick any $\epsilon \in (0, \epsilon_0]$. We have $\mathcal{N}(\sigma^{\chi+\epsilon\chi'}(r_i)) \subset \mathcal{N}(\sigma^{\chi}(r_i))$ for each $1 \le i \le k$ and so

$$\sigma^{\chi'}(\sigma^{\chi}(r_i)) = \sigma^{\chi+\epsilon\chi'}(r_i).$$

By Lemma 3.6, $\mathrm{gr}^{\chi+\epsilon\chi'}(R[t]h(I))$ is generated by $\{\sigma^{\chi+\epsilon\chi'}(r_i) | 1 \le i \le k\}$ and so $\mathrm{gr}^{\chi+\epsilon\chi'}(R[t]h(I)) \subset \mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(R[t]h(I)))$. If the containment were strict it would follow that

$$\mathrm{Exp}_{<^h}(\mathrm{gr}^{\chi+\epsilon\chi'}(R[t]h(I))) \subsetneq \mathrm{Exp}_{<^h}(\mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(R[t]h(I)))).$$

This is impossible due to Lemma 3.4, so we have

$$\mathrm{gr}^{\chi+\epsilon\chi'}(R[t]h(I)) = \mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(R[t]h(I))).$$

The result now follows by applying H to each side of this equation and using Lemma 3.7. ☐

4 Crossed products

Suppose that DA is a crossed product of a division ring D by a finitely generated free abelian group A of rank n. Let $\{a_1, \ldots, a_n\}$ be a generating set for A.

Proposition 4.1. *There is a strongly graded skew polynomial ring over D in $2n$ variables, $R = D[x_1, \ldots, x_n, y_1, \ldots, y_n]$, and a D-algebra homomorphism $\phi : R \to DA$ such that $\phi(x_i) = \overline{a_i}$ and $\phi(y_i) = \overline{a_i}^{-1}$ for each $1 \le i \le n$.*

Proof. First notice that the subring of DA generated by $\overline{a_1}, \ldots, \overline{a_n}$ and D is a strongly graded skew polynomial ring in n variables. So by Lemma 2.3 we may form $R_0 = D[x_1, \ldots, x_n]$ and a homomorphism $\phi_0 : R_0 \to DA$ such that $\phi_0(x_i) = \overline{a_i}$ for each i. Suppose we have constructed a strongly graded skew polynomial ring in $n+s$ variables, $R_s = D[x_1, \ldots, x_n, y_1, \ldots, y_s]$, and a homomorphism $\phi_s : R_s \to DA$ such that $\phi(x_i) = \overline{a_i}$ for $1 \le i \le n$ and $\phi(y_i) = \overline{a_i}^{-1}$ for $1 \le i \le s$. Notice that $S = \{x_{s+1}^k : k \in \mathbb{N}\}$ is an Ore set in R_s, and that conjugation by x_{s+1} inside $(R_s)_S$ induces an automorphism θ_s of R_s.

Now we set $R_{s+1} = R_s[y_{s+1}; \theta_s]$; that is R_{s+1} is the skew polynomial ring with indeterminate y_{s+1} and coefficients in R_s and with automorphism θ_s so that $r y_{s+1} = y_{s+1}\theta_s(r)$ (see section 1.2 of [10]). Since $\phi_s(r)\overline{a_{s+1}}^{-1} = \overline{a_{s+1}}^{-1}\phi_s(\theta_s(r))$ for each $r \in R_s$, we may extend ϕ_s to a map $\phi_{s+1} : R_{s+1} \to DA$ such that $\phi_{s+1}(y_{s+1}) = \overline{a_{s+1}}^{-1}$. ☐

We now fix DA and a pair (R, ϕ) given by Proposition 4.1.

There is a unique monoid map $\pi : \mathbb{N}^{2n} \to A$ with $\pi(e_i) = a_i$ and $\pi(e_{i+n}) = a_i^{-1}$ for $1 \le i \le n$. This defines an rational embedding of $\mathrm{Hom}(A, \mathbb{R})$ into $\mathrm{Hom}(\mathbb{N}^{2n}, \mathbb{R})$; $\chi \mapsto \tilde{\chi} = \chi \circ \pi$.

Proposition 4.2. *If I is a left ideal of DA then* $\mathrm{gr}^{\chi}(I) = \phi(\mathrm{gr}^{\tilde{\chi}}(\phi^{-1}(I)))$.

Proof. Suppose that $f \in R$ with $\phi(f) \in I$. Then either $\sigma^{\tilde{\chi}}(f)$ lies in the kernel of ϕ or $\phi(\sigma^{\tilde{\chi}}(f)) = \sigma^{\chi}(\phi(f))$. In either case $\phi(\sigma^{\tilde{\chi}}(f)) \in \mathrm{gr}^{\chi}(I)$.

