Algebraic Geometry 2017 Example Sheet 2 Answers

1) Easy that $A_f \subseteq \mathcal{O}_X(X_f)$. Suppose now $g \in \mathcal{O}_X(X_f)$. Given an affine cover $U_1, \ldots, U_p$ of X, have $g \in k[U_i]_f$ $\forall i \Rightarrow \exists\, n$ (wlog indep of i) s.t. $f^n g \in k[U_i]$ $\forall i \Rightarrow f^n g \in A = \mathcal{O}_X(X) \Rightarrow g \in A_f$.

2) Work in the C^∞ case — complex case entirely analogous. Given a C^∞ manifold X, we can define the structure sheaf (i.e. define a cts function f on $U \subset X$ to be C^∞ in terms of charts — clearly a well-defined concept). The resulting ringed space is clearly locally ringed over $\mathbb{R}$, and if $\phi : U \xrightarrow{\sim} \phi(U) = W \subset \mathbb{R}^n$ a chart, then by definition $\phi^\# : \mathcal{O}_W \longrightarrow \phi_* \mathcal{O}_U$ an IM of sheaves of rings & so $(U, \mathcal{O}_U)$ isomorphic to $(W, \mathcal{O}_W)$ as locally ringed spaces$/\mathbb{R}$.

Now the converse: Given $(X, \mathcal{O}_X)$, we have locally $(U, \mathcal{O}_U) \xrightarrow[(\phi,\, \phi^\#)]{\sim} (W, \mathcal{O}_W)$ for some open W in $\mathbb{R}^n$; an IM of locally ringed spaces$/\mathbb{R}$. Arguing as in Lemma 1.2, we see that $\phi^\# : \mathcal{O}_W \longrightarrow \phi_* \mathcal{O}_U$ is just given by composition with ϕ, $g \longmapsto g \circ \phi = \phi^*(g)$ (if we set $f_j = \phi^\#(x_j)$ & $f = (f_1, \ldots, f_n) : U \rightarrow \mathbb{R}^n$, then $\phi(P) = (b_1, \dots, b_m) \Rightarrow x_j - b_j \in m_{W, \phi(P)}$ $\forall j$

$\Rightarrow \phi^{\#}(x_j - b_j) = f_j - b_j \in m_{U,P} \Rightarrow f_j(P) = b_j$ $\forall j$

$\Rightarrow f(P) = \phi(P)$. Result then follows, since by construction $\phi^{\#}(x_j) = x_j \circ \phi = \phi^*(x_j)$ $\forall j$).

Thus $\mathcal{O}_U$ consists of pullbacks of C^∞ functions from W under the homeomorphism $\phi : U \to W$. Given two charts obtained in this way, say $\phi_1 : U \xrightarrow{\sim} W_1$ & $\phi_2 : U \xrightarrow{\sim} W_2$, we have a transition function (homeo) $\psi : W_1 \to W_2$. The C^∞ functions on W_2 pullback under ψ to C^∞ functions on W_1; in particular this is true for the coordinate functions $y_1, \dots, y_n$ on W_2, which pullback to C^∞ functions $\psi_1, \dots, \psi_n$ on W_1. Since ψ yields an IM of locally ringed spaces, above argument (or Ex Sheet 1, Q13) $\Rightarrow \psi = (\psi_1, \dots, \psi_n)$ & hence ψ is a C^∞ map.

The compatibility conditions for the transition functions obtained in this way are then clear:

e.g.

$$\begin{tikzcd}
& U_\alpha \cap U_\beta \cap U_\gamma \arrow[dl, "\phi_\alpha"'] \arrow[d, "\phi_\beta"] \arrow[dr, "\phi_\gamma"] & \\
W_\alpha \cap \phi_\alpha(U_\beta \cap U_\gamma) \arrow[r, "\psi_{\beta\alpha}"'] & W_\beta \cap \phi_\beta(U_\alpha \cap U_\gamma) \arrow[r, "\psi_{\gamma\beta}"'] & W_\gamma \cap \phi_\gamma(U_\alpha \cap U_\beta)
\end{tikzcd}$$

$\phi_\alpha(U_\alpha \cap U_\beta \cap U_\gamma) = W_\alpha \cap \phi_\alpha(U_\beta \cap U_\gamma)$

being IM of ringed spaces $\Rightarrow \psi_{\gamma\beta} \circ \psi_{\beta\alpha} = \psi_{\gamma\alpha}$ on $\phi_\alpha(U_\alpha \cap U_\beta \cap U_\gamma)$ etc.

