A Bernstein-type inequality for localisations of Iwasawa algebras of Heisenberg pro-p groups

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Abstract

We prove a result about the possible dimensions of modules over the completed $\mathbb{F}_p$ group algebra of a Heisenberg pro-p group that are not torsion qua modules over the centre. We explain why this result is analogous to a result of Bernstein for modules over Weyl algebras in characteristic 0.

1 Introduction

In this paper we study the representation theory of completed group algebras of pro-p groups of finite rank. These are complete Noetherian local rings and are of interest to number theorists who often call them Iwasawa algebras.

Finite rank pro-p groups can be thought of as p-adic Lie groups. The theory of these was first developed by Lazard in [12]. A good modern account of this can be found in [8] by Dixon, de Sautoy, Mann and Segal.

Completed group algebras have recently been studied from a representation theoretic point of view by Venjakob. In [16] he showed that if a pro-p group of finite rank has no p-torsion then its completed group algebra is Auslander regular with the associated canonical dimension function of a module the same as the Krull dimension of the associated graded module with respect to a natural filtration. For a good introduction to the theory of Auslander regular rings see Clark's survey [7].

Ardakov has also looked at the representation theory of Iwasawa algebras. In [1] he showed that if a pro-p group of finite rank is soluble then the Krull dimension and the global dimension of its completed group algebra take the same value, whereas if the group is associated to a split simple Lie algebra of finite rank not of type $\mathfrak{sl}_2$ then the Krull dimension is strictly smaller than the global dimension.

We start by studying the completed group algebras of abelian pro-p groups of finite rank; we prove similar results to those of Bieri and Groves in [2] for the

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the usual group algebras of finitely generated abelian groups. Bieri and Groves used geometric properties of an invariant Σ first developed by Bieri and Strebel in a series of papers including [3] and [4]. Whilst an analogue of Σ for abelian pro-p groups was developed by King in his thesis [10], and this invariant does have many of the properties one might hope for, it does not seem to have good geometry. As a result we prove our results directly rather than going via such an invariant.

We begin by proving

Theorem A. *If G is a uniform pro-p group and M is a finitely generated $\mathbb{F}_p[[G]]$-module with $d_G(M) \leq \dim(G_{ab}) - t$, then the set of $H \in \mathcal{G}_{G,t}$ such that M is finitely generated over $\mathbb{F}_p[[H]]$ is open and dense.*

Here G_{ab} is the unique maximal torsion free abelian quotient of G and $\mathcal{G}_{G,t}$ is the Grassmannian variety consisting of isolated subgroups of G of corank t containing G_{ab} equipped with its natural topology. For the definition of other terms see section 2.

This result enables us to prove our main theorem, a pro-p analogue of an inequality due to Bernstein which puts a lower bound on the Gelfand-Kirillov dimension of modules for Weyl algebras in characteristic 0.

Theorem B. *If G is a Heisenberg pro-p group of rank $2r+1$ and centre Z, and M is a finitely generated $\mathbb{F}_p[[G]]$-module such that $d_G(M) \leq r$, then*

$$\mathrm{Ann}_{\mathbb{F}_p[[G]]}(M) \cap \mathbb{F}_p[[Z]] \neq 0.$$

To see the analogy with Bernstein's inequality for Weyl algebras notice that if we localise $\mathbb{F}_p[[G]]$ at the set $\mathbb{F}_p[[Z]]\backslash 0$ then the localisation of every module of dimension smaller than r is 0. So there is a lower bound on the dimension of modules that are not annihilated under this localisation.

Or put another way,

Corollary C. *Suppose that G is a Heisenberg pro-p group of rank $2r+1$ with centre Z and S is the central multiplicatively closed set $\mathbb{F}_p[[Z]]\backslash 0$. Then for any finitely generated module M over $\mathbb{F}_p[[G]]_S$ and any finitely generated $\mathbb{F}_p[[G]]$-submodule N of M with $N_S = M$ we have $d_G(N) \geq r+1$.*

Notice that this really is a direct analogue of Bernstein's inequality as $d_G(N)$ measures the growth rate of N with respect to its natural filtration. It seems reasonable to also consider it as a measure of the growth rate of M — perhaps one should take this to be smallest possible value as N varies amongst submodules of the given type. Also whilst the number $r+1$ is one bigger than we might expect, we can explain this by noticing that we are measuring the growth rate over the base field $\mathbb{F}_p$ rather than over the whole of the 'central' subalgebra $\mathrm{gr}\,\mathbb{F}_p[[Z]]$ that is the true analogue of the base field of a Weyl algebra.

We also obtain,

Corollary D. *If G is a Heisenberg pro-p group of rank $2r+1$ and centre Z and S is the central multiplicatively closed set $\mathbb{F}_p[[Z]]\backslash 0$ then the global dimension of the localisation $\mathbb{F}_p[[G]]_S$ is r.*

One might hope to extend Theorem B by proving a similar result for more general nilpotent class 2 pro-p groups. The work of Brookes on crossed products of fields by discrete abelian groups in [5] might lead us to make the following conjecture:

Conjecture. *Suppose that G is a uniform nilpotent class 2 pro-p group of finite rank with centre Z. Let H be an abelian subgroup of G of maximal rank and let $k = \dim(G) - \dim(H)$. If M is a finitely generated $\mathbb{F}_p[[G]]$-module with $d_G(M) \leq k$ then $\mathrm{Ann}_{\mathbb{F}_p[[Z]]}(M) \neq 0$.*

Theorem A also has implications for certain homological properties of finitely generated pro-p groups; we discuss these in [17].

