Pt III Differential Geometry Example Sheet 1

Q1 (i) No: $x \longmapsto x^{1/3}$ is not smooth

(ii) Yes:

$$\begin{tikzcd} R_1 \arrow[r, "x \,\mapsto\, x^{1/3}"] \arrow[d, "r \,\mapsto\, r"'] & R_2 \arrow[d, "r \,\mapsto\, r^3"] \\ \mathbb{R} \arrow[r, "id"'] & \mathbb{R} \end{tikzcd}$$

shows that the inverse is smooth.

Q2 ρ^* is a metric (straightforward)

In this metric, a small enough open ball in X around $\overline{P}$ of radius ε, consists of N ε-balls in $\mathbb{R}^2$ round each P_i, with the P_i $(i=1,\dots,N)$ identified.

If now X were a topological manifold, we could find an open nbhd $U \supset \overline{P}$, $U \subset B(\overline{P}, \varepsilon)$ which is homeomorphic to a disc in $\mathbb{R}^2$.

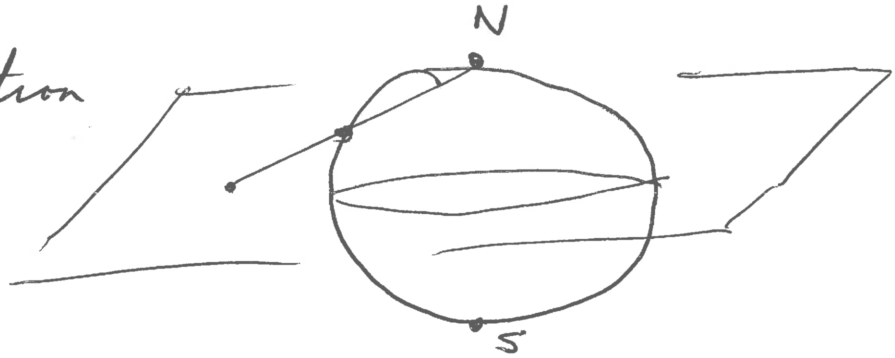
But $U \setminus \overline{P} = \coprod_{i=1}^{N} (U_i \setminus P_i)$, where U_i is an open nbhd of P_i in $B(P_i, \varepsilon)$, & so $U \setminus \overline{P}$ disconnected and not therefore homeo to a punctured disc. So X cannot be made into a topological manifold

<u>Q3</u> (i) Easy: given charts on $U_1 \subset M_1$, $U_2 \subset M_2$, have obvious chart on $U_1 \times U_2 \subset M_1 \times M_2$. The coordinate transformation functions are also clearly smooth.

(ii) S^n is a manifold by for instance taking standard projection charts on the various half-hyperspheres. Each of these descends to give a chart on an open subset of $\mathbb{R}P^n$ — there is a choice of sign which can be made but the transition functions are all smooth.

(iii) Points of $\mathbb{C}P^n$ given in terms of homogeneous coords $(z_0 : z_1 : \dots : z_n)$. Have chart on $U_0 = \{z_0 \neq 0\}$ given by $x_i = z_i/z_0$ for $i > 0$ and one on $U_1 = \{z_1 \neq 0\}$ by $y_1 = z_0/z_1$, $y_i = z_i/z_0$ for $i > 1$.
For these two charts, the coordinate transform is given by $y_1 = 1/x_1$, $y_i = x_i/x_j$ for $i > 0$.
Note that the charts identify U_i with $\mathbb{C}^n = \mathbb{R}^{2n}$ and the transformations are holomorphic rather than just smooth — i.e. $\mathbb{C}P^n$ is a <u>complex</u> manifold.
The above clearly works for any U_i & U_j.

Q4 (i) Map given by stereographic projection $(x, y, z) \mapsto \frac{x+iy}{1-z} = \zeta_1$



yields an identification of $S^2 \setminus \{N\}$ with $\mathbb{C}$ (chart)

and $(x, y, z) \mapsto \frac{x+iy}{1+z} = \zeta_2$ gives a chart $S^2 \setminus \{S\} \xrightarrow{\sim} \mathbb{C}$. Note that $\bar{\zeta}_1 \zeta_2 = \frac{x^2+y^2}{1-z^2} = 1$

i.e. transition function given by $\zeta_2 = 1/\bar{\zeta}_1$

In terms of homogeneous coordinates on $\mathbb{C}P^1$, the diffeomorphism is given on $S^2 \setminus \{N\}$ by

$(x, y, z) \mapsto (1 : \zeta_1) = (1-z : x+iy)$ and

on $S^2 \setminus \{S\}$ by $(\bar{\zeta}_2 : 1) = (x-iy : 1+z)$

Since $x^2+y^2 = 1-z^2$, these agree on overlap and in local coords it is just $\zeta_2 = 1/\bar{\zeta}_1$.