3) Basically just a topological argument. Suppose Z_1, Z_2 closed subvarieties with $X \times Y = Z_1 \cup Z_2$; wlog $Z_2 \neq X \times Y$.

RTP $Z_1 = X \times Y$.

For each $x \in X$, note that $\{x\} \times Y$ is homeo to Y

But $\{x\} \times Y = ((\{x\} \times Y) \cap Z_1) \cup ((\{x\} \times Y) \cap Z_2)$

Y irred $\Rightarrow$ either $\{x\} \times Y \subset Z_1$ or $\{x\} \times Y \subset Z_2$

Define $X_i = \{ x \in X : \{x\} \times Y \subset Z_i \}$, $i = 1, 2$

So $X = X_1 \cup X_2$. Since $Z_2 \neq X \times Y$, have $X_2 \neq X$

<u>Claim</u> Both X_1, X_2 closed:

For $y \in Y$, set $X_y := \{ x \in X : (x, y) \in Z_1 \}$

Since $X_y \times \{y\} = (X \times \{y\}) \cap Z_1$, we see X_y is closed in X

But $X_1 = \bigcap_{y \in Y} X_y \Rightarrow X_1$ closed. Similarly X_2 closed.

Thus X irreducible $\Rightarrow X_1 = X$ i.e. $X \times Y = Z_1$.

4) Suppose M a compact manifold, $\pi : M \times N \longrightarrow N$ & $Z \subset M \times N$ closed. If $\pi(Z)$ not closed, $\exists$ sequence $y_i \in \pi(Z)$, $y_i \longrightarrow y \notin \pi(Z)$. For each i, $\exists x_i \in M$ s.t. $(x_i, y_i) \in Z$. Since M compact, may assume wlog $x_i \longrightarrow x \in M$. Then $(x_i, y_i) \longrightarrow (x, y) \in Z \Rightarrow y \in \pi(Z)$ #

<u>Converse</u>: Note M is a metric space; if it is not compact, there is either an unbounded cts function on M, or a bounded cts function which doesn't attain its bounds.

In the first case, say $f: M \to \mathbb{R}$ unbounded above, we can construct a cts bounded function which doesn't attain its bounds e.g. $1/\max(f, 1)$. So can assume that we have a bounded cts function f not attaining bounds. Let $\Gamma = (f, id)^{-1}(\Delta) \subset M \times \mathbb{R}$ be graph of f (Δ = diagonal in $\mathbb{R} \times \mathbb{R}$), a closed subset with $\pi_2(\Gamma) = f(M)$ not closed in $\mathbb{R}$. $\therefore$ M not compact.

5) Since number of monomials in $X_0, \ldots, X_N$ of degree d is $\binom{d+N}{N}$, first part is clear. Now have morphism

$$\mathbb{P}^{\nu_j} \times \mathbb{P}^{\nu_{d-j}} \longrightarrow \mathbb{P}^{\nu_d} \qquad \left(\begin{array}{l}\nu_j + 1 = \text{dim of space of} \\ \text{forms of degree } i\end{array}\right)$$

for $1 \leq j \leq d-1$, corresponding to multiplying a form of degree j and one of degree $d-j$. Easy to see this is a morphism and hence image is closed. Union of finitely many of these corresponds to reducible forms.

6) Chow's Theorem $\Rightarrow$ $\exists$ complex proj variety Y and a birational morphism $\pi: Y \to X$. Since π birational, $\pi(Y)$ contains an open (dense) subset. Since Y proj & hence complete, $\pi(Y)$ is closed & hence π surjective. Since $Y \subset \mathbb{P}^N(\mathbb{C})$ a closed subset, need only show $\mathbb{P}^N(\mathbb{C})$ compact (since then Y compact & π cts in the classical topology $\Rightarrow X$ compact)
However, $\exists$ cts surjection $S^{2N+1} \to \mathbb{P}^N(\mathbb{C})$ & S^{2N+1} compact $\Rightarrow \mathbb{P}^N(\mathbb{C})$ compact as required.

