

1) Projective Case: If $V = \cup U_i$ with $U_i = V \setminus V^h(I_i)$, then $\phi = \bigcap V^h(I_i) = V^h(\sum I_i)$. But $\sum I_i \triangleleft k[x_0, \dots, x_n]$ fg. by Hilbert's Basis Thm $\Rightarrow \sum I_i = \langle f_1, \dots, f_N \rangle$, where each f_j involves only finitely many of the I_i . Thus $\sum I_i$ is a finite sum of some of the I_i & hence V covered by a finite subcollection of the U_i . Affine case entirely similar.

Comment Better to do it this way than use Hilbert 0-satz as it isn't really dependent on k being algebraically closed.

2) Suppose h an everywhere regular function on V ; for each $P \in V$, \exists representation $h = f_P/g_P$, with $f_P, g_P \in k[V]$, $g_P(P) \neq 0$. Consider $J \triangleleft k[V]$ generated by $\{g_P : P \in V\}$. Since $V(J) = \emptyset$,

Hilbert 0-satz $\Rightarrow J = k[V]$ $\therefore \exists P_1, \dots, P_N \in V$,

$$q_1, \dots, q_N \in k[V] \text{ s.t. } 1 = \sum_{i=1}^N q_i g_{P_i} \Rightarrow$$

$$h = \sum_{i=1}^N q_i g_{P_i} h = \sum q_i f_{P_i} \in k[V].$$

On \mathbb{P}^n , unique factorization in $k[x_0, \dots, x_n] \Rightarrow$

$\exists!$ representation of a rational function h as $h = F/G$,

F & G coprime & homogeneous of same degree \Rightarrow

G has no zeros $\Rightarrow G = \text{const} \Rightarrow h = \text{const}$.

3) $\phi = (f_1, \dots, f_m) : V \rightarrow W \subseteq A^m$, then $\phi^*(g) = g \circ \phi \in k[V]$ for $g \in k[W]$.

i.e. represent g by polynomial $G \in k[Y_1, \dots, Y_m]$ & then $\phi^*(g) = G(f_1, \dots, f_m) \in k[V]$.

(a) Suppose ϕ dominating: given $g \in W$ s.t. $\phi^*(g) = 0$, have G vanishing at all points of $\phi(V) \Rightarrow G \in I(W) \Rightarrow g = 0$

Conversely, suppose G vanishes at all points of $\phi(V)$, then $\phi^*(g) = 0 \in k[V] \Rightarrow g = 0 \Rightarrow G \in I(W)$ & so ϕ is dominating

(b) If $\phi: V \xrightarrow{\sim} W' \hookrightarrow W$, then $\phi^*: k[W] \rightarrow k[W'] \xrightarrow{\sim} k[V]$, where the first map is just $k[W] \rightarrow k[W]/I(W')$.

Conversely, if ϕ^* surjective, let $I = \ker \phi^* \triangleleft k[W]$ & $W' = V(I)$. Then ϕ^* induces an IM

$$k[W'] = k[W]/I \xrightarrow{\sim} k[V] \quad (\text{0-satz} \Rightarrow I(W') = I)$$

4) Both of them are isomorphic to $\text{Bil}_A(M \times N, P)$:

Given $\phi \in \text{Bil}_A(M \times N, P)$, universal property of \otimes $\Rightarrow \exists! \theta \in \text{Hom}_A(M \otimes N, P)$ s.t. $\phi = \theta \circ \gamma$ (where $\gamma: M \times N \rightarrow M \otimes_A N$ is universal bilinear map)

Conversely, given such a θ , have corresponding bilinear map $\theta \circ \gamma$. Clearly this identification gives an IM of A -modules.

However, given such a $\phi \in \text{Bil}_A(M \times N, P)$, may define $\tilde{\phi} \in \text{Hom}_A(M, \text{Hom}_A(N, P))$ by $\tilde{\phi}(m)(n) = \phi(m, n)$, also clearly an IM of A -modules.

5) $N \otimes_B P$ considered as an A -module via the
 A -module structure on $N \cong M \otimes_A N$ as a B -module
 via B -module structure on N .