Acknowledgements

This work was first done as part of the author's PhD thesis which was funded by EPSRC and supervised by Chris Brookes. The author would also like to thank the referee of an earlier version of this paper for his comments.

2 Preliminaries

2.1 Groups

Given a group G and elements $g, h \in G$, we write $[g, h]$ for $g^{-1}h^{-1}gh$. Then given subgroups $H, K \leq G$ we write $[H, K]$ for the subgroup generated by $\{[h, k] | h \in H, k \in K\}$. We also write G' for $[G, G]$.

We say that a normal subgroup H of a group G is *isolated* in G if G/H is torsion free. If H is any subgroup of G we write $i(H)$ for the unique minimal normal isolated subgroup of G that contains H.

If a group G acts on a set X we write $C_G(X)$ for the subgroup of G that fixes each element of X pointwise.

2.2 Pro-p groups

A *profinite group* is a compact Hausdorff topological group whose open subgroups form a base for the neighbourhoods of the identity.

A *pro-p group* is a profinite group in which every open normal subgroup has index equal to a power of p, a prime.

A pro-p group is *powerful* if $G/\overline{G^{p^e}}$ is abelian, where $e = 2$ if $p = 2$ and $e = 1$ otherwise.

A pro-p group G is *finitely generated* if it has a finite subset X such that the closure of the subgroup generated by X is G. We call X a *topological generating set* for G.

Given a profinite group G we set $d(G)$ to be the minimal cardinality of a topological generating set for G. We then define the *rank* of G to be

$$\mathrm{rk}(G) = \sup\{d(H) | H \text{ is a closed subgroup of } G\}.$$

If G is a finitely generated powerful pro-p group then $\mathrm{rk}(G) = d(G)$ holds.

Given a pro-p group G the *lower p-series of* G is defined as follows: $G_1 = G$ and for $i \geq 1$

$$G_{i+1} = \overline{G_i^p[G_i, G]}$$

A finitely generated powerful pro-p group is *uniform* if for each i

$$|G_i/G_{i+1}| = |G/G_2|.$$

We recall a useful result that occurs as Corollary 4.3 in [8]:

Lemma 2.1. *Every pro-p group of finite rank has a characterstic open uniform subgroup.*

Given a pro-p group G of finite rank the *dimension* of G, $\dim(G)$ is defined to be the rank of any open uniform subgroup of G. That this is well-defined is the content of Lemma 4.6 of [8].

2.3 Completed group algebras

Given a uniform pro-p group G the *completed group algebra* of G over $\mathbb{F}_p$ is

$$\Omega_G = \mathbb{F}_p[[G]] = \varprojlim_{N \lhd_o G} \mathbb{F}_p[G/N]$$

We will write J_G for the kernel of the natural map $\Omega_G \to \mathbb{F}_p$.

A *pro-p Ω_G-module* is a pro-p group M that is an abstract Ω_G-module such that the natural map $M \times \Omega_G \to M$ is continuous.

We now summarise some results of Chapter 8 of [18]:

Proposition 2.2. *Suppose that G is a uniform pro-p group.*

1. *Ω_G is a complete Noetherian local domain with maximal ideal J_G.*
2. *The sequence $(J_G^n)_{n\geq 0}$ is a filtration of Ω_G consisting of open ideals.*
3. *The associated graded ring of Ω_G with respect to this filtration, $\mathrm{gr}^{J_G}(\Omega_G)$ is a polynomial ring over $\mathbb{F}_p$ in $\dim(G)$ variables.*
4. *Every finitely generated Ω_G-module M is a pro-p Ω_G-module.*
5. *If we filter any pro-p Ω_G-module M by $(MJ_G^n)_{n\geq 0}$ then the filtration is separated. The associated graded module $\mathrm{gr}^{J_G}(M)$ is a $\mathrm{gr}^{J_G}(\Omega_G)$-module.*

We finish this section by recalling a useful little lemma that connects the question of whether a module is finitely generated with the question of whether its associated graded module is finitely generated.

Lemma 2.3. *Suppose that G is a uniform pro-p group and that M is a pro-p Ω_G-module such that every open neighbourhood of 0 in M contains MJ_G^n for sufficiently large n. Then M is a finitely generated Ω_G-module whenever $\mathrm{gr}^{J_G}(M)$ is a finitely generated $\mathrm{gr}^{J_G}(\Omega_G)$-module.*

Proof. See the proof of Lemma 8.6.2 of [18] □

2.4 Global dimension and Ext

Given a ring R and an R-module M we define

$$E_R^i(M) = \mathrm{Ext}_R^i(M, R)$$

Notice that, since R is an R-bimodule, if M is a left (right) R-module then $E_R^i(M)$ is right (left) R-module. The *grade* of M, $j_R(M)$ is defined by

$$j_R(M) = \min\{i \geq 0 | E^i(M) \neq 0\}$$

or ∞ if no such i exists. A ring R is said to satisfy Auslander's condition if for every Noetherian R-module M and every $i \geq 0$, every Noetherian submodule of $E^i(M)$ has grade at least i. A ring with finite global dimension that satisfies Auslander's condition is said to be Auslander regular.

