Algebraic Geometry 2017 — Example Sheet 3, Answers.

1) Have

$$\begin{array}{ccccccccc}
0 & \to & U_0 & \to & V_0 & \to & W_0 & \to & 0 \\
 & & \downarrow \alpha_0 & & \downarrow \beta_0 & & \downarrow \gamma_0 & & \\
0 & \to & \operatorname{Ker}\alpha_1 & \to & \operatorname{Ker}\beta_1 & \to & \operatorname{Ker}\gamma_1 & &
\end{array}$$

& the standard Snake Lemma (diagram chase) $\Rightarrow$

$$0 \to H^0(U_*) \to H^0(V_*) \to H^0(W_*) \to H^1(U_*) \to H^1(V_*) \to H^1(W_*)$$

exact.

But

$$\begin{array}{ccccccccc}
 & & U_1/\operatorname{Im}\alpha_0 & \to & V_1/\operatorname{Im}\beta_0 & \to & W_1/\operatorname{Im}\gamma_0 & \to & 0 \\
 & & \downarrow \alpha_1 & & \downarrow \beta_1 & & \downarrow \gamma_1 & & \\
0 & \to & \operatorname{Ker}\alpha_2 & \to & \operatorname{Ker}\beta_2 & \to & \operatorname{Ker}\gamma_2 & &
\end{array}$$

$$\Rightarrow \quad H^1(U_*) \to H^1(V_*) \to H^1(W_*) \to H^2(U_*) \to H^2(V_*) \to H^2(W_*)$$

exact

etc. (For lectures, need this result in category Abgp.)

2) For all $x \in X$, the sequence on stalks

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \quad \text{is exact.}$$

From definition of $\mathcal{D}$, its then clear that the sequences

$D(\mathcal{F})(U) \to D(\mathcal{G})(U) \to D(\mathcal{H})(U)$ are exact for all U open $\subseteq X$ (recalling that $D(\mathcal{F})(U) = \{ s : U \to \coprod_{P \in U} \mathcal{F}_P$ s.t. $s(P) \in \mathcal{F}_P \ \forall P \}$)

The reason for setting the question is that this is a case where just moving exactness on sections is the easiest option – the stalks of $D(\mathcal{F})$ are quite complicated.

3) In the manifold case, when $\mathcal{U}$ might be infinite, we'd better extend our definition of cochains $\check{C}^n(\mathcal{U}, \mathcal{F})$ to be the alternating elts of $\bigoplus_{i_0, \dots, i_n \text{ distinct}} \Gamma(U_{i_0 \dots i_n}, \mathcal{F})$.

Letting $\check{Z}^1(\mathcal{U}, \mathcal{O}_X^*) = \operatorname{Ker}(\check{C}^1(\mathcal{U}, \mathcal{O}_X^*) \xrightarrow{\delta_1} \check{C}^2(\mathcal{U}, \mathcal{O}_X^*))$

we can define a HM of multiplicative groups

$\check{Z}^1(\mathcal{U}, \mathcal{O}_X^*) \xrightarrow{\Theta} (\operatorname{Pic} X)_{\mathcal{U}}$, the subgp of $\operatorname{Pic} X$ consisting of IM classes which are trivialised wrt $\mathcal{U}$, via

$(g_{\alpha\beta}) \longmapsto$ IM class determined via Prop 3.1 & Ex Sht 2, Q11 by transition functions $g_{\alpha\beta}$, with $g_{\alpha\alpha} := 1 \ \forall \alpha$

(if $\mathcal{U}$ finite & cochain given by g_{ij} with $i<j$, also need to specify that $g_{ji} = g_{ij}^{-1}$ to define the IM class). Clearly Θ is surjective. Moreover, if $g_{\alpha\beta} = f_\alpha / f_\beta$ is a Čech coboundary ($f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X^*) \ \forall \alpha$), the corresponding invertible sheaf is trivial – more generally if