(3)

Given $z \in P$, have A -bilinear map

$$\begin{aligned} M \times N &\longrightarrow M \otimes_A (N \otimes_B P) \\ (\alpha, \beta) &\longmapsto \alpha \otimes (\beta \otimes z) \end{aligned}$$

where $N \otimes_B P$ considered as a A -module & $\alpha \otimes \beta$ defined
 to be image of (α, β) under appropriate bilinear universal map.

This induces a HM

$$\begin{aligned} M \otimes_A N &\xrightarrow{f_z} M \otimes_A (N \otimes_B P) \\ \alpha \otimes \beta &\longmapsto \alpha \otimes (\beta \otimes z) \end{aligned}$$

which we observe is in fact also B -linear.

Now consider the B -bilinear map

$$\begin{aligned} (M \otimes_A N) \times P &\longrightarrow M \otimes_A (N \otimes_B P) \\ (t, z) &\longmapsto f_z(t). \end{aligned}$$

This induces the required map $(M \otimes_A N) \otimes_B P \longrightarrow M \otimes_A (N \otimes_B P)$
 under which $(\alpha \otimes \beta) \otimes z \longmapsto \alpha \otimes (\beta \otimes z)$.

The converse is constructed similarly. Note that we have
 an IM of (A, B) -bimodules

Aliter Define $M \otimes_A N \otimes_B P$ to be universal w.r.t
 trilinear maps to (A, B) -bimodules, which is A -linear
 in the first factor, (A, B) -linear in the second factor
 & B -linear in the third.

(b) We saw in (0.4) that:

• if $\sigma \triangleleft A$, $A/\sigma \otimes_A M \cong M/\sigma M$

• if S mult subset of A , $S^{-1}A \otimes_A M \cong S^{-1}M$

(make sure you can move them).

$$(a) \quad A/I \otimes_A (M \otimes_A N) \underset{QS}{\cong} (A/I \otimes_A M) \otimes_A N$$

$$\cong M/IM \otimes_A N \cong (M/IM \otimes_{A/I} A/I) \otimes_A N$$

$$\underset{QS}{\cong} M/IM \otimes_{A/I} (A/I \otimes_A N) \cong M/IM \otimes_{A/I} N/IN$$

$$(b) \quad \text{LHS} \underset{QS}{\cong} S^{-1}A \otimes_A (M \otimes_A N) \underset{QS}{\cong} (S^{-1}A \otimes_A M) \otimes_A N$$

$$\cong S^{-1}M \otimes_A N \cong (S^{-1}M \otimes_{S^{-1}A} S^{-1}A) \otimes_A N$$

$$\underset{QS}{\cong} S^{-1}M \otimes_{S^{-1}A} (S^{-1}A \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N.$$

7) Given $U \cong X$ & $s \in \mathcal{F}(U)$, claim that

$$\phi(U)(s) = \gamma(U)(s). \quad \text{This follows since}$$

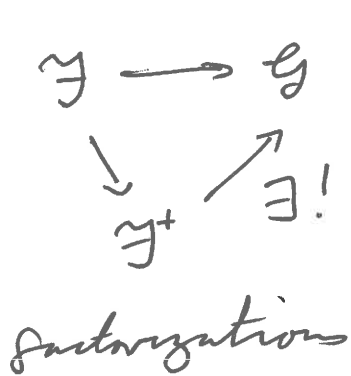
$$(\phi(U)(s))_x = \phi_x(s_x) = \gamma_x(s_x) = (\gamma(U)(s))_x \quad \forall x \in U$$

i.e. \exists open nbhd $x \in U_x \subset U$ s.t.