Lemma 2.4. *If R and M are Noetherian, and S is an Ore set in R, then $E^i_R(M)_S \cong E^i_{R_S}(M_S)$ for each i.*

Proof. Let $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M as an R-module consisting of Noetherian modules. Then

$$\cdots \to (P_n)_S \to (P_{n-1})_S \to \cdots \to (P_0)_S \to M_S \to 0$$

is a projective resolution of M_S as an R_S-module consisting of Noetherian R_S-modules. For each i, $(\mathrm{Hom}_R(P_i, R))_S \cong \mathrm{Hom}_{R_S}((P_i)_S, R_S)$. Since localisation is an exact functor the result follows. $\square$

Lemma 2.5 ((Remark 6.4 of [16])). *If R is a Noetherian ring and M is a finitely generated R-module of projective dimension n then $E^n_R(M) \neq 0$.*

It follows easily from this lemma if a Noetherian ring with finite global dimension, then the global dimension is precisely

$$\sup\{n | E^n(M) \neq 0 \text{ and } M \text{ is an } R\text{-module}\}.$$

2.5 Gelfand-Kirillov dimension

Suppose that R is an $\mathbb{F}_p$-algebra with finite generating set X. Set $V \subset R$ to be the $\mathbb{F}_p$-vector space spanned by X. Then we may define

$$d_X(n) = \dim_{\mathbb{F}_p}\left(\sum_{i=0}^{n} V^i\right)$$

We then define the *GK-dimension* of R by

$$\mathrm{GKdim}(R) = \overline{\lim} \log_n d_X(n).$$

Lemma 1.1 of [11] tells us that this definition is independent of the choice of generating set X.

Similarly given a finitely generated left R-module M with finite generating set F, we may define

$$d_{X,F}(n) = \dim_{\mathbb{F}_p}\left(\sum_{i=0}^{n} V^i F\right)$$

and the *GK-dimension* of M by

$$\mathrm{GKdim}(M) = \overline{\lim} \log_n d_{X,F}(n).$$

Again this is independent of the choice of F and X by Lemma 1.1 of [11].

If G is a uniform pro-p group, we define the *dimension* of a finitely generated Ω_G-module to be the GK-dimension of the associated graded module with respect to the J_G-adic filtration. i.e.

$$d_G(M) = \mathrm{GKdim}(\mathrm{gr}(M)).$$

We recall that because $\mathrm{gr}(\Omega_G)$ is a finitely generated commutative ring $d_G(M)$ is also equal to the Krull dimension of $\mathrm{gr}(M)$ as a $\mathrm{gr}(\Omega_G)$-module.

Lemma 2.6 ((Theorem 3.21 of [16])). *If G is a uniform pro-p group then Ω_G is Auslander regular. Moreover if M is a finitely generated Ω_G-module, then*

$$j_{\Omega_G}(M) + d_G(M) = \dim(G).$$

3 Grassmannians in uniform pro-p groups

We begin with a very useful lemma, the first part of which seems to be due to Brumer in [6] in a more general settting.

Lemma 3.1. *If G is a uniform pro-p group of with normal subgroup H and M is a finitely generated Ω_G-module, then M is a finitely generated Ω_H-module if and only if M/MJ_H is a finite dimensional vector space over $\mathbb{F}_p$. Moreover in this case $d_G(M) = d_H(M)$.*

Proof. Since $\Omega_H/J_H \cong \mathbb{F}_p$, M is a finitely generated Ω_H-module implies that M/MJ_H is a finite dimensional $\mathbb{F}_p$-vector space.

Conversely, suppose that M/MJ_H is a finite dimensional $\mathbb{F}_p$-vector space. Since the natural map $M/MJ_H \times J_H^n/J_H^{n+1} \to MJ_H^n/MJ_H^{n+1}$ is onto, $\mathrm{gr}^{J_H}(M)$ is a finitely generated $\mathrm{gr}^{J_H}(\Omega_H)$-module. It follows by Theorem 2.3 that M is a finitely generated Ω_H-module.

Now if these conditions hold then the Ω_G-module M/MJ_H is Artinian and so satisfies $(M/MJ_H)J_G^k = 0$ for some positive integer k, since J_G is the Jacobson radical of Ω_G. So $MJ_G^k \subseteq MJ_H \subseteq MJ_G$ and thus, since $J_HJ_G = J_GJ_H$, $MJ_G^{nk} \leq MJ_H^n \leq MJ_G^n$ for each $n \geq 1$. Now

$$\dim_{\mathbb{F}_p}(M/MJ_G^n) \leq \dim_{\mathbb{F}_p}(M/MJ_H^n) \leq \dim_{\mathbb{F}_p}\left(M/MJ_G^{nk}\right).$$

But,

$$d_G(M) = \overline{\lim}\log_n \dim_{\mathbb{F}_p}(M/MJ_G^n) = \overline{\lim}\log_n \dim_{\mathbb{F}_p}(M/MJ_G^{nk})$$

and $d_H(M) = \overline{\lim}\log_n \dim_{\mathbb{F}_p}(M/MJ_H^n)$. The result follows. □

Our goal for this section is to prove for abelian G that finitely generated Ω_G-modules M are actually finitely generated over 'most' subgroups of dimension $d_G(M)$. For modules of grade 1 and 2 this is essentially the content of Lemmas 1 and 2 of [9].

