Algebraic Geometry 2017 Ex Sheet 1 Answers

1) Projective Case: If $V = \bigcup U_i$ with $U_i = V \setminus V^h(I_i)$, then $\phi = \bigcap_i V^h(I_i) = V^h(\sum I_i)$. But $\sum_i I_i \lhd k[X_0,\ldots,X_n]$ f.g. by Hilbert's Basis Thm $\Rightarrow$ $\sum_i I_i = \langle f_1,\ldots,f_N\rangle$, where each f_j involves only finitely many of the I_i. Thus $\sum_i I_i$ is a finite sum of some of the I_i & hence V covered by a finite subcollection of the U_i. Affine case entirely similar.

Comment Better to do it this way than use Hilbert 0-satz as it isn't really dependent on k being algebraically closed.

2) Suppose h an everywhere regular function on V; for each $P \in V$, $\exists$ representation $h = f_P/g_P$, with $f_P, g_P \in k[V]$, $g_P(P) \neq 0$. Consider $J \lhd k[V]$ generated by $\{g_P : P \in V\}$. Since $V(J) = \phi$, Hilbert 0-satz $\Rightarrow J = k[V]$ $\therefore \exists P_1,\ldots,P_N \in V$, $q_1,\ldots,q_N \in k[V]$ s.t. $1 = \sum_{i=1}^N q_i g_{P_i}$ $\Rightarrow$
$h = \sum_1^N q_i g_{P_i} h = \sum q_i f_{P_i} \in k[V]$.

On $\mathbb{P}^n$, unique factorization in $k[X_0,\ldots,X_n] \Rightarrow$ $\exists!$ representation of a rational function h as $h = F/G$, F & G coprime & homogeneous of same degree $\Rightarrow$ G has no zeros $\Rightarrow G = \text{const} \Rightarrow h = \text{const}$.

3) $\phi = (f_1,\ldots,f_m) : V \to W \subseteq \mathbb{A}^m$, then $\phi^*(g) = g \circ f \in k[V]$ for $g \in k[W]$.

i.e. represent g by polynomial $G \in k[Y_1, \dots, Y_m]$ &

then $\phi^*(g) = G(f_1, \dots, f_m) \in k[V]$.

(a) Suppose ϕ dominating: given $g \in W$ s.t. $\phi^*(g) = 0$,

have G vanishing at all points of $\phi(V) \Rightarrow G \in I(W)$

$\Rightarrow g = 0$

Conversely, suppose G vanishes at all points of $\phi(V)$,

then $\phi^*(g) = 0 \in k[V] \Rightarrow g = 0 \Rightarrow G \in I(W)$

& so ϕ is dominating

(b) If $\phi : V \xrightarrow{\sim} W' \hookrightarrow W$, then

$\phi^* : k[W] \twoheadrightarrow k[W'] \xrightarrow{\sim} k[V]$, where the first

map is just $k[W] \to k[W]/I(W')$.

Conversely, if ϕ^* surjective, let $I = \ker \phi^* \lhd k[W]$

& $W' = V(I)$. Then ϕ^* induces an IM

$k[W'] = k[W]/I \xrightarrow{\sim} k[V]$ (0-sat $\Rightarrow I(W') = I$)

4) Both of these are isomorphic to $\mathrm{Bil}_A(M \times N, P)$:

Given $\phi \in \mathrm{Bil}_A(M \times N, P)$, universal property of $\otimes$

$\Rightarrow \exists!\ \theta \in \mathrm{Hom}_A(M \otimes N, P)$ s.t. $\phi = \theta \circ \alpha$

(where $\alpha : M \times N \to M \otimes_A N$ is universal bilinear map)

Conversely, given such a θ, have corresponding bilinear

map $\theta \circ \alpha$. Clearly this identification gives an IM of A-modules.

However, given such a $\phi \in \mathrm{Bil}_A(M \times N, P)$, may

define $\tilde{\phi} \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$ by

$\tilde{\phi}(m)(n) = \phi(m, n)$, also clearly an IM of A-modules.

5) $N \otimes_B P$ considered as an A-module via the A-module structure on N & $M \otimes_A N$ as a B-module via B-module structure on N.