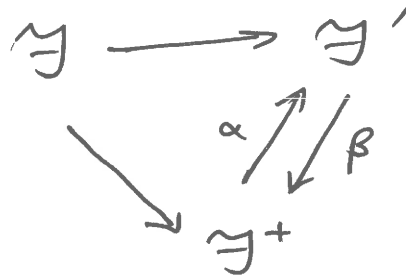
$$\phi(U)(s)|_{U_x} = \gamma(U)(s)|_{U_x}$$

Now check condition (A) for $\mathcal{G} \Rightarrow \phi(U)(s) = \gamma(U)(s).$

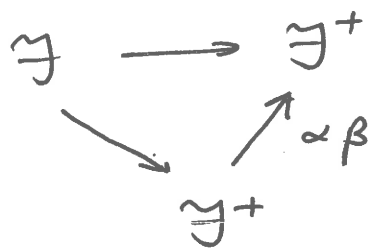
8) $\mathcal{F} \rightarrow \mathcal{F}^+$ has universal property that for any morphism $\mathcal{F} \rightarrow \mathcal{G}$ to a sheaf \mathcal{G} , $\exists!$ factorization



Suppose $\mathcal{F} \rightarrow \mathcal{F}'$ also has this property. Then get unique



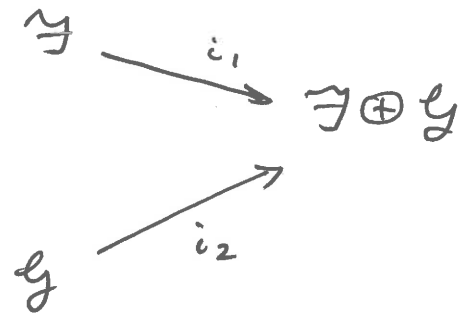
Applying uniqueness of factorization for $\mathcal{F} \rightarrow \mathcal{F}^+$ to diagram



, deduce $\alpha\beta = \text{id}_{\mathcal{F}^+}$

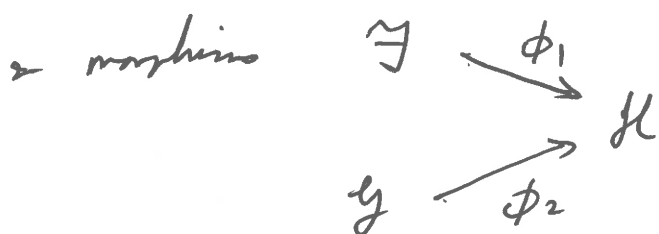
Similarly $\beta\alpha = \text{id}_{\mathcal{F}'}$.

9) (a) Have natural morphisms



given by $i_1(s) = (s, 0)$, $i_2(t) = (0, t)$.

This has universal property that for any sheaf / presheaf \mathcal{H}



, $\exists!$ morphism

$$\mathcal{F} \oplus \mathcal{G} \xrightarrow{\phi} \mathcal{H} \text{ s.t.}$$

$$\phi_1 = \phi \circ i_1, \phi_2 = \phi \circ i_2$$

(6)

Existence & uniqueness of ϕ is clear :

$$\phi(s, t) = (\phi_1(s), \phi_2(t)).$$

Suppose S also has this universal property, say

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{j_1} & S \\ \mathcal{G} & \xrightarrow{j_2} & S \end{array}, \text{ the } \exists \text{ morphisms } \mathcal{F} \oplus \mathcal{G} \rightarrow S \quad S \rightarrow \mathcal{F} \oplus \mathcal{G}$$

which are mutually inverse with

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{j_1} & \mathcal{F} \oplus \mathcal{G} \\ \mathcal{F} & \xrightarrow{i_1} & \mathcal{F} \oplus \mathcal{G} \\ \mathcal{G} & \xrightarrow{i_2} & \mathcal{F} \oplus \mathcal{G} \\ \mathcal{G} & \xrightarrow{j_2} & S \end{array}$$

Remark $\mathcal{F} \oplus \mathcal{G}$ is in fact a biproduct & so \exists morphisms

$$\begin{array}{ccc} \mathcal{F} \oplus \mathcal{G} & \xrightarrow{\pi_1} & \mathcal{F} \\ \mathcal{F} \oplus \mathcal{G} & \xrightarrow{\pi_2} & \mathcal{G} \end{array}, \text{ universal w.r.t. diagrams } \mathcal{H} \begin{array}{c} \xrightarrow{\theta_1} \mathcal{F} \\ \searrow \theta_2 \mathcal{G} \end{array}$$

i.e. factoring thru morphism $\mathcal{H} \rightarrow \mathcal{F} \oplus \mathcal{G}$.