We begin by defining a topology on the set of isolated subgroups of a given dimension.

Definition 3.2. *Given a free abelian pro-p group A of finite rank and $i \leq \dim(A)$, define the Grassmann space $\mathcal{G}_{A,i} = \{B \leq A | A/B \cong \mathbb{Z}_p^i\}$.*

Since $B \in \mathcal{G}_{A,i}$ is the kernel of a continuous map from A to $\mathbb{Z}_p^i$ and points are closed in $\mathbb{Z}_p^i$, $\mathcal{G}_{A,i}$ consists of closed subgroups of A. Recall that $A_n = A^{p^n}$ since A is abelian.

Definition 3.3. *Given $B, C \in \mathcal{G}_{A,i}$ let*

$$d_i(B,C) = p^{-\sup\{n | BA_n = CA_n\}}.$$

Lemma 3.4. *$(\mathcal{G}_{A,i}, d_i)$ is a metric space.*

Proof. Since $B, C \in \mathcal{G}_{A,i}$ are closed in A, if $B \neq C$ then there is an n such that $BA_n \neq CA_n$.

Suppose that $B, C, D \in \mathcal{G}_{A,i}$. If $BA_n \neq DA_n$, then $BA_n \neq CA_n$ or $CA_n \neq DA_n$. So

$$d_i(B, D) \leq \max(d_i(B, C), d_i(C, D)).$$

$\square$

Given $B \in \mathcal{G}_{A,i}$, we will write

$$\begin{aligned} \mathcal{B}_n(B) &= \{C \in \mathcal{G}_{A,i} | d_i(B, C) \leq p^{-n}\} \\ &= \{C \in \mathcal{G}_{A,i} | J_C\Omega_A + J_{A_n}\Omega_A = J_B\Omega_A + J_{A_n}\Omega_A\} \end{aligned}$$

and

$$\mathcal{B}'_n(B) = \{C \in \mathcal{G}_{A,i} | J_C\Omega_A + J_A^n = J_B\Omega_A + J_A^n\}.$$

Notice that $\{\mathcal{B}_n(B) | B \in \mathcal{G}_{A,i}, n \in \mathbb{N}\}$ and $\{\mathcal{B}'_n(B) | B \in \mathcal{G}_{A,i}, n \in \mathbb{N}\}$ each form a base for the topology on $\mathcal{G}_{A,i}$ induced by the metric since the two chains of ideals $\{J_{A_n}\Omega_A\}$ and $\{J_A^n\}$ are cofinal (see Lemma 7.1 of [8]).

Remark 3.5. *If A is a free abelian pro-p group of rank n then there is a natural correspondence between $\mathcal{G}_{A,i}$ and the p-adic Grassmann manifold of $n-i$ dimensional $\mathbb{Q}_p$-subspaces of $\mathbb{Q}_p^n$. Moreover this correspondence is a homeomorphism. The reader may prefer to view the topology in this way.*

Lemma 3.6. *If M is an Ω_A-module, then for each $i \leq \dim(A)$, the set of $B \in \mathcal{G}_{A,i}$ such that M is finitely generated over Ω_B is open.*

Proof. Suppose that $B \in \mathcal{G}_{A,i}$ and that M is a finitely generated Ω_B-module. It is sufficient to prove that there is a k such that M is finitely generated as an Ω_C-module for each $C \in \mathcal{B}'_k(B)$.

Now as M is finitely generated over Ω_B, M/MJ_B has finite $\mathbb{F}_p$-dimension and so there is a k such that $MJ_A^k \subseteq MJ_B$. So $MJ_A^k \subseteq MJ_B + MJ_A^{k+1} = MJ_C + MJ_A^{k+1}$ for each $C \in \mathcal{B}'_{k+1}(B)$. It follows inductively that $MJ_A^k \subseteq MJ_C + MJ_A^{k+n}$ for all $n \geq 0$. As MJ_C is closed in M,

$$MJ_C = \bigcap_{n \geq 0} (MJ_C + MJ_A^{k+n})$$

and so $MJ_A^k \subseteq MJ_C$ for each such C. We may now use Lemma 3.1 to deduce that M is finitely generated over Ω_C. $\square$

Lemma 3.7. *If M is a finitely generated torsion Ω_A-module, then the set of $B \in \mathcal{G}_{A,1}$ such that M is finitely generated over Ω_B is dense in $\mathcal{G}_{A,1}$.*