(ii) The natural map is given by $(z_0, z_1) \mapsto (z_0 : z_1)$

Take obvious atlas on $\mathbb{C}P^1$ with open sets $U_0, U_1 \cong \mathbb{C}^1$

& wrt these charts the map is given by:

$\mathbb{C}^2 \setminus \{z_0 = 0\} \longrightarrow U_0 \subset \mathbb{P}^1$ by $(z_0 : z_1) \mapsto \frac{z_1}{z_0}$

$\mathbb{C}^2 \setminus \{z_1 = 0\} \longrightarrow U_1 \subset \mathbb{P}^1$ by $(z_0 : z_1) \mapsto z_0/z_1$

With S^3 given by $|z_0|^2 + |z_1|^2 = 1$, this induces a smooth surjective map $S^3 \xrightarrow{\pi} \mathbb{C}P^1$.

In terms of real coords, the first map above (to U_0) may be written $(x_0, y_0, x_1, y_1) \mapsto \frac{1}{x_0^2+y_0^2}(x_0x_1+y_0y_1, x_0y_1-x_1y_0)$ and can check derivative is surjective $T_{S^3,P} \to T_{\mathbb{C}P^1,\pi(P)}$ (to cut down on algebra, choose coords on $\mathbb{C}^2$ so that $P = (1,0,0,0)$ & then Jacobian matrix acting on $T_{S^3,P} = \{x_0 = 0\}$ given by $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$) and so preimage theorem $\Rightarrow$ fibre is a 1-dim^l submfld (as it is compact, it has to be an S^1).

Explicitly, since the fibre of $S^3 \to \mathbb{C}P^1$ (given by $\underset{|z_0|^2+|z_1|^2=1}{(z_0, z_1)} \longmapsto (z_0 : z_1)$) consists of all points $e^{i\theta}(z_0, z_1)$ for some choice of point (z_0, z_1) in the fibre, $0 \le \theta < 2\pi$, the fibre is just the image of $e^{i\theta} \mapsto e^{i\theta}(z_0, z_1)$.

The above map easily checked to be both an immersion and a homeomorphism, and so every fibre is diffeomorphic to S^1.

Q5 (i) $SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \text{ s.t. } |a|^2+|b|^2=1 \right\}$ (5

and this is identified with $S^3 \subset \mathbb{C}^2$ as in Q4.

(ii) For any α, we have a chart on $S^1 \setminus \{e^{i\alpha}\}$ $\xrightarrow[\theta]{} (\alpha, \alpha+2\pi)$ with inverse $\theta \mapsto e^{i\theta}$.

$\frac{d}{d\theta}$ then defines a nowhere vanishing v field on $S^1 \setminus \{e^{i\alpha}\}$. However, for different choices of α, the tgt vector $\frac{d}{d\theta}$ is unchanged (since coord transformation is just a translation). Hence $\partial/\partial\theta$ is a nowhere vanishing v field $\Rightarrow TS^1 \cong S^1 \times \mathbb{R}$ trivial

(iii) $\exists$ left invariant global frame for TG (i.e. d everywhere indep v fields) & hence it's the trivial bundle (of course (ii) is a special case for this)

(iv) $S^3 \underset{\text{diffeo}}{\cong} SU(2) \Rightarrow TS^3 \cong TSU(2)$
$\cong SU(2) \times \mathbb{R}^3 \cong S^3 \times \mathbb{R}^3$

Q6 Choose a ball $B = B(\underline{0}, \varepsilon)$ contained in given nbhd

For any smooth function f on B and $\underline{a} \in B$, consider $h_{\underline{a}}(t) = f(t\underline{a})$, smooth function in variable t defined for $0 \le t \le 1$.

So $f(\underline{a}) - f(\underline{0}) = \int_0^1 h_{\underline{a}}'(t)\, dt$

$$= \int_0^1 \sum_{i=1}^n a_i \, {\partial f}/{\partial x_i} (t\underline{a})\, dt.$$

Now set $g_i(\underline{a}) = \int_0^1 {\partial f}/{\partial x_i} (t\underline{a})\, dt$

Clearly $g_i(\underline{x})$ is smooth in $\underline{x}$ with $g_i(\underline{0}) = {\partial f}/{\partial x_i}(\underline{0})$. Starting from

$$f(\underline{x}) = f(\underline{0}) + \sum_{i=1}^n x_i \, g_i(\underline{x})$$

and repeating previous argument for each g_i, we achieve desired statement.