7) Suppose $X = \bigcup_{i=1}^{N} X_i$ is the decomposition of X into its irred components. Recall that if $U \subset X$ dense open then $\mathrm{Rat}(X) \xrightarrow{\sim} \mathrm{Rat}(U)$. Now set

$$U_i = X_i \setminus (X_i \cap \bigcup_{j \neq i} X_j) \quad \& \quad U = \bigcup_{i=1}^{N} U_i$$

Note that $U_i = X \setminus \bigcup_{j \neq i} X_j$ is open in X & dense in X_i
$\Rightarrow$ U open & dense in X. Since the U_i are disjoint by construction, have IMs of k-algebras

$$\mathrm{Rat}(X) \cong \mathrm{Rat}(U) \cong \bigoplus_{i=1}^{N} \mathrm{Rat}(U_i) \cong \bigoplus_{i=1}^{N} k(X_i)$$

Clear now from definition that X birationally equiv to Y $\Rightarrow$ $\mathrm{Rat}(X) \cong \mathrm{Rat}(Y)$ as k-algebras.

We prove the converse first for <u>irreducible</u> varieties. So wlog X, Y affine irred, $k(X) \cong k(Y)$, $Y \subset \mathbb{A}^m$ say. Let $f_i \in k(X)$ correspond to coord functions $y_i \in k(Y)$. Then

$\phi := (f_1, \ldots, f_m) : X \longrightarrow \mathbb{A}^m$ defines a <u>rational</u> map; taking affine piece of X, may assume wlog $f_i \in k[X]$ & so ϕ is a morphism

If now $F \in I(Y) \triangleleft k[Y_1, \dots, Y_m]$, then

$F(y_1, \dots, y_m) = 0$ in $k(Y) \Rightarrow$

$F(f_1, \dots, f_m) = 0$ in $k[X] \Rightarrow$

$F(f_1(P), \dots, f_m(P)) = 0 \quad \forall P \in X$

Since this is true $\forall F \in I(Y)$, have $\phi(P) \in Y \ \forall P \in X$.

So $\phi : X \to Y$ gives rise to $\alpha = \phi^* : k[Y] \hookrightarrow k[X]$, inducing the given IM $k(Y) \xrightarrow{\sim} k(X)$.

Now choose $h \in k[Y]$ s.t. generators x_i of $k[X]$ are in the image of $k[Y]_h$ under α, & hence α induces an IM $\alpha : k[Y]_h \xrightarrow{\sim} k[X]_{\alpha(h)} \Rightarrow$

$\phi : D(\alpha(h)) \xrightarrow{\sim} D(h)$ i.e. X & Y birat equiv.

In general, if $Rat(X) \cong Rat(Y)$ as k-algebras, first part of question $\Rightarrow$ have $X = \bigcup_{i=1}^{N} X_i$, $Y = \bigcup_{i=1}^{N} Y_i$ with $k(X_i) \cong k(Y_i)$ wlog $\forall i$ (check).

Then $\exists$ open dense subsets $U_i \subset X_i$ & $V_i \subset Y_i$ for $i = 1, \dots, N$ which are IM as varieties by the irreducible case. By first part of question, may take the U_i & V_j to be disjoint in X (resp. Y)

Set $U = \bigcup_{i=1}^{N} U_i$ open dense in X & $V = \bigcup_{i=1}^{N} V_i$ open dense in Y. Since $U_i \cong V_i \ \forall i$, we deduce that $U \cong V$, and hence that X & Y are birat. equiv.

8) $X = \bigcup_{i=1}^{N} X_i$ with X_i irred components

By assumption $F_i = (\bigcup_{j \neq i} X_j) \cup Z$ closed in X, $F_i \not\supset X_i$;

Since $F_i \cap X_i \neq X_i$ can choose $P_i \in X \setminus F_i$ & then choose $f_i \in I(F_i)$ st. $f_i(P_i) \neq 0$. So $f = \sum_{i=1}^{N} f_i$ vanishes on Z but not at any of the P_i.

Now recall that $D(g) \subset X$ is dense $\overset{\text{Ex Sht 1, Q3}}{\Longleftrightarrow}$ $A \to A_g$ an injection $\Leftrightarrow$ g a non-zero divisor

Let $T \subset A$ be mult subset of non-zero divisors

Under obvious map $A \to Rat(X)$ of k-algebras, the image of $t \in T$ is invertible in $Rat(X)$ (inverse represented by pair $(D(t), 1/t)$, $D(t)$ dense open as above).

Hence universal property of rings of fractions yields factorization $\phi : T^{-1}A \to Rat(X)$, a HM of k-algebras.

