Part III Differential Geometry   Example Sheet 4   1

Q1) Only point at issue is the fact that $\rho(P,Q) \neq 0$ for $P \neq Q$. This is proved as in Ch 4 of my book (in case $n=2$) – wrt local coords, metric given as $\sum g_{ij}\, dx_i \otimes dx_j$ for some +ve def symmetric matrix of functions (g_{ij}). So locally in some Euclidean ball $B(\phi(P), \delta)$ in image of chart ϕ, $\exists\ \varepsilon^2 > 0$ s.t. $g_{ij} - \varepsilon^2 \delta_{ij}$ still +ve def. So if $d(P,Q)$ denotes the Euclidean length (on image of chart ϕ), easy argument shows $\rho(P,Q) \geq \varepsilon \min\{\delta, d(P,Q)\}$
> 0.

Q2) $M^m \hookrightarrow N^n$. Locally have coords $x_1, \ldots, x_n$ on $U \subset N$ s.t. $M \cap U$ given by $x_{m+1} = \cdots = x_n = 0$.
So TN locally trivialized by $\partial/\partial x_1, \ldots, \partial/\partial x_n$ with local sections $\sum f_i\, \partial/\partial x_i$, $f_i \in C(U)$

Have

$$\begin{array}{ccc} TN|_M = \pi^{-1}(M) & \subset & TN \\ \downarrow & & \downarrow \pi \\ M & \subset & N \end{array}$$

locally trivialized (on $U \cap M$) by $\partial/\partial x_1, \ldots, \partial/\partial x_n$ (or $\partial/\partial x_i|_M$) with local sections of form

$\sum g_i \partial/\partial x_i$, with $g_i \in \mathcal{O}(U \cap M)$.

Note that inclusion $M \subset N$ induces an inclusion of TM as a subbundle of $TN|_M$ and since M has local coords $x_1, \dots, x_m$ & hence TM locally trivialized by $\partial/\partial x_1, \dots, \partial/\partial x_m$, the inclusion is the obvious one).

Now any local section X of $T_N|_M$ over $V \subset M$, say $\sum g_j \partial/\partial x_j$, extends to a local section $\tilde{X} = \sum f_j \partial/\partial x_j$ of T_N over some $U \subset N$ with $U \cap M = V$.

We define $\nabla_i(X) = \nabla_i(\tilde{X})|_M$ — this is well-defined: if $Z = \sum f_j \partial/\partial x_j$ is zero on M, ie. all the f_j vanish on $M \cap U = V$, then

$$\nabla_i(Z) = \nabla_i\left(\sum_j f_j \partial/\partial x_j\right) = \sum_j \partial f_j/\partial x_i \, \partial/\partial x_j + \sum_j f_j \nabla_i(\partial/\partial x_j)$$

$$= 0 \text{ on } V = M \cap U \text{ for } 1 \le i \le m.$$

But in obvious notation,

$$\nabla_i(fX) = \nabla_i(\tilde{f}\tilde{X})|_M = \frac{\partial \tilde{f}}{\partial x_i}\Big|_M X + \tilde{f}\,\nabla_i(\tilde{X})|_M$$

$$= \partial f/\partial x_i \, X + f\,\nabla_i(X)$$

& $$\nabla_{f\partial/\partial x_i}(X) = \nabla_{\tilde{f}\partial/\partial x_i}(\tilde{X})|_M = \tilde{f}\,\nabla_{\partial/\partial x_i}(\tilde{X})|_M$$

$$= f\,\nabla_i(X) \text{ for } 1 \le i \le m$$

we deduce that ∇ is a connection on $TN|_M$

Now the metric g on TN restricts to a metric on $TN|_M$ & hence to a metric on $TM \subset TN|_M$.

Denote the orthogonal projection map $\pi : TN|_M \to TM$

(each $T_P N$ for $P \in M$ decomposes as an orthogonal direct sum $T_P N = T_P M \oplus (T_P M)^\perp$ i.e. $(TM)^\perp$ is the bundle given by $v \in T_P N$ $(P \in M)$ s.t. $g(v,w) = 0 \ \forall w \in T_P M$).