Proof. The case $\dim(A) = 1$ is easy so we assume that $\dim(A) \geq 2$. Suppose for contradiction that the result does not hold, so there is a $B \in \mathcal{G}_{A,1}$ and $k \in \mathbb{N}$ such that for all $C \in \mathcal{B}_k(B)$, M is not finitely generated over Ω_C. Lemma 3.1 tells us that for each such C, M/MJ_C is not finite dimensional over $\mathbb{F}_p$. But $\Omega_A/J_C\Omega_A \cong \Omega_{A/C}$ is such that every proper quotient is finite dimensional and so M/MJ_C is not a torsion $\Omega_A/J_C\Omega_A$-module. So $\mathrm{Ann}_{\Omega_{A/C}}(M/MJ_C) = 0$. Hence $\mathrm{Ann}_{\Omega_A}(M/MJ_C) \subseteq J_C\Omega_A$, and $\mathrm{Ann}_{\Omega_A}(M) \subseteq J_C\Omega_A$.

Now,

$$\bigcap_{C \in B_k(B)} J_C \Omega_A \subseteq \bigcap_{D \in \mathcal{G}_{B,1}} \left(\bigcap_{\substack{C \in B_k(B) \\ D \leq C}} J_C \Omega_A \right)$$

As $\{C \in B_k(B) | D \leq C\}$ is infinite for each $D \in \mathcal{G}_{B,1}$, and the image of $J_C \Omega_A$ in the local domain $\Omega_{A/D}$ is a height 1 prime for each $D \in \mathcal{G}_{B,1}$,

$$\bigcap_{\substack{C \in B_k(B) \\ D \leq C}} J_C \Omega_A = J_D \Omega_A.$$

By a similar argument,

$$\bigcap_{D \in \mathcal{G}_{B,i}} J_D \Omega_A = \bigcap_{E \in \mathcal{G}_{B,i+1}} J_E \Omega_A,$$

for each $i \leq \dim(B) - 1$.

It follows that

$$\bigcap_{C \in B_k(B)} J_C \Omega_A = 0,$$

our desired contradiction. $\square$

Lemma 3.8. *Suppose that M is a finitely generated Ω_A-module and $d_A(M) \leq \dim(A) - t$. The set of $B \in \mathcal{G}_{A,t}$ with M finitely generated as an Ω_B-module is dense.*

Proof. We prove this by induction on t. The case $t = 1$ is Lemma 3.7.

Suppose that $B \in \mathcal{G}_{A,t}$ and pick $C > B$ with $C \in \mathcal{G}_{A,t-1}$. By the induction hypothesis, every open ball around C contains a subgroup D of A such that M is finitely generated over Ω_D. In other words, for each positive integer k, there is a $D \in \mathcal{B}_k(C)$ such that M is finitely generated over Ω_D. Observe, using Lemma 3.1, that $d_D(M) = d_A(M) < \dim(D)$ and so M is Ω_D-torsion. It follows from Lemma 3.7 that the set of $E \in \mathcal{G}_{D,1}$ such that M is finitely generated over Ω_E is dense in $\mathcal{G}_{D,1}$. So it suffices to show that $U = \mathcal{B}_k(B) \cap \mathcal{G}_{D,1}$ is a non-empty open subset of $\mathcal{G}_{D,1}$ since then it must contain a subgroup F of A with M finitely generated over Ω_F.

That U is open follows by seeing that the restriction of the metric on $\mathcal{G}_{A,t}$ to $\mathcal{G}_{D,1}$ is just the usual metric on $\mathcal{G}_{D,1}$.

Without loss of generality, we may assume that A has topological generators $\{a_1, \dots, a_n\}$, B is the closed subgroup of A generated by $\{a_1, \dots, a_{n-t}\}$, $C = \overline{< a_1, \dots, a_{n-t+1} >}$ and $D = \overline{< a_1 + \epsilon_1, \dots, a_{n-t+1} + \epsilon_{n-t+1} >}$ with $\epsilon_i \in A_k$ for each i. It follows that $\overline{< a_1 + \epsilon_1, \dots, a_{n-t} + \epsilon_{n-t} >}$ lies in U. $\square$

Definition 3.9. *Given G a finitely generated pro-p group let $G_{ab} = G/i(G')$ where $i(G')$ is the isolator of G' in G, and let π be the natural projection of G onto G_{ab}. We set $\mathcal{G}_{G,t} = \{\pi^{-1}(B) | B \in \mathcal{G}_{G_{ab},t}\}$ for $t \leq \dim(G_{ab})$ and give it the induced metric.*

The following theorem should be compared with Lemma 5.1 of [2].

Theorem 3.10. *If G is a uniform pro-p group of finite rank, and M is a finitely generated Ω_G-module with $d_G(M) \leq \dim(G_{ab})-t$, then the set of $H \in \mathcal{G}_{G,t}$ such that M is finitely generated over Ω_H is open and dense.*

Proof. Notice, using Lemma 3.1, that if $G' \leq H \leq G$ then M is finitely generated over Ω_H if and and only $M/MJ_{i(G')}$ is finitely generated over $\Omega_{H/i(G')}$ and that $d_{G/i(G')}(M/MJ_{i(G')}) \leq d_G(M)$. The result now follows by Lemmas 3.6 and 3.8. □

4 Representations of Heisenberg groups

Recall that $e = 2$ if $p = 2$ and $e = 1$ otherwise.