Given $z \in P$, have A-bilinear map

$$M \times N \longrightarrow M \otimes_A (N \otimes_B P)$$
$$(x, y) \longmapsto x \otimes (y \otimes z)$$

where $N \otimes_B P$ considered as a A-module & $x \otimes y$ defined to be image of (x, y) under appropriate bilinear universal map.

This induces a HM $\quad M \otimes_A N \xrightarrow{f_z} M \otimes_A (N \otimes_B P)$
$$x \otimes y \longmapsto x \otimes (y \otimes z)$$

which we observe is in fact also B-linear.

Now consider the B-bilinear map

$$(M \otimes_A N) \times P \longmapsto M \otimes_A (N \otimes_B P)$$
$$(t, z) \longmapsto f_z(t).$$

This induces the required map $(M \otimes_A N) \otimes_B P \longrightarrow M \otimes_A (N \otimes_B P)$ under which $(x \otimes y) \otimes z \longmapsto x \otimes (y \otimes z)$.

The inverse is constructed similarly. Note that we have an IM of (A, B)-bimodules

<u>Aliter</u> Define $M \otimes_A N \otimes_B P$ to be universal wrt trilinear maps to (A, B)-bimodules, which is A-linear in the first factor, (A, B)-linear in the second factor & B-linear in the third.

(6) We saw in (0.4) that:

- if $\mathfrak{a} \triangleleft A$, $A/\mathfrak{a} \otimes_A M \cong M/\mathfrak{a}M$
- if S mult subset of A, $S^{-1}A \otimes_A M \cong S^{-1}M$

(make sure you can prove them).

(a) $A/_I \otimes_A (M \otimes_A N) \underset{Q5}{\cong} (A/_I \otimes_A M) \otimes_A N$

$\cong M/_{IM} \otimes_A N \cong (M/_{IM} \otimes_{A/_I} A/_I) \otimes_A N$

$\underset{Q5}{\cong} M/_{IM} \otimes_{A/_I} (A/_I \otimes_A N) \cong M/_{IM} \otimes_{A/_I} N/_{IN}$

(b) $LHS \cong S^{-1}A \otimes_A (M \otimes_A N) \underset{Q5}{\cong} (S^{-1}A \otimes_A M) \otimes_A N$

$\cong S^{-1}M \otimes_A N \cong (S^{-1}M \otimes_{S^{-1}A} S^{-1}A) \otimes_A N$

$\underset{Q5}{\cong} S^{-1}M \otimes_{S^{-1}A} (S^{-1}A \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N.$

7) Given U open in X & $s \in \mathcal{F}(U)$, claim that $\phi(U)(s) = \psi(U)(s)$. This follows since

$$(\phi(U)(s))_x = \phi_x(s_x) = \psi_x(s_x) = (\psi(U)(s))_x \quad \forall x \in U$$

i.e. $\exists$ open nbhd $x \in U_x \subset U$ s.t.

$$\phi(U)(s)|_{U_x} = \psi(U)(s)|_{U_x}$$

Now sheaf condition (A) for $\mathcal{G}$ $\Rightarrow$ $\phi(U)(s) = \psi(U)(s)$.

8) $\mathcal{F} \longrightarrow \mathcal{F}^+$ has universal property that for any morphism $\mathcal{F} \longrightarrow \mathcal{G}$ to a sheaf $\mathcal{G}$, $\exists!$ factorization

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \\ \mathcal{F}^+ \arrow[ur, "\exists!"'] & \end{tikzcd}$$

factorizations

Suppose $\mathcal{F} \longrightarrow \mathcal{F}'$ also has this property. Then get unique

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}' \arrow[d, shift left, "\beta"] \\ & \mathcal{F}^+ \arrow[u, shift left, "\alpha"] \end{tikzcd}$$

Applying uniqueness of factorization for $\mathcal{F} \longrightarrow \mathcal{F}^+$ to diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^+ \\ & \mathcal{F}^+ \arrow[u, "\alpha\beta"'] \end{tikzcd}$$

, deduce $\alpha\beta = id_{\mathcal{F}^+}$

Similarly $\beta\alpha = id_{\mathcal{F}'}$.

9) (a) Have natural morphisms

$$\begin{tikzcd} \mathcal{F} \arrow[dr, "i_1"] & \\ & \mathcal{F} \oplus \mathcal{G} \\ \mathcal{G} \arrow[ur, "i_2"'] & \end{tikzcd}$$

given by $i_1(s) = (s, 0)$, $i_2(t) = (0, t)$.