(b) Give \mathcal{O}_X -modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$, a bilinear

map of \mathcal{O}_X -modules is a morphism of sheaves of

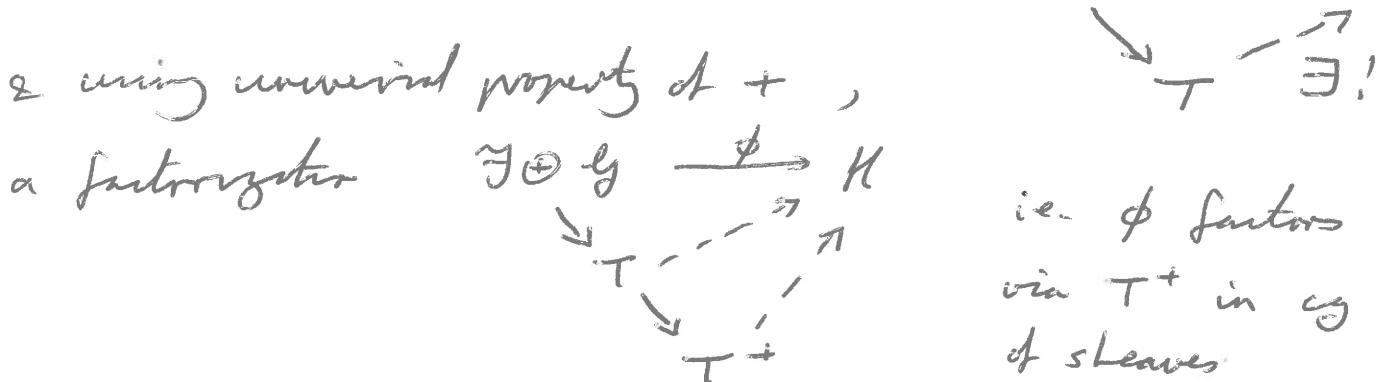
algebras $\mathcal{F} \oplus \mathcal{G} \xrightarrow{\phi} \mathcal{H}$ s.t. for U open,

the corresponding map $\phi(U)$ is a bilinear map of $\mathcal{O}_X(U)$ -modules

Set $T(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$, the product tensor-product.

We define the sheaf tensor product to be T^+ & denote it by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. Clearly $\exists \mathcal{O}_X$ -bilinear map of presheaves $\mathcal{F} \oplus \mathcal{G} \rightarrow T$ & composing with sheafification, a bilinear map of \mathcal{O}_X -modules $\mathcal{F} \oplus \mathcal{G} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$

Given any bilinear map $\phi : \mathcal{F} \oplus \mathcal{G} \rightarrow \mathcal{H}$ of sheaves, $\exists!$ factorization in cg of presheaves $\mathcal{F} \oplus \mathcal{G} \xrightarrow{\phi} \mathcal{H}$



Given a factorization of ϕ via T^+ , the composite $T \rightarrow T^+ \rightarrow \mathcal{H}$ is unique by universal property of sheaf tensor product, & the factorization $T^+ \rightarrow \mathcal{H}$ unique by universal property of sheafification. Standard argument then shows uniqueness of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ up to IM

10) Suppose $\phi : \mathcal{F} \rightarrow \mathcal{G}$ a morphism of sheaves.

- If $\theta : \mathcal{K} \rightarrow \mathcal{F}$ s.t. $\phi \theta = 0$, then $\theta(U) \cap \mathcal{K}(U) \subseteq (\text{Ker } \phi)(U) \quad \forall U$ & so $\exists!$ map $\bar{\theta} : \mathcal{K} \rightarrow \text{Ker } \phi$ s.t.
-
- commutes
- If $\theta : \mathcal{G} \rightarrow \mathcal{Q}$ s.t. $\theta \phi = 0$, then for U open in X , $\exists!$ factorization of $\theta(U)$

$$\begin{array}{ccc} \exists(u) \xrightarrow{\phi(u)} \mathcal{Y}(u) & \longrightarrow & \mathcal{E}(u) := \mathcal{Y}(u)/\phi(u)\exists(u) \\ & \searrow \theta(u) & \downarrow \\ & & \mathcal{Q}(u) \end{array}$$

through the quotient $\mathcal{E}(u)$. Universal property of localization \Rightarrow quotient map $\mathcal{E} \rightarrow \mathcal{Q}$

factors uniquely through $\text{coker } \phi = \mathcal{E}^+ \rightarrow \mathcal{Q}$