a trivialization $\mathcal{L}|_{U_\alpha} = \mathcal{O}_{U_\alpha}\sigma_\alpha$ has transition functions $h_{\alpha\beta} \in \mathcal{O}_X^*(U_{\alpha\beta})$, and if $f_\alpha \in \mathcal{O}_X^*(U_\alpha)$ $\forall \alpha$, then the trivialization $\mathcal{L}|_{U_\alpha} = \mathcal{O}_{U_\alpha} f_\alpha \sigma_\alpha$ has transition functions $f_\beta/f_\alpha\, h_{\alpha\beta}$ (since $\sigma_\beta = h_{\alpha\beta}\sigma_\alpha$ $\Rightarrow f_\beta\sigma_\beta = (h_{\alpha\beta} f_\beta/f_\alpha)(f_\alpha\sigma_\alpha)$). So its clear that $g_{\alpha\beta} = f_\alpha/f_\beta$ maps to trivial bundle. Conversely, if $(g_{\alpha\beta})$ maps to trivial bundle, there is a global generator of line bundle (transition functions all 1), & hence any trivialization wrt $\mathcal{U}$ has transition functions corresponding to a coboundary.

4) Interpreting $\mathcal{K}^*/\mathcal{O}_V^*$ in terms of $(\mathcal{K}^*/\mathcal{O}_V^*)^+$, we see that a section $\Gamma(\mathcal{K}^*/\mathcal{O}_V^*)$ is given by an open cover $\mathcal{U} = \{U_\alpha\}$ (wlog finite) and elts $h_\alpha \in \mathcal{K}^*(U_\alpha) = k(V)^*$ (mod $\mathcal{O}_V^*(U_\alpha)$) which agree (mod $\mathcal{O}_V^*(U_{\alpha\beta})$) on the overlaps $U_\alpha \cap U_\beta$. Thus $h_\alpha/h_\beta = g_{\beta\alpha} \in \mathcal{O}_V^*(U_{\alpha\beta})$ & compatibility cond$^{=}$s clearly satisfied, and hence this info determines an LM class in $\mathrm{Pic}(V)$ as in Q3. Note that the h_α may be replaced by $h'_\alpha = h_\alpha f_\alpha$ for any $f_\alpha \in \mathcal{O}_V^*(U_\alpha)$, but then $h'_\alpha/h'_\beta = f_\alpha/f_\beta\, g_{\beta\alpha}$ which defines the same elt of $\mathrm{Pic}(V)$. Compare this with construction of

$\mathcal{O}_V(W)$ in lectures (W a codim 1, locally principal subvariety of V, locally given by $h_\alpha \in \mathcal{O}_V(U_\alpha) \subset k(V)^*$ modulo elts of $\mathcal{O}_V^*(U_\alpha)$) — the elements of $H^0(\mathcal{K}^*/\mathcal{O}_V^*)$ are known as <u>Cartier divisors</u>, which can be thought of as generalizations of locally principal subvarieties. The image of $H^0(\mathcal{K}^*)$ in $H^0(\mathcal{K}^*/\mathcal{O}_V^*)$ is determined by a rational function $h \in k(V)^*$, which then clearly determines the 1M class of the trivial invertible sheaf with $g_{\beta\alpha} = 1 \ \forall \alpha, \beta$. Conversely, if $g_{\beta\alpha} = h_\alpha/h_\beta$ determines the trivial invertible sheaf in $\mathrm{Pic}(V)_{\mathcal{U}}$, we saw in Q3 that $g_{\beta\alpha} = f_\beta/f_\alpha$ for suitable $f_\alpha \in \mathcal{O}_V^*(U_\alpha)$.
So setting $h'_\alpha = f_\alpha h_\alpha$, $h'_\beta = f_\beta h_\beta$, have

$h'_\beta/h'_\alpha = g_{\beta\alpha} f_\alpha/f_\beta = 1 \quad \forall \alpha, \beta$ & the only way this can occur is for $h'_\alpha = h \in k(V)^* \quad \forall \alpha$
ie. the line bundle comes from an elt of $\mathrm{Im}(H^0(\mathcal{K}^*))$.
Thus we have an injective HM $H^0(\mathcal{K}^*/\mathcal{O}_V^*)/\mathrm{Im}\, H^0(\mathcal{K}^*)$

$$\hookrightarrow \mathrm{Pic}(X)$$

This is surjective, since an elt of $\mathrm{Pic}(V)$ is in $\mathrm{Pic}(V)_{\mathcal{U}}$ for some $\mathcal{U} = \{U_\alpha\}$, whose 1M class is determined by transition functions $g_{\alpha\beta} \in \mathcal{O}_X^*(U_{\alpha\beta}) \subset k(V)^*$
— recall that the variety V is assumed irreducible