This has universal property that for any sheaf / presheaf $\mathcal{H}$ & morphisms

$$\begin{tikzcd} \mathcal{F} \arrow[dr, "\phi_1"] & \\ & \mathcal{H} \\ \mathcal{G} \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

, $\exists!$ morphism $\mathcal{F} \oplus \mathcal{G} \xrightarrow{\phi} \mathcal{H}$ s.t. $\phi_1 = \phi \circ i_1$, $\phi_2 = \phi \circ i_2$.

Existence & uniqueness of ϕ is clear:

$$\phi(s, t) = (\phi_1(s), \phi_2(t)).$$

Suppose $\mathcal{S}$ also has this universal property, say

$$\begin{tikzcd} \mathcal{F} \arrow[dr, "j_1"] & \\ & \mathcal{S} \\ \mathcal{G} \arrow[ur, "j_2"'] & \end{tikzcd}$$

, then $\exists$ morphisms $\mathcal{F} \oplus \mathcal{G} \to \mathcal{S}$,
$\mathcal{S} \to \mathcal{F} \oplus \mathcal{G}$
which are mutually inverse with

$$\begin{tikzcd} \mathcal{F} \arrow[dr, "i_1"] \arrow[drr, bend left, "j_1"] & & \\ & \mathcal{F} \oplus \mathcal{G} \arrow[r] & \mathcal{S} \\ \mathcal{G} \arrow[ur, "i_2"'] \arrow[urr, bend right, "j_2"'] & & \end{tikzcd}$$

<u>Remark</u> $\mathcal{F} \oplus \mathcal{G}$ is in fact a biproduct & so $\exists$ morphisms

$$\begin{tikzcd} & \mathcal{F} \\ \mathcal{F} \oplus \mathcal{G} \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & \mathcal{G} \end{tikzcd}$$

, universal wrt diagrams

$$\begin{tikzcd} & \mathcal{F} \\ \mathcal{H} \arrow[ur, "\theta_1"] \arrow[dr, "\theta_2"'] & \\ & \mathcal{G} \end{tikzcd}$$

i.e. factoring thru morphism $\mathcal{H} \to \mathcal{F} \oplus \mathcal{G}$.

(b) Given $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$, a bilinear map of $\mathcal{O}_X$-modules is a morphism of sheaves of abgps $\mathcal{F} \oplus \mathcal{G} \xrightarrow{\phi} \mathcal{H}$ s.t. for U open, the corresponding map $\phi(U)$ is a bilinear map of $\mathcal{O}_X(U)$-modules

Set $T(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$, the <u>presheaf</u> tensor-product.

We define the sheaf tensor product to be T^+ & denote it by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. Clearly $\exists$ $\mathcal{O}_X$-bilinear map of presheaves $\mathcal{F} \oplus \mathcal{G} \to T$ & composing with sheafification, a bilinear map of $\mathcal{O}_X$-modules $\mathcal{F} \oplus \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$

Given any bilinear map $\phi : \mathcal{F} \oplus \mathcal{G} \to \mathcal{H}$ of sheaves, $\exists!$ factorization in cag of presheaves

$$\begin{tikzcd} \mathcal{F} \oplus \mathcal{G} \arrow[r, "\phi"] \arrow[dr] & \mathcal{H} \\ & T \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

& using universal property of $+$, a factorization

$$\begin{tikzcd} \mathcal{F} \oplus \mathcal{G} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{H} \\ & T \arrow[ur, dashed] \arrow[d] & \\ & T^+ \arrow[uur, dashed] & \end{tikzcd}$$

i.e. ϕ factors via T^+ in cag of sheaves

Given a factorization of ϕ via T^+, the composite $T \to T^+ \to \mathcal{H}$ is unique by universal property of presheaf tensor product, & the factorization $T^+ \to \mathcal{H}$ unique by universal property of sheafification. Standard argument then shows uniqueness of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ up to IM

10) Suppose $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves.