Any elt $h \in Rat(X)$ is represented by a pair (U, g) with U dense open & $g \in \mathcal{O}_X(U)$. By first part, $\exists$ dense open $D(f) \subset U$, f a non-zero divisor

So $g|_{D(f)} \in \mathcal{O}_X(D(f)) = A_f$ of form a/f^r some $r \geq 0$

$\Rightarrow h = \phi(a/f^r)$. So ϕ surjection

If however $\phi(a/s) = \phi(b/t)$, then again by first part $\exists$ open dense $D(f) \subset X$ s.t. $a/s = b/t \in \mathcal{O}_X(D(f)) = A_f$

$\Rightarrow a/s = b/t$ in $T^{-1}A$. $\therefore$ $tot(A) \cong Rat(X)$

~~Finally, it's clear that any elt in $T^{-1}A$ comes from an elt of A_g for some $g \in T$ & hence corresponds to a rational function on X.~~ $\therefore$ $Rat(X) \cong tot(A)$

9) If $k[V]$ generated by $x_1, \ldots, x_n$ and $k[W]$ by $y_1, \ldots, y_m$ as k-algebras, then it's clear that A is generated by elts $x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_m$.
So A a f.g. k-algebra.
P = k-algebra of all functions $V \times W \to k$
$\phi : A \longrightarrow P$ given by $f \otimes g \mapsto \theta$, where $\theta(x, y) = f(x)\, g(y)$, & extended linearly. Note that ϕ is a k-algebra HM. Claim ϕ is <u>injective</u>:
Suppose $\phi\left(\sum_{i=1}^{r} f_i \otimes g_i\right) = 0$; wlog we can assume that $g_1, \ldots, g_r$ are linearly independent. Let $x \in V$; then $\sum_{i=1}^{r} f_i(x)\, g_i(y) = 0 \;\; \forall y \Rightarrow \sum_{i=1}^{r} f_i(x)\, g_i = 0$ in $k[W]$.

But linear independence $\Rightarrow f_i(x) = 0 \;\; \forall i$ & x.
$\Rightarrow f_i = 0$ in $k[V]$. In particular, it follows that A is reduced as it injects into a reduced k-algebra P.

Have morphisms
$$\begin{tikzcd} & V \times W \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V & & W \end{tikzcd}$$
. Universal property says

that if $\phi_1 : U \to V$, $\phi_2 : U \to W$ morphisms of affine varieties, $\exists!$ factorization $\phi : U \longrightarrow V \times W$ where $\phi(x) = (\phi_1(x), \phi_2(x))$ with $\phi_i = \pi_i \circ \phi$ for $i = 1, 2$. (in fact, this generalizes in obvious way to product of arbitrary varieties)

Now let Z be affine variety with coordinate ring $k[V] \otimes_k k[W]$ (f.g. & reduced).

Given $k[V] \xrightarrow{\phi_1^*} k[U]$, $k[W] \xrightarrow{\phi_2^*} k[U]$, $\exists !$ factorization through $k[Z] = k[V] \otimes_k k[W]$ by universal property of tensor product of algebras. Thus, given $\phi_1 : U \to V$, $\phi_2 : U \to W$, $\exists !$ factorization through Z $\Rightarrow$ $Z \cong X \times Y$ (using standard argument via universal property).

10) Have diagram

$$\begin{tikzcd}
& \mathcal{O}^r_{U_{ij}} \arrow[dd, "\psi_{ji}"] \arrow[dr] & \\
\mathcal{M}|_{U_{ij}} \arrow[ur, "\alpha_i"] \arrow[dr, "\alpha_j"'] & & \mathcal{O}_{U_{ij}} \\
& \mathcal{O}^r_{U_{ij}} \arrow[ur] &
\end{tikzcd}$$

Standard result that if A a commutative ring and $A^r \longrightarrow A^r$ given by matrix H with coeffs in A, the dual map (given by composite with H) is given by matrix H^t, where dualizing is a contravariant functor

So in our case, dualizing takes us from the jth trivialization to the ith trivialization ie. the transition function is given by the transpose of $\psi_{ij} = \psi_{ji}^{-1}$.