\exists so $\exists!$ factorization

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\tau} & \text{coker } \phi \\ \theta \searrow & & \downarrow \bar{\theta} \\ & \mathcal{Q} & \end{array}$$

Given a factorization of ϕ , $\mathcal{Y} \xrightarrow{\tau} \mathcal{K} \xrightarrow{\theta} \mathcal{Y}$,

clearly $\mathcal{Y} \xrightarrow{\phi} \mathcal{Y} \rightarrow \text{coker } \theta$ is zero so

$\exists!$ factorization

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\tau} & \mathcal{K} & \xrightarrow{\theta} & \mathcal{Y} & \xrightarrow{\tau_1} & \text{coker } \phi \\ & & & & & \searrow \tau_2 & \downarrow \alpha \\ & & & & & & \text{coker } \theta \end{array}$$

Thus $\text{Ker } \tau_1 \subseteq \text{Ker } \tau_2$ i.e.

$\text{Im } \phi \subseteq \text{Im } \theta$. Clearly α an IM \Rightarrow equality

Conversely, equality $\Rightarrow (\text{Im } \phi)_P = \text{Im } \phi_P = \text{Im } \theta_P = (\text{Im } \theta)_P \forall P$

On passing to stalk maps

$$\begin{array}{ccc} \mathcal{Y}_P \longrightarrow \text{coker } \phi_P = \mathcal{Y}_P / \phi_P \mathcal{Y}_P & & \\ & \downarrow & \\ & \text{coker } \theta_P = \mathcal{Y}_P / \theta_P \mathcal{K}_P & \end{array}$$

deduce that $\alpha_P : (\text{coker } \phi)_P \xrightarrow{\sim} (\text{coker } \theta)_P \forall P$

$\Rightarrow \alpha$ is an isomorphism

11) Can define an exact sequence of sheaves

$$0 \rightarrow \mathbb{Z} \hookrightarrow \mathcal{O}_U \rightarrow \mathcal{O}_U^* \rightarrow 1$$

$$f \longmapsto \exp(2\pi i f)$$

This gives an exact sequence on stalks — valid for $P \in V \subset U$ with V connected & simply conn open set,

we have an exact sequence on sections over V , since any $g \in \mathcal{O}_U^*(V)$ comes from $\frac{1}{2\pi i} \int_{P_0}^P \frac{g'(w) dw}{g(w)}$

i.e. $\frac{1}{2\pi i} \log g$. Wlog assume U is connected —

otherwise work with connected components. If U also simply connected, we've just proved that the map

$$\Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(U, \mathcal{O}_U^*) \text{ is surjective}$$

If U not simply connected, $\exists a \notin U$ for which we cannot define $\log(z - a)$ (\exists closed curve γ in U with winding number $n(\gamma, a) \neq 0$) and then

$$\Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(U, \mathcal{O}_U^*) \text{ not surjective. So map}$$

on sections is surjective \iff every connected component of U is simply connected.

12) We embed $\mathbb{P}^n \xrightarrow{\nu_d} \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$

defined by all the monomials of degree d . This

defines an IM of \mathbb{P}^n with its image $\subset \mathbb{P}^N$ (see below);

cf Q14 for fact that regular maps \equiv morphisms.

So the Veronese map ν_d defined by all monomials of degree $d \in X_0, \dots, X_n$. We take homogeneous coords on \mathbb{P}^n given by $x_{i_0 \dots i_n}$ with $\sum i_k = d$. (10)

Define a k -algebra HM

$\Theta: k[X_{i_0 \dots i_n} : \sum i_k = d] \longrightarrow k[X_0, \dots, X_n]$ by $X_{i_0 \dots i_n} \mapsto X_0^{i_0} \dots X_n^{i_n}$ & set $I = \ker \Theta$, a prime ideal since $k[X_{i_0 \dots i_n}] / I \hookrightarrow k[X_0, \dots, X_n]$ an ID.