<u>Q7</u> RTP $f_*[X, Y](h) = [f_*X, f_*Y](h)$

for any smooth function h i.e. for any $P \in M$

$$(f_*[X,Y])_{f(P)}(h) = [f_*X, f_*Y]_{f(P)}(h)$$

$$\forall h \in \mathcal{C}_{N, f(P)}$$

But $[f_*X, f_*Y]_{f(P)}(h)$

$= (f_*X)_{f(P)} f_*Y(h) - (f_*Y)_{f(P)} f_*X(h)$

$= (f_*X)\, Y(h\circ f)\circ f^{-1} - (f_*Y)\, X(h\circ f)\circ f^{-1}$

$= X_P(Y(h\circ f)) - Y_P(X(h\circ f)) = [X,Y]_P(h\circ f)$

$= (f_*[X,Y])_{f(P)}(h)$

8) $d_p x$, $d_p y$, $d_p z$ is just the dual basis to $\partial/\partial x|_p$, $\partial/\partial y|_p$, $\partial/\partial z|_p$ and so the claim is a trivial check. From this one spots that the given distribution comes from the level sets (integral submanifolds)

$$z = x^2 + y^2 + c \qquad (c \in \mathbb{R})$$

i.e. tangent space at point on a level set is generated by X and Y; X and Y are therefore an involutive smooth distribution on $\mathbb{R}^3 \setminus \{x = y = 0\}$. Clearly however this extends to one on all of $\mathbb{R}^3$ (since for points $(0,0,c)$, we just have (tangent) space generated by $\partial/\partial x$ & $\partial/\partial y$.

9) Choose coord nbhds

$$\begin{array}{ccc} \overset{\ni p}{U} & \hookrightarrow & V \\ \cap & & \cap \\ M & & N \end{array}$$

(embedding $\underset{wlog}{\Rightarrow}$ $U = V \cap M$) with coordinates

$(u_1, \ldots, u_m)$ on U & $(v_1, \ldots, v_n)$ on V.

Wlog also $\left(\partial v_i/\partial u_j\right)_{\substack{1 \le i \le m \\ 1 \le j \le m}}$ " $\ast$ non-singular

Hence if we set $y_i = v_i|_M$ for $i \le m$, we have $\det\left(\partial y_i/\partial u_j\right) \neq 0$ in some nbhd of $P \in U$

i.e. $\underline{y} \circ \underline{u}^{-1}$ is a local diffeo by inverse function thm.

$$\begin{tikzcd} & U' \arrow[dl, "{(u_1, \dots, u_m)}"'] \arrow[dr, "{(y_1, \dots, y_m)}"] & \\ \mathbb{R}^m & & \mathbb{R}^m \end{tikzcd}$$

Since $\underline{y} = (\underline{y} \circ \underline{u}^{-1}) \circ \underline{u}$, we deduce that $y_1, \dots, y_m$ a local coord system on M, wlog on U, say $\phi = (y_1, \dots, y_m)$. The map $\alpha \circ \phi^{-1}$ on ϕU is therefore given locally in coords by

$$(v_1, \dots, v_n) = (y_1, \dots, y_m, h_{m+1}(y_1, \dots, y_m), \dots, h_n(y_1, \dots, y_m))$$

& M given locally by equations $v_i = h_i(v_1, \dots, v_m)$ for $i > m$

Now choose new local coord system $x_1, \dots, x_n$ in a nbhd of $P \in N$ by $x_i = v_i$ for $i \le m$

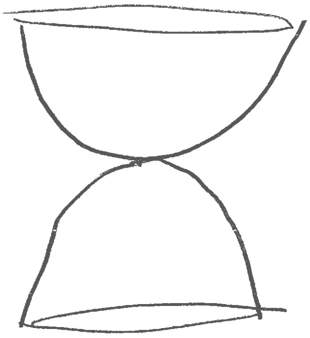
$x_i = v_i - h_i(v_1, \dots, v_m)$ for $i > m$

$$\therefore \left(\partial x_i / \partial v_j\right) = \left(\begin{array}{c|c} I_m & 0 \\ \hline /// & I_{n-m} \end{array}\right) \underset{\text{IFT}}{\Longrightarrow} x_1, \dots, x_n$$

a local coord system on some nbhd V of $P \in N$; by assumption the restrictions of $x_1, \dots, x_m$ form a local coord system on $U = V \cap M$ and U is given by eqn=s

$$x_{m+1} = \dots = x_n = 0$$

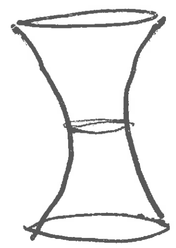
Q10 $c = 0$:



we have an immersed copy of two paraboloids identified at origin

For $c < 0$, it is the embedded copy of two paraboloids.