<u>CLAIM</u> Setting $\nabla^* := \pi \circ \nabla$ (i.e. for v fields X, Y on M, $\nabla^*_X(Y) = \pi(\nabla_X(\tilde{Y}))$ in usual notation), we have ∇^* is the Levi-Civita connection on M.

<u>Pf</u> Easy check that ∇^* is a connection on M: note that $\nabla_i^*(Y) = \pi(\nabla_i(\tilde{Y})|_M)$ for $i = 1, \dots, m$

& so $\nabla_i^*(\partial/\partial x_j) = \pi((\nabla_i \partial/\partial x_j)|_M)$

$$= \pi((\nabla_j \partial/\partial x_i)|_M) = \nabla_j^*(\partial/\partial x_i)$$

for $i = 1, \dots, m$, ∇ torsion free $\Rightarrow \nabla^*$ torsion free.

Moreover for v. fields X, Y on M have

$$\partial/\partial x_i \langle X, Y \rangle_M = \partial/\partial x_i \langle \tilde{X}, \tilde{Y} \rangle_N \Big|_M \quad \text{for } i = 1, \dots, m$$

$$= (\langle \nabla_i \tilde{X}, \tilde{Y} \rangle_N + \langle \tilde{X}, \nabla_i \tilde{Y} \rangle)|_M$$

$$= \langle \nabla_i X, Y \rangle + \langle X, \nabla_i Y \rangle \quad \text{where } \nabla_i X \ \& \ \nabla_i Y$$

are local sections of $TN|_M$

$$= \langle \pi(\nabla_i X), Y \rangle_M + \langle X, \pi(\nabla_i Y) \rangle_M$$

$$= \langle \nabla_i^* X, Y \rangle_M + \langle X, \nabla_i^* Y \rangle_M$$

$\Rightarrow \nabla^*$ is the Levi-Civita connection on M (by uniqueness of the connection).

Q3) Formulae for Γ^k_{ij} in terms of g show that the Christoffel symbols are invariant under scaling of metric & so Levi-Civita connection is invariant

Alter If ∇ the Levi-Civita connection associated with g, it is symmetric by defn : since it's compatible with g, it's clearly also compatible with λg $(\lambda = r^2)$

i.e. $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$ is still true $\Rightarrow \nabla$ is Levi-Civita connection wrt λg.

Thus $\nabla_X \nabla_Y Z$ doesn't depend on scaling

$\Rightarrow R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

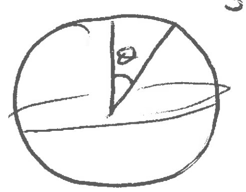
is indep of scaling of metric

So $R(U, V, U, V) = \langle U, R(U,V)V \rangle$ scales like r^2

whilst $\langle U, U \rangle \langle V, V \rangle - \langle U, V \rangle^2$ scales like r^4

$\Rightarrow$ sectional curvatures scale like $1/r^2$.

Q4) In case $n=2$, can use spherical polars & unit sphere has parametrization



S^2

$$\sigma(\theta,\phi) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$

$$\Rightarrow \sigma_\theta = (-\sin\phi\sin\theta,\ \sin\phi\cos\theta,\ 0)$$

$$\sigma_\phi = (\cos\phi\cos\theta,\ \cos\phi\sin\theta,\ -\sin\phi)$$

$\Rightarrow$ metric $= d\phi^2 + \sin^2\phi\, d\theta^2$ on S^2.