- If $\theta : \mathcal{H} \to \mathcal{F}$ s.t. $\phi\theta = 0$, then $\theta(U)\,\mathcal{H}(U) \leq (\operatorname{Ker}\phi)(U)$ $\forall\, U$ & so $\exists!$ map $\bar{\theta} : \mathcal{H} \to \operatorname{Ker}\phi$ s.t.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\theta"] \arrow[dr, "\bar{\theta}"'] & \mathcal{F} \arrow[r, "\phi"] & \mathcal{G} \\ & \operatorname{Ker}\phi \arrow[u] & \end{tikzcd}$$

commutes

- If $\theta : \mathcal{G} \to Q$ s.t. $\theta\phi = 0$, then for U open in X, $\exists!$ factorization of $\theta(U)$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi(U)"] & \mathcal{G}(U) \arrow[r] \arrow[dr, "\theta(U)"'] & \mathcal{C}(U) := \mathcal{G}(U)/\phi(U)\mathcal{F}(U) \arrow[d] \\ & & Q(U) \end{tikzcd}$$

through the presheaf $\mathcal{C}(U)$. Universal property of sheafification $\Rightarrow$ presheaf morphism $\mathcal{C} \to Q$ factors uniquely through $\operatorname{coker}\phi = \mathcal{C}^+ \to Q$ & so $\exists!$ factorization

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\pi"] \arrow[dr, "\theta"'] & \operatorname{Coker}\phi \arrow[d, "\bar\theta"] \\ & Q \end{tikzcd}$$

Given a factorization of ϕ, $\mathcal{F} \xrightarrow{\pi} \mathcal{H} \xrightarrow{\theta} \mathcal{G}$, clearly $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \to \operatorname{Coker}\theta$ is zero & so $\exists!$ factorization

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\pi"] & \mathcal{H} \arrow[r, "\theta"] & \mathcal{G} \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & \operatorname{Coker}\phi \arrow[d, "\alpha"] \\ & & & \operatorname{Coker}\theta \end{tikzcd}$$

Thus $\operatorname{Ker}\pi_1 \subseteq \operatorname{Ker}\pi_2$ i.e.

$\operatorname{Im}\phi \subseteq \operatorname{Im}\theta$. Clearly α an IM $\Rightarrow$ equality

Conversely, equality $\Rightarrow (\operatorname{Im}\phi)_P = \operatorname{Im}\phi_P = \operatorname{Im}\theta_P = (\operatorname{Im}\theta)_P$ $\forall P$

On passing to stalk maps

$$\begin{tikzcd} \mathcal{G}_P \arrow[r] \arrow[dr] & \operatorname{Coker}\phi_P = \mathcal{G}_P/\phi_P\mathcal{F}_P \arrow[d] \\ & \operatorname{Coker}\theta_P = \mathcal{G}_P/\theta_P\mathcal{H}_P \end{tikzcd}$$

deduce that $\alpha_P : (\operatorname{Coker}\phi)_P \xrightarrow{\sim} (\operatorname{Coker}\theta)_P$ $\forall P$

$\Rightarrow \alpha$ is an isomorphism

11) Can define an exact sequence of sheaves

$$0 \to \mathbb{Z} \hookrightarrow \mathcal{O}_U \to \mathcal{O}_U^* \to 1$$
$$f \longmapsto \exp(2\pi i f)$$

This gives an exact sequence on stalks — indeed for $P \in V \subset U$ with V connected & simply conn open nbhd, we have an exact sequence on sections over V, since any $g \in \mathcal{O}_U^*(V)$ comes from $\frac{1}{2\pi i}\int_{P_0}^{P} \frac{g'(w)\,dw}{g(w)}$

ie. $\frac{1}{2\pi i}\log g$. Wlog assume U is connected — otherwise work with connected components. If U also simply connected, we've just observed that the map $\Gamma(U,\mathcal{O}_U) \to \Gamma(U,\mathcal{O}_U^*)$ is surjective

If U not simply connected, $\exists\, a \notin U$ for which we cannot define $\log(z-a)$ ($\exists$ closed curve γ in U with winding number $n(\gamma, a) \neq 0$) and then $\Gamma(U,\mathcal{O}_U) \to \Gamma(U,\mathcal{O}_U^*)$ not surjective. So map on sections is surjective $\Longleftrightarrow$ every connected component of U is simply connected.