An elt is in $\ker \Theta \iff$ all its homog parts are 0 & so I is a homogeneous ideal. Set $V = V^h(I) \subset \mathbb{P}^n$ irreducible

From definition of I , have $\nu_d(\mathbb{P}^n) \subset V$.

Note that $\Theta(X_{i_0 \dots i_n}^d - (X_{d_0 \dots 0})^{i_0} \dots (X_{0 \dots d})^{i_n}) = 0$, & so for $P = (x_{i_0 \dots i_n}) \in V$, have $x_{i_0 \dots i_n}^d = (x_{d_0 \dots 0})^{i_0} \dots (x_{0 \dots d})^{i_n}$

Since some coord $x_{i_0 \dots i_n} \neq 0$, at least one of the coords $x_{d_0 \dots 0}, x_{0 \dots d_0 \dots 0}, \dots, x_{0 \dots 0 \dots d}$ non-zero. (1)

Note that $\Theta(X_{i_0 \dots i_n} X_{j_0 \dots j_n} - X_{i'_0 \dots i'_n} X_{j'_0 \dots j'_n}) = 0$ whenever $i_k + j_k = i'_k + j'_k \forall k$, i.e. their quadratics in I (2)

Define inverse morphism $\phi: V \longrightarrow \mathbb{P}^n$ by

$(X_{d_0 \dots 0} : X_{d-1 \ 1 \ 0 \dots 0} : \dots : X_{d-1 \ 0 \dots 0 \ 1}) \sim (X_{1 \ d-1 \ 0 \dots 0} : \dots : X_{0 \ d-1 \ 0 \dots 0 \ 1})$
 $\sim \dots \sim (X_{1 \ 0 \dots 0 \ d-1} : X_{0 \ 1 \ 0 \dots 0 \ d-1} : \dots : X_{0 \dots 0 \ d})$,

equivalent representations of same map by (2) & regular at every point of V by (1).

Clearly now $\phi \circ \nu_d = \text{id}_{\mathbb{P}^n}$. However given (ii)

$P = (x_{i_0} \dots x_{i_n}) \in V$, can assume wlog $x_{d_0 \dots d_0} \neq 0$ by (i) & so morphism given at P by first representation.

Observe $X_{d_0 \dots d_0}^{d-1} X_{i_0 \dots i_n} - X_{d_0 \dots d_0}^{i_0} X_{d-1 \ i_0 \dots d_0}^{i_1} \dots X_{d-1 \ 0 \dots 0}^{i_n}$

& so coords of P satisfy $x_{d_0 \dots d_0}^{d-1} x_{i_0 \dots i_n} = x_{d_0 \dots d_0}^{i_0} \dots x_{d-1 \ 0 \dots 0}^{i_n} \in I$

So $\nu_d \circ \phi(P)$ has coords $x_{d_0 \dots d_0}^{d-1} x_{i_0 \dots i_n}$

ie. $\nu_d \circ \phi(P) = P$. $\therefore \nu_d \circ \phi = \text{id}_V$ and

$\nu_d: \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ an embedding with image V as claimed.

The hypersurface $F = 0 \subset \mathbb{P}^n$ corresponds under ν_d to a hyperplane section of image $V = \nu_d(\mathbb{P}^n)$, since F a linear combination of monomials of degree d .

Then $\mathbb{P}^n \setminus V^h(F)$ has natural structure of an affine variety, namely $V \setminus$ corresponding hyperplane section

13) Work in the C^∞ case - complex manifold case analogous

Suppose $f: X \rightarrow Y$ a continuous map of manifolds with $\phi = (f, f^\#): (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ a morphism of the corresponding locally ringed spaces / \mathbb{R} .

A chart on X of the form $\theta: U \xrightarrow{\sim} \theta(U) = W \subset \mathbb{R}^n$ for some θ inducing an IM of locally ringed spaces / \mathbb{R} .

Similarly a chart on Y of the form $\psi: U' \xrightarrow{\sim} \psi(U') = W' \subset \mathbb{R}^m$ inducing an IM of locally ringed spaces / \mathbb{R} .