On $\mathbb{R}^3 \setminus \{0\}$, have coords
$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

$$\Rightarrow dx = dr\sin\phi\cos\theta - r\sin\phi\sin\theta\, d\theta + r\cos\phi\cos\theta\, d\phi$$

$$dy = dr\sin\phi\sin\theta + r\sin\phi\cos\theta + r\cos\phi\sin\theta\, d\phi$$

$dz = dr\cos\phi - r\sin\phi\, d\phi$. So Euclidean metric

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\phi^2 + \sin^2\phi\, d\theta^2).$$

In general, we have local coords $u_1, \dots, u_{n-1}$ on unit sphere S^{n-1} i.e. $\sigma(u_1, \dots, u_{n-1})$ a local parametrization, say $\sigma(u_1, \dots, u_{n-1}) = (x(u_1, \dots, u_{n-1}), y(u_1, \dots, u_{n-1}), z(u_1, \dots, u_{n-1}))$.

We then consider local parametrization

$$(r, u_1, \dots, u_{n-1}) \longmapsto r\,\sigma(u_1, \dots, u_{n-1}) \text{ of } \mathbb{R}^n \setminus \{0\}.$$

Note that $\sigma, r\sigma_{u_1}, \dots, r\sigma_{u_{n-1}}$ are lin indep (considered as tgt vectors on $\mathbb{R}^n$ at $P = r\sigma(u_1, \dots, u_{n-1})$) & moreover

$\langle \underset{\sim}{\sigma}, r\sigma_{u_i}\rangle = 0 \quad \forall i$

(since $\langle \underset{\sim}{\sigma}, \underset{\sim}{\sigma}\rangle = 1$ on S^{n-1}, on differentiating wrt u_i we get $\langle \sigma, \sigma_{u_i}\rangle = 0 \quad \forall i$). So metric on $\mathbb{R}^n \setminus \{0\}$ wrt this parametrization is given by

$$\langle \sigma, \sigma\rangle \, dr^2 + r^2 \sum_{i,j} \langle \sigma_{u_i}, \sigma_{u_j}\rangle \, du_i du_j$$

$$= dr^2 + r^2 dS^2 \quad \text{as claimed.}$$

5) Let D denote the Levi-Civita connection on M — by Q2 this is given by the orthogonal projection $\pi : TN \to TM$ i.e. $D_V = \pi \circ \nabla_V$.

Now $\nabla_V (fW) = V(f)\, W + f \nabla_V W \implies$

$\mathrm{II}(V, fW) = f\, \mathrm{II}(V, W)$ for V, W tgt to M.

Moreover $\mathrm{II}(V, W) - \mathrm{II}(W, V) = (\nabla_V W - \nabla_W V)^{\perp}$

$= [V, W]^{\perp} = 0$

(∇ torsion free).

So $\mathrm{II} : TM \times TM \to (TM)^{\perp}$ symmetric bilinear

Extending the tgt vectors v, w, x, y to local vector fields V, W, X, Y with $V_p = v$, etc RTP

$$\langle R(V, W)X, Y\rangle = \langle \bar{R}(V, W)X, Y\rangle$$

$$+ \langle \mathrm{II}(V, X), \mathrm{II}(W, Y)\rangle - \langle \mathrm{II}(V, Y), \mathrm{II}(W, X)\rangle$$

Now $R(V,W)X = \nabla_V \nabla_W X - \nabla_W \nabla_V X - \nabla_{[V,W]} X$

$$\Rightarrow \langle R(V,W)X, Y\rangle = \langle \nabla_V \nabla_W X, Y\rangle - \langle \nabla_W \nabla_V X, Y\rangle - \langle \nabla_{[V,W]} X, Y\rangle$$

Recalling that for $D = \overline{\nabla}$ the Levi-Civita connection on M,

Q2 $\Rightarrow$ $D_W X$ is tangential component of $\nabla_W X$,

have
$$\langle \nabla_V \nabla_W X, Y\rangle = \langle \nabla_V D_W X, Y\rangle + \langle \nabla_V \mathrm{II}(W,X), Y\rangle$$

$$= \langle D_V D_W X, Y\rangle + \cancel{\langle \mathrm{II}(V, D_W X), Y\rangle} + \langle \nabla_V \mathrm{II}(W,X), Y\rangle$$

$$= \langle D_V D_W X, Y\rangle + \cancel{V\langle \mathrm{II}(W,X), Y\rangle} - \langle \mathrm{II}(W,X), \nabla_V Y\rangle$$ metric connection

$$= \langle D_V D_W X, Y\rangle - \langle \mathrm{II}(W,X), \mathrm{II}(V,Y)\rangle$$