12) We embed $\mathbb{P}^n \overset{v_d}{\hookrightarrow} \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$ defined by all the monomials of degree d. This defines an IM of $\mathbb{P}^n$ with its image $\subset \mathbb{P}^N$ (see below); cf Q14 for fact that regular maps $\equiv$ morphisms.

So the <u>Veronese</u> map v_d defined by all monomials of degree d in $X_0, \dots, X_n$. We take homogeneous coords on $\mathbb{P}^N$ given by $x_{i_0 \dots i_n}$ with $\sum i_k = d$.

Define a k-algebra HM

$\theta : k[X_{i_0 \dots i_n} : \sum i_k = d] \longrightarrow k[X_0, \dots, X_n]$ by

$X_{i_0 \dots i_n} \longmapsto X_0^{i_0} \dots X_n^{i_n}$ & set $I = \ker \theta$, a <u>prime</u> ideal since $k[X_{i_0 \dots i_n}]/I \hookrightarrow k[X_0, \dots, X_n]$ an ID.

An elt is in $\ker \theta \iff$ all its homog parts are & so I is a homogeneous ideal. Set $V = V^h(I) \subset \mathbb{P}^N$ irreducible

From definition of I, have $v_d(\mathbb{P}^n) \subset V$.

Note that $\theta\left(X_{i_0 \dots i_n}^d - (X_{d 0 \dots 0})^{i_0} \dots (X_{0 \dots 0 d})^{i_n}\right) = 0$, & so for $P = (x_{i_0 \dots i_n}) \in V$, have $x_{i_0 \dots i_n}^d = (x_{d 0 \dots 0})^{i_0} \dots (x_{0 \dots 0 d})^{i_n}$

Since <u>some</u> coord $x_{i_0 \dots i_n} \neq 0$, at least one of the coords $x_{d 0 \dots 0}, x_{0 d 0 \dots 0}, \dots, x_{0 \dots 0 d}$ non-zero. (1)

Note that $\theta(X_{i_0 \dots i_n} X_{j_0 \dots j_n} - X_{i_0' \dots i_n'} X_{j_0' \dots j_n'}) = 0$ whenever $i_k + j_k = i_k' + j_k'$ $\forall k$, i.e. these quadratics in I (2)

Define birational morphism $\phi : V \longrightarrow \mathbb{P}^n$ by

$(X_{d 0 \dots 0} : X_{d-1\, 1\, 0 \dots 0} : \dots : X_{d-1\, 0 \dots 0\, 1}) \sim (X_{1\, d-1\, 0 \dots 0} : \dots : X_{0\, d-1\, 0 \dots 0\, 1})$

$\sim \dots \sim (X_{1\, 0 \dots 0\, d-1} : X_{0\, 1\, 0 \dots 0\, d-1} : \dots : X_{0 \dots 0 d})$, equivalent representations of same map by (2) & regular at every point of V by (1).

Clearly now $\phi \circ v_d = id_{\mathbb{P}^n}$. However given $P = (x_{i_0 \dots i_n}) \in V$, can assume wlog $x_{d0\dots0} \neq 0$ by (1) & so morphism given at P by first representation.

Observe $X_{d0\dots0}^{d-1} X_{i_0 \dots i_n} - X_{d0\dots0}^{i_0} X_{d-1\,1\,0\dots0}^{i_1} \cdots X_{d-1\,0\dots0\,1}^{i_n} \in I$

& so coords of P satisfy $x_{d0\dots0}^{d-1} x_{i_0 \dots i_n} = x_{d0\dots0}^{i_0} \cdots x_{d-1\,0\dots0\,1}^{i_n}$

So $v_d \circ \phi(P)$ has coords $x_{d0\dots0}^{d-1} x_{i_0 \dots i_n}$

ie. $v_d \circ \phi(P) = P$. $\therefore$ $v_d \circ \phi = id_V$ and $v_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ an embedding with image V as claimed.

The hypersurface $F = 0$ in $\mathbb{P}^n$ corresponds under v_d to a hyperplane section of image $V = v_d(\mathbb{P}^n)$, since F a <u>linear</u> combination of monomials of degree d.

Then $\mathbb{P}^n \setminus V^h(F)$ has natural structure of an affine variety, namely $V \setminus$ corresponding hyperplane section

13) Work in the C^∞ case – complex manifold case analogous

Suppose $f : X \to Y$ a continuous map of manifolds with $\phi = (f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a morphism of the corresponding locally ringed spaces / $\mathbb{R}$.