11) Let $\alpha_i : U_i \to X$: define a subsheaf $\mathcal{M}$ of $\bigoplus_i \alpha_{i*}\mathcal{O}_{U_i}^r$ by

$$\mathcal{M}(U) = \left\{ \{\sigma_i\} \in \bigoplus_i \mathcal{O}_X(U_i \cap U)^r \text{ s.t. } \sigma_j = \psi_{ji}\sigma_i \ \forall i,j \right\}$$

This is easily seen to be a sheaf (cover is finite), a locally free $\mathcal{O}_X$-module ($\exists$ obvious isom $\mathcal{M}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^r$ given by projection onto ith factor, other factors determined by condition $\sigma_j = \psi_{ji}\sigma_i$ for $j \neq i$, with transition functions as given, since

$$\begin{tikzcd}
 & \mathcal{M}|_{U_{ij}} \arrow[dl] \arrow[dr] & \\
\mathcal{O}_{U_{ij}}^r \arrow[rr, "\psi_{ji}"] & & \mathcal{O}_{U_{ij}}^r
\end{tikzcd}$$

12) Essentially repeat arguments in (1.6).

<u>Sheaf condition (A)</u> Wlog reduce to case $x \in \tilde{M}(V) = M$

Suppose $V = \bigcup_{i=1}^{N} D(f_i)$ and $x|_{D(f_i)} = 0 \quad \forall i$

ie $x/1 = 0$ in M_{f_i} $\forall i$; $\therefore \exists r_i$ s.t. $f_i^{r_i}x = 0$ in M

But $V(f_1^{r_1}, \dots, f_N^{r_N}) = \emptyset \underset{0\text{-satz}}{\Longrightarrow}$

$1 = \sum a_i f_i^{r_i}$ for some $a_i \in A \implies x = \sum a_i f_i^{r_i} x = 0$ in M.

<u>Sheaf Condition (B)</u> Again reduce down to open set being whole variety. Given $\mathcal{B}$-cover $V = \bigcup_i D(f_i)$ and sections $\sigma_i \in M_{f_i}$ s.t. $\sigma_i = \sigma_j$ on $M_{f_i f_j}$.

If we take a finite subcover $V = \bigcup_{i=1}^{N} D(f_i)$ and show that the sections $\sigma_1, \dots, \sigma_N$ glue to a global section m with $m = \sigma_i \in M_{f_i}$ for $i = 1, \dots, N$, then for $k \notin \{1, \dots, N\}$, have $D(f_k) = \bigcup_{i=1}^{N} D(f_i f_k)$ & since $\sigma_i = \sigma_k \in M_{f_i f_k}$ $\forall i$, deduce that $m = \sigma_k$ in $M_{f_i f_k}$ $\forall i \Rightarrow m = \sigma_k$ in M_f by sheaf condition (A).

Thus for $i = 1, \dots, N$, can set $\sigma_i = m_i / f_i^r$ for some $r >> 0$ indep of i, in M_{f_i}.

Since $m_i/f_i^r = m_j / f_j^r$ in $M_{f_i f_j}$, $\exists\, n \geq 0$ s.t. $(f_i f_j)^n (f_j^r m_i - f_i^r m_j) = 0$ in M.

Taking $n >> 0$ indep of i, j, we have wlog

$$f_j^{n+r} f_i^n m_i - f_i^{n+r} f_j^n m_j = 0 \quad \text{in } M. \ \forall i, j$$

Since $V(f_1^{n+r}, \dots, f_N^{n+r}) = \phi$, 0-satz $\Rightarrow$ can write $1 = \sum_1^N e_i f_i^{n+r} \in A$. Set $m = \sum_{i=1}^N e_i f_i^n m_i$;

then $f_j^{n+r} m = \sum_i e_i f_i^n f_j^{n+r} m_i = \sum_i e_i f_i^{n+r} f_j^n m_j$

$= f_j^n m_j$ in M.

i.e. $f_j^{n}(f_j^{r} - m_j) = 0$ in M

$\therefore$ $m/1 = m_j/f_j^{r} = \sigma_j$ in M_{f_j} for $j = 1, \dots, N$

13) Recall that the morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ was given by

$$\{s : U \to \coprod_{P \in U} \mathcal{O}_{Y, f(P)} \text{ s.t. } \dots\dots\} \longmapsto g$$

where $g(P) = s(P)(f(P))$ $\forall P$

Locally $s(Q) = h_{f(Q)}$ for some $h \in \mathcal{O}_Y(W)$ & so locally $g(Q) = h(f(Q)) = f^{\#}(h)(Q)$

i.e. $g = f^{\#}(h) = h \circ f \in \mathcal{O}_X(V)$, where $P \in V \subset f^{-1}(W)$

- if f constant, $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ has image consisting of constant functions — so not in general surjective
- if $X = \{P\}$, $f^{-1}\mathcal{O}_Y \cong \mathcal{O}_{Y, f(P)} \xrightarrow[\text{eval at } f(P)]{} k = \mathcal{O}_{X, P}$ & so not in general injective.