Using the identity $\overline{R}(V,W)X = D_V D_W X - D_W D_V X - D_{[V,W]} X$

& $\langle \nabla_{[V,W]} X, Y\rangle = \langle D_{[V,W]} X, Y\rangle$

& $\langle \nabla_W \nabla_V X, Y\rangle = \langle D_W D_V X, Y\rangle - \langle \mathrm{II}(V,X), \mathrm{II}(W,Y)\rangle$

we obtain
$$\langle R(V,W)X, Y\rangle = \langle \overline{R}(V,W)X, Y\rangle + \langle \mathrm{II}(V,X), \mathrm{II}(W,Y)\rangle - \langle \mathrm{II}(W,X), \mathrm{II}(V,Y)\rangle$$

as required

Q6) $M \subset N = \mathbb{R}^n$ defined by $f = 0$ & the tgt space at $P \in M$ is $\operatorname{Ker} d_P f$, where

$d_P f \left(\sum v_i \partial/\partial x_i \right) = \sum \partial f/\partial x_i (P)\, v_i$

[usually we identify $T_{N,P} \cong \mathbb{R}^n$ i.e. $d_P f \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum \frac{\partial f}{\partial x_i}(P) v_i$

Let $\underline{X} := \sum \partial f/\partial x_i \, \partial/\partial x_i$ be the v field on $\mathbb{R}^n$ corresponding to $\operatorname{grad} f = \begin{pmatrix} \partial f/\partial x_1 \\ \vdots \\ \partial f/\partial x_n \end{pmatrix}$, & then

$\langle \underline{X}, v \rangle_P = \sum \partial f/\partial x_i (P)\, v_i \quad \forall\, v = \sum v_i \partial/\partial x_i \in T_{\mathbb{R}^n, P}$

On M, $\underline{X}$ is non-vanishing & $\langle \underline{X}, v \rangle = 0$ $\forall v \in T_{M,P} \Rightarrow$ it is a normal vector $\forall P \in M$.

Setting $h = \left(\sum_i (\partial f/\partial x_i)^2 \right)^{1/2}$, then have a smooth field of normal unit vectors $\underset{\sim}{N} = 1/h\, \underline{X}$ on M, with $h \underset{\sim}{N} = \sum \partial f/\partial x_i \, \partial/\partial x_i$. Since both the tangent space to S^{n-1} at $\underset{\sim}{N}(P)$ and to M at P are given by the same condition of being orthogonal to $\underset{\sim}{N}(P)$, the Gauss map $\underset{\sim}{N} : M \to S^{n-1}$ induces derivative

$d_P \underset{\sim}{N} : T_{M,P} \longrightarrow T_{S^{n-1}, N(P)} = T_{M,P}$.

We now let ∇ denote the standard (trivial) connection on $\mathbb{R}^n$ (= Levi-Civita wrt Euclidean metric)

So $\nabla(\underline{X})(\partial/\partial x_k) = \nabla_{\partial/\partial x_k}(\underline{X})$

$= \sum_i \frac{\partial^2 f}{\partial x_k \partial x_i} \partial/\partial x_i = d(\underline{X})(\partial/\partial x_k) \quad \forall k$

i.e. $d(\underline{X})(v) = \nabla(\underline{X})(v) = \nabla_v(\underline{X}) \quad \forall v \in T_p\mathbb{R}^n$

Given now $v, w \in T_{M,p}$ & local v fields V, W on M with $V_p = v$, $W_p = w$, recall that $\nabla_v(W)$ was well-defined (as $\nabla_v(\tilde{W})|_M$ for any extension $\tilde{W}$ of W).

