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A note on some properties of the least common multiple of conjugacy class sizes

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Abstract. We present some comments about the least common multiple of the sizes of conjugacy classes. It is noted how the problems related to this least common multiple connect to questions concerning the existence of regular orbits of linear groups.

For Professor K P Shum on the occasion of his 70th birthday

1. Introduction

Let $\mathrm{cs}(G)$ be the set of conjugacy class sizes of a finite group G and $\mathrm{lc}(G)$ be the least common multiple of $\mathrm{cs}(G)$. In this note we wish to consider when $\mathrm{lc}(G)$ is equal to the order of G and what happens when it is not. We will use the language of referring to the size of a conjugacy class containing an element $x \in G$ as the *index* of x, and the notation x^G for the conjugacy class containing x. Clearly, if the centre of G, denoted $Z(G)$, is not trivial, then $\mathrm{lc}(G)$ cannot be equal to the order of G, so we concentrate on groups with trivial centre. Using these ideas we introduce the following:-

Definition 1.1. *We say a group G is a full index group if its order $|G|$ is given by the least common multiple of the indices of the elements of G i.e.*

$$|G| = \mathrm{lcm}\{|x^G| : x \in G\}.$$

We denote the centraliser of x in G by $C_G(x)$ and note that $|x^G|$ is given by the index of $C_G(x)$ in G. Clearly if G is not a full index group there must be at least one prime which divides $|C_G(x)|$ for all $x \in G$. We also give this property a name:-

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Definition 1.2. *Let G be a group without centre. We say that a prime p is quasi-central in a group G if $|C_G(x)|$ is divisible by p for all $x \in G$.*

One might hope that there are no such primes, however in the next section we give some examples to show that this is not true. We are left with the following question:-

Question 1.3. *Can we describe those groups without centre that do not have full index? (Equivalently, those groups with a quasi-central prime.)*

It would seem from the examples given in the next section that the prime 2 and Fermat and Mersenne primes might play a role in the solution to this problem.

We observe that for a p-group G the breadth, $b(G)$ is defined to be such that $p^{b(G)}$ is the size of the largest conjugacy class in G. Note that $p^{b(G)} = \text{lcm}\{|x^G| : x \in G\}$. The breadth has been the subject of many papers, we mention the most recent [4].

2. Some groups with quasi-central primes

Let G be a group with a faithful irreducible representation over GF(q) for some prime q. Let the module which gives rise to this representation be V. There has been considerable research into when V contains a regular orbit for G. That is there is an element $v \in V$ so that $|v^G| = |G|$.

Consider the situation when G is a p-group, $p \neq q$. We have the following simple lemma:

Lemma 2.1. *Let G be a p-group with an irreducible faithful module V. Then the extension $H = VG$ will be a full index group if and only if there is a regular orbit.*

Proof. We begin by showing that there is always a p-element whose centraliser is a p-group. Since G is a p-group it has a non-trivial centre. Since G has a faithful irreducible representation any non-trivial element, say z, of the centre acts fixed-point freely on V, in particular $C_V(z) = 1$. Thus we only need to consider whether p is quasi-central.

Assume first that there is no regular orbit. Let $v \in V$. Then $|v^G| < |G|$. So $C_G(v)$ is not trivial. Thus p divides the order of the centraliser of every element of H as all other elements have order divisible by p. The converse is straightforward since if $|v^G| = |G|$ then p does not divide $|C_G(v)|$. □

The existence of regular orbits has been studied by many authors. In particular Huppert and Manz [5] constructed some example of groups which do not have regular orbits. In Example 5 (a), (b) and (c) they find groups $H = VG$ where G is a p-group and V is an irreducible GF(q) module for G and V has no regular orbit. We use the notation C_n to denote the cyclic group of order n.

Example 2.2. *Using the examples above we obtain three classes of groups without centre which are not of full index.*
a) In this example p is a Mersenne prime $p = 2^f - 1$. We have $G = C_p \wr C_p$ with

$|V| = 2^{fp}$ and 2 is a quasi-central prime.
b) In this example $q = 2^f + 1$ is a Fermat prime. We have $G = C_{2^f} \wr C_2$ with $|V| = q^{2f+1}$ and q is a quasi-central prime.
c) In this example q is a Mersenne prime $q = 2^f - 1$. We have G is a group of order $2^f + 1$ with $|V| = q^2$ and q is a quasi-central prime.

We can vary Lemma 2.1 a little.

Lemma 2.3. *Let G be a group with an irreducible faithful module V over $GF(q)$ for some prime q. Let $H = VG$. If there is a regular orbit and H does not have full index then q is quasi-central in H.*

Proof. Since there is a regular orbit, there is an element $v \in V$ so that $|v^G| = |G|$. Then $C_H(v)$ is q-group. So the only possible quasi-prime is q. □

We end this section with a slightly more complex example of a group where 2 is quasi-central.

Example 2.4. *Let G be a group isomorphic to a direct product of the symmetric group of degree 3 and a group of order 2. G can be realised as a group of 2×2 matrices over $GF(5)$. Specifically it can be generated by the following matrices:*

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let V be a the vector space of dimension 2 over $GF(5)$. Then V becomes a G-module with the action defined by the matrices. Let $H = VG$ then a straightforward calculation shows that 2 is a quasi-central prime for H.