Choose $\alpha_0 \in A$ & set $h_{\alpha_0} = 1$, & then set $h_\beta = g_{\alpha_0 \beta}$ $\forall \beta \in A$. Thus

$$h_\alpha / h_\beta = g_{\alpha_0 \alpha} / g_{\alpha_0 \beta} = 1/g_{\alpha\beta} = g_{\beta\alpha} \quad \forall \alpha, \beta$$

as required.

Finally, the constant sheaf $\mathcal{K}^*$ is flabby & so $H^1(\mathcal{K}^*) = 0$ (using V irred) $\Rightarrow$ $\operatorname{Pic} V \cong H^1(\mathcal{O}_V^*)$ from l.e.s.

5) Given $\phi = (f, f^\#)$ with $f^{-1}(U_i) \to U_i$ homeo $\forall i$, clear topologically that f is a homeo.
Since $f^{-1}U_i \to U_i$ an IM of varieties $\forall i$, $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ yields an IM on stalks $\forall P \in Y$ $\Rightarrow f^\#$ an IM of sheaves of k-algebras. So $\exists$ inverse morphism $\phi^{-1} = (g, g^\#)$ where $g = f^{-1}$ & $g^\# : \mathcal{O}_X \to g_* \mathcal{O}_Y$ given by setting $\mathcal{O}_X(W) \to \mathcal{O}_Y(g^{-1}W)$ for W open $\subset X$ be the inverse of IM $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$, where $U = g^{-1}W$

Given $Y \subset \mathbb{A}^m$ & a k-algebra HM $\alpha : B \to A$, can let $h_i = \alpha(y_i) \in A$ for $i = 1, \ldots, m$, $y_i =$ ith coordinate function on Y. Define regular map $\phi : X \to \mathbb{A}^m$ by $\phi = (h_1, \ldots, h_m)$. If $F \in I(Y)$, then $F(y_1, \ldots, y_m) = 0$ in $B \Rightarrow F(h_1, \ldots, h_m) = 0$ in A i.e. $\phi : X \to Y$.
The corresponding morphism of sheaves $\phi^\# = \phi^*$ induces $\phi^* : B \to A$ with $\phi^*(y_i) = y_i \circ \phi = h_i = \alpha(y_i)$ $\forall i$ i.e. $\phi^* = \alpha$.

If we start with $\phi : X \to Y$, this induces a k-algebra HM $B \to A$ on global sections & above recipe recovers ϕ.

Wlog may assume that $U_i = D(f_i)$, $f_i \in k[Y]$ generating unit ideal in B, & hence in $A = \mathcal{O}_X(X)$. Identifying f_i with its image in A, we have an open affine cover of X by sets $\phi^{-1}U_i = X_{f_i} := \{x \in A : f_i(x) \neq 0\}$ (for simplicity miss out i for which $\phi^{-1}U_i = \emptyset$ i.e. f_i has image 0 in A & so have cover by X_{f_i} for $1 \le i \le r$ say.)

By assumption the X_{f_i} are affine & Ex Sht 2, Q1
$\Rightarrow k[X_{f_i}] = A_{f_i}$, $i = 1, \dots, r$. Write

$k[X_{f_i}] = B[g_{i1}/f_i^{n_1}, \dots, g_{is(i)}/f_i^{n_{s(i)}}]$ &

$C = B[g_{i1}, \dots, g_{is(i)} : 1 \le i \le r]$. Clearly $C \subset A$ is reduced & f.g. over k $\Rightarrow \exists$ affine variety Z with $k[Z] \cong C$.

