

## IB Groups, Rings and Modules: Example Sheet 2

All rings in this course are commutative with a multiplicative identity.

1. Let  $\omega = (1 + \sqrt{-3})/2$ , and let  $R$  be the set  $\{a + b\omega : a, b \in \mathbb{Z}\}$ . Show that  $R$  is a subring of the ring  $\mathbb{C}$ . What are the units of  $R$ ?
2. An element  $r$  of a ring  $R$  is *nilpotent* if  $r^n = 0$  for some  $n$ .
  - (i) What are the nilpotent elements of  $\mathbb{Z}/6\mathbb{Z}$ ? Of  $\mathbb{Z}/8\mathbb{Z}$ ? Of  $\mathbb{Z}/24\mathbb{Z}$ ?
  - (ii) Show that if  $r$  is nilpotent then  $r$  is not a unit, but  $1 + r$  and  $1 - r$  are units.
  - (iii) Show that the nilpotent elements form an ideal in  $R$ .
3. Let  $r$  be an element of a ring  $R$ . Show that, in the polynomial ring  $R[X]$ , the polynomial  $1 + rX$  is a unit if and only if  $r$  is nilpotent. Is it possible for the polynomial  $1 + X$  to be a product of two non-units?
4. Let  $I_1 \subset I_2 \subset I_3 \subset \dots$  be ideals in a ring  $R$ . Show that the union  $I = \bigcup_{n=1}^{\infty} I_n$  is also an ideal. If each  $I_n$  is proper, explain why  $I$  must be proper.
5. (i) Show that if  $I$  and  $J$  are ideals in the ring  $R$ , then so is  $I \cap J$ , and the quotient  $R/(I \cap J)$  is isomorphic to a subring of the product  $R/I \times R/J$ .  
 (ii) Show that if  $p$  and  $q$  are coprime integers, then  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$  is isomorphic to  $\mathbb{Z}/pq\mathbb{Z}$ .
6. Let  $R$  be the ring  $C[0, 1]$  of continuous real-valued functions on  $[0, 1]$ , and let  $I = \{f \in R : f(x) = 0 \text{ for } 0 \leq x \leq \frac{1}{2}\}$ . Show that  $I$  is an ideal. What is  $R/I$ ?
7. Let  $R$  be an integral domain and  $F$  be its field of fractions. Suppose that  $\phi : R \rightarrow K$  is an injective ring homomorphism from  $R$  to a field  $K$ . Show that  $\phi$  extends to an injective homomorphism  $\Phi : F \rightarrow K$  from  $F$  to  $K$ . What happens if we do not assume that  $\phi$  is injective?
8. Let  $R$  be any ring. Show that the ring  $R[X]$  is a principal ideal domain if and only if  $R$  is a field.
9. An element  $r$  of a ring  $R$  is *idempotent* if  $r^2 = r$ .
  - (i) What are the idempotent elements of  $\mathbb{Z}/6\mathbb{Z}$ ? Of  $\mathbb{Z}/8\mathbb{Z}$ ? Of  $\mathbb{Z}/24\mathbb{Z}$ ?
  - (ii) Show that if  $r$  is idempotent then so is  $r' = 1 - r$ , and  $r \cdot r' = 0$ .
  - (iii) Show also that if  $r$  is idempotent then the ideal  $(r)$  is naturally a ring, and that  $R$  is isomorphic to  $(r) \times (r')$ .
10. (i) Show that the set  $P(S)$  of all subsets of a given set  $S$  is a ring with respect to the operations of symmetric difference and intersection - the power-set ring. Note that in this ring  $A^2 = A$  for all elements  $A$ . Describe the principal ideals in this ring. Describe the ideal  $(A, B)$  generated by elements  $A, B$ .  
 (ii) The ring  $R$  is *Boolean* if every element of  $R$  is idempotent. Prove that every finite Boolean ring is isomorphic to a power-set ring  $P(S)$  for some set  $S$ .  
 Give an example to show that this need not remain true for infinite Boolean rings.

### Additional Questions

11. A sequence  $\{a_n\}$  of rational numbers is a *Cauchy sequence* if  $|a_n - a_m| \rightarrow 0$  as  $m, n \rightarrow \infty$ , and  $\{a_n\}$  is a *null sequence* if  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . Quoting any standard results from Analysis, show that the Cauchy sequences with componentwise addition and multiplication form a ring  $C$ , and that the null sequences form a maximal ideal  $N$ .  
Deduce that  $C/N$  is a field, with a subfield which may be identified with  $\mathbb{Q}$ . Explain briefly why the equation  $x^2 = 2$  has a solution in this field.
12. Let  $\varpi$  be a set of prime numbers. Write  $\mathbb{Z}_{\varpi}$  for the collection of all rationals  $m/n$  (in lowest terms) such that the only prime factors of the denominator  $n$  are in  $\varpi$ .
- (i) Show that  $\mathbb{Z}_{\varpi}$  is a subring of the field  $\mathbb{Q}$  of rational numbers.
  - (ii) Show that any subring  $R$  of  $\mathbb{Q}$  is of the form  $\mathbb{Z}_{\varpi}$  for some set  $\varpi$  of primes.
  - (iii) Given (ii), what are the maximal subrings of  $\mathbb{Q}$ ?
13. Let  $F$  be a field, and let  $R = F[X, Y]$  be the polynomial ring in two variables.
- (i) Let  $I$  be the principal ideal generated by the element  $X - Y$  in  $R$ . Show that  $R/I \cong F[X]$ .
  - (ii) What can you say about  $R/I$  when  $I$  is the principal ideal generated by  $X^2 + Y$ ?
  - (iii) [Harder] What can you say about  $R/I$  when  $I$  is the principal ideal generated by  $X^2 - Y^2$ ?

Comments and corrections should be sent to `rdc26@dpms.cam.ac.uk`.