Which suggests the following question:-

Question 2.5. *Does there exist a group G of odd order with trivial centre which does not have full index?*

3. Some positive results

First we comment that if $\gcd(|C_G(x)|, |C_G(y)|) = 1$ for two elements x, y then G has full index. Now we give some examples of groups with full index.

Example 3.1. *The full symmetric group S_n for $n \geq 3$. Consideration of the conjugacy class sizes of an n-cycle and an $(n-1)$-cycle yield the result.*

Example 3.2. *The Alternating groups A_n for $n \geq 4$. For n odd consider the n-cycle and $(n-2)$-cycle. For n even consider the $(n-1)$-cycle and the $(n-3)$-cycle, this is sufficient if $n - 1$ is not divisible by 3. If 3 does divide $n - 1$ also consider the element consisting of an $(n-5)$-cycle and two transpositions.*

Example 3.3. *Frobenius groups, of the form $G = K \rtimes H$, where H and K have coprime orders.*

In the next proposition we show ways to construct more full index groups.

Proposition 3.4. *(i) Let $G = H \times K$. Then G is a full index group if and only if both H and K are full index groups.*

(ii) Suppose $N \trianglelefteq G$, both N and G/N are full index groups and the orders of N and G/N are coprime. Then G is a full index group.

PROOF. (i) Assume that H and K are full index groups. Let p be a prime. There exist $h \in H$ and $k \in K$ so that $p \nmid |C_H(h)|$ and $p \nmid |C_K(k)|$. Let $g = (h, k) \in G$, then $C_G(g) = C_H(h) \times C_K(k)$ and the result follows.

Suppose that H is not a full index group. So there is a prime, say p, so that p divides $|C_H(h)|$ for all $h \in H$. But for $g = (h, k) \in H \times K$, we have $C_G(g) = C_H(h) \times C_K(k)$ and thus p divides $|C_G(g)|$ for all $g \in G$. Hence G is not a full index group.

(ii) From the hypotheses we have

$$|N| = \mathrm{lcm}\{|x^N| : x \in N\}$$

and

$$|G/N| = \mathrm{lcm}\{|xN^{G/N}| : xN \in G/N\}.$$

Recalling that $|x^N|$ divides $|x^G|$ and $|x^{G/N}|$ divides $|x^G|$ we get the result $|G| = \mathrm{lcm}\{|x^G| : x \in G\}$. □

We would like to thank Jan Saxl for showing us how to prove the next theorem.

Theorem 3.5. *Simple groups are full index groups.*

PROOF. The proof is a case-by-case study using the classification. For Alternating groups see Example 3.2 above. For G a simple group of Lie type consider a regular unipotent element, such an element is a p-element, where p is the characteristic and has centraliser a p-group, thus class size divisible by the full p'-part of $|G|$ see [2, §5.1]. Furthermore, the centraliser of a regular semi-simple element is a torus, so its class size is divisible by the full power of p, [2, §1.14].

A check through the Atlas [3], yields the result for the 26 sporadic groups. □

4. CONCLUDING REMARKS

It would be very satisfactory if we could classify groups with a quasi-central prime or, equivalently, full index groups. As a small contribution to this we prove the following:-

Proposition 4.1. *Suppose G is a group with trivial centre and with a cyclic Sylow p-subgroup. Then p is not quasi-central in G.*

PROOF. Suppose p divides $\mathrm{C}_G(x)$ for all elements x of G. We show that the centre of G is not trivial. Let $x \in G$ and denote by y_x an element of order p in $\mathrm{C}_G(x)$. Note that, as the Sylow p-subgroup of G is cyclic and hence has a unique cyclic subgroup of order p, it follows that subgroups of order p in G are conjugate.

Thus, if y is an element of order p in G the centraliser of y contains a conjugate of any element of G and so, by Burnside, [1, §26], is central.□

Note from this that the class of full index groups contains the simple groups and groups with trivial centre all of whose Sylow subgroups are cyclic. This suggests the next question.

Question 4.2. *Let G be a supersoluble group with trivial centre. Does G have full index?*

Note that $\mathrm{lc}(G)$ always divides the order of G. Consider $|G|/\mathrm{lc}(G)$ and observe that G has full index if and only if $|G|/\mathrm{lc}(G) = 1$. We ask

Question 4.3. *If G is a group with trivial centre, how big can $\frac{|G|}{\mathrm{lc}(G)}$ be?*

References

[1] W. Burnside. *Theory of groups of finite order.* Dover Publications Inc., New York, 1955.
[2] Roger W Carter. *Finite groups of Lie type:Conjugacy classes and complex characters* A Wiley-Interscience Publication, John Wiley & Sons Ltd, Chichester,1993
[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups.* Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.
[4] Giovanni Cutolo, Howard Smith, and James Wiegold. On finite p-groups with subgroups of breadth 1. *Bulletin of the Australian Mathematical Society*, 82:84–98, 2010.
[5] Bertram Huppert. and Olaf Manz. Orbit sizes of p-groups. *Arch. Math (Basel)*, 54:105–110, 1987.