Thus $B \to C \hookrightarrow A$ induces morphisms of varieties

$X \xrightarrow{g} Z \to Y$, where $X_{f_i} = g^{-1}Z_{f_i}$ $\forall i$

& g restricts to morphisms $X_{f_i} \to Z_{f_i}$. But $C_{f_i} \xrightarrow{\sim} A_{f_i}$ $\forall i$

$\Rightarrow$ morphisms $X_{f_i} \to Z_{f_i}$ are IMs of affine varieties $\forall i$

& so first part of the question $\Rightarrow$ g an IM and

$X \cong Z$ is affine

6) Consult Reid [R], §6 or my Part II notes [W] from the web. Need to use Nullstellensatz for the fact that codim 1 $\Rightarrow$ hypersurface (for both affine & projective cases)

7) An invertible sheaf corresponds to a "Cartier divisor" ie. a section of $H^0(\mathcal{K}^*/\mathcal{O}_V^*)$ as described in Q4. ie by data $\mathbb{P}^n = \bigcup_\alpha U_\alpha$ & rational functions $h_\alpha \in k(\mathbb{P}^n)$ (modulo $\mathcal{O}_{\mathbb{P}^n}^*(U_\alpha)$). The rational function h_α may be written (uniquely) as F_α / G_α, F_α, G_α coprime homogeneous polys of the same degree. For a given α, we consider the irred components V_i of $\{F_\alpha = 0\}$ which intersect U_α and assign multiplicities a_i according to the multiplicity of the corresponding factor in F_α, and similarly with the irred components W_j of $\{G_\alpha = 0\}$ with multiplicities b_j. Thus $h_\alpha = 0$ on U_α determines a formal sum. (a "divisor") of the form $\sum a_i V_i - \sum b_j W_j$ with V_i & W_j corresponding to hypersurfaces in $\mathbb{P}^n$. Considering h_β on U_β may give a different divisor $\sum a_i' V_i' - \sum b_j W_j'$, but there will be

consistent on the overlap $U_\alpha \cap U_\beta$. Doing this for all the U_α may add some V_i & W_j (not intersecting original piece), and in this way we obtain a divisor $D = \sum_i a_i V_i - \sum_j b_j W_j$ with V_i, W_j irred hypersurfaces in $\mathbb{P}^n$, where $D|_{U_\alpha}$ defined by h_α (essentially "zeros - poles" with multiplicities).

We now define $\mathcal{O}_{\mathbb{P}^n}(D)$ by analogy with $\mathcal{O}_{\mathbb{P}^n}(V)$ or $\mathcal{O}_{\mathbb{P}^n}(-V)$ when V is an irred hypersurface, namely

$$\Gamma(U, \mathcal{O}_{\mathbb{P}^n}(D)) = \{ g \in k(\mathbb{P}^n) \text{ s.t. } h_\alpha g \in \mathcal{O}_{\mathbb{P}^n}(U \cap U_\alpha) \ \forall \alpha \}$$

i.e. $\mathcal{O}_{\mathbb{P}^n}(D)|_{U_\alpha} = {}^1\!/_{h_\alpha} \mathcal{O}_{U_\alpha}$ — i.e. this is the invertible sheaf whose transition functions correspond to those of the given elt $\mathcal{L} \in \operatorname{Pic} \mathbb{P}^n$ we started with.

Wlog we assume that no V_i or W_j is a coordinate hyperplane $X_i = 0$; then on the affine pieces $U_0, \dots, U_n$ of $\mathbb{P}^n$, the divisor on U_i is defined by a rational function $\frac{F}{G X_i^m} = h_i$, with F homogeneous of degree $\sum a_i \deg V_i$, G homogeneous of degree $\sum b_j \deg W_j$ & $m = \deg F - \deg G$. So wrt cover $U_0, \dots, U_n$, $\mathcal{O}_{\mathbb{P}^n}(D)|_{U_i} = {}^1\!/_{h_i} \mathcal{O}_{U_i} \Rightarrow$ transition functions $\phi_{ji} = (X_i/X_j)^m$ i.e. $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^n}(m)$.