Definition 4.1. *We say that a torsion-free pro-p group G of rank $2r+1$ is a* Heisenberg *pro-p group if it has centre Z isomorphic to $\mathbb{Z}_p$ and $G' \subseteq Z^{p^e}$. Notice that a Heisenberg pro-p group is necessarily uniform.*

Our goal for the rest of the paper is to understand the finitely generated modules for Heisenberg pro-p groups. Before we do that we make our work easier with the following definition:

Definition 4.2. *We say that a Heisenberg pro-p group is* clean *if it has a topological generating set $\{x_1, \ldots, x_r, y_1, \ldots, y_r, z\}$ such that $Z = \overline{<z>}$, $[x_i, y_i] = z^{p^e}$ for each i, and $[x_i, x_j] = [x_i, y_j] = 1$ for each pair of distinct i and j.*

We now show that, provided we don't mind passing to finite index subgroups, restricting our attention to clean Heisenberg groups doesn't do us any harm.

Lemma 4.3. *Every Heisenberg pro-p group contains a clean Heisenberg subgroup of finite index.*

Proof. Suppose that G is a Heisenberg pro-p group. There is a non-degenerate alternating $\mathbb{Z}_p$-bilinear form

$$\begin{array}{rcl} G/Z \times G/Z & \to & Z_{1+e} \\ (gZ, hZ) & \mapsto & [g,h] \end{array}$$

so we may choose a topological generating set $\{x_1, \ldots, x_r, y_1, \ldots, y_r, z\}$ such that $[x_i, y_i] = z^{\lambda_i p^{n_i+e}}$ for each i and $[x_i, x_j] = [x_i, y_j] = 1$ for each distinct pair i, j, where $\lambda_i \in \mathbb{Z}_p^\times$ and $n_i \in \mathbb{N}$.

By replacing each x_i by $x_i^{\lambda_i^{-1}}$ we may assume that $\lambda_i = 1$ for each i. Similarly by passing to the subgroup of finite index topologically generated by $\{x_i^{p^{k-n_i}}, y_i, z | 1 \leq i \leq r\}$ where $k = \max_{1 \leq i \leq r}\{n_i\}$ we may assume that $n_i = k$ for each i.

Finally passing to the subgroup topologically generated by $\{x_i, y_i, z^{p^{k-1}}\}$ we obtain a clean Heisenberg subgroup of G of finite index. □

The point of the definition of clean is that it enables us to prove Proposition 4.6. The following two lemmas are merely tools to that end.

Lemma 4.4. *Let G be a clean Heisenberg group. For each $g \in G \backslash ZG_2$ and each $z \in Z_{n+e}$ there exists $\delta \in G_n$ such that $[g, \delta] = z$.*

Proof. Fix $g \in G \backslash ZG_2$. Notice that for each $k \in \mathbb{N}$ there is a map

$$G_{k+1} \to Z_{k+1+e};\, y \mapsto [g, y].$$

The definition of a clean Heisenberg group ensures that this map is onto; in effect we have a $\mathbb{Z}_p$-bilinear form $\mathbb{Z}_p^{2r} \times \mathbb{Z}_p^{2r} \to \mathbb{Z}_p$ given on a basis by

$$< e_i, e_j >= \begin{cases} [x_i, x_j] \text{ for } 1 \leq i, j \leq r \\ [x_i, y_j] \text{ for } 1 \leq i, j - r \leq r \\ [x_i, y_j] \text{ for } 1 \leq i - r, j \leq r \\ [y_i, y_j] \text{ for } r + 1 \leq i, j \leq 2r \end{cases},$$

i.e. it is represented by the matrix $p^e J$ where

$$J = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

and we are asserting that given elements $\mathbf{x} \in \mathbb{Z}_p^{2r} \backslash p\mathbb{Z}_p^{2r}$ and $\lambda \in p^{k+e}\mathbb{Z}_p$ there is an element $\mathbf{y} \in p^k \mathbb{Z}_p^{2r}$ such that $< \mathbf{x}, \mathbf{y} >= \lambda$. This is follows from simple computation. $\square$

Lemma 4.5. *Let G be a clean Heisenberg group. If $g \in G \backslash ZG_2$ and $\epsilon \in G_n$, then $C_G(g\epsilon) \subseteq C_G(g)G_n$.*

Proof. Suppose that $h \in C_G(g\epsilon)$. As $[h, g\epsilon] = [h, \epsilon][h, g]^\epsilon$, we get $[h, g] \in Z_{n+e}$. Using Lemma 4.4 we may find $\delta \in G_n$ such that $[g, \delta] = [g, h]^{-1}$. It follows that $h\delta \in C_G(g)$. $\square$

Proposition 4.6. *Let G be a clean Heisenberg group and let $(-)^\perp : \mathcal{G}_{G,r} \to \mathcal{G}_{G,r}$ such that if $H \in \mathcal{G}_{G,r}$, $H^\perp = C_G(H)$. Then $(-)^\perp$ is an isometry.*