Thus $(\theta, \theta^\#) : (U, \mathcal{O}_U) \xrightarrow{\sim} (W, \mathcal{O}_W)$ is an MM⁽¹²⁾ of locally rigid spaces / \mathbb{R} , where $\theta^\#$ is the standard pullback map $\theta^\# : \mathcal{O}_W \rightarrow \theta_* \mathcal{O}_U$ given by $h \mapsto h \circ \theta$.

Similarly $(\gamma, \gamma^\#) : (U', \mathcal{O}_{U'}) \xrightarrow{\sim} (W', \mathcal{O}_{W'})$ an MM.

Suppose now that $f(U) \subseteq U'$ & so $U \subseteq f^{-1}U'$.

Setting $g = \gamma \circ f \circ \theta^{-1} : W \rightarrow W'$, we have a (commute) morphism $(g, g^\#) : (W, \mathcal{O}_W) \rightarrow (W', \mathcal{O}_{W'})$ of locally rigid spaces / \mathbb{R} .

RTP g is smooth & $g^\#$ is the standard pullback of smooth functions given by composing with g .

Let y_j be the coordinate functions on \mathbb{R}^m , $1 \leq j \leq m$; as in the proof of Lemma 1.2, we set $g_j = g^\#(y_j)$, $j=1, \dots, m$ with $g_j \in \mathcal{O}_W(W)$. We define a smooth map $G : W \rightarrow \mathbb{R}^m$ by $G = (g_1, \dots, g_m)$

If $g(P) = (b_1, \dots, b_m)$, then $y_j - b_j \in \mathfrak{m}_{W', g(P)} \forall j$

Since $g^\#$ a local map on the local rings,

$$g^\#(y_j - b_j) = g_j - b_j \in \mathfrak{m}_{W, P} \Rightarrow g_j(P) = b_j \quad \forall j$$

Thus $g(P) = G(P)$ on $W \Rightarrow g = G$ is smooth.

Moreover, by construction $y_j \circ g = y_j \circ G = g_j = g^\#(y_j)$

for $j=1, \dots, m \Rightarrow g^\#$ is the standard pullback of smooth functions

Since $\theta^\#$ & $\psi^\#$ were given by pulling back functions, (13)
 we deduce that $f^\# : \mathcal{O}_Y(U') \rightarrow \mathcal{O}_X(U)$ given by
 $h \mapsto h \circ f$. Since smoothness is a local property, it
 follows that $f : X \rightarrow Y$ is smooth; since
 $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ locally given by composition with f ,
 this is true for any open subset of Y .

(14) We start with an easy lemma; let U be an open
 subset of $V \subset \mathbb{P}^n$, V under projection

Lemma Suppose $h \in k(V)$ is regular on U , then
 $\{h=0\} \subset U$ is Zariski closed.

Pf Can cover U by finitely many opens $U = \bigcup_{i=1}^N U_i$
 for which we can write $h = F_i/G_i$ on U_i , with F_i, G_i
 homogeneous polynomials of same degree & G_i nowhere vanishing on U_i .
 Then $\{h=0\} \cap U_i = \{F_i=0\} \cap U_i$ closed in $U_i \forall i$
 $\Rightarrow \{h=0\} \subset U$ is Zariski closed. \square

Suppose now $\phi : V \rightarrow W$ as given & let f denote the
 corresponding map on topological spaces. We show first that f
 is continuous. Given $P \in V$, we can find a regular $(m+1)$ -tuple
 at P , $(h_0 : h_1 : \dots : h_m)$ s.t. $f(P) = (h_0(P) : \dots : h_m(P))$

The same $(m+1)$ -tuple will then give the image $f(Q)$ for Q in some Zariski open subset U of P . If we can show that $f: U \rightarrow W$ is étale in the Zariski topology, it will follow (continuity being a local property) that $f: V \rightarrow W$ is étale. But given a closed subset $Z \subseteq W$, given by the vanishing of homogeneous polynomials $F_i(Y_0, \dots, Y_m)$, $i=1, \dots, M$, $(f|_U)^{-1}(Z)$ is given by the vanishing of regular functions $F_i(h_0, \dots, h_m)$ on U , the intersection of closed sets (by above Lemma) is hence closed.