(8) The result is true more generally, but for varieties we commented in lectures that direct sums of sheaves may be defined singlemindedly, since the presheaf direct sum defined in terms of direct sums of sections is a sheaf (any open subset $U \subset X$ is compact, so glueing cond= (B) cannot introduce infinitely many non-zero entries). With this comment, the question is easy: take flabby resolutions $\mathcal{F}_i \to \mathcal{F}_i^*$ for each i & then observe that

$\bigoplus_i \mathcal{F}_i \to \bigoplus_i \mathcal{F}_i^*$ gives a flabby resolution of $\bigoplus_i \mathcal{F}_i = \mathcal{F}$

Taking sections and then cohomology of the complex yields $H^r(\mathcal{F}) = \bigoplus_{i \in I} H^r(\mathcal{F}_i)$ as required.

(9) Recall that $D^h(F)$ with F homogeneous form a basis for open subsets of $\mathbb{P}^n$. Then $\pi^{-1}(D^h(F)) = D(F) \subset \mathbb{A}^{n+1}$ is affine, where $D^h(F) \subset \mathbb{P}^n$ affine by Ex Sht 1, Q 12

Thus $\pi : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$ is an affine map – cf Q 5

<u>CLAIM</u> $\pi_* \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}} \cong \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)$.

Proof $\pi_* \mathcal{O}_{\mathbb{A}^{n+1}\setminus\{0\}}(D^h(F)) = \mathcal{O}_{\mathbb{A}^{n+1}\setminus\{0\}}(D(F))$

$$= \left\{ \frac{G}{F^l} \; ; \; l \geq 0 \, , \; G \in k[X_0, \dots, X_n] \right\}$$

$$\cong \bigoplus_{d \in \mathbb{Z}} A_d \, , \quad A_d = \left\{ \frac{G}{F^l} \; ; \; l \geq 0 \ \& \ G \text{ homogeneous with } \deg G - l \deg F = d \right\}$$

Note that under this map, $\mathcal{O}_{\mathbb{A}^{n+1}\setminus\{0\}}(D(F))$ really is IM to the <u>sum</u> of the A_d, since any elt has non-zero image in A_d for only finitely many $d \in \mathbb{Z}$

Now $A_d \cong \Gamma(D^h(F), \mathcal{O}_{\mathbb{P}^n}(d)) =$

$$\left\{ \frac{G}{F^l X_0^d} \in k(\mathbb{P}^n) \right\}$$

via $G/F^l \longmapsto G/F^l X_0^d$.

Use of Coroll 1.3 yields required IM. of sheaves on $\mathbb{P}^n$

10) G acts on $\mathcal{F}$ means that for all open $U \subset X$, it acts on sections $\mathcal{F}(U)$, and these actions are consistent with restrictions:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{g} & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{g} & \mathcal{F}(V) \end{array} \quad \text{commutes}$$

If we take our construction of cohomology via the flabby resolution $0 \to \mathcal{F} \to D^* \mathcal{F}$, the induced action on the stalks $\mathcal{F}_x$ for $x \in X$ of G yields an action of G on $D^0 \mathcal{F}$ with

$$\begin{array}{ccccc} 0 & \to & \mathcal{F} & \longrightarrow & D^0 \mathcal{F} \\ & & \downarrow g & & \downarrow g \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & D^0 \mathcal{F} \end{array} \qquad \text{commuting}$$

Iterating this argument, we get an action of G on the whole resolution $0 \to \mathcal{F} \to D^* \mathcal{F}$, with action commuting with all morphisms $D^i \mathcal{F} \to D^{i+1} \mathcal{F}$.
This then yields an action of G on the cohomology groups (obtained by taking sections of resolution $D^* \mathcal{F}$ and cohomology of the resulting complex).

If $G = k^*$ acts on $\mathcal{F}$ by $\lambda \mapsto \theta_\lambda$ where $\theta_\lambda(s) = \lambda^r s$ for $s \in \mathcal{F}(U)$, then the action on all stalks is the same, and hence the action on sections of $D^0(\mathcal{F})$ given by same formula. Thus have induced action on whole complex given by $\theta_\lambda(s) = \lambda^r s$ for $s \in D^i \mathcal{F}(U)$. Passing to cohomology, k^* acts on the sheaf cohomology in the same way.

11) We now let k^* act on $\mathbb{A}^{n+1} \setminus \{0\}$ in obvious way, and hence on $\pi_* \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}}$. The factor in the sum found in Q 9 then just corresponds to the part on which $\lambda \in k^*$ acts by mult^n by λ^d.