A chart on X of the form $\theta : U \xrightarrow{\sim} \theta(U) = W \subset \mathbb{R}^n$ for some θ inducing an IM of locally ringed spaces / $\mathbb{R}$.

Similarly a chart on Y of the form $\psi : U' \xrightarrow{\sim} \psi(U') = W' \subset \mathbb{R}^m$ inducing an IM of locally ringed spaces / $\mathbb{R}$.

Thus $(\theta, \theta^\#) : (U, \mathcal{O}_U) \xrightarrow{\sim} (W, \mathcal{O}_W)$ is an IM of locally rigid spaces / $\mathbb{R}$, where $\theta^\#$ is the standard pullback map $\theta^\# : \mathcal{O}_W \to \theta_* \mathcal{O}_U$ given by $h \mapsto h \circ \theta$.
Similarly $(\psi, \psi^\#) : (U', \mathcal{O}_{U'}) \xrightarrow{\sim} (W', \mathcal{O}_{W'})$ an IM.

Suppose now that $f(U) \subseteq U'$ & so $U \subseteq f^{-1}U'$.

Setting $g = \psi f \theta^{-1} : W \to W'$, we have a (composite) morphism $(g, g^\#) : (W, \mathcal{O}_W) \to (W', \mathcal{O}_{W'})$ of locally rigid spaces / $\mathbb{R}$.

<u>RTP</u> g is smooth & $g^\#$ is the standard pullback of smooth functions given by composing with g.

Let y_j be the coordinate functions on $\mathbb{R}^m$, $1 \le j \le m$; as in the proof of Lemma 1.2, we set $g_j = g^\#(y_j)$, $j = 1, \dots, m$ with $g_j \in \mathcal{O}_W(W)$. We define a smooth map $G : W \to \mathbb{R}^m$ by $G = (g_1, \dots, g_m)$

If $g(P) = (b_1, \dots, b_m)$, then $y_j - b_j \in m_{W', g(P)}$ $\forall j$

Since $g^\#$ a local map on the local rings,
$g^\#(y_j - b_j) = g_j - b_j \in m_{W,P} \Rightarrow g_j(P) = b_j \quad \forall j$

Thus $g(P) = G(P)$ on $W \Rightarrow g = G$ is smooth.

Moreover, by construction $y_j \circ g = y_j \circ G = g_j = g^\#(y_j)$ for $j = 1, \dots, m \Rightarrow g^\#$ is the standard pullback of smooth functions

Since $\theta^\#$ & $\psi^\#$ were given by pulling back functions, we deduce that $f^\# : \mathcal{O}_Y(U') \to \mathcal{O}_X(U)$ given by $h \mapsto h \circ f$. Since smoothness is a local property, it follows that $f : X \to Y$ is smooth; since $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ locally given by composition with f, this is true for <u>any</u> open subset of Y.

14) We start with an easy lemma; let U be an open subset of $V \subset \mathbb{P}^n$, V quasi-projective

<u>Lemma</u> Suppose $h \in k(V)$ is regular on U, then $\{h = 0\} \subset U$ is Zariski closed.

<u>Pf</u> Can cover U by finitely many opens $U = \bigcup_{i=1}^N U_i$ for which we can write $h = F_i/G_i$ on U_i, with F_i, G_i homogeneous polynomials of same degree & G_i nowhere vanishing on U_i. Then $\{h = 0\} \cap U_i = \{F_i = 0\} \cap U_i$ closed in U_i $\forall i$ $\Rightarrow \{h = 0\} \subset U$ is Zariski closed. $\square$

Suppose now $\phi : V \to W$ as given & let f denote the corresponding map on topological spaces. We show first that f is continuous. Given $P \in V$, we can find a regular $(m+1)$-tuple at P, $(h_0 : h_1 : \dots : h_m)$ s.t. $f(P) = (h_0(P) : \dots : h_m(P))$

The same $(m+1)$-tuple will then give the image $f(Q)$ for Q in some Zariski open subset U of P. If we can show that $f : U \to W$ is cts in the Zariski topology, it will follow (continuity being a local property) that $f : V \to W$ is cts. But given a closed subset $Z \subseteq W$, given by the vanishing of homogeneous polynomials $F_i(Y_0, \dots, Y_m)$, $i = 1, \dots, M$, $(f|_U)^{-1}(Z)$ is given by the vanishing of regular functions $F_i(h_0, \dots, h_m)$ on U, the intersection of closed sets (by above Lemma) & hence closed.