For $c > 0$, it is an embedded surface of revolution of the smooth curve $z^2 = x^4 - c$ in the (x, z) plane



Q11 We have induced maps

$$\begin{tikzcd} S^2 \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & \mathbb{R}^6 \\ & \mathbb{RP}^2 \arrow[ur, "\rho"'] & \end{tikzcd}$$

given by $\sigma(x, y, z) =$

$$\begin{array}{cccccc} (x^2, & y^2, & z^2, & xy, & yz, & zx) \\ X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \end{array}$$

since $\sigma(-x, -y, -z) = \sigma(x, y, z)$

Need to check that σ (and hence ρ) is an immersion and that ρ is a homeomorphism onto its image.

Consider chart on S^2 given by $z > 0$; we have x, y local coords and $z = \sqrt{1 - x^2 - y^2}$

The map σ is given in these coordinates by

$$(x^2, y^2, 1-x^2-y^2, xy, y\sqrt{1-x^2-y^2}, x\sqrt{1-x^2-y^2})$$

Then

$$\partial\sigma/\partial x = \left(2x, 0, -2x, y, \frac{-xy}{\sqrt{1-x^2-y^2}}, \sqrt{1-x^2-y^2} - \frac{x^2}{\sqrt{1-x^2-y^2}}\right)$$

$$\partial\sigma/\partial y = \left(0, 2y, -2y, x, \sqrt{1-x^2-y^2} - \frac{y^2}{\sqrt{1-x^2-y^2}}, \frac{-xy}{\sqrt{1-x^2-y^2}}\right)$$

are by inspection indep $\forall\ (x,y)$ with $x^2+y^2<1$

$\therefore$ σ a local immersion on S^2 and ρ on $\mathbb{R}P^2$.

However we can define the inverse map on the image by

$$\frac{\pm 1}{\sqrt{X_1^2+X_4^2+X_6^2}}(X_1, X_4, X_6) \quad \text{for } x \neq 0$$

$$\frac{\pm 1}{\sqrt{X_4^2+X_2^2+X_5^2}}(X_4, X_2, X_5) \quad \text{for } y \neq 0$$

$$\frac{\pm 1}{\sqrt{X_6^2+X_5^2+X_3^2}}(X_6, X_5, X_3) \quad \text{for } z \neq 0$$

— these are only locally defined maps to S^2 and involve choices of signs but are well-defined inverses to ρ with image $\mathbb{R}P^2$.

The required embedding in $\mathbb{R}^4$ will be given by

$(x, y, z) \longmapsto (x^2, y^2, y(x+z), x(y+z))$,

an immersion by looking at charts e.g. (x, y) on

$\{z>0\} \cap S^2$, etc

Want to prove this is now a homeo

i.e. reconstruct (x, y, z) up to sign (will

$x^2+y^2+z^2=1$)

From the first part, sufficient to reconstruct

z^2, xy, yz, zx from $x^2, y^2, y(x+z), x(y+z)$.

But $z^2 = 1 - x^2 - y^2$, $z(x-y) = x(y+z) - y(x+z)$

$$\Rightarrow \quad xz = \left\{ \frac{(y(x+z))^2}{y^2} + y^2 - 1 \right\} / 2$$

(works so long $y \neq 0$)

$\rightsquigarrow$ xy & yz too

Similarly $$yz = \left\{ \frac{(x(y+z))^2}{x^2} + x^2 - 1 \right\} / 2$$

(works so long $x \neq 0$)

& then get xy & zx

Similarly $$xy = -\left\{ \frac{(z(x-y))^2}{z^2} + z^2 - 1 \right\} / 2$$

(works so long $z \neq 0$)

& then get yz & zx.