However, examination of the proof of (4.6) shows that the IMs

$$H^r(\mathbb{A}^{n+1} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}}) \xrightarrow{\sim} H^r(\mathbb{P}^n, \pi_* \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}})$$

respect the action of $G = k^*$, since $H^r(\mathbb{P}^n, \pi_* \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}})$ was obtained by taking π_* of the standard flabby resolution of $\mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}}$ on $\mathbb{A}^{n+1} \setminus \{0\}$.

The the part of $H^r(\mathbb{A}^{n+1} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}})$ on which $\lambda \in k^*$ acts by mult^n by λ^d, denoted $H^r(\mathbb{A}^{n+1} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}})_{\deg d}$, corresponds to the part of $H^r(\mathbb{P}^n, \pi_* \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}}) =$

$$H^r(\mathbb{P}^n, \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)) \overset{Q8}{=} \bigoplus_{d \in \mathbb{Z}} H^r(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$$

on which k^* acts in this way, namely (by Q 10) precisely $H^r(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d))$. Thus (4.8) follows from description of $H^r(\mathbb{A}^{n+1} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^{n+1} \setminus \{0\}})$ & action of k^* on it.

12) Have open cover $\mathcal{U}_1, \ldots, \mathcal{U}_d$ with $\mathcal{U}_l = X$ for some l

Have $\delta_n : \check{C}^n(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^{n+1}(\mathcal{U}, \mathcal{F})$ where $\delta_n(\alpha) = \beta$,

with $$\beta_{i_0, \ldots, i_{n+1}} = \sum_{0 \le j \le n+1} (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_{n+1}} \Big|_{\mathcal{U}_{i_0 \ldots i_{n+1}}}.$$

& $k_n : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^{n-1}(\mathcal{U}, \mathcal{F})$, where $k_n(\alpha) = \beta$

with $$\beta_{i_0 \ldots i_{n-1}} = \begin{cases} (-1)^k \alpha_{i_0 \ldots l \ldots i_{n-1}} & \text{if } i_{k-1} < l < i_k \\ 0 & \text{otherwise} \end{cases}$$

this includes cases $k=0$ ie. $l < i_0$ & $k = n$ ie. $l > i_{n-1}$

Suppose now we consider $i_0, \ldots, i_n$

Case 1 $\quad i_{k-1} < l < i_k$, then

$$(k_{n+1} \delta_n \alpha)_{i_0 \ldots i_n} = (-1)^k (\delta_n \alpha)_{i_0 \ldots l \ldots i_n}$$

$$= \sum_{j<k} (-1)^{j+k} \alpha_{i_0 \ldots \hat{i}_j \ldots l \ldots i_n} + \alpha_{i_0 \ldots i_n}$$

$$+ \sum_{j \geq k} (-1)^{j+k+1} \alpha_{i_0 \ldots l \ldots \hat{i}_j \ldots i_n}$$

& $$(\delta_{n-1} k_n \alpha)_{i_0 \ldots i_n} = \sum (-1)^j (k_n \alpha)_{i_0 \ldots \hat{i}_j \ldots i_n}$$

$$= \sum_{k \le j} (-1)^{j+k} \alpha_{i_0 \ldots l \ldots \hat{i}_j \ldots i_n} + \sum_{k > j} (-1)^{j+k+1} \alpha_{i_0 \ldots \hat{i}_j \ldots l \ldots i_n}$$

Summing these two expressions just yields $\alpha_{i_0 \ldots i_n}$.

Case 2 $\quad l = i_j$ for some j. Then

$(k_{n+1}\delta_n \alpha)_{i_0 \cdots i_n} = 0$ by definition of k_{n+1} &

$(\delta_{n-1} k_n \alpha)_{i_0 \cdots i_n} = (-1)^{2j} \alpha_{i_0 \cdots i_n} = \alpha_{i_0 \cdots i_n}$

So in both cases $((k_{n+1}\delta_n + \delta_{n+1} k_n)\alpha)_{i_0 \cdots i_n} = \alpha_{i_0 \cdots i_n}$

$\Rightarrow \quad k_{n+1}\delta_n + \delta_{n-1}k_n = id_{\check{C}^n(\mathcal{U},\mathcal{F})}$

(13) Case $f : X \to Y$ with both X & Y affine, say $k[X] = A$, $k[Y] = B$. If $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, then $\mathcal{F} = \widetilde{M}$ for some A module M, and then regarding M as a B-module ${}_BM$ via the HM $f^* : B \to A$, we've seen that $f_* \widetilde{M} = ({}_BM)^\sim$ & hence is quasi-coherent.