14) If $U \ni P$, define map

$$(f^{-1}\mathcal{F})(U) = \{s : U \to \coprod_{P \in U} \mathcal{F}_{f(P)} \text{ s.t. } \dots\} \to \mathcal{F}_{f(P)}$$

by $s \mapsto s(P)$. Taking limit, this clearly induces a HM $(f^{-1}\mathcal{F})_P \longrightarrow \mathcal{F}_{f(P)}$

Given (V, σ) defining a given germ of $\mathcal{F}$ at $f(P)$, $(f^{-1}V, s)$ where $s(Q) = \sigma_{f(Q)}$ $\forall Q \in f^{-1}V$ defines a section of $(f^{-1}\mathcal{F})(f^{-1}V)$ whose germ at P

has image the given germ $\sigma_{f(P)}$ of $\mathcal{F}_{f(P)}$. So above HM is surjective.

Conversely, suppose (U, s) has image 0 in $\mathcal{F}_{f(P)}$; shrinking U, may assume $\exists\ W \ni f(P)$ s.t. $f(U) \subseteq W$ & $\exists\ \sigma \in \mathcal{F}(W)$ s.t. $s(Q) = \sigma_{f(Q)}\ \forall\ Q \in U$.

By assumption $\sigma_{f(P)} = 0 \implies \exists\ f(P) \in W' \subseteq W$ s.t. $\sigma|_{W'} = 0$. Shrinking U to U' s.t. $f(U') \subseteq W'$, deduce that $s|_{U'} = 0$ & so $s_P = 0$.

Hence map on stalks is an IM.

Now $\mathcal{G}(U) := \varinjlim_{V \supseteq f(U)} \mathcal{F}(V)$, so an elt of $\mathcal{G}(U)$ is represented by an equiv. class of sections $\sigma_V \in \mathcal{F}(V)$, where $V \supseteq f(U)$ & V sufficiently small, where $\sigma_V \sim \sigma_{V'} \iff \exists\ V \cap V' \supseteq W \supseteq f(U)$ s.t. $\sigma_V|_W = \sigma_{V'}|_W$.

Note that for any open $V_0 \supseteq f(U)$, we can also write $\mathcal{G}(U) = \varinjlim_{V_0 \supseteq V \supseteq f(U)} \mathcal{F}(V)$, since any elt of $\varinjlim_{V \supseteq f(U)} \mathcal{F}(V)$, given by an equiv class of sections $\sigma_V \in \mathcal{F}(V)$, is also given by an equivalence class of sections $\sigma_V|_{V \cap V_0} \in \mathcal{F}(V \cap V_0)$.

So for $U \supseteq U'$ open in X & $V \supseteq f(U)$,
we can define a map $\mathcal{F}(V) \to \mathcal{G}(U') = \varinjlim_{V \supseteq V' \supseteq f(U')} \mathcal{F}(V')$
just by restriction.

So given an equivalence class of sections defining an elt of $\mathcal{G}(U)$, we note that $(V_1, \sigma_1) \sim (V_2, \sigma_2)$
$\Rightarrow \exists\ V_1 \cap V_2 \supseteq W \supseteq f(U)$ s.t. $\sigma_1|_W = \sigma_2|_W$
& the image in $\mathcal{G}(U')$ under above map is well-defined
— so $\mathcal{G}$ is a <u>presheaf</u> on X.

We now define a morphism of presheaves
$\theta : \mathcal{G} \to f^{-1}\mathcal{F}$ as follows: given an equivalence class of sections (V, σ_V), $\sigma_V \in \mathcal{F}(V)$, $V \supseteq f(U)$,
we can define the image $s : U \to \coprod_{P \in U} \mathcal{F}_{f(P)}$
to be given by $s(P) = (\sigma_V)_{f(P)}$ $\forall P \in U$.
Clearly this depends only on the equiv. class of (V, σ_V), and as the function s is given by germs of a section of $\mathcal{F}$ (in fact on all of U, not just locally), we have $s \in f^{-1}\mathcal{F}(U)$.