Now suppose that $\alpha \in I$. We may choose $f \in R$ such that $\phi(f) = \alpha$ and no two elements of $\mathcal{N}(f)$ map to the same element of A under π. Then $\sigma^{\chi}(\alpha) = \phi(\sigma^{\tilde{\chi}}(f))$, so $\sigma^{\chi}(\alpha) \in \phi(\mathrm{gr}^{\tilde{\chi}}(\phi^{-1}(I)))$. ☐

Suppose that M is a module over DA with finite generating set X. Recall from Proposition 2.1 that

$$\Delta(M) = \{\chi \in \mathrm{Hom}(A, \mathbb{R}) | 1 \notin \mathrm{gr}^{\chi}(\mathrm{ann}_{DA}(x)) \text{ for some } x \in X\}.$$

Theorem 4.3. *If M is a finitely generated DA-module then $\Delta(M)$ is a closed rational polyhedral cone.*

Proof. That $\Delta(M)$ is closed follows easily from the equivalence of conditions (1) and (5) in Proposition 2.1.

Suppose that X is a generating set for M. For each $x \in X$ let M_x be the DA-submodule of M generated by x. By Lemma 2.2, $\Delta(M) = \bigcup_{x \in X} \Delta(M_x)$. Since the finite union of rational polyhedral cones is always a rational polyhedral cone, we may assume without loss of generality that M is a cyclic left DA-module $M = DA/I$, say.

Now, $\Delta(M) = \{\chi \in \mathrm{Hom}(A, \mathbb{R}) | \mathrm{gr}^{\chi}(I) \ne DA\}$. But by Proposition 4.2 $\mathrm{gr}^{\chi}(I) = DA$ if and only if $\phi(\mathrm{gr}^{\tilde{\chi}}(\phi^{-1}(I))) = DA$. Now using Theorem 3.9 we see that the set

$$\{\tilde{\chi} | \phi(\mathrm{gr}^{\tilde{\chi}}(\phi^{-1}(I))) \ne DA\} \subset \mathrm{Hom}(\mathbb{N}^{2n}, \mathbb{R})$$

is the rational polyhedral cone constisting of the intersection of the rational subspace $\pi^{-1}(\mathrm{Hom}(\mathbb{N}^n, \mathbb{R}))$ and a rational polyhedral cone consisting of the union of some of the rational polyhedral cones that are equivalence classes with respect to the left ideal $\phi^{-1}(I)$ of R.

The result follows. ☐

Lemma 4.4. *If I is a left ideal of DA and $\chi, \chi' \in \mathrm{Hom}(A, \mathbb{R})$ then for $\epsilon > 0$ sufficiently small*

$$\mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(I)) = \mathrm{gr}^{\chi + \epsilon\chi'}(I).$$

Proof. By Lemma 3.10 applied to the left ideal $\phi^{-1}(I)$ of R there is an $\epsilon_0 > 0$ such that

$$\mathrm{gr}^{\tilde{\chi'}}(\mathrm{gr}^{\tilde{\chi}}(\phi^{-1}(I))) = \mathrm{gr}^{\widetilde{\chi+\epsilon\chi'}}(\phi^{-1}(I)).$$

for each $\epsilon \in (0, \epsilon]$. Applying ϕ to each side of this equation and using Proposition 4.2 we obtain:

$$\begin{aligned}
\mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(I)) &= \phi(\mathrm{gr}^{\tilde{\chi'}}(\phi^{-1}(\mathrm{gr}^{\chi}(I))) \\
&= \phi(\mathrm{gr}^{\tilde{\chi'}}(\mathrm{gr}^{\tilde{\chi}}(\phi^{-1}(I)))) \\
&= \phi(\mathrm{gr}^{\widetilde{\chi+\epsilon\chi'}}(\phi^{-1}(I))) \\
&= \mathrm{gr}^{\chi+\epsilon\chi'}(I)
\end{aligned}$$

as required. □

Theorem 4.5. *If M is a DA-module generated by the finite set X and $\chi \in \Delta(M)$ then*

$$LC_{\chi}(\Delta(M)) = \Delta(\mathrm{gr}^{\chi}(M)).$$

where M is given the χ-filtration $F^{\chi}_{\mu}M = F^{\chi}_{\mu}(DA).X$.

Proof. Suppose that $\chi \in \Delta(M)$. Recall that

$$LC_{\chi}(\Delta(M)) = \{\chi' \in \mathrm{Hom}(A, \mathbb{R}) | (\exists \epsilon_0 > 0)(\forall \epsilon \in (0, \epsilon_0])\chi + \epsilon\chi' \in \Delta(M)\}.$$

Now $\chi + \epsilon\chi' \in \Delta(M)$ precisely if $\mathrm{gr}^{\chi+\epsilon\chi'}(\mathrm{ann}_{DA}(x)) \neq DA$ for some $x \in X$. By Lemma 4.4, for ϵ sufficiently small,

$$\mathrm{gr}^{\chi+\epsilon\chi'}(\mathrm{ann}_{DA}(x)) = \mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(\mathrm{ann}_{DA}(x))).$$

So $\chi' \in LC_{\chi}(\Delta(M))$ precisely if $\mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(\mathrm{ann}_{DA}(x))) \neq DA$ for some $x \in X$.

But $\mathrm{gr}^{\chi}(M)$ is generated by the set $\sigma^{\chi}(X)$ and $\chi' \in \Delta(\mathrm{gr}^{\chi}(M))$ precisely if $\mathrm{gr}^{\chi'}(\mathrm{gr}^{\chi}(\mathrm{ann}_{DA}(x))) = \mathrm{gr}^{\chi'}(\mathrm{ann}_{DA}(\sigma^{\chi}(x))) \neq DA$ for some $x \in X$. The result follows.

□

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