Then $\langle \nabla_v W, \underset{\sim}{N} \rangle_p = 1/h(p) \langle \nabla_v W, \underline{X} \rangle_p$

$= -1/h(p) \langle W, \nabla_v(\underline{X}) \rangle_p = -1/h(p) \langle W, d_p(\underline{X}) v \rangle_p$

$= -\langle W, d_p \underset{\sim}{N}(v) \rangle_p$ as $\underline{X} = h\underset{\sim}{N}$ & $\langle W, \underset{\sim}{N} \rangle_p = 0$.

& so $\mathrm{II}(v, w) = -\langle d_p\underset{\sim}{N}(v), w \rangle \underset{\sim}{N}$ as claimed.

For sphere $S^{n-1} \subset \mathbb{R}^n$ of radius r, clearly have $\underset{\sim}{N}(x) = 1/r\, x \quad \forall x \in S^{n-1}$ i.e. $\underset{\sim}{N} = 1/r\ \mathrm{id}$

So $\mathrm{II}(v, w) = -1/r \langle v, w \rangle \underset{\sim}{N} \quad \forall v, w \in T_pM$.

But sectional curvature for plane spanned by $v, w \in T_pM$

is $\langle \overline{R}(v, w) w, v \rangle / (\langle v,v \rangle \langle w,w \rangle - \langle v,w \rangle^2)$

$$= \frac{(\langle \mathrm{II}(v,v), \mathrm{II}(w,w) \rangle - \langle \mathrm{II}(v,w), \mathrm{II}(v,w) \rangle)}{\langle v,v \rangle \langle w,w \rangle - \langle v,w \rangle^2} = 1/r^2$$

Q7) Suppose $J: V \to V$ an EM of a v space V s.t. $J^2 = -1$. Then $\exists$ basis of V wrt which the matrix of J is :

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & & 0 \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

In particular, $\dim V$ is even. So M has an almost complex structure $\Rightarrow \dim M$ even $= 2n$ say.

Now for any $\Theta \in \operatorname{End} TM$, we have

$$\nabla_X(\Theta(Y)) = (\nabla_X \Theta)(Y) + \Theta(\nabla_X Y) \quad \& \text{ so}$$

∇ compatible with J means that $\nabla_X(JY) = J\nabla_X Y$ $\forall X, Y$ (ie. $\Leftrightarrow \nabla J = 0$)

Now for any Koszul connection ∇, we can define an associated connection ∇' by $\nabla'(JX) = \frac{1}{2}(\nabla(JX) + J(\nabla X))$

Note then, by substituting $X \Leftrightarrow JX$, we get

$$J\nabla' X = \tfrac{1}{2}(J(\nabla X) + \nabla(JX)) = \nabla'(JX)$$

ie. ∇' compatible with the almost complex structure J.

Observe $\quad \nabla' X = \frac{1}{2}(\nabla X - J\nabla(JX))$ $\quad (*)$

May check directly this is a connection, or alternatively convert that $\nabla' X = \nabla X - \frac{1}{2} J \nabla(JX) - \frac{1}{2}\nabla X$ is the connection $\nabla - \frac{1}{2} J \circ \nabla J$ where 2nd term $\in \Omega^1(\operatorname{End} TM)$

We should now impose the missing condition that the almost complex str J is also compatible with metric g

ie. $\quad g(JX, JY) = g(X, Y) \quad \forall X, Y$

It will turn out that the connection we find is not unique as claimed in original version of question, but only that a distinguished choice may be made.