Given now an open subset $U \subset W$ and $r \in \mathcal{O}_W(U) \subset k(W)$ a regular function on U, claim that the composite $r \circ \phi = r \circ f$ is regular on $f^{-1}(U) \subset V$ — this is clear: given any $P \in f^{-1}(U)$, can choose $h_0, \dots, h_m$ regular functions at P with $\phi(P) = (h_0(P) : \dots : h_m(P))$.

We choose a representation $r = F/G$, quotient of homogeneous polynomials of same degree for which $G(\phi(P)) \neq 0$.

Thus $\phi^*(r) := r \circ \phi$ is represented by

$F(h_0, \dots, h_m)/G(h_0, \dots, h_m) \in k(V)$ which is clearly regular at P.

So required morphism of sheaves of rings $f^\# : \mathcal{O}_W \to f_* \mathcal{O}_V$ is given by $f^\#(U) : \mathcal{O}_W(U) \to \mathcal{O}_V(f^{-1}U)$

$$r \longmapsto \phi^*(r) = r \circ \phi.$$

Then $(f, f^{\#}) : (V, \mathcal{O}_V) \longrightarrow (W, \mathcal{O}_W)$ is a morphism of locally rigid spaces / k.

Conversely given the rigid spaces $(V, \mathcal{O}_V)$ & $(W, \mathcal{O}_W)$, we show that a morphism of locally rigid spaces $(f, f^{\#})$ between them comes from an everywhere regular rational map $\phi : V \to W$ in the way described above.

Let W_0 = affine piece of W given by $Y_0 \neq 0$, wlog non-empty

Set $h_i = f^{\#}(Y_i/Y_0) \in \mathcal{O}_V(f^{-1}W_0)$ for $i \leq m$

(so $h_0 = 1$) — cf proof of Lemma 1.2 in Lectures

h_i for $i \leq m$ a rational function on V, regular on $f^{-1}W_0$.

Recall that a rational function is determined by its values on any open (dense) subset of its domain of definition (= points where it is regular). If we take another non-empty affine piece W_i given by $Y_i \neq 0$, then the $(m+1)$-tuple $(g_0 : \dots : g_m)$ obtained analogously to the $(h_0 : \dots : h_m)$ (where this time $g_i = 1$) differs from $(h_0 : \dots : h_m)$ by multiplying through by a non-zero rational function h determined by $f^{\#}(Y_0/Y_i) \in \mathcal{O}_V(f^{-1}W_i)$. So we have a well-defined rational map $\phi = (h_0 : h_1 : \dots : h_m)$ from V to $\mathbb{P}^m$, and from above discussion this is regular on each $f^{-1}(W_i)$, and hence is an everywhere regular map $\phi : V \longrightarrow W$. So RTP $f(P) = \phi(P)$ $\forall P \in V$

This however is the familiar argument, using fact that $(f, f^{\#})$ a morphism of locally ringed spaces $/k$.

Suppose $f(P) = Q = (a_0 : a_1 : \dots : a_m) \in W$, where wlog we assume $a_0 = 1$. Set $y_i = Y_i/Y_0$ the coordinate functions on the affine piece W_0 and $h_i = f^{\#}(y_i)$ as above — so $h_0 = 1$ and $h_1, \dots, h_m \in \mathcal{O}_{V,P}$.

Moreover $f^{\#}(y_i - a_i) \in m_{V,P} \Rightarrow h_i(P) = a_i \quad 1 \le i \le m$

$\Rightarrow f(P) = \phi(P)$ as required.

Finally, for any open set $U \subset W$, have

$\mathcal{O}_W(U) \subseteq k(W)$ where $k(W)$ generated by $y_1, \dots, y_m$.

The fact that $f^{\#}(y_i) = h_i = y_i \circ \phi = \phi^*(y_i)$ for $1 \le i \le m$

$\Rightarrow f^{\#}$ and ϕ^* agree on $\mathcal{O}_W(U)$ as required.

Rk Compare this answer to answer to Q13 !