Q12 (i) Set $f : GL(n, \mathbb{R}) \xrightarrow[\det]{} \mathbb{R}^*$,

so that $SL(n, \mathbb{R})$ is fibre $f^{-1}(1)$.
If $A \in SL(n, \mathbb{R})$ then $f \circ R_A = f$ $\quad$ R_A = right mult^n by A

$\Rightarrow d_A f \circ R_A = d_I f$. Identifying $T_{\mathbb{R}^*, 1} = \mathbb{R}\, \partial/\partial r$ with $\mathbb{R}$, $d_I f : M_{n \times n}(\mathbb{R}) \to \mathbb{R}$

given by $H \longmapsto \frac{d}{dt}\Big|_0 \det(I + tH) = \operatorname{Tr} H$

In particular $d_I f$ & hence $d_A f$ for all $A \in SL(n, \mathbb{R})$ surjective.
Thm from lectures $\Rightarrow SL(n, \mathbb{R}) = f^{-1}(1)$ a manifold. Lie algebra $T_I\, SL(n, \mathbb{R})$ is identified as the matrices H with $\operatorname{Tr}(H) = 0$

(ii) Take $f : GL(n, \mathbb{C}) \longrightarrow$ Hermitian matrices $\mathbb{C}^{n(n-1)/2} \oplus \mathbb{R}^n = \mathbb{R}^{n^2}$

given by $f(A) = AA^*$. Since
$f(I + tH) = I + t(H + H^*) + t^2 HH^*$
& general hermitian matrix is of form $H + H^*$, df_I is surjective & the previous argument shows that df_A surjective $\forall A \in U(n)$ $[f \circ R_A = f]$
So $U(n)$ a manifold: Lie algebra = skew-hermitian matrices

(iii) Similar. Lie algebra = skew hermitian & trace free matrices.

<u>Q13</u> Recall if $P \in M \overset{\text{submfld}}{\subset} N$ & $f : N \to \mathbb{R}$ smooth then $(d_P f)|_{T_P M} = d_P(f|_M)$. (†)

Know that $O(n) \subset GL(n, \mathbb{R}) \subset M_{n\times n}(\mathbb{R})$ is a submanifold

Tgt space at I to $GL(n, \mathbb{R}) \longleftrightarrow M_{n\times n}(\mathbb{R})$

$$H \longleftrightarrow \sum H_{pq} \partial/\partial x_{pq}$$

Tgt space at I to $O(n) \longleftrightarrow$ antisymmetric matrices

For any local chart $\phi : U \to \mathbb{R}^{n(n-1)/2}$ on a nbhd of $I \in O(n)$, have $\phi^{-1} : \phi(U) \to U$ is locally a smooth map inducing an IM from $\mathbb{R}^{n(n-1)/2}$ to tgt space to $O(n)$ at I (tautology)

Choose now map f given locally on $GL(n, \mathbb{R})$ by $A \mapsto \frac{1}{2}(A - A^t)$. This induces

$$d_I f : M_{n\times n}(\mathbb{R}) \to \text{antisymmetric matrices}$$
$$H \longmapsto \tfrac{1}{2}(H - H^t)$$

Setting $\psi = f|_{O(n)}$, (†) $\Rightarrow d_I \psi = $ identity (once canonical identifications are made). Thus

the map $\psi \circ \phi^{-1} = f \circ \phi^{-1}$ is a smooth map inducing an IM on tgt spaces (i.e. derivative is an IM) $\underset{IFT}{\Longrightarrow}$ it is a local diffeo $\Rightarrow$ α locally a chart.

$\exists$ standard map from the unit quaternions $\Theta : Sp(1) \to SO(3)$ via $q \mapsto \Theta_q$, where Θ_q acts on the imaginary quaternions ($\cong \mathbb{R}^3$) by $\Theta_q(x) = q x q^{-1} = q x q^*$. Note that $\|\Theta_q(x)\| = \|x\|$, $Sp(1) = S^3$ is connected $\Rightarrow$ given map is to $SO(3)$ (& $\Theta_1 = I_3$).

Mechanical check verifies that $\mathrm{Ker}(Sp(1) \to SO(3)) = \{\pm 1\}$ and that Θ is a 2-1 HM, $\Theta_{-q} = \Theta_q$.

Clearly Θ given by polys so is a smooth map.

The derivative of Θ is an IM on tangent spaces — the tangent space to $Sp(1)$ at 1 may be seen to be quaternions $h_1 i + h_2 j + h_3 k \Leftrightarrow (h_1, h_2, h_3) \in \mathbb{R}^3$ and $d\Theta_1 : \mathbb{R}^3 \to \mathbb{R}^3$ then checked to be

$$(h_1, h_2, h_3) \mapsto \begin{pmatrix} 0 & -2h_3 & 2h_2 \\ 2h_3 & 0 & -2h_1 \\ -2h_2 & 2h_1 & 0 \end{pmatrix} \text{ an IM}$$

$\Rightarrow$ Θ locally an IM & hence we have a diffeo $\mathbb{R}P^3 = S^3/\{\pm 1\} \xrightarrow{\sim} SO(3)$. So $\mathbb{R}P^3$ is diffeo to a Lie group & so trivial tgt bundle.