Suppose in the above that ∇ is a metric connection: then

$$(\nabla' X, Y) + (X, \nabla' Y) = \underline{\tfrac{1}{2}(\nabla X, Y)} + \tfrac{1}{2}(\nabla JX, JY)$$

$$+ \underline{\tfrac{1}{2}(X, \nabla Y)} + \tfrac{1}{2}(JX, \nabla JY) \qquad \text{from } (*)$$

$$= \tfrac{1}{2}(X, Y) + \tfrac{1}{2}(JX, JY) = g(X, Y)$$

(assuming J is metric compatible) $\therefore$ ∇' also a metric connection

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Any other connection can be written in the form $\nabla' + \theta$ with $\theta \in \Omega^1(\mathrm{End}\, TM)$: condition for this connection to be compatible with J and the metric are

(i) $\theta J = J\theta$ & (ii) $(\theta X, Y) + (X, \theta Y) = 0$

There are many choices for such a θ (essentially we are just saying that it is skew-hermitian), but there is however a distinguished choice for the connection, as we can always start with ∇ being the Levi-Civita connection & take the corresponding ∇'.

If we have a connection which is compatible with almost complex structure J, the parallel transport τ_t is also compatible with J, i.e. $J\tau_t = \tau_t J$

On any given tgt space, J defines the structure of a v space over $\mathbb{C}$ (namely $i.v := J(v)$) & thus the says that τ_t is an IM of complex v spaces.

Argument in notes $\Rightarrow$ it is also an isometry

Thus parallel transport around a closed loop yields an elt of $GL(n,\mathbb{C}) \cap O(2n) = U(n)$.

Q8) Recall $R(X,Y,Z,T) = R(Z,T,X,Y)$ and is antisymmetric in first two & last two entries

$\Rightarrow$ R defines a symmetric bilinear form on $\Lambda^2 TM$

i.e. $R \in S^2(\Lambda^2 TM)^*$

When $\dim V = 3$ for a real v space V, have that

$V \times \Lambda^2 V \longrightarrow \Lambda^3 V = k$ is a perfect pairing, namely

$$(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) \wedge (\mu_1 e_2 \wedge e_3 + \mu_2 e_3 \wedge e_1 + \mu_3 e_1 \wedge e_2)$$
$$\longmapsto (\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3)\, e_1 \wedge e_2 \wedge e_3$$

i.e. we have an induced IM $\Lambda^2 V \xrightarrow{\sim} V^*$

A metric (i.e. inner product) on V determines an IM $V^* \to V$ & hence an identification $\Lambda^2 V \xrightarrow{\sim} V$ (depending only on metric & orientation) & so $S^2(\Lambda^2 V)^* \cong S^2 V^*$

Now apply this to the tangent space to M at a point P.

We have a map $c : S^2(\Lambda^2 V)^* \longrightarrow S^2 V^*$ given by $c(R) = r$ where $r(v,w) = \sum_i R(e_i, v, e_i, w)$ where $\{e_1, e_2, e_3\}$ an o-n basis

<u>CLAIM</u> When $\dim V = 3$, $\operatorname{Ker} c = 0$

<u>Pf</u> Take e_1, e_2, e_3 o-n basis

If image $r = c(R)$ is zero, then

$r(e_1, e_2) = 0 \Rightarrow R(e_3, e_1, e_3, e_2) = 0$

$r(e_1, e_1) = 0 \Rightarrow R(e_2, e_1, e_2, e_1) + R(e_3, e_1, e_3, e_1) = 0$

Similarly $R(e_1, e_2, e_1, e_2) + R(e_3, e_2, e_3, e_2) = 0$

$\therefore \quad R(e_3, e_1, e_3, e_1) = R(e_3, e_2, e_3, e_2)$

But $R(e_1, e_3, e_1, e_3) + R(e_2, e_3, e_2, e_3) = 0$

$\Rightarrow R(e_1, e_3, e_1, e_3) = 0$ & similar symmetrised statements $\Rightarrow R(e_1, e_3, e_2, e_3) = 0$ & similar statements $\Rightarrow R = 0$.

Since c is linear & $\dim S^2(\Lambda^2 V)^* = \dim S^2 V^*$, have c an IM. So for $r \in S^2 V^*$, $\exists !$ $R \in S^2(\Lambda^2 V)^*$ s.t. $c(R) = r$.