Assume now that X is arbitrary & Y is affine.

Note that $U_i \cap U_j \cong \Delta_X \cap (U_i \times U_j)$ is IM to a closed subvariety of the affine variety $U_i \times U_j$ (Δ_X closed) $\Rightarrow U_i \cap U_j$ affine for all i, j.

Thus $f_*(\mathcal{F}|_{U_i})$ quasi-coherent $\forall i$ & likewise $f_*(\mathcal{F}|_{U_i \cap U_j})$ quasi-coherent $\forall i,j$, from first part of question.

Define a morphism between quasi-coherent sheaves

$$\bigoplus_i f_*(\mathcal{F}|_{U_i}) \xrightarrow{\theta} \bigoplus_{i,j} f_*(\mathcal{F}|_{U_i \cap U_j}) \quad \text{by}$$

$$\theta(\sigma_i)_{i \in I} = \pm(\sigma_i|_{U_i \cap U_j} - \sigma_j|_{U_i \cap U_j})_{i,j \in I}$$

(with + sign if $i<j$ and − sign if $i>j$).

For V open in Y, $\operatorname{Ker}\theta(V)$ consists of families $\sigma_i \in \mathcal{F}(f^{-1}V \cap U_i)$ for which

$$\sigma_i|_{f^{-1}V \cap U_i \cap U_j} = \sigma_j|_{f^{-1}V \cap U_i \cap U_j} \qquad \forall i,j$$

ie by the sheaf cond=s coming from a unique $\sigma \in \mathcal{F}(f^{-1}V)$
$= f_*\mathcal{F}(V)$

$\therefore$ $\operatorname{Ker}\theta \cong f_*\mathcal{F}$ is quasi-coherent using Thm 3.4 & Coroll 3.5 — cf. proof of (4.5)

Finally, since quasi-coherence is a local property, result holds for morphisms of arbitrary varieties.

14) Note first that $\mathcal{O}_{\mathbb{P}^N}(m)|_{U_i} \cong \mathcal{O}_{U_i}$ for the standard affine pieces $U_0, \dots, U_N$ of $\mathbb{P}^N$ via

$$f(X_0/X_i, \ldots, X_N/X_i) \big/ (X_0/X_i)^m \longmapsto f(X_0/X_i, \ldots, X_N/X_i)$$

and restriction to $V \subset \mathbb{P}^N$ gives IM $\mathcal{O}_V(m)|_{U_i \cap V} \cong \mathcal{O}_{U_i \cap V}$.

We just give the proof for $V = \mathbb{P}^N$ & restricting everything to V gives result on V. (alter:

$i_*(\mathcal{F}(m)) = (i_*\mathcal{F})(m)$, $\mathcal{F}$ a quasi-coherent sheaf on $\mathbb{P}^N$, using Ex Sht 4, Q1).

So $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^N}(m)|_{U_i} \longleftarrow \mathcal{F}|_{U_i}$

$$\|$$

$\mathcal{F}(m)|_{U_i}$

via $\quad s_i \otimes (X_i/X_0)^m \longleftarrow\!\shortmid\, s_i \in \mathcal{F}|_{U_i}(U_i)$

These patch together to give a section of $\mathcal{F}(m)$

$$\Longleftrightarrow \quad s_i \otimes (X_i/X_0)^m |_{U_i \cap U_j} = s_j \otimes (X_j/X_0)^m |_{U_i \cap U_j}$$

$$\|$$

$$s_i \cdot (X_i/X_j)^m \otimes (X_j/X_0)^m$$

$$\Longleftrightarrow \quad s_j = (X_i/X_j)^m s_i \quad \text{in } \mathcal{F}(U_i \cap U_j) \quad \forall i,j$$