Proof. Supppose B, C are in $\mathcal{G}_{G,r}$ and $d_r(B, C) = p^{-n}$. Pick a finite topological generating set $\{g_1, \dots, g_r, z\}$ for B with $z \in Z$, then

$$B^\perp = \bigcap_{1 \leq i \leq r} C_G(g_i).$$

By Lemma 4.5, if $\epsilon_i \in G_n$, then $C_G(g_i\epsilon_i) \subseteq C_G(g_i)G_n$, so $C_G(C) \subseteq C_G(B)G_n$. It follows by symmetry that $C_G(C)G_n = C_G(B)G_n$ and so $(-)^\perp$ is a contraction mapping. But $(-)^{\perp\perp} = \mathrm{id}$ so $(-)^\perp$ is an isometry as asserted. $\square$

Now we are ready to prove our first important result: that for a Heisenberg group all Noetherian modules of sufficiently small dimension have non-trivial annihilator.

We remark at this point that (non-trivial) general results of this form are still out of reach. For example, for $\Omega_{SL_2(\mathbb{Z}_p)}$ it is not even known whether or not the only Noetherian modules with non-trivial annihilator are those of finite dimension over $\mathbb{F}_p$.

Theorem 4.7. *Let G be a Heisenberg pro-p group and let M be a finitely generated module over Ω_G such that $d_G(M) \leq r$ then $\mathrm{Ann}_{\Omega_G}(M) \neq 0$.*

Proof. Firstly suppose that H be a clean Heisenberg subgroup of G of finite index. Notice that $d_H(M) = d_G(M)$, by Lemma 3.1, and $\mathrm{Ann}_{\Omega_H}(M) \subset \mathrm{Ann}_{\Omega_G}(M)$. Consequently, it suffices to prove the result when G is clean. We suppose now that this is the case.

Let S_1 be the set of $H \in \mathcal{G}_{G,r}$ such that M is finitely generated over Ω_H, and S_2 be the set of $H \in \mathcal{G}_{G,r}$ such that M is finitely generated over $\Omega_{H^\perp}$.

By Theorem 3.10 and Lemma 4.6, S_1 and S_2 are open and dense in $\mathcal{G}_{G,r}$ and so have non-empty intersection.

Now, if $H \in S_1$ then M is a torsion Ω_H-module since M is finitely generated and $d_H(M) < \dim(H)$. Also, if H is in S_2 then M is finitely generated over its endomorphism ring as an Ω_H-module, $\mathrm{End}_{\Omega_H}(M)$, since $\Omega_{H^\perp}$ acts on M by Ω_H-endomorphisms. Now for H in the intersection $S_1 \cap S_2$, let X be a finite generating set for M over $\mathrm{End}_{\Omega_H}(M)$. Then $\mathrm{Ann}_{\Omega_H}(M) = \bigcap_{x\in X} \mathrm{ann}_{\Omega_H}(x) \neq 0$, because Ω_H has no non-trival zero divisors. □

Remark 4.8. *This result is best possible. For example, if we take an abelian subgroup A that trivially intersects Z, then the Ω_G-module obtained by inducing the trivial Ω_A-module to Ω_G has dimension $r+1$ and can be shown to have trivial annihilator.*

Our next goal is to prove that when G is a Heisenberg pro-p group, any non-zero ideal in Ω_G must meet Ω_Z. This result should be compared with a theorem for discrete group algebras which states that any non-zero ideal in the group algebra of a finitely generated torsion free nilpotent group must intersect the centre of the group algebra non-trivially, see section 11.4 of [14] for example for a stronger formulation of this result. It would be nice to prove an exact analogue here for all nilpotent uniform pro-p groups but at present we can only handle Heisenberg groups. Before we prove the result we give a preparatory lemma.

Notice first that if H is a normal subgroup of G then G acts on the set of ideals of Ω_H by conjugation.

Lemma 4.9. *Suppose that G is a Heisenberg pro-p group and H is an isolated subgroup of G properly containing Z. Whenever I is a non-zero prime ideal of Ω_H with finite G-orbit, the set of $K/Z \in \mathcal{G}_{H/Z,1}$ such that $I \cap \Omega_K \neq 0$ contains an open and dense subset of $\mathcal{G}_{H/Z,1}$.*

Proof. We let $X = \bigcap_{g\in G} I^g$, a non-zero G-invariant ideal in Ω_H since Ω_H has no non-trivial zero-divisors. Notice that every minimal prime ideal above X is of the form I^g for some $g \in G$. Since $J_Z\Omega_H$ is a height 1 prime ideal there are now two cases: firstly $X = I = J_Z\Omega_H$; secondly the image of $\bar{X}$ of X in $\Omega_{H/Z}$ is a non-zero ideal. In the first case the result is trivial since every $K \in \mathcal{G}_{H/Z,1}$ satisfies the required property so we assume from now on that we are in the second case.

Now Theorem 3.10 guarantees that the set of subgroups $K/Z \in \mathcal{G}_{H/Z,1}$ such that $\Omega_{H/Z}/\bar{X}$ is finitely generated over $\Omega_{K/Z}$ is open and dense so it suffices to prove that for each such K, $X \cap \Omega_K \neq 0$. Suppose we have such a K. Using Lemma 3.1 we see that Ω_H/X is a finitely generated Ω_K-module. It follows that there is an $h \in H\backslash K$ and that there are $\alpha_0, \ldots, \alpha_n \in \Omega_K$ not all zero such that $\sum_{i=0}^n \alpha_i h^i \in X$. Suppose that n is minimal subject to this.