For $V = (TM)_p$, this ensures that the Ricci tensor determines the full Riemannian curvature tensor

29) Clearly $T(M \times N)_{(x,y)} = TM_x \oplus TN_y$. Given inner-products on TM_x & TN_y, the sum is an i.p. on the direct sum. From this it is clear that $g + h$ defines a metric $\langle \, , \rangle$ on $M \times N$

Clearly $\langle X, Y \rangle = 0$ for X a v field on M
Y a v field on N

Moreover writing $X = \sum X_i \partial/\partial x_i$, $Y = \sum Y_j \partial/\partial y_j$,
for any $f = f(x_1, \dots, x_m, y_1, \dots, y_n)$,

$$XYf = \sum_{i,j} X_i \partial/\partial x_i \, Y_j \partial/\partial y_j f = \sum X_i Y_j \frac{\partial^2 f}{\partial x_i \partial y_j} = YXf.$$

Moreover, for v fields X_1, X_2 on M, the bracket (on $M \times N$) $[X_1, X_2]$ comes from a v field on M & similar statement holds for v fields Y_1, Y_2 on N. Now use the formula for the Levi-Civita connection (cf Fundamental Lemma of Riemannian Geometry) to ~~deduce~~ that $\langle \nabla_X Y, Z \rangle = 0$ for any v field Z coming from M or N $\Rightarrow \langle \nabla_X Y, Z \rangle_p = 0$ pointwise for all $Z_p \in T(M \times N)_p \Rightarrow \nabla_X Y = 0$
(alternatively argue directly from Christoffel symbols).

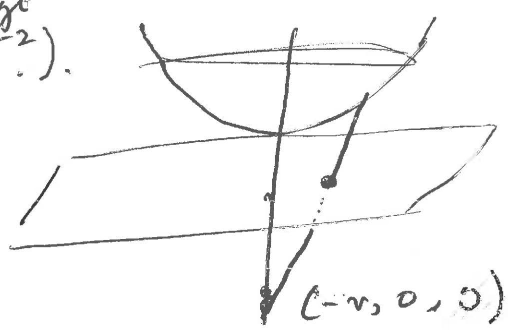
Note that for v fields Y, Y' on N, $\nabla_Y Y'$ (for above L-C connection on $M \times N$) comes from $\nabla_Y Y'$ for L-C connection on N. So for X v field on M, Y a v field on N,

$$R(X,Y)Y = \nabla_X \underbrace{(\nabla_Y Y)}_{Y'} - \nabla_Y \underbrace{(\nabla_X Y)}_{\overset{\shortparallel}{0}} - \underbrace{\nabla_{[X,Y]}}_{\overset{\shortparallel}{0}} Y$$

$$\Rightarrow R(X, Y, X, Y) = \langle X, R(X,Y)Y \rangle = 0.$$

Q10) Isometries of Lorentzian form in $O^+(n, 1)$ include elts $\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right)$ for $A \in SO(n)$ & elts $\left(\begin{array}{cc|ccc} \cosh d & -\sinh d & & 0 & \\ -\sinh d & \cosh d & & & \\ \hline & & 1 & & 0 \\ & 0 & & \ddots & \\ & & 0 & & 1 \end{array}\right)$ ✓

Now use elts of 1st type to send any point on $H(r)$ to a point of the form $(r\cosh d, r\sinh d, 0, \ldots, 0)$ & then use 2nd type of symmetry to send it to $Q = (r, 0, \ldots, 0)$ (cf similar argument in case of S^n). Isometries of 1st type act transitively on planes in tangent space at Q & so sectional curvatures of $H(r)$ are all constant = Gaussian curvature of $H_2(r)$ (2-dim' case).
(tgt space to $H(r)$ split into orthogonal direct sum of tgt space to $H_2(r)$ & $\mathbb{R}^{n-2}$).