Given now $U \ni P$, we have obvious map
$\mathcal{G}(U) \to \mathcal{F}_{f(P)}$ given by taking equiv. class of

pairs (V, σ) with $\sigma \in \mathcal{F}(V)$, $V \supseteq f(U)$ to the germ $\sigma_{f(P)}$. Clearly this factors via the above defined map $\mathcal{G}(U) \longrightarrow (f^{-1}\mathcal{F})(U) \longrightarrow \mathcal{F}_{f(P)}$ & if $U \supseteq U' \ni P$, then

$$\begin{tikzcd} \mathcal{G}(U) \arrow[rr] \arrow[dr] & & \mathcal{F}_{f(P)} \\ & \mathcal{G}(U') \arrow[ur] & \end{tikzcd}$$

commutes.

So as induced map $(f^{-1}\mathcal{F})_P \xrightarrow{\sim} \mathcal{F}_{f(P)}$ on stalks is an IM, RTP induced map $\mathcal{G}_P \longrightarrow \mathcal{F}_{f(P)}$ an IM.

Given a germ in RHS represented by (V, σ) say, for any $U \subset f^{-1}V$ this defines a section of $\mathcal{G}(U)$ with image $\sigma_{f(P)}$ — so map is <u>surjective</u>.

Similarly if we have a germ in $\mathcal{G}_P$, say repr. by a local section (U, s), $U \ni P$, then this corresponds to an equivalence class of sections (V, σ_V), $\sigma_V \in \mathcal{F}(V)$ with $V \supseteq f(U)$, V suff small.

If image of σ_V in $\mathcal{F}_{f(P)}$ under above map is zero, then we have $W \subseteq V$ s.t. $\sigma_V|_W = 0$

Now choose U' s.t. $f(U') \subseteq W$; so $s|_{U'} = 0$ & so germ $s_P = 0$ in $\mathcal{G}_P$. Hence injectivity.

So the map of presheaves $\mathcal{G} \xrightarrow{\theta} f^{-1}\mathcal{F}$ induces IMs on stalks & so induced morphism of sheaves $\mathcal{G}^+ \to f^{-1}\mathcal{F}$ is an IM of sheaves.

15) ($\Rightarrow$) is standard, and for ($\Leftarrow$) need

Lemma 1 If M an A-module, then $M_m = 0$ for all maximal ideals $m \triangleleft A$ $\Rightarrow$ $M = 0$

Pf Suppose $M \neq 0$ and let x be a non-zero elt of M.

Let $I = \mathrm{Ann}(x) := \{a \in A \text{ s.t. } ax = 0\} \triangleleft A$

Since $1 \notin I$, choose a maximal ideal $m \supseteq I$.

Since $M_m = 0$, $x/1 = 0$ in M_m $\Rightarrow$ x is killed by some elt of $A \setminus m$. But $I \cap (A \setminus m) = \phi$ $\#$. $\square$

The required result follows by repeated application of:

Lemma 2 Given a sequence $M \xrightarrow{\theta} N \xrightarrow{\phi} P$ of A-modules s.t. for all maximal ideals m, we have $M_m \xrightarrow{\theta_m} N_m \xrightarrow{\phi_m} P_m$ is exact at N_m, the original sequence exact at N.

Pf First show that $\psi = \phi \circ \theta = 0$. Let $K = \mathrm{Ker}\,\psi$ & so $0 \to K \hookrightarrow M \to P$ exact. Localizing at m, deduce that $K_m = M_m$ $\forall m$, since $\psi_m = \phi_m \circ \theta_m = 0$

ie. $(M/K)_m = 0 \ \forall m \underset{\text{Lemma 1}}{\Longrightarrow} M/K = 0$.

So $K = M$ & $\psi = 0$

So $\mathrm{Im}\,\theta \subset \mathrm{Ker}\,\psi$. Standard arguments $\Rightarrow (\mathrm{Im}\,\theta)_m = \mathrm{Im}\,\theta_m$ & $(\mathrm{Ker}\,\phi)_m = \mathrm{Ker}\,\phi_m$. Thus $(\mathrm{Ker}\,\phi / \mathrm{Im}\,\theta)_m = \mathrm{Ker}\,\phi_m / \mathrm{Im}\,\theta_m = 0$ $\forall$ max ideals m $\Rightarrow \mathrm{Im}\,\theta = \mathrm{Ker}\,\phi$

(ie. $\mathrm{Ker}\,\phi / \mathrm{Im}\,\theta = 0$) $\Rightarrow$ $M \to N \to P$ exact at N $\square$

16) $\phi : Y \to X$, $\phi^* : k[X] = A \to B = k[Y]$

Given A-module M, we know that $\phi^* \tilde{M} = (B \otimes_A M)^\sim$ on Y.