As the alternating bilinear form in Lemma 4.4 is non-degenerate, we have $\dim(C_G(H)/Z)+\dim(H/Z)=\dim(G/Z)$ for all isolated $Z\le H\le G$ so we may pick $g\in C_G(K)\backslash C_G(H)$. Then if $n>0$

$$0\neq(\sum_{i=0}^{n}\alpha_i h^i)^g-[h^n,g](\sum_{i=0}^{n}\alpha_i h^i)\in X,$$

contrary to the minimality of n. It follows that $n=0$ and so $0\neq\alpha_0\in X\cap\Omega_K$ as required. $\square$

We now prove Theorem A.

Theorem 4.10. *Suppose that G is a Heisenberg pro-p group with centre Z and $0\neq I\lhd\Omega_G$. Then $I\cap\Omega_Z\neq 0$.*

Proof. Let H be a minimal isolated subgroup of G containing Z such that $I_H:=I\cap\Omega_H\neq 0$. Suppose that $H\neq Z$. Because I_H is a G-invariant ideal, G acts on the (finite) set $\{P_1,\dots,P_k\}$ of minimal primes above I_H. Using Lemma 4.9 we may find $K/Z\in\mathcal{G}_{H/Z,1}$ such that $P_i\cap\Omega_K\neq 0$ for each $1\le i\le k$. Since $(P_1\cdots P_k)^N\subseteq I_H$ for sufficiently large N, it follows that $I_K=I_H\cap\Omega_K\neq 0$ contradicting the minimality of H. It follows that $H=Z$ as required. $\square$

The following theorem that combines Theorem B and Corollary D of the introduction is an analogue of the Bernstein inequality for the representations of Weyl algebras.

Corollary 4.11. *If G is a Heisenberg pro-p group of rank $2r+1$ with centre Z and M is a finitely generated module over Ω_G such that $d_G(M)\le r$, then*

$$\mathrm{Ann}_{\Omega_G}(M)\cap\Omega_Z\neq 0.$$

Moreover if $S=\Omega_Z\backslash 0$ then $\mathrm{gldim}((\Omega_G)_S)=r$

Proof. By Lemma 4.3 G contains a clean Heisenberg pro-p group H of finite index. Now M is a finitely generated Ω_H-module with $d_H(M)=d_G(M)$ so Theorem 4.7 tells us that $\mathrm{Ann}_{\Omega_H}(M)\neq 0$. It follows from Theorem 4.10 that $\mathrm{Ann}_{\Omega_H}(M)\cap\Omega_{Z(H)}\neq 0$. Now $\Omega_{Z(H)}\subseteq\Omega_Z$ and the first part follows.

Now let A be a maximal abelian subgroup of G disjoint from Z, so $A\cong\mathbb{Z}_p^r$. If M is the left Ω_G-module taken by inducing the trivial Ω_A-module to G, so $M=\Omega_G/\Omega_G J_A$ and $E^r_{\Omega_G}(M)\cong\Omega_G/J_A\Omega_G$. By Lemma 2.4 $E^r_{(\Omega_G)_S}(M_S)\cong E^r_{\Omega_G}(M)_S\neq 0$ and so $\mathrm{gldim}((\Omega_G)_S)\ge r$.

The global dimension of $(\Omega_G)_S$ is certainly finite since it is a localisation of a ring of finite global dimension. It follows that we may find N a finitely generated $(\Omega_G)_S$-module such that $\mathrm{pd}(N)=\mathrm{gldim}((\Omega_G)_S)=n$. There is a finitely generated Ω_G-module M such that $M_S\cong N$. By Lemmas 2.5 and 2.4 we have $0\neq E^n_{(\Omega_G)_S}(N)\cong E^n_{\Omega_G}(M)_S$. But by Lemma 2.6, if $n>r$ then $d_G(E^n_{\Omega_G}(M))<r+1$ and so, by the first part, $E^n_{\Omega_G}(M)$ is Ω_Z-torsion, a contradiction. So we have $n\le r$ as required. $\square$

We finish by making clear the analogy between this result and Bernstein's inequality for Weyl algebras by restating and proving Corollary C:

Corollary 4.12. *Suppose that G is a Heisenberg pro-p group of rank $2r+1$ with centre Z and S is the central multiplicatively closed set $\Omega_Z \backslash 0$. Then for any finitely generated module M over $(\Omega_G)_S$ and any finitely generated Ω_G-submodule N of M with $N_S = M$ we have $d_G(N) \geq r+1$.*

Proof. Let M and N be as in the statement. Let j be the grade of M as a $(\Omega_G)_S$-module. By 4.11 we see that $j \leq r$. Since $E^j_{\Omega_G}(N)_S = E^j_{(\Omega_G)_S}(M)$, we may deduce that $j_{\Omega_G}(N) \leq j$. Recalling Lemma 2.6 we easily obtain the result. □

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