Now stereographic projection of $H_2(r)$ from point $(-r, 0, 0)$ onto plane $x_0 = 0$, identifies $H_2(r)$ isometrically with disc of radius r in plane with metric

$$\frac{4r^4 |dw|^2}{(r^2 - |w|^2)^2}$$

(cf calculation in Ch5 of my book for case $r = 1$), which in turn is isometric to the Poincaré disc model of hyperbolic plane with metric scaled by r^2.

So, using result from Q3, need only show that hyperbolic plane has constant curvature -1

Let us work with upper half-plane model H, with metric $g = \dfrac{|dz|^2}{y^2}$ ie. $g_{11} = g_{22} = 1/y^2$, $g_{12} = g_{21} = 0$

Thus $g^{11} = g^{22} = y^2$, $g^{12} = 0$ for inverse matrix.

So $\Gamma^1_{11} = \frac{1}{2} g^{11}(g_{11,1} + \dots) = 0$

$\Gamma^2_{11} = \frac{1}{2} g^{22}(g_{12,1} + g_{12,1} - g_{11,2}) = 1/y$

$\Gamma^1_{12} = \frac{1}{2} g^{11}(g_{11,2} + \cancel{g_{21,1}} - \cancel{g_{12,1}}) = -1/y$

$\Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{22} = \frac{1}{2} g^{22}(g_{22,2}) = -1/y$.

$\therefore \nabla_1 \partial/\partial x = \frac{1}{y}\partial/\partial y$, $\nabla_1 \partial/\partial y = -\frac{1}{y}\partial/\partial x$, $\nabla_2 \partial/\partial y = -\frac{1}{y}\partial/\partial y$

$\therefore R(\partial/\partial x, \partial/\partial y)\partial/\partial y = \nabla_1\nabla_2 \partial/\partial y - \nabla_2\nabla_1 \partial/\partial y$

$= -\nabla_1 \frac{1}{y}\partial/\partial y + \nabla_2 \frac{1}{y}\partial/\partial x$

$= \cancel{+\frac{1}{y^2}\partial/\partial x} - \cancel{\frac{1}{y^2}\partial/\partial x} - \frac{1}{y^2}\partial/\partial x$.

So $\langle \partial/\partial x, R(\partial/\partial x, \partial/\partial y)\partial/\partial y\rangle = -\frac{1}{y^2}\cdot\frac{1}{y^2} = -\frac{1}{y^4}$.

$\therefore K = -\frac{1}{y^4} \Big/ \frac{1}{y^2}\cdot\frac{1}{y^2} = -1$.

Q11) Second Bianchi (in normal coords at a point P)

reads $\partial/\partial x^h R_{klij} + \partial/\partial x^k R_{lhij} + \partial/\partial x^l R_{hkij} = 0$

Contract with g^{lj} & use fact that coords are normal

to get $\partial/\partial x^h r_{ki} - \partial/\partial x^k r_{hi} + \sum_j \partial/\partial x^j R_{hkij} = 0$

Setting $h = i$, get $\partial/\partial x^i r_{ki} - \partial/\partial x^k r_{ii} + \sum_j \partial/\partial x^j R_{ikij}$

$= 0$ (*)

Now assume that Ricci tensor $r = c\,g$

for some function c on M, where g denotes the metric i.e. that the Ricci curvatures are constant at each point (but maybe vary as the point varies).

With normal coords as above (so that $g_{ij}(P) = \delta_{ij}$ & first order terms vanish), we can sum equation (*) over i, obtaining

$$\frac{\partial c}{\partial x^k} - n\frac{\partial c}{\partial x^k} + \underbrace{\sum_j \partial/\partial x_j \, c \, g_{kj}}_{= \partial/\partial x^k c} = 0$$

$$\Rightarrow \quad (2-n)\frac{\partial c}{\partial x^k} = 0 \quad \Rightarrow \quad \frac{\partial c}{\partial x^k} = 0 \quad \forall k$$

$(n > 2)$.

$\therefore$ $dc = 0$ everywhere & c is locally constant.

Assuming M is <u>connected</u>, deduce $c =$ const.