Given B-module, we remarked that $\phi_* \tilde{N} = ({}_A N)^\sim$ on X

where ${}_A N$ is just N considered as an A-module via ϕ^*.

Setting $\mathcal{F} = \tilde{M}$ by (3.4), we have

$\phi_* \phi^* \tilde{M} = ({}_A(B \otimes_A M))^\sim$. Now $\phi_* \mathcal{O}_Y = ({}_A B)^\sim$

By remark in Fact 2 after (3.6),

$$\tilde{M} \otimes_{\mathcal{O}_X} \phi_* \mathcal{O}_Y = (M \otimes_A ({}_A B))^\sim .$$

But $M \otimes_A ({}_A B) = {}_A(M \otimes_A B)$ i.e. $M \otimes_A B$ considered as an A-module, & so $\tilde{M} \otimes_{\mathcal{O}_X} \phi_* \mathcal{O}_Y = ({}_A(M \otimes_A B))^\sim$

$$= \phi_* \phi^* \tilde{M} .$$

In general, given morphism $\phi : Y \to X$ and affine piece U of X, by assumption $\phi^{-1}(U) = V$ is an affine piece of Y and we have morphism of affine varieties $\phi_U : V \to U$ & Hom on coord rings $\phi_U^* : A = k[U] \to B = k[V]$

For $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, have $\mathcal{F}|_U = \tilde{M}$ for some A-module M, and then

$$(\phi^* \mathcal{F})|_V = \phi_U^* (\mathcal{F}|_U) = \phi_U^* \tilde{M} = (B \otimes_A M)^\sim$$

But for $\mathcal{G}$ a quasi-coherent $\mathcal{O}_Y$-module, say $\mathcal{G}|_V = \tilde{N}$ for some B-module N, we have

$$(\phi_* \mathcal{G})|_U = (\phi_U)_* (\mathcal{G}|_V) = ({}_A N)^\sim \text{ on } U$$

and from previous part, then deduce $\phi_* \phi^* \mathcal{F}|_U \cong \mathcal{F} \otimes_{\mathcal{O}_X} \phi_* \mathcal{O}_Y|_U$.

<u>Mechanical check</u>   Consistency of above IMs :

Let us suppose $X = \bigcup_{i=1}^{N} U_i$ an open affine cover &
$\theta_i : \mathcal{F}|_{U_i} \otimes_{\mathcal{O}_{U_i}} \phi_* \mathcal{O}_V|_{U_i} \cong \phi_* \phi^* \mathcal{F}|_{U_i}$ denote the IMs found above. If $U \subseteq U_i$ an open affine, induced IM of sheaves $\theta_i|_U$ comes from IM of modules

$$M \otimes_A ({}_A B) \xrightarrow{\sim} {}_A(M \otimes_A B), \text{ where } A = k[U],\ B = k[V]$$

($V = \phi^{-1}(U)$) & $M = \mathcal{F}(U)$ — this is certainly true for $U \in \mathcal{B}_i$ = basis of open affines of form $D(f)$ with $f \in k[U_i]$ since localization $(\ )_f$ commutes with everything, and for more general open affine U, the IM of modules induces correct IMs $\theta_i|_V$ for $V \in \mathcal{B}_i$ with $V \subseteq U$, and that is sufficient to imply claim via + construction. So if U open affine, $U \subset U_i$ and U_j, then $\theta_i|_U = \theta_j|_U$.

We'll remark in lectures that U, V affine $\Rightarrow$ $U \cap V$ affine (under the IM $X \xrightarrow{\sim} \Delta \subset X \times X$, note that $U \cap V \xrightarrow{\sim} \Delta \cap (U \times V)$). Thus $\mathcal{B} = \{\text{open affine } U \text{ s.t. } U \subseteq U_i \text{ some } i\}$ is a basis of open (affine) sets of X, closed under finite intersection.

For all $P \in X$, have well-defined IM on stalks, and this is given by θ_i (whenever $P \in U_i$). Composing with these IMs of stalks defines an IM

$(\mathcal{F} \otimes_{\mathcal{O}_X} \phi_* \mathcal{O}_Y)^+ \xrightarrow{\sim} (\phi_* \phi^* \mathcal{F})^+$, and hence an IM $\mathcal{F} \otimes_{\mathcal{O}_X} \phi_* \mathcal{O}_Y \xrightarrow{\sim} \phi_* \phi^* \mathcal{F}$ as required