Given now an open subset $U \subseteq W$ and $r \in \mathcal{O}_W(U) \subseteq k(W)$ a regular function on U , claim that the composite $r \circ \phi = r \circ f$ is regular on $f^{-1}(U) \subseteq V$ — this is clear: given any $P \in f^{-1}(U)$, can choose h_0, \dots, h_m regular functions at P with $\phi(P) = (h_0(P) : \dots : h_m(P))$.

We choose a representation $r = F/G$, quotient of homogeneous polynomials of same degree for which $G(\phi(P)) \neq 0$.

Thus $\phi^*(r) := r \circ \phi$ is represented by

$$\frac{F(h_0, \dots, h_m)}{G(h_0, \dots, h_m)} \in k(V) \quad \text{which is clearly regular at } P.$$

So required morphism of sheaves of rings $f^\# : \mathcal{O}_W \rightarrow f_* \mathcal{O}_V$ is given by $f^\#(U) : \mathcal{O}_W(U) \rightarrow \mathcal{O}_V(f^{-1}U)$

$$r \longmapsto \phi^*(r) = r \circ \phi.$$

The $(f, f^\#) : (V, \mathcal{O}_V) \rightarrow (W, \mathcal{O}_W)$ is a morphism of locally ringed spaces / k .

Conversely given the ringed spaces (V, \mathcal{O}_V) & (W, \mathcal{O}_W) , we show that a morphism of locally ringed spaces $(f, f^\#)$ between them comes from an everywhere regular rational map $\phi : V \rightarrow W$ in the way described above.

Let $W_0 =$ affine piece of W given by $Y_0 \neq 0$, wlog non-empty. Set $h_i = f^\#(Y_i/Y_0) \in \mathcal{O}_V(f^{-1}W_0)$ for $i \leq m$ (so $h_0 = 1$) — cf proof of Lemma 1.2 in lectures. h_i for $i \leq m$ a rational function on V , regular on $f^{-1}W_0$.

Recall that a rational function is determined by its values on any open (dense) subset of its domain of definition (= points where it is regular). If we take another non-empty affine piece W_i given by $Y_i \neq 0$, then the $(m+1)$ -tuple $(g_0 : \dots : g_m)$ obtained analogously to the $(h_0 : \dots : h_m)$ (where this time $g_i = 1$) differs from $(h_0 : \dots : h_m)$ by multiplying through by a non-zero rational function h determined by $f^\#(Y_0/Y_i) \in \mathcal{O}_V(f^{-1}W_i)$. So we have a well-defined rational map $\phi = (h_0 : h_1 : \dots : h_m)$ from V to \mathbb{P}^m , and from above discussion this is regular on each $f^{-1}(W_i)$, and hence is an everywhere regular map $\phi : V \rightarrow W$. So RTP $f(P) = \phi(P) \quad \forall P \in V$

(18)

This however is the familiar argument, using fact that $(f, f^\#)$ a morphism of locally ringed spaces / k .

Suppose $f(P) = Q = (a_0 : a_1 : \dots : a_m) \in W$, where
why we assume $a_0 = 1$. Set $y_i = Y_i / Y_0$ the coordinate
functions on the affine piece W_0 and $h_i = f^\#(y_i)$ as above
— so $h_0 = 1$ and $h_1, \dots, h_m \in \mathcal{O}_{V, P}$.

Moreover $f^\#(y_i - a_i) \in \mathfrak{m}_{V, P} \Rightarrow h_i(P) = a_i \quad 1 \leq i \leq m$
 $\Rightarrow f(P) = \phi(P)$ as required.

Finally, for any open set $U \subset W$, have

$\mathcal{O}_W(U) \subseteq k(W)$ where $k(W)$ generated by y_1, \dots, y_m .

The fact that $f^\#(y_i) = h_i = y_i \circ \phi = \phi^*(y_i)$ for
 $1 \leq i \leq m$

$\Rightarrow f^\#$ and ϕ^* agree on $\mathcal{O}_W(U)$ as required.

Rk Compare this answer to answer to Q13!